UNIVERSITY OF COPENHAGEN

Mandatory Bachelor Project

Math-Econ Programme

Financial Modeling with Continuous-Time Models

Theory and Applications

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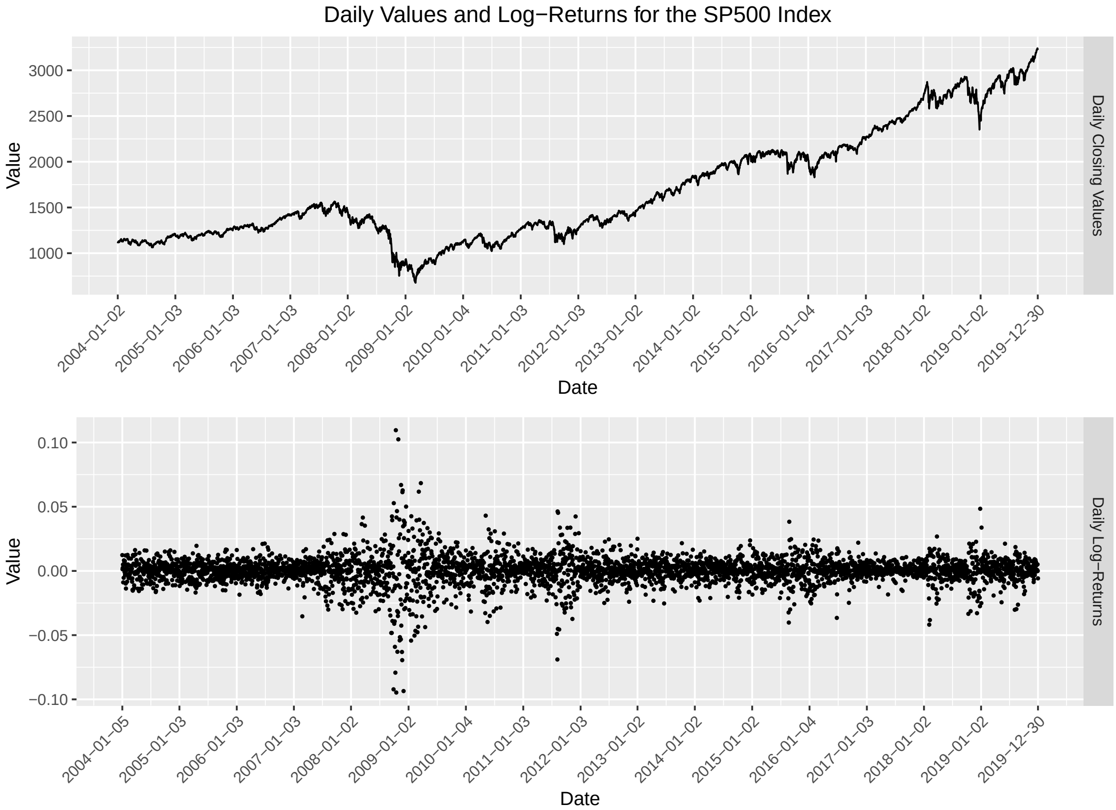

Figure 1: The top plot shows daily closing values for the SP500 index between 2004/01/02-2019/12/30. Corresponding log-returns are represented by the black points in the lower plot. Source: https://finance.yahoo.com/quote/%5EGSPC/.

1 Introduction

The financial market consists of a risky asset (a stock) and a risk-free asset (e.g., zero-coupon bond), both traded continuously up to some fixed time horizon T. As usual, the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{0 \leq t \leq T}, P\right)$ is introduced to capture the information flow: Ω contains all possible states of the market, $\mathcal{F}$ is the corresponding σ-algebra, P is the physical measure, and $(\mathcal{F}_t^W)$ is the σ-field generated by the Brownian motion $(W_t)_{0 \leq t \leq T}$, i.e.,

$$\mathcal{F}_t^W = \sigma \left\{ W_s \mid 0 \leq s \leq t \right\}. \tag{1}$$

In the remainder of this project, S_t denotes the value of the risky asset at time t, and the process is assumed to solve the stochastic differential equation (SDE)

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t \tag{2}$$

$$S_0 = s > 0 \tag{3}$$

where $\mu(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are deterministic functions possibly depending on both time (t) and the state of the underlying (S_t). Moreover, $\sigma(t,s) > 0$ for all $(t,s) \in [0,T] \times \mathbb{R}^+$, and both $\mu(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are sufficiently well-behaved such that a solution to the above SDE exists. We will use following terminology: $\mu(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are called the drift and volatility, respectively, of (S_t), whereas dS_t is referred to as the dynamics of the process.

The risk-free asset's value through time is given by the process $(B_t)_{0\le t\le T}$ with dynamics

$$dB_t = rB_t dt \tag{4}$$
$$B_0 = 1 \tag{5}$$

where r is constant. The risk-free asset is interpreted as the money account with short rate of interest r.

Questions: Model Assumptions

The economic interpretation of the above setting is crucial for working with continuous-time models in practice. Answering the questions below will provide some intuition behind the model assumptions. I would recommend you to read chapters 2 and 4 in Björk (2009).

1.1

Discuss the term *risk*. What is the difference between a risk-free and a risky asset? What would a locally risk-free asset be? How many sources of risk appear in this model setting? Is $\mathcal{F}_t^W = \mathcal{F}_t^S$? What is the financial interpretation of the $\mathcal{F}_t^S$.

1.2

Explain the economic interpretation of $D_t := B_t/B_T$. Is $S_0 > 0$ a strong assumption? Illustrate B_t, D_t, S_t and $D_t S_T$. You can use the following pseudo-code for generating (S_t) (for fun: adjust n as well as the *seed*):

```
# Parameters
S <-              # Starting value of process
mu <- 0.05        # drift
sigma <- 0.2      # volatility

# Time grid
T <- 1            # End time
n <-              # Number of evaluations
dt <- T/(n)       # Equidistant time step

S_vec <- numeric(n)

for (i in 1:n){

  Z <-            # Generate value from normal distribution with mean=0 and var=1
```

```
  dS <-          # Compute S_i - S_{i-1}

  S <- S + dS # Update S

  S_vec[i] <- S
}
```

1.3

Is it reasonable to assume that stock prices are represented by continuous processes? Discuss the (dis)advantage of using continuous-time models for representing stock prices (e.g., discuss the trade off between numerical tractability and modeling reality/financial markets).

2 Geometric Brownian Motion

Assume that

$$S_t = S_0 e^{\left(\mu - \sigma^2/2\right)t + \sigma W_t} \tag{6}$$

where $S_0 = s > 0$ is the (time-0) starting value of the process. The process in equation (6) is called a Geometric Brownian Motion (GBM) and is closely linked to equations (2)-(3), although we can't prove that until we get Ito's formula in our toolbox (next topic). The Geometric Brownian Motion is defined and discussed by Björk (2009) in chapter 5.

Questions: Geometric Brownian Motion

2.1

Is (S_t) continuous? Illustrate (S_t) using pseudo-code from the box. Also compare this simulation technique with Euler's method and build-in function from sde-package respectively.

```
for (i in 1:n){
  Z <- # Generate value from normal dist. with mean=0 and var=1
  X <- # log(S_i/S_{i-1})
  S <- # Update S
  S_vec[i] <- S
}

# Or without the for-loop....
Z <- # Generate n values from multi dim. normal dist.
X <- cumsum(...) # Argument should be a n-dim vector
S <- # Generate GBM
```

2.2

Compute $E(S_t)$ and $V(S_t)$. What is the distribution of S_t? Also compute $E(S_t \mid \mathcal{F}_s^S)$ for $s < t$ (hint: realize that $S_t = S_s \times S_t/S_s$). Is (S_t) a martingale under P?

2.3

Let $R_t := (S_t - S_{t-1})/S_t$ denote the *return* over one period. Determine the distribution of the associated log-return $r_t := \log(S_t/S_{t-1})$. How is R_t linked with r_t? Explain the convenience of using log-returns when working with financial time-series.

2.4

Would you prefer holding the risk-free asset or the risky-asset?

2.5

Produce figure 1. Are data a realization of a Geometric Brownian Motion (i.e., discuss whether model assumptions for GBM are violated)?

2.6

Derive the maximum likelihood estimator of μ and σ. Estimate μ and σ for the sample used to generate figure 1. Finally, discuss a significant shortfall for the maximum likelihood estimator of the drift μ (e.g., you may support your arguments with data).

3 Itǒ Calculus and the Black-Scholes model

Itǒ calculus and in particular Itǒ's formula play a crucial role in mathematical finance. The following results are from chapter 4 in Björk (2009) (p. 49-62, read it!).

Theorem 3.1 (Itǒ's formula) *Assume that the process $X = (X_t)_{t\geq 0}$ has dynamics given by*

$$dX_x = \mu_t dt + \sigma_t dW_t, \tag{7}$$

where μ_t and σ_t are adapted processes, and let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be $C^{1,2}$-function. Introduce a new process by $Z_t = f(t, X_t)$. Then $Z = (Z_t)_{t\geq 0}$ has dynamics given by

$$dZ_t = df(t, X_t) = \left(f_t(t, X_t) + \mu_t f_x(t, X_t) + \frac{1}{2}\sigma_t^2 f_{xx}(t, X_t) \right) dt + \sigma_t f_x(t, X_t) dW_t, \tag{8}$$

where f_t, f_x and f_{xx} denote the derivative with respect to time, the derivative with respective to space and the second-derivative with respective to space, respectively.

Another formulation of Ito's formula is stated in the following proposition — you may find this version useful!

Proposition 3.2 *With the assumptions as in 3.1, dZ_t is given by*

$$dZ_t = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, X_t)(dX_t)^2, \tag{9}$$

where

$$(dt)^2 = 0, \qquad dt \cdot dW_t = 0 \qquad \textit{and} \qquad (dW_t)^2 = dt.$$

Finally, the multidimensional version of Ito's formula is given in definition 4.16 in Björk (2009).

Questions: Ito Calculus

Itŏ

3.1

State a condition such that the process $(X_t)_{t\geq 0}$ with dynamics given by equation (7) becomes a P-martingale. What do we call X when $\mu_t \geq 0$ P-almost surely? Finally, derive/state the condition such that $Z_t = f(t, X_t)$ becomes a P-martingale.

3.2

Compute the dynamics of $(S_t)_{t\geq 0}$ where S_t is given by equation (6). Moreover, apply Ito's formula to compute $E(S_t)$.

From now on, we will specify $(S_t)_{t\geq 0}$ by its stochastic differential and not equation (6).

3.3

Compute the dynamics of $(Z_t)_{t\geq 0}$ where $Z_t = S_t/B_t$. Hint: Use the multidimensional Ito formula from Björk (2009). Is the process a martingale?

4 Black-Scholes Model

The Black-Scholes model consists of two assets with dynamics given by

$$dB_t = rB_t dt, \tag{10}$$
$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{11}$$

where $B_0 = 1$ and $S_0 = s > 0$, and r, μ and $\sigma > 0$ are constants. This famous model was (and to some degree still is) used for pricing T-claims. Following definition 7.1 in Björk (2009), the stochastic variable $\mathcal{X}$ is called a T-claim if it's $\mathcal{F}_T^S$-measurable. Here T is the date of maturity (or exercise date). A simple T-claim is on the form $\mathcal{X} = \Phi(S_T)$, where $\Phi(\cdot)$ is the contract function.

The focal point is to price $\mathcal{X}$ at any point in time $t < T$, i.e. determine the pricing function $\pi(t; \mathcal{X})$ for $t < T$ with terminal condition $\pi(T; \mathcal{X}) = \mathcal{X}$. The Black-Scholes model is free of arbitrage as well as complete (we will discuss both concepts next time), and we will also assume that

$$\pi(t; \mathcal{X}) = F(t, S_t),$$

where F is a $C^{1,2}$-function.

Pricing in this type of models is conducted under a *risk-neutral* probability measure (also called equivalent martingale measure (EMM), see definition) Q.

Questions: Black-Scholes Model

Itǒ

5 Hedging and Pricing

6 Monte Carlos Methods

A) The Fundamental Theorem of Derivative Trading

B) Stochastic Volatility

References

Björk, T. (2009). *Arbitrage Theory in Continuous Time* (3 ed.). Oxford University Press.