The aim of the problem is to establish the delta-hedging rule and deriving the Black-Scholes PDE for European call options.

Let $T > 0$ and $(W_t)_{0\leq t\leq T}$ be a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$. In the Black-Scholes model the dynamics of the riskless asset is given as the unique solution of the ordinary differential equation

$$dB_t = rB_t dt, \qquad B_0 = 1,$$

where $r > 0$, that is $B_t = e^{rt}$ for any $0 \leq t \leq T$. The dynamics of the risky asset is given as the unique solution of the stochastic differential equation (SDE), for any $0 \leq t \leq T$,

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad S_0 = s,$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.

1 Risk-Neutral Valuation

Consider the $\mathcal{F}_T$-measurable variable $\mathcal{X} = (S_T - K)^+$, where $T > 0$ is the maturity and $K > 0$ is the strike (i.e., European option).

There Exists a Pricing Measure Q

A risk-neutral pricing measure Q is a measure on $\mathcal{F}_T$ satisfying the following two conditions

(i) Q is equivalent to P on $\mathcal{F}_T$.

(ii) The discounted process $(\tilde{S}_t)_{t\in[0,T]}$, $\tilde{S}_t = S_t/B_t$, is a martingale under Q on $[0, T]$.

First I show the existence of an equivalent measure on $\mathcal{F}_T$ and secondly I specify the measure such that $(\tilde{S}_t)_{t\in[0,T]}$ becomes a martingale under the new measure.

Let $\phi \in \mathbb{R}$ be a constant and define the *likelihood process* $(L_t)_{t\in[0,T]}$ (non-negative) by

$$L_t = e^{\phi W_t - \frac{1}{2}\phi^2 t},$$

where $E(L_t) = 1$ since $(W_t)_{t\geq 0}$ is Brownian motion under P, i.e. $L_t \sim \log \mathcal{N}\left(-\frac{1}{2}\phi^2 t, \phi^2 t\right)$. Especially L_T is integrable with respect to the Lebesgue measure, and Radon-Nikodým's theorem yields that I can define a measure Q on $\mathcal{F}_T$ by $dQ = L_T dP$. In fact Q is a probability measure by construction: $Q(\Omega) = E(L_T) = 1$. Note that P and Q are absolute continuous with respect to each other and hence equivalent by definition, and condition (i) is satisfied.

Condition (ii) is only satisfied for a specific choice of the Girsanov kernel ϕ. Using Ito's formula, the dynamics of $\tilde{S}_t$ is given by

$$d\tilde{S}_t = (\mu - r + \phi\sigma)\, \tilde{S}_t dt + \sigma \tilde{S}_t dW_t^*,$$

where $W_t = W_t^* + t\phi$ by definition, and $(W^*)_{0\le t\le T}$ is a Brownian motion under Q (Girsanov). Now choose $\phi = (r-\mu)/\sigma)$ to get

$$d\tilde{S}_t = \sigma \tilde{S}_t dW_t^*,$$

i.e. $(\tilde{S}_t)_{t\in[0,T]}$ is an martingale under Q since $(W_t^*)_{t\in[0,T]}$ is a Brownian motion under Q (Girsanov), and the Ito integral with respect to a Brownian motion is a martingale. I conclude that there exists a risk-neutral measure Q.

Dynamics of S_t under Q

By substituting $dW_t = \frac{r-\mu}{\sigma}dt + dW_t^*$ realize that

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ &= (\mu + r - \mu)\, S_t dt + \sigma S_t dW_t^* \\ &= r S_t dt + \sigma S_t dW_t^*, \end{aligned}$$

and $S_0 = s$. In particular $(S_t)_{t\in[0,T]}$ is also a geometric Brownian motion under Q with initial value s, drift r and volatility σ, i.e. the solution to the above SDE is for $t < T$ given by

$$S_t = s e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^*},$$

where (W_t^*) is a Brownian motion under Q. I will in the following subproblem use the representation

$$\begin{aligned} S_T &= \frac{S_T}{S_t} S_t \\ &= S_t e^{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W_T^* - W_t^*\right)} \end{aligned}$$

and use that S_t is measurable with respect to $\mathcal{F}_t$ while $W_T^* - W_t^*$ is independent of $\mathcal{F}_t$.

Price of European Option

Risk-neutral valuation yields that the arbitrage-free price of $\mathcal{X}$ at time $t < T$ is given by

$$\begin{aligned} C(t, S_t) &= e^{-r(T-t)} E^Q\left(\mathcal{X} \mid \mathcal{F}_t\right) \\ &= e^{-r(T-t)} E^Q\left(1_{(S_T > K)}(S_T - K) \mid S_t\right) \\ &= \underbrace{e^{-r(T-t)} E^Q\left(1_{(S_T > K)} S_T \mid S_t\right)}_{(\star)} - \underbrace{e^{-r(T-t)} K E^Q\left(1_{(S_T > K)} \mid S_t\right)}_{(\star\star)}. \end{aligned}$$

The $(\star\star)$-term is easily computed since I know the explicit form of S_t under Q. Let $S_t = s$ and realize that

$$\begin{aligned}
(\star\star) &= e^{-r(T-t)} KQ\left(S_T/S_t > K/S_t \mid S_t\right) \\
&= e^{-r(T-t)} KQ\left(e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma(W_T^*-W_t^*)} > K/S_t \mid S_t\right) \\
&= e^{-r(T-t)} KQ\left(\frac{W_T^*-W_t^*}{\sqrt{T-t}} > \frac{\log(K/S_t) - \left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= e^{-r(T-t)} KN(d_2),
\end{aligned}$$

where $N(\cdot)$ is the CDF for the (standard) normal distribution, while

$$\begin{aligned}
d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left(\log(S_t/K) + \left(r+\frac{1}{2}\sigma^2\left(\right. T-t\right)\right) \\
d_2 &= d_1 - \sigma\sqrt{T-t}.
\end{aligned}$$

Note that I in the above computations have used independent and stationary increments of the Q-Brownian motion, the fact that $\frac{W_T^*-W_t^*}{\sqrt{T-t}} \sim \mathcal{N}(0,1)$ under Q as well as the equality $N(-z) = 1 - N(z)$.

For the $(\star)$-term introduce $Z = \alpha + \beta U$ where $\alpha = \left(r - \frac{1}{2}\sigma^2\right)(T-t)$, $\beta = \sigma\sqrt{T-t}$ and $U \sim \mathcal{N}(0,1)$ with density f_U. Thus Z is a scale-location transformation of U with density $f_Z(z) = \frac{1}{\beta} f_U\left(\frac{z-\alpha}{\beta}\right)$ and CDF $F_Z(z) = N\left(\frac{z-\alpha}{\beta}\right)$, where $N(\cdot)$ is the CDF for the standard normal distribution. Note $\alpha + \frac{1}{2}\beta^2 = r(T-t)$ and see that

$$\begin{aligned}
(\star) &= e^{-r(T-t)} S_t \int 1_{(S_t e^Z > K)} e^Z dQ \\
&= e^{-r(T-t)} S_t \int_{\log(K/S_t)}^{\infty} e^z f_Z(z) dz \\
&= e^{-r(T-t)} S_t \int_{\log(K/S_t)}^{\infty} \frac{e^z}{\beta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-\alpha}{\beta}\right)} dz \\
&= S_t \frac{1}{\beta} \int_{\log(K/S_t)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-(\alpha+\beta^2)}{\beta}\right)^2} dz \\
&= S_t N\left(\frac{-\log(K/S_t) + \left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right),
\end{aligned}$$

where I have used $\frac{1}{2\beta^2}\left(z^2 - 2z\alpha + \alpha^2 - 2\beta^2\alpha\right) = \frac{1}{2\beta^2}\left(z - (\alpha+\beta^2)\right)^2 + \alpha + \frac{1}{2}\beta^2$, $F_Z(z) = N\left(\frac{z-\alpha}{\beta}\right)$ and $N(-z) = 1 - N(z)$. I conclude that the arbitrage-free price for the European call at time t is given by

$$C(t, S_t) = S_t N(d_1) - e^{-r(T-t)} KN(d_2),$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left(\log(S_t/K) + \left(r + \frac{1}{2}\sigma^2\left(T-t\right)\right)\right.$$
$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

This price is unique

The first theorem of mathematical finance states that the Black-Scholes model is free of arbitrage since there exists an equivalent martingale measure (Q). Thus, I can use this measure to price options in the model. The second fundamental theorem states an arbitrage-free model is complete if and only if the pricing measure is unique. Since the Black-Scholes model is complete Q is the unique equivalent martingale measure. Consequently the arbitrage-fee price in the previous section is unique: the price is determined by a conditional expectation under a unique measure.

2 Dynamics of the self-financing portfolio

Consider the self-financing trading strategy $h(t) = (h_t^0, h_t^1)_{0\leq t\leq T}$ with value process $(V_t = V_t^h)_{0\leq t\leq T}$. The value of the self-financing portfolio is given by

$$V_t = h_t^0 B_t + h_t S_t$$

for $t \in [0, T]$.

Dynamics of V_t under the self-financing condition

Note that

$$\begin{aligned} V_t &= h_t^0 B_t + h_t^1 S_t \\ &= h_0^0 B_0 + h_0^1 S_0 + \int_0^t h_u^0 dB_u + \int_0^t H_u^1 dS_u \qquad a.s. \end{aligned}$$

Thus, the P-dynamics of V_t is given by

$$\begin{aligned} dV_t &= h_t^0 dB_t + h_t^1 dS_t \\ &= h_t^0 r B_t dt + h_t^1 \left(\mu S_t dt + \sigma S_t dW_t\right) \\ &= (h_t^0 r B_t + h_t^1 \mu S_t)dt + h_t^1 \sigma S_t dW_t \end{aligned}$$

with initial value $V_0 = h_0^0 B_0 + h_0^1 S_0$.

Dynamics of the self-financing portfolio under Q

By substituting $dW_t = \frac{r-\mu}{\sigma}dt + dW_t^*$ realize that

$$\begin{aligned} dV_t &= (h_t^0 rB_t + H_t \mu S_t)dt + h_t^1 \sigma S_t dW_t \\ &= (h_t^0 rB_t + h_t^1 rS_t)dt + h_t^1 \sigma S_t dW_t^* \\ &= rV_t dt + h_t^1 \sigma S_t dW_t^*. \end{aligned}$$

where (W_t^*) is a Brownian motion under Q.

3 Delta-Hedging and Black-Scholes PDE

I will consider the scenario where we acquire the option and want to Delta-hedge this position. The money account will be chosen such that net value of the portfolio is zero, i.e. the strategy becomes self-financing by construction.

Dynamics of $e^{-rt}C(t, S_t)$ under Q

Ito's formula on the process $e^{-rt}C(t, S_t)$ using the Q-dynamics of S_t yields that

$$\begin{aligned} d\left(e^{-rt}C(t,S_t)\right) &= e^{-rt}\Big\{ \left(C_t(t,S_t) - rC(t,S_t)\right)dt + C_s(t,S_t)dS_t + \frac{1}{2}C_{ss}(t,S_t)(dS_t)^2 \Big\} \\ &= e^{-rt}\Big\{ \left(C_t(t,S_t) + rS_tC_s(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{ss}(t,S_t) - rC(t,S_t)\right)dt \\ &\qquad + \sigma S_t C_s(t,S_t)dW_t^* \Big\}, \end{aligned}$$

where C_t is the derivative with respect to time, C_s is the derivative with respect to state and C_{ss} is the second-derivative with respect to the state.

Dynamics of $e^{-rt}V_t$ under Q

The P^*-dynamics of $e^{-rt}V_t$ is given by

$$\begin{aligned} d\left(e^{-rt}V_t\right) &= -re^{-rt}V_t dt + e^{-rt}dV_t \\ &= -re^{-rt}V_t dt + e^{-rt}\left(rV_t dt + h_t^1 \sigma S_t dW_t^*\right) \\ &= e^{-rt}h_t^1 \sigma S_t dW_t^*, \end{aligned}$$

i.e. the discounted value-process is a P^*-martingale.

Determine h_t^1

I want to choose h_t^1 such that the self-financing strategy $h(t)$ replicates the option. One approach is to consider a long position in $\mathcal{X}$ and h_t^1 short in S_t. This portfolio must satisfy the following two conditions:

i) The portfolio must me riskless (no dW_t^*-term)

ii) The growth-rate of the portfolio must be equal to the risk-free rate r.

Therefore, consider the dynamics of the process $\left(e^{-rt}(C(t,S_t)-V_t)\right)$

$$
\begin{aligned}
d\left(e^{-rt}(C(t,S_t)-V_t)\right) &= d\left((e^{-rt}C(t,S_t)\right) - d\left((e^{-rt}V_t\right) \\
&= e^{-rt}\Bigg\{ \left(C_t(t,S_t) + rS_tC_s(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{ss}(t,S_t) - rC(t,S_t)\right) dt \\
&\quad + \sigma S_t \left(C_s(t,S_t) - H_t\right) dW_t^* \Bigg\}.
\end{aligned}
$$

In order to satisfy condition i), I must choose h_t^1 such that the dW_t^*-term cancels out, i.e. $h_t^1 = C_s(t,S_t)$ as promised. Moreover, since h_t^0 is chosen such the net position is zero,

$$h_t^0 = \frac{C_s(t,S_t)S_t - C(t,S_t)}{B_t}.$$

Derive the Black-Scholes partial differential equation with terminal condition

Let $h_t^1 = C_s(t,S_t)$ and realize that $d\left(e^{-rt}(C(t,S_t)-V_t)\right) = 0$ in order to satisfy ii) stated in the previous subsection (another approach would be to compute the Q-dynamics of $C(t,S_t) - h_t^1 S_t$ and equate the expression with $r(C(t,S_t) - h_t^1 S_t)dt$ where $h_t^1 = C_s(t,S_t)$). This immediately yields that

$$e^{-rt}\left(C_t(t,S_t) + rS_tC_s(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{ss}(t,S_t) - rC(t,S_t)\right) dt = 0$$

and dropping the dt-term yields that

$$C_t(t,S_t) + rS_tC_s(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{ss}(t,S_t) - rC(t,S_t) = 0$$

for all $t \in [0,T]$ with probability 1. Furthermore, since $h(t)$ is replicating $\mathcal{X}$, no-arbitrage implies that $C(T,S_t) = V_T = (S_T - K)^+$ with probability 1. Note that S_t has support on $(0,\infty)$ implying that the above PDE should hold for all $t \in [0,T]$ and $s \in (0,\infty)$ for $S_t = s$. Summarizing, the arbitrage-free price of the European call is determined by the unique function $C : [0,T] \times (0,\infty)$ solving the PDE (boundary problem)

$$
\begin{aligned}
C_t(t,s) + rsC_s(t,s) + \frac{1}{2}\sigma^2 s^2 C_{ss}(t,s) - rC(t,s) &= 0 \\
C(T,s) &= (s-K)^+
\end{aligned}
$$

as promised.