WWW-NOTES, UNNAMED BUT PROBABLY BY MARK DAVIS
— ADDED COMMENTS BY ROLF POULSEN

Robustness of Black-Scholes Hedging

OR: HEDGING W/ MISSPECIFIED VOLATILITY

If we assume the Black-Scholes price model

$$dS_t = \mu S_t dt + \sigma S_t dw_t \tag{1}$$

then the price at time t of an option with exercise value $h(S_T)$ is $C_h(S_t, r, \sigma, t) = C(t, S_t)$ where $C(t, s)$ satisfies the Black-Scholes PDE

$$\frac{\partial C}{\partial t} + rs\frac{\partial C}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 C}{\partial s^2} - rC = 0, \tag{2}$$

with boundary condition

$$C(T, s) = h(s).$$

NOTE ANY PRICE @ WHICH WE SELL OPTION CAN BE EXPRESSED AS AN IMPLIED VOLATILITY

Suppose we sell an option at implied volatility $\hat{\sigma}$, i.e. we receive at time 0 the premium $C_h(S_0, r, \hat{\sigma}, 0)$, and we hedge under the assumption that the model (1) is correct with $\sigma = \hat{\sigma}$. The hedging strategy is then 'delta hedging': the number of units of the risky asset held at time t is the so-called option 'delta' $\partial C/\partial s$:

$$\phi_t = \frac{\partial C}{\partial s}(t, S_t). \tag{3}$$

Suppose now that the model (1) is *not* correct, but the 'true' price model is

$$dS_t = \alpha(t, \omega) S_t dt + \beta(t, \omega) S_t dw_t, \tag{4}$$

where w_t is an $\mathcal{F}_t$-Brownian motion for some filtration $\mathcal{F}_t$ (not necessarily the natural filtration of w_t) and α_t, β_t are $\mathcal{F}_t$-adapted, say bounded, processes. It is no loss of generality to write the drift and diffusion in (4) as $\alpha S, \beta S$: since $S_t > 0$ a.s. we could always write a general diffusion coefficient γ as $\gamma_t = (\gamma_t/S_t)S_t \equiv \alpha_t S_t$. In fact the model (4) is saying little more than that S_t is a positive process with continuous sample paths.

Using strategy (3) the value X_t of the hedging portfolio is given by $X_0 = C(0, S_0)$ and

(*1)

$$dX_t = \frac{\partial C}{\partial s} dS_t + \left(X_t - \frac{\partial C}{\partial s} S_t\right) r\, dt$$

WHY exactly? USE (*1) TO IMPLICITLY DEFINE X. PUT φ_t IN STOCK AND $\psi_t = (X_t - \varphi_t S_t)/B_t$ in RISK-FREE. $V(\varphi, \psi) = X$ and self-fin holds by construction

where S_t satisfies (4). By the Ito formula, $Y_t \equiv C(t, S_t)$ satisfies

$$dY_t = \frac{\partial C}{\partial s} dS + \left(\frac{\partial C}{\partial t} + \frac{1}{2}\beta^2 S_t^2 \frac{\partial^2 C}{\partial s^2}\right) dt.$$

FOR $t \in]0,T[$ THE HEDGING ERROR INTERPRETATION IS POSSIBLY DUBIOUS — NOT SO FOR $t = 0, T$.

Thus the hedging error $Z_t \equiv X_t - Y_t$ satisfies

$$\frac{d}{dt} Z_t = rX_t - rS_t \frac{\partial C}{\partial s} - \frac{\partial C}{\partial t} - \frac{1}{2}\beta^2 S_t^2 \frac{\partial^2 C}{\partial s^2}.$$

Using (2) and denoting $\Gamma_t = \Gamma(t, S_t) = \partial^2 C(t, S_t)/\partial s^2$, we find that

$$dZ_t = rZ_t dt + \frac{1}{2} S_t^2 \Gamma_t^2 (\hat{\sigma}^2 - \beta_t^2) dt$$

Since $Z_0 = 0$, the final hedging error is (TO CHECK: LOOK AT $N_t = Z_t/B_t$)

(*2)

$$\boxed{Z_T = X_T - h(S_T) = \int_0^T e^{r(T-s)} \frac{1}{2} S_t^2 \Gamma_t^2 (\hat{\sigma}^2 - \beta_t^2) dt.}$$

value of hedge PF @ T

as $Y_T = h(S_T)$ by construction

(*2) is called (by some) "THE FUNDAMENTAL THEOREM OF DERIVATIVE TRADING"

Comments:

This is a key formula, as it shows that successful hedging is quite possible even under significant model error. It is hard to imagine that the derivatives industry could exist at all without some result of this kind. Notice that:

- Successful hedging depends entirely on the relationship between the Black-Scholes implied volatility $\hat{\sigma}$ and the true 'local volatility' β_t. For example, if we are lucky and $\hat{\sigma}^2 \geq \beta_t^2$ a.s. for all t then the hedging strategy (3) makes a profit *with probability one* even though the true price model is substantially different from the assumed model (1), as long as $\Gamma_t \geq 0$, which holds for standard puts and calls.

- The hedging error also depends on the option convexity Γ. If Γ is small then hedging error is small even if the volatility has been underestimated.

- THE FINAL HEDGING ERROR IN (*2) DEPENDS ON THE WHOLE PATH OF THE STOCK PRICE, NOT JUST ON S_T.

- THE RESULT IN (*2) — TO THE BEST OF MY KNOWLEDGE — FIRST APPEARED IN EL KAROUI, JEANBLANC-PIQUE & SHREVE (19 (eqn (6.7) & THEREABOUTS). (ALTHOUGH I'M SURE SOME MAY SEE IT IN BRUNO DUPIRE'S EARLY 90'IES WORK)
- INTERESTING FOLLOW-UP: Ahmad & Wilmott (2005)

REFERENCES

Ahmad, R & Wilmott, P 2005 Which free lunch would you like today, Sir? Delta-hedging, volatility arbitrage and optimal portfolios, *Wilmott magazine* (November), pp. 64–79.

El Karoui, N, Jeanblanc-Pique, M & Shreve, S (1998), Robustness of the Black and Scholes Formula, *Mathematical Finance*, vol. 8, pp. 93-126.