FACULTY OF SCIENCE
UNIVERSITY OF COPENHAGEN

**Thesis for the Master degree in Mathematics-Economics**
**Department of Mathematical Sciences**

**University of Copenhagen**

Laurs Randbøll Leth

# The Fundamental Theorem of Derivative Trading

Theory and Applications

**Academic advisor**

Professor Rolf Poulsen, IMF University of Copenhagen

Submitted: August 3, 2015

## Acknowledgement

I am especially grateful to my supervisor Professor Rolf Poulsen for his support and help during my work on this thesis. Prior, I was among the very few math-econ students at the Department of Mathematical Sciences who have never written a thesis supervised by Rolf. It has been a pleasure. I would like to thank my fellow student Mikael Kristoffersen for all the useful discussions we have had relating to the work within this thesis. Finally, I will express a heartfelt thanks to Silas Furu Friby, Nicolai Bjerre Pedersen and Simon Hartvig Rasmussen; My course mates throughout the study. For friendship, help and memorable moments.

Laurs Randbøll Leth

Department of Mathematical Sciences
University of Copenhagen
Copenhagen, Denmark
August, 2015

**Abstract**

It is an empirical fact that trading European options yields a non-zero volatility risk-premium and thereby potential positive profits. One could try to collect the premium developing a volatility trading strategy involving the option and the underlying stock. This approach leads to a non-zero profit-loss at the expiry of the option, and in the case of a delta hedged option the profit-loss is referred to as the Fundamental Theorem of Derivative Trading. We have derived the Fundamental Theorem of Derivative Trading for three different benchmark models in the financial literature: the Black-Scholes model, Merton's jump-diffusion model and Heston's stochastic volatility model. The latter two are incomplete implying a non-zero risk-premium. Motivated by this, we derive risk-minimizing strategies. In addition, we derive a valuation formula and hedging strategies for Bates' stochastic volatility jump-diffusion model. Forecasts of actual volatility are made in order to investigate the empirical performance of the strategies using data from the S&P500 index. Positive risk-premiums are detected, and results show that risk-minimizing strategies for Heston's model respectively Bates' model are preferred due to the standard deviations of terminal hedge errors and the behaviour of the hedging portfolios' quadratic variations during their lifetimes.

# Contents

# List of Figures

# List of Tables

# Chapter 1

# Introduction

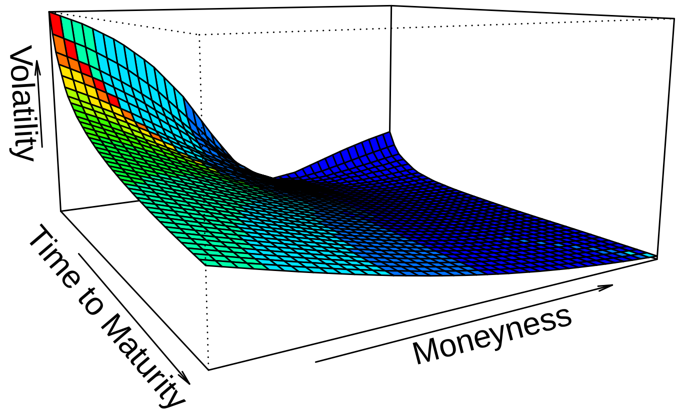
The aim of this thesis is to investigate the outcome of hedging *European options* under a significant model error. A European option is one of the most famous members in the family of financial instruments that is defined as the obligation of one party to transfer something of value to another party. The transferred value is often money and within the agreement lies a future date $T$ where the payment will be made as well as some conditions under which the payment will be made. Pricing and hedging of financial instruments play a major role within the area of mathematical finance and are closely related to the development of stochastic integration theory during the 20th century. The center cornerstone in both disciplines is a mathematical modelling of the Brownian motion named after the botanist Robert Brown who in 1827 observed random movements of pollen particles floating in water. It was within physics the Brownian motion first became influential through one of Albert Einsteins four famous papers in the year 1905, but actually the French mathematician Louis Jean-Baptiste Alphonse Bachelier already in 1900 derived a model in his PhD thesis "The Theory of Speculation" to describe fluctuating prices over time of the Paris stock market using independent Gaussian variables. For inscrutable reasons, the impact of Bachelier's work first came in the last half of the 1900s where the American economist Paul Samuelson adopted the ideas of Bachelier. Samuelson proposed (Samuelson, 1965) that the famous geometric Brownian motion was a good choice for modelling stock prices and furthermore, he derived valuation formulas for European and American options assuming that the *Axiom of mean expectation* is fulfilled, i.e. that the discounted pay-off of an option is a martingale. The major breakthrough came with the paper (Black and Scholes, 1973) deriving a risk-neutral valuation of European options using a no-arbitrage argument in terms of dynamically adjusted hedging portfolios. The model consists of one risk-free asset, $B$, with constant rate of return (for example a bond) and one risky asset (typically a stock), $S$, following a geometric Brownian motion, i.e. the log return of the risky asset follows a normal distribution with mean and variance determined by the drift respectively the volatility, both assumed constant. However, the basic assumptions in Black-Scholes are violated by the *volatility skew* (or the volatility smile): The correspondence between the Black-Scholes price and the volatility is bijective; Thus, the pricing formula can be inverted to obtain an expression for the volatility as a function of the price, the rate of return of the risk-free asset, the strike price and the time to expiry for the option. Hence observing a market price and market parameters for a given European option yields the *implied* volatility - *wrong number which, plugged into the wrong formula, gives the right answer* as it is said in (Rebonato, 2009). The implied volatility should not be confused with the *actual volatility* associated with the amount of noise in the risky asset. We would - under the assumption that the Black-Scholes model is correct - expect the implied volatility to be equal among European options written on a risky-asset with same exercise date but with different strike prices. This was more or less the situation before the stock market crash in 1987/10/19 where the S&P500 index dropped 22.6%! Subsequently, empirical observations have shown that the implied volatility is not constant as a function of the strike price. In addition, we also observe characteristic differences in the implied volatility for options with same strike price but different maturities; A term structure of volatility. It is often useful to express implied volatility as a function of both maturity and strike for a fixed time $t$ - the implied volatility surface at time $t$. Three main results about the implied volatility surface are stated in (Cont and Tankov, 2004):

*Smiles and skews* For European options with fixed expiration, the implied volatility shows a strong dependence with respect to strike price. The curve combining implied volatility and the strike price is either decreasing (skew) or shaped as a smile. The latter arises when at-the-money options have higher implied volatility than out-the-money options.

*Flattening of the smile* The above dependence between implied volatility and the strike price decreases with maturity. Thus, the skew (or smile) is more clearly seen for short-term options than for long-term options.

*Floating smiles* Define the moneyness of the option as strike price relative to the stock price at a fixed time $t$. The described volatility patterns vary less if implied volatility is expressed in terms of moneyness instead of the absolute strike price.

(a) Volatility surface for the S&P500 index (2004/01/29)




(b) Contour plot of implied volatilities for the S&P500 index (2004/01/29)

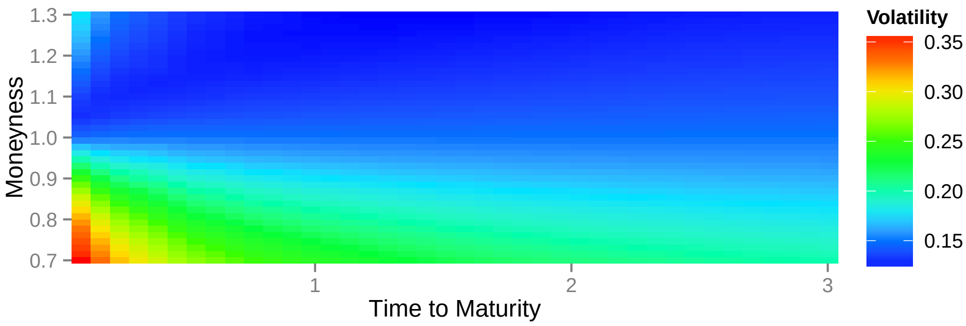



Figure 1.1: *Figure (a) illustrates the implied volatility surface on the date (2004/01/29) while figure (b) is the contour plot corresponding to values in figure (a). Implied volatilities are expressed through the expiry $T$ and the moneyness $K/S$ where $K$ is the strike price and $S = 1134.11$ is the spot value. We consider options with expiries ranging between* 1 *and* 36 *months (annualised) and relative strikes ranging between* 1.3 *and* 0.7 *leading to a total number of* 182 *observed implied volatilities. A bivariate liner interpolation of the volatility surface is done using the R-package Akima (Gebhardt, 2015).*

After 1987, a massive increase was seen in the complexity of financial models used to describe the financial market. Among those was the class of local volatility models, where both drift and volatility of the stock can be expressed through sufficiently well-behaved functions depending on time and the price of the stock. The local volatility model is sometimes referred to as the generalized Black-Scholes model and it is a well-known result that this model implies completeness (Björk, 2009, Chapter 8), which means that the European option can be hedged perfectly by a self-financing portfolio containing the stock and the risk-free asset. However, this is not the market reality. Hedging options involves some amount of risk, which cannot be eliminated by the underlying, and trading options is used to transfer risk among the investors - the equity market is not complete. A second critical point is the model's lack of ability to describe *tail events* of the empirical distribution of log returns in stock prices, which exhibit much heavier tails than the Gaussian distribution can model[1]

Another approach to modelling the financial market is to assume that the volatility itself is stochastic, as in the class of stochastic volatility models. These models enable us to describe a correlation between the stock and its volatility, and furthermore they can produce realistic implied volatility patterns for longer maturities. Especially the family of generalized autoregressive conditional heteroskedasticity models (GARCH) have become popular the last decade in order to forecast actual volatility. A forecast is of great importance for risk management as well as for the private investor: The former needs to estimate future losses of portfolios while the latter wants to build a proper trading strategy involving the actual volatility. As in the Black-Scholes universe, a (simple) stochastic volatility model has difficulties modelling the observed tail events of log returns. Moreover, neither of the models can describe large sudden moves in the price - extreme unpredictable market movements, interpreted as jumps, which the Brownian motion cannot capture due the continuity[2]. And during financial crises, the price behaviour of the stock exhibits large downward jumps.

[1]The local volatility model is **not** a Gaussian process but the noise is driven by a Brownian motion. Hence, we may find a non-linear volatility function that can model the heavy tails observed for log returns of the stock but also implies major fluctuations for the diffusion coefficient.

[2]More correctly, due to the existence of a continuous version of the Brownian motion.

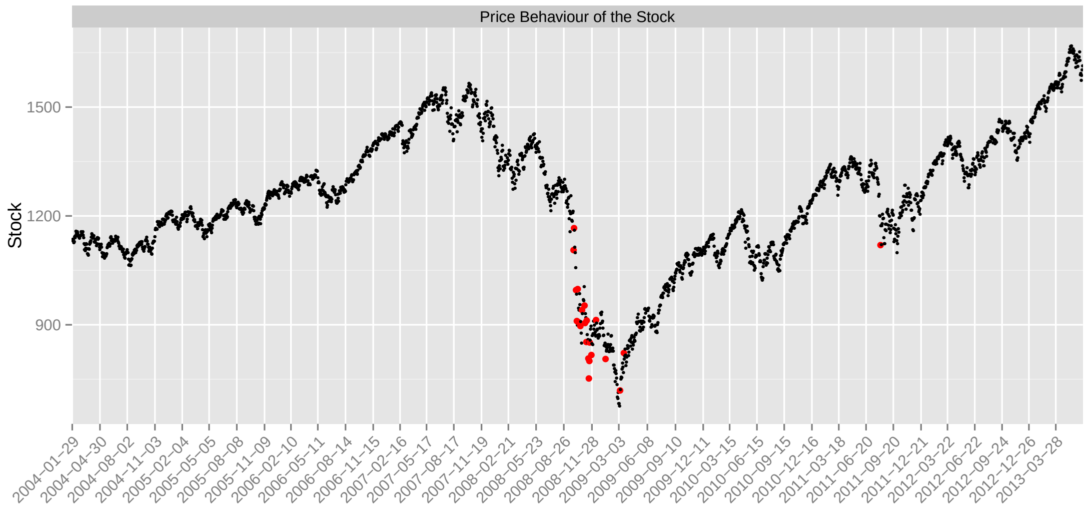



Figure 1.2: *The price behaviour of the stock in the period 2004/01/29 - 2013/07/27 in the S&P500 index. Red points correspond to a daily change in the value of at least* 5 *percentage.*

According to figure 1.2 extreme movements in prices do occur. We observe 22 daily movements of at least 5% in a period of 13 years - a change of this size translates to four times a Black-Scholes volatility equal to 0.2, which is expected to appear once every 63rd year.

Such observations have contributed to the gained ground of Lévy-processes in the financial literature. Especially the class of exponential Lévy-models seems to be a good choice to model empirical observations of the price behaviour from a theoretical point of view. These models can be approximated by a jump-diffusion model (a decomposition of a diffusion model and a pure jump model). It is argued in (Cont and Tankov, 2004, Chapter 15) that an implied volatility pattern as the one seen in figure 1.1 can be generated by a model including both stochastic volatility and a jump component. However, the fundamental question in this thesis is whether or not we benefit from increasing the complexity of the financial model in a risk-minimizing hedging sense. Introducing stochastic volatility as well as jumps in financial models violates completeness and hedging strategies will thus lead to a non-zero residual risk with probability one. And will a strategy intended to hedge for example the jump-risk outperform the usual $\Delta$-hedge in the simple Black-Scholes model?

The thesis is structured as follows.

**Chapter 2** A brief review of basic definitions, assumptions and notations that are used in the thesis. Furthermore, the generalized Black-Scholes model is revisited; A definition of the model and well-known results are stated. The chapter is completed with a formulation of the *Fundamental Theorem of Derivative Trading* and related numerical examples (*Wilmott's Hedge Experiment*).

**Chapter 3** A theoretical review of Merton's jump-diffusion model. The Fundamental Theorem of Derivative Trading is derived under this model. A valuation formula as well as the $\Delta$-hedge of the European call is derived under specific assumptions about the financial market. In addition, we state the locally risk-minimizing strategy related to the minimal martingale measure. The chapter is completed with simulation studies illustrating the performance of hedging strategies within this model.

**Chapter 4** A theoretical review of Heston's stochastic volatility model. The Fundamental Theorem of Derivative Trading is derived under this model. A detailed derivation of the valuation formula of the European call is given. This formula will lead to both the $\Delta$-hedge and the locally risk-minimizing hedge related to the minimal martingale measure. Simulation studies similar to those in chapter 3 are made. The chapter is completed with a derivation of a valuation formula as well as hedging strategies for the Bates model.

**Chapter 5** A design of the hedge experiment based on data from the S&P500 index over the period 2004/01/29 - 2013/06/26. This design is used for an empirical investigation of the hedging strategies (Chapter 6). Interpolation of implied volatilities for 3-months at-the-money European options is carried out with splines as well as a simple bilinear approach. Moreover, forecasts of actual volatility are made with 1) The EGARCH(1,1)-model and 2) A Bayesian parameter estimation using Markov Chain Monte Carlo methods.

**Chapter 6** An empirical investigation of the hedging strategies described in chapter 2-4 based on the hedge experiment designed in chapter 5. The chapter is completed with a discussion of the empirical findings.

**Chapter 7** A conclusion of this thesis. The empirical findings are synthesized with respect to the theoretical analysis of the four financial models under consideration. The chapter also contains suggestions for further investigations within this topic.

# Chapter 2

# Some Preliminary Concepts

To begin with, we will give some notion describing the universe, where trading finds place. This includes basic assumptions and notations used throughout this thesis. Moreover, we will present and describe the generalized Black-Scholes model. A simple example will illustrate the methods used for deriving the Fundamental Theorem of Trading. In addition, we state the Fundamental Theorem of Derivative Trading under the generalized Black-Scholes model and discuss some basic properties related to the theorem. The chapter is completed with a simulation study known as *Wilmott's hedge experiment.*

## 2.1 The Financial Market

In the following chapters, we will consider a financial market consisting of a risky asset $S$ and a risk-free asset $B$ traded continuously within some fixed time horizon $T$. To capture the information flow of this financial market we introduce the filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t^S)_{t\in[0,T]}, \mathbb{P}\big)$ where:

1. $\Omega$ represents all possible states of the market.
2. $\mathcal{F}$ is the corresponding $\sigma$-algebra on $\Omega$.
3. $\mathbb{P}$ is the physical probability measure.
4. $(\mathcal{F}_t^S)_{t\in[0,T]}$ is the natural filtration of the stochastic process $S$: An increasing sequence of sub-$\sigma$-algebras of $\mathcal{F}$ such that

$$\mathcal{F}_t^S = \sigma\left\{S_s\,,\, 0 \leq s \leq t\right\} \vee \mathcal{N}(\mathbb{P}),$$

where $\sigma\{S_s\,,\,0 \leq s \leq t\}$ is the smallest $\sigma$-algebra making $S_t$ $\mathcal{F}_t^S$-measurable for all $t \in [0,T]$, while $\mathcal{N}(\mathbb{P})$ is the set of all $\mathbb{P}$-null-sets. We will assume $\mathcal{F} = \mathcal{F}_T^S$ and that $(F_t^S)_{t\in[0,T]}$ is right-continuous. From now on think of $\{S_s\,,\,0 \leq s \leq t\}$ as the observed path of the asset between 0 and $t$ and drop the notation $S$ in the superscript $\mathcal{F}_t^S$.

We are particularly interested in pricing and hedging a *European Call Option*; a member of the family of financial derivatives (or contingent claims). In general, one should think of a financial derivative as an asset with value expressed in terms of the underlying asset $S$, but we need a formal mathematical definition.

**Definition 2.1** *A contingent claim with maturity $T$ is defined as an $\mathcal{F}_T$-measurable random variable $\mathcal{X}$. If we can write*

$$\mathcal{X} = \Phi(S_T)$$

*for some function $\Phi$, we will refer to the claim $\mathcal{X}$ as a simple $T$-claim and $\Phi$ as the contract function of the claim.*

A contingent claim can be interpreted as a contract that pays out the random amount $\mathcal{X}$ to the holder at time $T$, and an option is a contract which gives the holder the right, but not the obligation, to buy or sell the underlying asset at a specified strike price $K$ on or before the specified date $T$. In the specific case of the European call definition 2.1 reads

$$\mathcal{X} = \max(S_T - K, 0) = \Phi(S_T),$$

where $\Phi(x) = (x-K)^+$, so $\mathcal{X}$ is simple. How should we value $\mathcal{X}$ at time $t \in [0,T]$? The majority of the literature in mathematical finance approaches this problem assuming that the underlying asset $(S_t)_{t\geq 0}$ follows some type of diffusion process defined on $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\big)$. Or more specifically: The asset price can be modelled by a stochastic process with local dynamics on the time interval $[t, t+\Delta t]$ that can be approximated by a stochastic difference equation. From the stochastic difference equation, the stochastic differential equation is derived, which is a short-hand notation for a stochastic integral representation. An example is the simple Black-Scholes model claiming that $(S_t)_{t\geq 0}$ is of the form

$$S_t = S_0 + \mu \int_0^t S_u du + \sigma \int_0^t S_u dW_u,$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion under $\mathbb{P}$, while $\mu$ and $\sigma > 0$ are constants. The dynamic representation is then written as $dS_t = \mu S_t dt + \sigma S_t dW_t$. Simply put, a specification of the dynamics of the underlying $(S_t)_{t\geq 0}$ as well as a specification of the dynamics of the risk-free asset $(B_t)_{t\geq 0}$ is necessary in order to determine a price of $\mathcal{X}$. The only component that differs in the financial models considered in this thesis is the specification of $(S_t)_{t\geq 0}$, and we can state the definition of the risk-free asset as follows.

**Definition 2.2** *The price process $B = (B_t)_{t\geq 0}$ of the locally risk-free asset has dynamics*

$$dB(t) = r_t B_t dt \tag{2.1}$$

*where $(r_t)_{t\geq 0}$ is any adapted process.*

By equation (2.1) we deduce that

$$B_t = B_0 e^{\int_0^t r_s ds},$$

and we will think of the risk-free asset as a money account with short rate of interest $r_t$ and in the special case $r_t = r$ for all $t \geq 0$, we can interpret $(B_t)_{t\geq 0}$ as the price process of a bond. In reality the short rate of interest is **not** constant but assuming it to be so simplifies the numerous derivations and computations in the following chapters. The same applies to the continuous *dividend* yields related to the risky asset (if you for example purchase a share $(S_t)_{t\geq 0}$ in an S&P500 fund, you will during the time $dt$ receive dividends $\gamma_t S_t dt$ for some adapted process $\gamma_t$). In addition, we will make some assumptions about trading on the financial market as well as the form of the pricing function $C$:

**Assumption 2.3** *We will throughout this thesis assume (unless otherwise stated) that*

1. *The European call $\mathcal{X}$ as well as the risky asset $(S_t)_{t\geq 0}$ can be bought and sold without any financial frictions. From now on $\mathcal{X}$ will denote the European call, and $S_t$ is the stock price.*

2. *The risky asset $(S_t)_{t\geq 0}$ is perfectly divisible as to the amount which is held.*

3. *The locally risk-free asset has dynamics $dB_t = rB_t dt$ for some constant $r \geq 0$, and an investor can borrow and lend at the rate $r$ without any financial frictions.*

4. *The continuous dividend yields related to the $(S_t)_{t\geq 0}$ equal zero.*

5. *Continuous trading is feasable.*
6. *The financial market is free of arbitrage.*
7. *The value of $\mathcal{X}$ at time $t \in [0, T]$ is given by $C_t = C(\cdot)$ for some sufficiently smooth function in the arguments.*

The last assumption is a bit technical and ensures that we can use Itǒ's formula to obtain partial (integro)-differential equations for pricing purposes in the financial models considered. Before proceeding to the generalized Black-Scholes model, we introduce some notation that will be used throughout this thesis.

**Notation 2.4** *We will use the following notations (unless otherwise stated)*

1. *Let $\partial_t$ denote the derivative with respect to time, $\partial_s$ the derivative with respect to a direction $s$ and a similar interpretation of the two terms $\partial_{ts}$ and $\partial_{ss}$.*
2. *Let the image measure $X(\mathbb{P})$ of the real random variable $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ denote the distribution of $X$. Here $\mathcal{B}$ is the Borel-$\sigma$-algebra. The notation is equivalent to the probabilistic formulation $\mathbb{P}(X \in A)$ for all $A \in \mathcal{B}$.*
3. *Let $m$ denote the Lebesgue measure on $(\mathbb{R}, \mathcal{B})$.*
4. *Let $\mu$ be a $\sigma$-finite measure with density $f$ with respect to another $\sigma$-finite measure $\nu$. We then write $\mu = f \cdot \nu$.*
5. *Let $\mathbb{Q}$ denote a equivalent martingale measure.*
6. *Let $E[X] = \int X d\mathbb{P}$ denote the expectation of a real random variable $X$ under $\mathbb{P}$ while $E^{\mathbb{Q}}[X] = \int X d\mathbb{Q}$ is the expectation under $\mathbb{Q}$.*
7. *Let $E_t[\cdot]$ denote the conditional expectation given the filtration $\mathcal{F}_t$. We intend to use both the $E_t[\cdot]$-notation and the $E[\cdot|\mathcal{F}_t]$-notation. When $(S_t)_{t\geq 0}$ is Markovian we may write $E[\cdot]|_{s=S_t}$.*
8. *Let $(\hat{S}_t)_{t\geq 0} = (S_t/B_t)_{t\geq 0}$ denote the discounted price process.*

Knowledge of the (generalized) Black-Scholes model is a necessary prerequisite for reading this thesis. It serves as a foundation for the Merton model respectively the Heston model described in chapter 3 and 4. Both of these models can be seen as an extension of Black-Scholes. Furthermore, the Bates model is just a combination of the Merton model and the Heston model. The methods used to derive the Fundamental Theorem of Derivative Trading (FTODT) in the Black-Scholes model are similar to the methods used in the three other models - and increasing the complexity of the model will increase the number of technical details in the FTODT. In particular, the Fundamental Theorem of Derivative Trading under the Black-Scholes model will provide us with methods and intuition in the following chapters.

## 2.2 The Generalized Black-Scholes Model

Consider a financial market consisting of a risky asset $(S_t)_{t\geq 0}$ and the money account $(B_t)_{t\geq 0}$. We will assume that the two assets have $\mathbb{P}$-dynamics

$$dB_t = rB_t dt \tag{2.2}$$

$$dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t \tag{2.3}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion under $\mathbb{P}$ while $\mu$ and $\sigma$ are deterministic functions depending on time and the underlying. We will also require that $\sigma(t, s) > 0$ for all $(t, s)$, that $\mu(\cdot, \cdot), \sigma(\cdot, \cdot)$ are sufficiently well-behaved such that a solution to the above SDE exists and finally that $E[\int_s^t |S_u \sigma(u, S_u)|^2] < \infty \ \forall s \leq t \in [0, \infty)$. This set-up is often referred to as the *generalized Black-Scholes Model.* Itŏ's formula shows that the solution to the stochastic differential equation 2.3 is given by

$$S_t = S_0 e^{\int_0^t \mu(u,S_u) - \frac{1}{2}\sigma^2(u,S_u)du + \int_0^t \sigma(u,S_u)dW_u}$$

where $S_0$ is the starting value of the process. When $\sigma(t, S_t)$ is time-varying but deterministic, the integral $\int_0^t \sigma(u, S_u)dW_u$ has a normal distribution with zero mean and variance $\int_0^t \sigma(u, S_u)du$. This follows from the fact that the stochastic integral is a linear combination of independent normally distributed increments. The simple Black-Scholes model assumes that $\mu(t, S_t) = \mu$ and $\sigma(t, S_t) = \sigma > 0$ implying that log-returns of the risky asset, $\log(S_t/S_s)$, follow a normal distribution with mean $(\mu - \frac{1}{2}\sigma^2)(t - s)$ and variance $\sigma^2(t - s)$ for $s < t$.

Now, the First Fundamental Theorem of Asset Pricing implies the existence of an equivalent martingale measure $\mathbb{Q}$, while the Second Fundamental Theorem of Asset Pricing implies that $\mathbb{Q}$ is unique and is in fact the equivalent martingale measure using $B_t$ as numeraire. Thus, the process $(S_t/B_t)_{t\geq 0}$ is a martingale under $\mathbb{Q}$ and it can be deduced that $(S_t)_{t\geq 0}$ has $\mathbb{Q}$-dynamics

$$dS_t = rS_tdt + \sigma(t, S_t)dW_t^{\mathbb{Q}},$$

where $(W_t^{\mathbb{Q}})_{t\geq 0}$ is a standard Brownian motion under $\mathbb{Q}$. These statements are trivial consequences of an interaction between Itŏ's formula applied on the discounted price process, $(\hat{S}_t)_{t\geq 0}$, and Girsanov's theorem used for the change-of-measure: The process $(\hat{S}_t)_{t\geq 0}$ has $\mathbb{P}$-dynamics

$$d\hat{S}_t = (\mu(t, S_t) - r)\,\hat{S}_tdt + \sigma(t, S_t)\hat{S}_tdW_t$$

implying that $\mu(t, S_t) = r$ is a necessary and sufficient condition in order for the discounted price process to be a martingale. This condition is violated under $\mathbb{P}$ - the world is **not** risk-neutral. Let $\phi_t$ be any $\mathcal{F}_t$-measurable process, choose a fixed $T$ (the expiry for $\mathcal{X}$) and a define a Likelihood process $(L_t)_{t\in[0,T]}$ by

$$L_t = e^{\int_0^t \phi_s dW_s - \frac{1}{2}\int_0^t \phi_s^2 ds}.$$

The assumption $E[L_T] = 1$ yields that $L_T$ is integrable with respect to the Lebesgue measure and more importantly - using Radon-Nikodým's theorem - we can define a new measure on $\mathcal{F}_T$ by $d\mathbb{Q} = L_Td\mathbb{P}$. In fact $\mathbb{Q}$ is a probability measure: $\mathbb{Q}(\Omega) = \int d\mathbb{Q} = \int L_TdP = E[L_T] = 1$ by assumption. The two measures are absolute continuous with respect to each other and hence equivalent by definition. Girsanov's theorem leads to the dynamics

$$d\hat{S}_t = (\mu(t, S_t) - r + \phi_t\sigma(t, S_t))\,\hat{S}_tdt + \sigma(t, S_t)\hat{S}_tdW_t^{\mathbb{Q}}$$

under $\mathbb{Q}$. Choosing the Girsanov kernel

$$\phi_t = \frac{r - \mu(t, S_t)}{\sigma(t, S_t)}$$

is equivalent to choosing the measure $\mathbb{Q}$ such that $(\hat{S}_t)_{t\geq 0}$ becomes a $\mathbb{Q}$-martingale. The technical assumptions for the functions $\mu$ and $\sigma$ will imply that $E[L_T] = 1$. Applying Itŏ's formula on the process $(S_t)_{t\geq 0}$ confirms our claim. Girnanov's theorem can be found in (Björk, 2009, Theorem 11.3).

With the $\mathbb{Q}$-dynamics specified, we will state the Black-Scholes partial differential equation for pricing purposes. A proof of the PDE is straightforward, and the equation is furthermore a special case of Merton's partial integro-differential equation, which is proved in the next chapter.

**Proposition 2.5 (The Black-Scholes Equation)** *Assume that the financial model is given by* (2.2) *and* (2.3)*. The arbitrage-free price of the European call $\mathcal{X}$ is determined by the unique function $C : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ solving the PDE*

$$\partial_t C(t,s) + \frac{1}{2}\sigma^2(t,s)S_t^2 \partial_{ss}C(t,s) + rS_t\partial_s C(t,s) - rC(t,s) = 0 \tag{2.4}$$

$$C(T,s) = (s-K)^+. \tag{2.5}$$

Applying the Feynman-Kač theorem (Björk, 2009, Proposition 5.6) on proposition 2.5 immediately yields the risk-neutral valuation of $\mathcal{X}$

$$C(t,S_t) = e^{-r\tau}E_t^{\mathbb{Q}}[\mathcal{X}] = e^{-r\tau}E_t^{\mathbb{Q}}\left[\left(S_t e^{\left(r-\frac{1}{2}\sigma^2(t,S_t)\right)\tau + \sigma(t,S_t)W_\tau^{\mathbb{Q}}} - K\right)^+\right], \tag{2.6}$$

where $\tau = T - t$ is time to maturity at time $t$. Let $\mu(t,S_t) = \mu$ and $\sigma(t,S_t) = \sigma$ be constants[1]. The Black-Scholes arbitrage-free price (Björk, 2009, Proposition 7.10) for the European option is obtained by computing equation (2.6).

**Proposition 2.6** *Consider the financial model given by the $\mathbb{P}$-dynamics*

$$dB_t = rB_t dt$$
$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

*where $(W_t)_{t\geq 0}$ is a standard Brownian motion under $\mathbb{P}$. The arbitrage-free price of $\mathcal{X}$ at time $t \leq T$ is given by*

$$C(t,S_t) = S_t\psi\left(d_1(t,S_t)\right) - e^{-r\tau}K\psi\left(d_2(t,S_t)\right) \tag{2.7}$$

*where $\psi$ is the distribution function for a standard normally distributed variable and*

[1] The pricing formula is also valid for time-varying but non-random volatility.

$$d_1(t, S_t) = \frac{1}{\sigma\sqrt{\tau}}\left\{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau\right\} \tag{2.8}$$

$$d_2(t, S_t) = d_1(t, S_t) - \sigma\sqrt{\tau}. \tag{2.9}$$

Given knowledge of market parameters, proposition 2.6 determines the value of $\mathcal{X}$ at time $t$ in the simple Black-Scholes model. This formula and the corresponding *Delta* hedge will be used in the numerical examples. Consider the portfolio with a long option position and short $\Delta_t$ units of the stock. The corresponding value process is given by $\Pi_t = C(t, S_t) - \Delta_t S_t$, and we require $\Pi_t = dC(t, S_t) - \Delta_t dS_t = 0$ in order to obtain a risk-neutral portfolio. Applying Itǒ's formula on the pricing function $C(t, S_t)$ shows that $\Delta_t = \partial_s C(t, S_t)$ will imply the risk-neutrality. This is called to $\Delta$-hedge the option. Investigating the pricing formula in 2.6 yields that $C(t, S_t)$ is homogeneous of degree one with respect to the underlying and the strike price understood that $C$ is also depends of $K$. Euler's Homogeneous Function Theorem implies

$$\begin{aligned} C(t, S_t) &= S_t \partial_s C(t, S_t) + K \partial_K C(t, S_t) && \Leftrightarrow \\ \partial_s C(t, S_t) &= \frac{C(t, S_t) - K\partial_K C(t, S_t)}{S_t}, \end{aligned}$$

where $\partial_K C(t, S_t) = -e^{-r\tau}\psi(d_2(t, S_t))$. Combining the above equation with proposition 2.6 yields

$$\Delta_t = \partial_s C(t, S_t) = \psi\left(d_1(t, S_t)\right). \tag{2.10}$$

Before stating the FTODT for the generalized Black-Scholes model, we will consider an enlightening example.

**Example 2.1** Assume that the risky asset follows the process $(S_t)_{t\geq 0}$, which is the solution to the stochastic differential equation

$$dS_t = \sigma_r S_t dW_t^{\mathbb{Q}} \tag{2.11}$$

$$S_0 = s, \tag{2.12}$$

where $(W_t^{\mathbb{Q}})_{t\geq 0}$ is a standard Brownian motion under the risk neutral measure $\mathbb{Q}$, $\sigma_r > 0$ is the volatility assumed constant, while the price process, $(B_t)_{t\geq 0}$, of the risk-free asset has dynamics $dB_t = 0$, i.e. we assume that the interest rate is zero. Consider now the European call $\mathcal{X}$ written on $(S_t)_{t\in[0,T]}$ with pay-off $(S_T - K)^+$ at maturity $T$. Introduce the implied volatility $\sigma_i(t, S_t)$ which is uniquely determined by how the market is pricing the option at time $t \in [0, T)$. Assume that $\mathcal{X}$ is bought or sold for the price $\hat{C}_t$ at time $t$. The implied volatility is then solution to the equation $C(t, S_t; \sigma) = \hat{C}_t$, and can be found by inverting the Black-Scholes formula. We will in general find that $\sigma_i(t, S_t) \neq \sigma_r$, since the market does not have perfect information about the true process of the underlying, and $\sigma_i(t, S_t)$ can be interpreted as a wrong parameter in a wrong model yielding the correct price. Thus, the option is mispriced by the market and maybe it is possible to find a volatility arbitrage, i.e. make profit by hedging the option. We therefore consider three volatilities: The actual volatility $\sigma_r$ which we cannot observe (in contrast to the other market parameters), the implied volatility $\sigma_i(t, S_t) = \sigma_i$ determined by the market price of the option and finally a hedging volatility $\sigma_h(t, S_t) = \sigma_h$ chosen by the trader. All volatilities are assumed constant in this example. For convenience we will use the notation $C^i(t, S_t)$ as the price at time $t$ corresponding to the implied volatility $\sigma_i$. Likewise, the notation $C^h(t, S_t)$ and $C^r(t, S_t)$ will be used. Now, take the scenario where we buy the option at time $t = 0$ for the market price $C^i(0, S_0)$ corresponding to the implied volatility and we want to $\Delta$-hedge the position. If we choose the money account $B_t$ such that the net value of the portfolio is zero,

$$\Pi_t^h = C^i(t, S_t) + B_t - \Delta_t^h S_t = 0,$$

the strategy is by definition self-financing, which implies ($dB_t = 0$)

$$d\Pi_t^h = dC^i(t, S_t) - \Delta_t^h dS(t). \tag{2.13}$$

Apply Itŏ's formula on $C^h(t, S_t)$ to obtain

$$dC^h(t, S_t) = \left(\partial_t C^h(t, S_t) + \frac{1}{2}\sigma_r^2 S_t^2 \partial_{ss} C^h(t, S_t)\right) dt + \Delta_t^h dS_t.$$

On the other hand, we must have

$$\partial_t C^h(t, S_t) = -\left(\frac{1}{2}\sigma_h^2(t, S_t) S_t^2 \partial_{ss} C^h(t, S_t)\right)$$

due to the Black-Scholes PDE. Combining the two above equation yields

$$0 = -dC^h(t, S_t) + \left\{ \frac{1}{2} \left( \sigma_r^2 - \sigma_h^2 \right) S_t^2 \partial_{ss} C^h(t, S_t) \right\} + \Delta_t^h dS_t,$$

which is added to equation (2.13)

$$d\Pi_t^h = dC^i(t, S_t) - dC^h(t, S_t) + \left\{ \frac{1}{2} \left( \sigma_r^2 - \sigma_h^2 \right) S_t^2 \partial_{ss} C^h(t, S_t) \right\}.$$

The profit-and-loss holding the portfolio, $(\Pi_t)_{t \in [0,T]}$, over $[0, T]$ is defined as $P\&L_T^h \equiv \int_0^T e^{-rt} d\Pi_t$. Thus,

$$\begin{aligned} P\&L_T^h &= \int_0^T d\Pi_t^h \\ &= C^h(t, S_t) - C^i(t, S_t) + \int_0^T \frac{1}{2} \left( \sigma_r^2 - \sigma_h^2 \right) S_t^2 \partial_{ss} C^h(t, S_t) dt \end{aligned}$$

using the boundary conditions $C^h(T, S_T) = C^i(T, S_T) = \mathcal{X}$. This is the Fundamental Theorem of Derivative Trading related to the simple model set in this example. ∘

Example 2.1 illustrates the consequences for hedging in misspecified models. Indeed successful hedging depends on the relationship between the actual volatility $\sigma_r$ and the hedging volatility $\sigma_h$ chosen by the trader as well as up-front premium $C^h(0, S_0) - C^i(0, S_0)$. Note that the $\partial_{ss} C^h(t, S_t)$ is the Black-Scholes gamma:

$$\begin{aligned} \Gamma(t, S_t) &\equiv \partial_{ss} C^h(t, S_t) \\ &= \frac{\psi'(d_1^h(t, S_t)}{S_t \sigma_h \sqrt{\tau}} \\ &> 0, \end{aligned}$$

obtained by differentiating $\psi(d_1^h(t, S_t))$ with respect to $S_t$. Here $\psi'$ is the probability density function for a normally distributed random variable. Thus, the sign of the hedging error only depends on the relationship between $\sigma_r$ and $\sigma_h$. Example 2.1 also indicates that there are two natural candidates for the chosen hedging volatility; The implied volatility $\sigma_i$ and the actual volatility $\sigma_r$. We will in the next section discuss important properties related to the Fundamental Theorem of Derivative Trading in the generalized Black-Scholes model.

## 2.3 FTODT: The Generalized Black-Scholes model

We will return to the model defined in equations 2.2 and 2.3: Assume that the price process of the risky asset, $(S_t)_{t\geq 0}$, is solution to the stochastic differential equation

$$dS_t = \mu_r(t, S_t)S_t dt + \sigma_r(t, S_t)S_t dW_t \tag{2.14}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion under $\mathbb{P}$ adapted to filtration and $\mu : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ are deterministic functions sufficiently well-behaved such that (2.14) has an unique solution. As in example 2.1 we refer to $\sigma_r(\cdot,\cdot)$, $\sigma_i(\cdot,\cdot)$ and $\sigma_h(\cdot,\cdot)$ as the actual, the implied and respectively the hedging volatility process. All of these are now assumed to satisfy the regularity condition

$$E\left[\int_s^t \left|S_u \sigma(u, S_u)\right|^2 du\right] < \infty$$

for all $s \leq t \in [0,\infty)$. As in example 2.1 perfect hedging in real life practice relies on the trader having perfect information about the dynamics of the underlying, which is not the case for empirical data. The hedging error is determined in the following theorem.

**Theorem 2.7 (The Fundamental Theorem of Derivative Trading)** *Consider the European call $\mathcal{X}$ written on the underlying $S_t$ satisfying* (2.14)*. Let $C_t = C(t, S_t) \in C^{1,2}([0,\infty) \times \mathbb{R})$ be the corresponding price process with pay-off $C_T = \mathcal{X}$. Assume now that we buy the option at time $t = 0$ for the market-price $C^i(0, S_0)$ associated with the implied volatility $\sigma_i(0, S_0)$ and decide to $\Delta$-hedge this position assuming that $\sigma_h(0, S_0)$ is the true volatility leading to the fair price $C^h(0, S_0)$. The present value of the* ***Profit-and-Loss*** *obtained by holding this portfolio over the time interval $[0, T]$ is given by*

$$P\&L_T = C^h(0, S_0) - C^i(0, S_0) + \int_0^T e^{-rt}\frac{1}{2}\left(\sigma_r^2(t, S_t) - \sigma_h^2(t, S_t)\right) S_t^2 \partial_{ss} C^h(t, S_t) dt. \tag{2.15}$$

The theorem will be proved in a more general set-up in chapter 3. We will not waste time and paper showing theorem 2.7. The reader should just be aware that the proof follows example 2.1 using Itŏ's formula on the pricing function, the Black-Scholes partial differentiation equation and the self-financing condition related to the portfolio long $\mathcal{X}$, short $\partial_s C^h(t, S_t)$ in the underlying and the money account chosen as in example 2.1.

**Remark 2.1** Theorem 2.7 can be generalized in the case of a financial market consisting of $n$ risky assets. The result (in a more general set-up) can be found in the article (Poulsen, Ellersgaard, and Jönsson, 2015). ◦

**Remark 2.2** We will define $\Gamma(t, S_t) \equiv \partial_{ss}C^h(t, S_t)$ as the *Gamma* for the generalized Black-Scholes model, which is strictly positive for standard puts and calls. Hence, the hedging error

$$\int_0^T e^{-rt}\left(\sigma_r^2(t, S_t) - \sigma_h^2(t, S_t)\right) S_t^2 \Gamma_t dt, \tag{2.16}$$

shows that even though the model differs from the true model, successful hedging is quite possible. Especially if $\sigma_r(t, S_t) \geq \sigma_h(t, S_t)$ almost-surely for all $t \in [0, T)$ the hedging error is positive and - if we ignore the up front premium - we can make profit with probability one, which we will refer to as volatility arbitrage. The size of this profit depends on the convexity of the option. If $\Gamma(t, S_t)$ is approximately zero, the hedging error is small, while a higher $\Gamma(t, S_t)$ implies a larger hedging error. ◦

**Remark 2.3** The hedging error in theorem 2.7 depends on the whole path of $S_t$. ◦

**Remark 2.4** If we instead short the option, $P\&L_T$ will change sign. This strategy is more appropriate in conjunction with market realities typically showing that $\sigma_i(t, S_t) > \sigma_r(t, S_t)$. Choosing $\sigma_h(t, S_t) = \sigma_i(t, S_t)$ yields that (2.15) is uniquely determined by the hedge error. At each $t$ (assuming discrete time - for example daily observations) the sign of the integrand in (2.15) depends on the term $\sigma_r(t, S_t) - \sigma_i(t, S_t)$. Choosing $\sigma_h(t, S_t) = \sigma_r(t, S_t)$ yields that (2.15) becomes deterministic and equals the up-front premium $C^r(0, S_0) - C^i(0, S_0)$, which is negative under the assumption $\sigma_i(t, S_t) > \sigma_r(t, S_t)$. As Jesper Andreasen puts it: "In essence this is what all trading is all about: buy low - sell high." ◦

As mentioned earlier, we have two obvious choices for $\sigma_h(t, S_t)$: The implied and the actual volatility process. Both strategies can according to Theorem 2.7 lead to volatility arbitrage choosing the right position. The former relies on the assumption that $\sigma_i(t, S_t) > \sigma_r(t, S_t)$ or $\sigma_i(t, S_t) < \sigma_r(t, S_t)$ for all $t \in [0, T)$ with probability one, while the latter requires perfect information of the true volatility process. Both scenarios are illustrated in the following simulation study known as *Wilmot's hedge experiment* Wilmott and Ahmad (2005).

### 2.3.1 Wilmot's hedge experiment

In the Fundamental Theorem of Derivative Trading we have used an arbitrary volatility to hedge the European option. for an everyday-life trader, a natural choice is hedging with the implied volatility. The second interesting case is hedging with actual volatility. The $P\&L$ is investigated for both scenarios through a simulation study. Such experiments may sound ridiculous in practice since actual volatility is an unobservable quantity, but we will keep in mind that the conclusion should emphasize the importance of a good forecast of actual volatility. We will in the following assume that actual volatility exceeds implied volatility during the option's lifetime. Thus, we consider an underpriced option and intuitively, we must create a long hedging portfolio to make volatility arbitrage confirmed by equation (2.15). For convenience, we will also assume that actual and implied volatility as well as the drift in the $\mathbb{P}$-dynamics for $(S_t)_{t\geq 0}$ are constant. Furthermore, trading takes place in discrete time over $[0, T]$ for fixed $T > 0$ with corresponding time grid $0 = t_0 < t_1 < t_2 < ... < t_n = T$. This can be considered as daily observations. To simplify computations, we will use the Greeks $\Delta_t = \partial_s C(t, S_t)$, $\Gamma_t = \partial_{ss} C(t, S_t)$ and $\Theta_t = \partial_t C(t, S_t)$.

**Example 2.2 (Hedging with actual volatility)** Assume that the hedging volatility, $\sigma_h$, is chosen as the actual volatility $\sigma_r$. The Profit-and-Loss over $[0, T]$ is in continuous time then reduced to

$$P\&L_T^h = C^r(0, S_0) - C^i(0, S_0),$$

which is positive under the assumption that $\sigma_i < \sigma_r$, and we are guaranteed total profit by expiration - buy low, sell high. However, the profit is achieved randomly in a mark-to-market sense. Table 2.1 shows the value of our components in the hedge portfolio today and tomorrow

| **Component** | **Value today** | **Value tomorrow** |
|---|---|---|
| Option | $C^i(t, S_t)$ | $C^i(t, S_t) + dC^i(t, S_t)$ |
| Stock | $-\Delta^r S_t$ | $-\Delta^r S_t - \Delta^r dS_t$ |
| Money account | $-C^i(t, S_t) + \Delta^r S_t$ | $(-C^i(t, S_t) + \Delta^r S_t)(1 + rdt)$ |

Table 2.1: The mark-to-market value of the portfolio long one option, short $\Delta$ in the underlying and the money account chosen in accordance with the self-financing condition today.

The hedging portfolio has dynamics $d\Pi_t = dC^i(t, S_t) - \Delta_t^r dS_t - r\left(C^i(t, S_t) - \Delta_t^r S_t\right) dt$ due to the self-financing condition. Applying Itǒ's formula as well as the Black-Scholes PDE, the mark-to-market profit over one time step is given by

$$\begin{aligned}
d\Pi_t &= \Theta_t^i dt + \Delta^i(t, S_t) dS_t + \frac{1}{2}\sigma_r^2 S_t^2 \Gamma_t^i dt - \Delta_t^r dS_t - r\left(C^i(t, S_t) - \Delta_t^r S_t\right) dt \\
&= \Theta_t^i dt + \left(\Delta_t^i - \Delta_t^r\right) dS_t + \frac{1}{2}\sigma_r^2 S_t^2 \Gamma_t^i dt - r\left(C^i(t, S_t) - \Delta_t^r S_t\right) dt \\
&= \left(\Delta_t^i - \Delta_t^r\right) dS_t + \frac{1}{2}\left(\sigma_r^2 - \sigma_i^2\right) S_t^2 \Gamma_t^i dt - r\left(\Delta_t^i - \Delta_t^r\right) S_t dt \\
&= \left(\Delta_t^i - \Delta_t^r\right)\left((\mu - r) S_t dt + \sigma_r S_t dW_t\right) + \frac{1}{2}\left(\sigma_r^2 - \sigma_i^2\right) S_t^2 \Gamma_t^i dt
\end{aligned}$$

which is random by the $dW_t$-term. The profit-and-loss over $[0, T]$ is a process with deterministic initial and final value. The final profit is solely determined by the relationship between actual and implied volatility at initiation, but on a mark-to-market basis we may lose before we gain, which is not satisfactory from a risk management perspective. This is analogous to owing a bond. Figure 2.1 illustrates simulations of the $P\&L$ on a day-to-day basis.

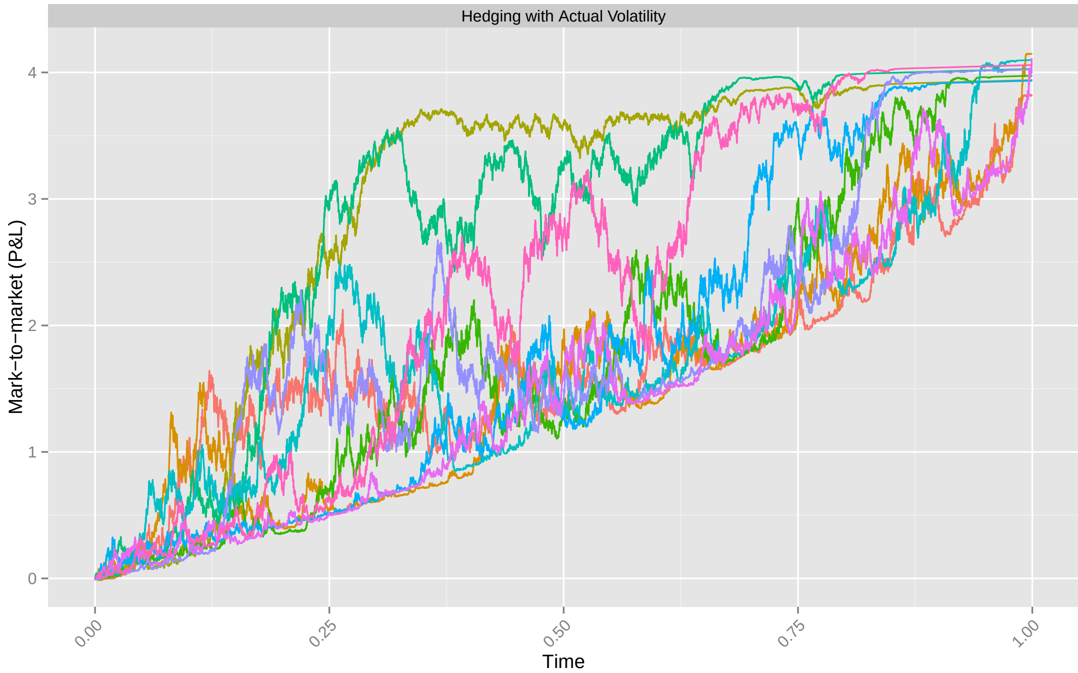



Figure 2.1: *We have simulated* 10 *paths of a geometric Brownian motion* $(S_t)_{t\in[0,T]}$ *with initial value* $S_0 = 100$*, drift* $\mu = 0.1$ *and volatility* $\sigma_r = 0.3$ *The relevant market parameters are specified as* $r = 0.05$*,* $\sigma_i = 0.2$*,* $K = 100$ *and* $T = 1$*. The expiry value of the portfolio is deterministic and equals* $\Pi_T^h = e^{rT}(C^r(0, S_0) - C^i(0, S_0)) = 3.975$ *using the Black-Scholes formula. The portfolio is rebalanced* $n = 10000$ *times, and the discretization of time explains why the terminal value only approximately hits* $\Pi_T^h$.

$\circ$

**Example 2.3 (Hedging with implied volatility)** The usual choice for the trader would be to hedge with the implied volatility. Equation (2.7) then reads

$$P\&L_T = \int_0^T \frac{1}{2} e^{-rt} \left(\sigma_r^2 - \sigma_i^2\right) S_t^2 \Gamma_t^i dt,$$

which is highly path dependent. Volatility arbitrage is possible if the actual volatility is always going to be higher than the implied volatility. However, the terminal value of the hedging portfolio is stochastic due to the presence of $(S_t)_{t\in[0,T]}$. Hence a perfect forecast of actual volatility is not strictly necessary in order to make profit. This confirms some of the strengths hedging with implied volatility.

Furthermore, the daily mark-to-market profit can now be expressed as

$$\begin{aligned} d\Pi_t &= \Theta_t^i dt + \frac{1}{2}\sigma_r^2 S_t^2 \Gamma_t^i dt - r\left(C^i(t,S_t) - \Delta_t^i S_t\right) dt \\ &= \frac{1}{2}\left(\sigma_r^2 - \sigma_i^2\right) S_t^2 \Gamma_t^i dt, \end{aligned}$$

which does not contain any $dW_t$-terms and hence the profit is deterministic. A more relaxing result from the risk management perspective. Figure 2.2 illustrates simulations of the $P\&L$ on the day-to-day basis.

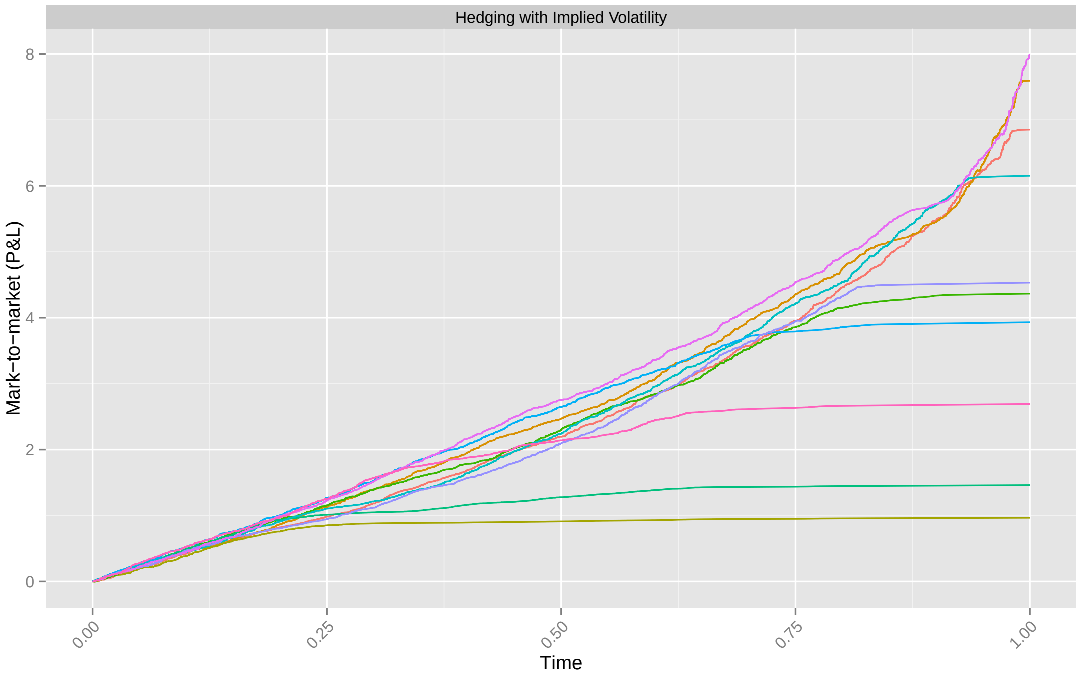



Figure 2.2: *The parameter specifications are the same as those specified in relation to figure 2.1, but now we choose $\sigma_h = \sigma_i$. One should note that all ten curves are smooth. This is caused by the fact that the daily mark-to-market profit does not depend on $dW_t$.*

Comparing figure 2.1 and figure 2.2 clearly indicates that hedging with implied volatility implies smoothness of the $P\&L$-paths in contrast to hedging with actual volatility. The paths of $P\&L$ in figure 2.1 exhibit major fluctuations during the option's lifetime equivalent to occurrences of big losses in the hedging portfolio. This is avoided by hedging with implied volatility. As a consequence, the *quadratic variation* of the hedging portfolio (a measure we later on will use for riskiness of hedging strategies) will be smaller, when hedging with implied volatility[2] - a less bumpy road avoiding big losses during the option's lifetime. It is possible to investigate statistical properties of the final random profit obtained by hedging with implied volatility. For an example you will find derivations of closed-form formulas for the expected total profit and for the variance of this profit in the article Wilmott and Ahmad (2005). ◦

We conclude that successful hedging of European options is possible under the generalized Black-Scholes model. According to theorem 2.7, the final hedging error is expressed through

$$\int_0^T e^{-rt}\left(\sigma_r^2(t,S_t)-\sigma_h^2(t,S_t)\right)S_t^2\Gamma_t dt,$$

and depends on the relationship between the actual and the hedging volatility process as well as the convexity of the generalized Gamma. Many choices for hedging volatility can be made, and we have treated the two most interesting scenarios - hedging with actual and implied volatility - assuming perfect information of both model and market parameters. The first mentioned hedging volatility shows that the final profit is deterministic but is achieved randomly on a day-to-day basis, while the latter leads to the opposite conclusion. Both strategies imply volatility arbitrage under the strict assumption that implied and actual volatility are constants different from each other. The preferred strategy depends on the purposes of hedging. In the empirical investigation (Chapter 6), we will use the following values for determining the optimal trading strategy

1. Mean and standard deviation of the terminal value of $P\&L$. These two values are inputs in a modified *Sharp-ratio.*

2. The quadratic variation of the portfolio during the option's lifetime.

As pointed out in chapter 1, the Black-Scholes model is far from market realities. A more accurate presentation of the price process would involve a correlation between the stock and the stochastic volatility process as well as include random jumps in the price process. Especially the conclusions from Wilmott's hedge example cannot be directly transferred to the financial market. However, the hedge experiment provides some indications in the business of hedging: 1) hedging with implied volatility will decrease the quadratic variation and 2) forecasting actual volatility is relevant in order to predict the final outcome of $P\&L$ as well as developing a volatility trading strategy.

[2] *If you are in the business of hedging, then the use of implied volatility should make you sleep better at night.* (Poulsen, Ellersgaard, and Jönsson, 2015)