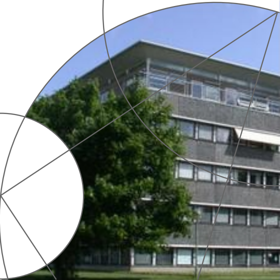
Faculty of Science

The Fundamental Theorem of Derivative Trading: Exposition, Extensions, Empirics

Rolf Poulsen
Department of Mathematical Sciences



Aarhus, December, 2014

Joint with Simon Ellersgaard Nielsen.

Basic question: What happens if we use a simple model to hedge/replicate some financial derivative/an option?

Answer: Not too much if we get the volatility about right. That is the Fundamental Theorem of Derivative Trading. About to become a text-book result.

We give proofs with juuuuust the right balance between rigor and intuition, some historical accounts, some extensions (that I won't go much into), and a empirical application.

The set-up

Dynamics of the underlying: 0-rates for simplicity and

$$dS(u) = \sigma_{actual}(u)S(u)dW(u),$$

where σ_{actual} is some stochastic process. Think of this as something we can't really observe/an illusive quantity (Wilmott)/*das Ding an sich* (Kant).

At time 0 we are given/observe a market price of some plain vanilla option expiring at time T. This price could be expressed in terms of an implied volatility, $\sigma_{implied}$.

Suppose our bank has sold the option and we are now told to hedge the position.

Imagine we do that as if we were in a Black/Scholes model with volatility σ_{hedge}, where this Δ-hedging would work perfectly. (Switch to R and do an experiment.)

The hedge volatility σ_{hedge} is something we the modelers choose.

The punchline: The Fundamental Theorem of Derivative Trading

Our total portfolio value, our profit-and-loss, our P&L at option expiry breaks down as

$$\begin{aligned}\text{P\&L}_T &= V_{implied}(0) - V_{hedge}(0) \\ &\quad + \frac{1}{2}\int_0^T S^2(u)\Gamma(S(u),u)(\sigma^2_{hedge} - \sigma^2_{actual}(u))du,\end{aligned}$$

where Γ is the Black/Scholes-Gamma of the option being hedged.

Proof: `Blackboard.` Careful (if straightforward) application of Ito's formula, the Black-Scholes pricing PDE and the self-financing condition.

The moniker *FToDT* was coined (we think) by Jesper Andreasen.

The first term on the right-hand side is what we get when we sell the option less what it costs to set up the hedge.

The second term, which is "non-explosive", is the hedge error of the dynamic strategy (if you insist on self-financing strategies). Or, alternatively viewed how much our strategy "bleeds" (if you insist on replication).

The theorem says — in quite a quantitative way — that if we Δ-hedge dynamically, then successful derivative trading comes down to estimating/predicting/guessing volatility.

In the Black/Scholes case we get — as we should — arbitrage if $\sigma_{implied} \neq \sigma_{actual}$. (Sanity check.)

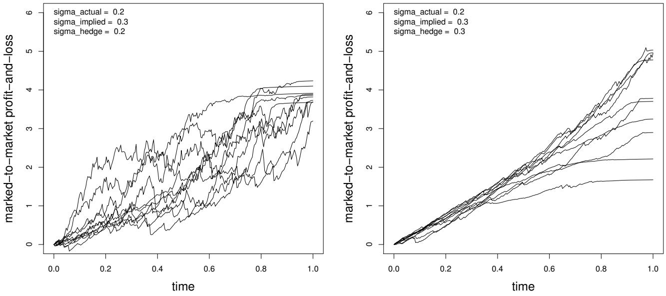
If we can bound volatility, we can super-replicate options (with single-signed Γ — which would be a technical definition of plain-vanilla).

In all commonly used stochastic volatility models, there are no (non-trivial) bounds on volatility. Therefore, El Karoui *et al.* (1998) viewed the result as negative.

Wilmott's Experiment

Suppose $\sigma_{actual} \equiv 0.2$, $\sigma_{implied} = 0.3$ for a(n initially) 1-year at-the-money call-option.

The graph on the left shows paths of (marked-to-market) P&L when hedging with actual volatility, the graph on right shows hedging with implied volatility:



When hedging with actual volatility, our terminal P&L is deterministic.

(Or it would be with continuous hedging. With daily rebalancing the effect of discretization effect is clearly visible.)

But we get to it in an erratic way.

Technically, there's a dW-term in the marked-to-market P&L-dynamics.

When hedging with implied volatility, we don't know exactly which terminal P&L we will get.

But it will be positive (again: up to discretization) and we will get there in a smooth way. Only *du*-terms.

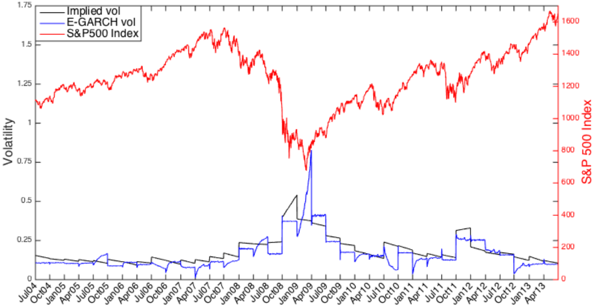
Which situation would you rather be in?

Empirics: What happens if we . . .

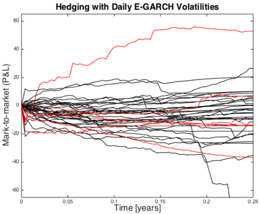
- . . . run the Wilmott-experiment on on real data? I have not seen that. And certainly not on post-2008 data. (Eeeeexcept a talk by Hans Bühler.) "Just" requires good option-price data, not a lot of "model fudging".
- . . . leave the constant hedge-volatility? Then we get complicated (model-dependent) dependence on dynamics of and relation between $\sigma_{implied}$, σ_{actual}, and σ_{hedge}. But still easy to run tests on data.
- . . . use a locally risk-minimizing Δ? (akin to Andreasen's "Wots me Δelδa?")

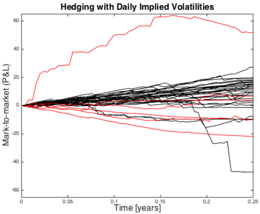
A lot to do; results in the following just deal with the first bullet point.

The data at a glance



The P&L paths





`old numbers --- get new ones from Simon`

The difference in path ruggedness is not as obvious as in the Wilmott experiment . . .

... but let's look at some numbers:

Quantity	Mean (m)	Std. Dev. (sd)	Hypothesis Tests
Hedge error, act. vol.	-6.7	26	**Q:** $m = 0$? **A:** Possibly; p-val. = 13%.
Hedge error, imp. vol.	-6.6	25	**Q**: $m = 0$? **A**: Possibly; p-val. = 13%. **Q**: $sd_{act} = sd_{imp}$? **A**: Yes; p-val. = 89%
Quad. variation, act. vol.	1.6	3.3	**Q**: $m_{QV_{act}} = m_{QV_{imp}}$? **A**: No; p-val. $< 0.1\%$.
Quad. variation, imp. vol.	1.2	2.8	**Q**: $m_{\ln(QV_{act})} = m_{\ln(QV_{imp})}$? **A**: No; p-val. $< 0.1\%$.

The table tells us that . . .

There is no simple way to do volatility arbitrage. I would be surprised if there were.

With only at-the-money options in play, it is even hard to pick up a volatility risk-premium.

If you are concerned only about the terminal hedge error, then hedging with actual and implied volatility are not significantly different in riskiness.

But when hedging with implied volatility, the quadratic variation of the P&L, the ruggedness of the path of the hedge error, is more than halved compared to using (forecasted) actual volatility. I.e. you sleep better at night.

Extension to jumps

In case of jumps, the local behaviour of P&L of a Δ-hedged position is thus:

$$
\begin{aligned}
d\mathsf{P\&L}_t &= \frac{1}{2}(\sigma^2_{hedge} - \sigma^2_{actual})S^2\Gamma dt \\
&\quad + \mathcal{J}\mathsf{P\&L}_t dN(t) - \mathsf{E}^Q_t(\mathcal{J}\mathsf{P\&L}_\tau)\lambda^Q dt
\end{aligned}
$$

If we sell options ("we are short Γ"), then $\mathcal{J}\mathsf{P\&L}_\tau < 0$ by convexity so that in plain English:

- As long as there is no jump, our P&L increases steadily.
- When that is a jump, our P&L takes a hit. And this is irrespective of whether the jump is up or down.

We are picking up pennies in front of a steam train.

That's all she wrote folks!