The Fundamental Theorem of Derivative Trading - Exposition, Extensions, and Experiments

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Abstract

When estimated volatilities are not in perfect agreement with reality, delta hedged option portfolios will incur a non-zero profit-and-loss over time. There is, however, a surprisingly simple formula for the resulting hedge error, which has been known since the late 90s. We call this The Fundamental Theorem of Derivative Trading. This paper is a survey with twists of that result. We prove a more general version of it and discuss various extensions (including jumps) and applications (including deriving the Dupire-Gyöngy-Derman formula). We also consider its practical consequences both in simulation experiments and on empirical data thus demonstrating the benefits of hedging with implied volatility.

Keywords: Delta Hedging, Model Uncertainty, Volatility Arbitrage.

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1 A Meditation on the Art of Derivative Hedging

Introduction. Of all possible concepts within the field mathematical finance, that of *continuous time derivative hedging* indubitably emerges as the central pillar. First used in the seminal work by Black & Scholes (1973)[1], it has become the cornerstone in the determination of no-arbitrage prices for new financial products. Yet a disconnect between this body of abstract mathematical theory and real world practise prevails. Specifically, successful hedging relies crucially on us having near perfect information about the model that drives the underlying asset. Even if we boldly adopt the standard stochastic differential equation paradigm of asset pricing, it remains to make exact specifications for the degree to which the price process reacts to market fluctuations (i.e. to specify the diffusion term, the volatility). Alas, volatility blatantly transcends direct human observation, being, as it were, a Kantian *Ding an sich.*[2], of which we only have approximate knowledge.

One such source comes from measuring the standard deviation of past log returns over time (this is tantamount to assuming that the model can at least *locally* be approximated as a geometric brownian motion). Yet this process raises uncomfortable questions pertaining to statistical measurement: under ordinary circumstances, increasing the sample space should narrow the confidence interval around our sample parameter. Only here, there is no a priori way of telling when a model undergoes a drastic structural change. [3] Inevitably, this implies that extending the time series of log returns too far into the past might lead to a less accurate estimator, as we might end up sampling from a governing dynamics that is no longer valid. Of course, we may take *some* measures against this issue, by trying our luck with ever more intricate time series analyses until we stumble upon a model the parameters of which satisfy our arbitrary tolerance for statistical significance. Nevertheless, in practise this procedure invariably boils down to checking some finite basket of models and selecting the best one from the lot. Furthermore, unknown structural breaks continue to pose a problem no matter what.

Alternatively, we might try to extract an implied volatility from the market by fitting our model to observed option prices. Nevertheless the inadequacy of the methodology quickly becomes apparent: first, implied volatility might be ill-defined as it is the case for certain exotic products such as barrier options. Secondly, it is quite clear that the market hysteria which drives the prices of traded options need not capture the market hysteria which drives the corresponding market for the underlying asset. Fair pricing ultimately boils down to understanding the true nature of the underlying product: not to mimic the collective madness of option traders.

Whilst volatility at its core remains elusive to us, the situation is perhaps not as dire as one might think. Specifically, we can develop a formal understanding of the profit-&-loss we incur upon hedging a portfolio with an erroneous volatility - at least insofar as we make some

[1] That Black and Scholes along with Merton were the first is the general consensus, although the paper Haug & Taleb (2011) shows that the view is not universal.

[2] Literally, *thing in itself* or the *noumenon*. Kant held that there is a distinction between the way things appear to observers (phenomena) and the way reality actually is construed (noumena).

[3] This scenario is not at all implausible. Unlike physics where the fundamental laws are assumed to have no *sufficient reason* to change (in the Leibnizian or Occamian sense), this philosophical principle would hardly withstand scrutiny in a social science context. Asset price processes are fundamentally governed by market agents and their reactions to various events (be they self-induced or exogenous). There is really no reason to assume that these market players will not drastically change their opinions at some point (for one reason or the other).

moderate assumptions of the dynamical form of the underlying assets. To give a concrete example of this, consider the simple interest rate free framework presented in Andreasen (2003) where the price process of a single, non-dividend paying asset is assumed to follow the real dynamics $dX_t = \mu_r(t)X_t dt + \sigma_r(t)X_t dW_t$. Let V_t^i be the value of an option that trades in the market at a certain implied volatility σ_i (possibly quite different from the epistemically inaccessible $\sigma_r(t)$). Now if we were to set up a hedge of a long position on such an option, using σ_i as our hedge volatility, an application of Itō's formula, coupled with the Black-Scholes equation, shows that the infinitesimal value change in the hedge portfolio $\Pi_t = V_t^i - \partial_x V_t^i \cdot X_t$ is

$$d\Pi_t = \tfrac{1}{2}(\sigma_r^2(t) - \sigma_i^2)X_t^2 \partial_{xx}^2 \Pi_t dt, \tag{1}$$

which generally is non-zero unless $\sigma_i = \sigma_r(t)$. For reasons that will become clearer below, the importance of this result is of such magnitude that Andreasen dubs it *The Fundamental Theorem of Derivative Trading*. Indeed, a more abstract variation of it will be the central object of study in this paper.

To the best of our knowledge, quantitative studies into erroneous delta-hedging leading to a result like (1) first appeared in a paper on the robustness of the Black-Scholes formula by Karoui, Jeanblanc-Picque & Shreve (1998). They viewed the result as a largely negative one: unless volatility is bounded (which it is not in any stochastic volatility model) then there is no simple super-replication strategy. Subsequently, various sources have re-derived the result (with various tweaks) - most prominently the works of Gibson, Lhabitant, Pistre & Talay (1999), Henrard (2001), Mahayni, Schlögl & Schlögl (2001), Rasmussen (2001) Carr (2002), and Ahmad & Wilmott (2005). Today, the gravity of erroneous Δ-hedging is unquestionably more widely appreciated, yet the Fundamental Theorem of Derivative Trading continues to fly largely under the radar in academia and industry.

Overview. The structure of this paper is as follows: in section two we state and prove a new, generalised version of the Fundamental Theorem of Derivative Trading and discuss its various implications for hedging strategies and applications (some of which might prove surprising). In section three we expose the implications of adding a jump process to the framework, thus emphasising the relative ease with which the original proof can be adapted. Finally, section four presents an empirical investigation into what actually happens to our portfolio when we hedge using various volatilities.

2 The Fundamental Theorem of Derivative Trading

2.1 Derivation

Model Set-up. Consider a financial market comprised of a risk-free money account as well as n risky assets, each of which pays out a continuous dividend yield. We assume all assets to be infinitely divisible as to the amount which may be held, that trading takes place continuously in time and that no trade is subject to financial friction. Formally, we imagine the information flow of this world to be captured by the stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where Ω represents all possible states of the economy, and $\mathbb{F} = \{\mathscr{F}_t\}_{t \geq 0}$ is a filtration which satisfies the *usual conditions*.The price processes of the risky assets, $\boldsymbol{X}_t = (X_{1t}, X_{2t}, ..., X_{nt})^\intercal$, are assumed to follow the real dynamics

$$d\boldsymbol{X}_t = \mathbf{D}_{\boldsymbol{X}}[\boldsymbol{\mu}_r(t, \widetilde{\boldsymbol{X}}_t)dt + \boldsymbol{\sigma}_r(t, \widetilde{\boldsymbol{X}}_t)d\boldsymbol{W}_t], \tag{2}$$

where $\mathbf{D}_{\boldsymbol{X}}$ is the $n \times n$ diagonal matrix $\text{diag}(X_{1t}, X_{2t}, ..., X_{nt})$, $\boldsymbol{W}_t = (W_{1t}, W_{2t}, ..., W_{nt})^\intercal$ is an n-dimensional standard Brownian motion adapted to $\mathbb{F}$, and $\boldsymbol{\mu}_r : [0, \infty) \times \mathbb{R}^{n+m} \mapsto \mathbb{R}^n$ and $\boldsymbol{\sigma} : [0, \infty) \times \mathbb{R}^{n+m} \mapsto \mathbb{R}^{n \times n}$ are deterministic functions sufficiently well-behaved for the SDE to have an appropriately unique solution. Furthermore, we define $\widetilde{\boldsymbol{X}}_t$ as the $n + m$ dimensional vector $(\boldsymbol{X}_t; \boldsymbol{\chi}_t)$ where $\boldsymbol{\chi}_t = (\chi_{1t}, \chi_{2t}, ..., \chi_{mt})^\intercal$ has the interpretation of an m-dimensional state variable, the dynamics of which we need not specify.

Remark 1. In what follows we consider the scenario of what happens when we hedge an option on $\boldsymbol{X}_t$, ignorant of the existence of the state variable $\boldsymbol{\chi}_t$, as well as the form of $\boldsymbol{\mu}_r(\cdot, \cdot)$ and $\boldsymbol{\sigma}_r(\cdot, \cdot)$. Specifically, we shall imagine that we are misguided to the extent that we would model the dynamics of $\boldsymbol{X}_t$ as a local volatility SDE with diffusion matrix $\boldsymbol{\sigma}_h(t, \boldsymbol{X}_t)$. Similar assumptions pertain to the market, although here we label the "implied" diffusion matrix $\boldsymbol{\sigma}_i(t, \boldsymbol{X}_t)$ to distinguish it from our personal belief. For technical reasons, all $\boldsymbol{\sigma}(t, \boldsymbol{X}_t)$s are assumed to obey the regularity condition[4] $\mathbb{E} \int_t^s |\mathbf{D}_{\boldsymbol{X}}\boldsymbol{\sigma}(u, \boldsymbol{X}_u)|^2 du < \infty$, $\forall t \leq s \in [0, \infty)$. Finally, the reader is encouraged to keep a firm eye on the indices throughout these pages. We use r and i to emphasise that the volatility is *real* and *implied* respectively. Similarly h refers to an arbitrary *hedge* volatility.

Theorem 1. ***The Fundamental Theorem of Derivative Trading.*** *Let $V_t = V(t, \boldsymbol{X}_t) \in \mathcal{C}^{1,2}([0, \infty) \times \mathbb{R}^n)$ be the price process of a European option with terminal pay-off $V_T = g(\boldsymbol{X}_T)$. Assume we at time $t = 0$ acquire such an option for the market-price V_0^i, with the associated (not necessarily uniquely determined) implied volatility $\boldsymbol{\sigma}_i(0, \boldsymbol{X}_0)$. Furthermore, suppose we set out to Δ-hedge our position, but remain under the impression that the correct volatility ought, in fact, to be $\boldsymbol{\sigma}_h(0, \boldsymbol{X}_0)$, leading to the fair price V_0^h. Then the present value of the* ***profit-&-loss*** *we incur from holding such a portfolio over the interval $\mathbb{T} = [0, T]$ is*

$$P\&L_{\mathbb{T}}^h = V_0^h - V_0^i + \tfrac{1}{2} \int_0^T e^{-\int_0^t r_u du} \, \text{tr}[\mathbf{D}_{\boldsymbol{X}} \boldsymbol{\Sigma}_{rh}(t, \widetilde{\boldsymbol{X}}_t) \mathbf{D}_{\boldsymbol{X}} \nabla^2_{\boldsymbol{xx}} V_t^h] dt, \tag{3}$$

where $r_u = r(u, \boldsymbol{X}_u)$ is the locally risk free rate, $\nabla^2_{\boldsymbol{xx}}$ is the Hessian operator, and

$$\boldsymbol{\Sigma}_{rh}(t, \widetilde{\boldsymbol{X}}_t) \equiv \boldsymbol{\sigma}_r(t, \widetilde{\boldsymbol{X}}_t)\boldsymbol{\sigma}_r^\intercal(t, \widetilde{\boldsymbol{X}}_t) - \boldsymbol{\sigma}_h(t, \boldsymbol{X}_t)\boldsymbol{\sigma}_h^\intercal(t, \boldsymbol{X}_t). \tag{4}$$

is a matrix which takes values in $\mathbb{R}^{n \times n}$. ∎

Proof: Let $\{\Pi_t^h\}_{t \in [0,T]}$ be the value process of the hedge portfolio long one option valued according to the implied market conception, $\{V_t^i\}_{t \in [0,T]}$, and short $\Delta_t^h = \nabla_{\boldsymbol{x}} V_t^h$ units of the underlying $\{\boldsymbol{X}_t\}_{t \in [0,T]}$, where $\nabla_{\boldsymbol{x}}$ is the gradient operator. We suppose the money account B is chosen such that the net value of the position is zero:

$$\Pi_t^h = V_t^i + B_t - \nabla_{\boldsymbol{x}} V_t^h \bullet \boldsymbol{X}_t = 0$$

[4] Here we define the matrical norm $|\mathbf{D}_{\boldsymbol{X}}\boldsymbol{\sigma}| \equiv \text{tr}[\mathbf{D}_{\boldsymbol{X}}\boldsymbol{\sigma}\boldsymbol{\sigma}^\intercal\mathbf{D}_{\boldsymbol{X}}^\intercal]^{1/2}$.

where $\bullet$ is the dot product. Now consider the infinitesimal change to the value of this portfolio over the interval $[t, t+dt]$, where $t \in [0, T)$. From the *self-financing condition* we have that

$$d\Pi_t^h = dV_t^i + r_t B_t dt - \nabla_{\boldsymbol{x}} V_t^h \bullet (d\boldsymbol{X}_t + \boldsymbol{q}_t \circ \boldsymbol{X}_t dt),$$

where $\boldsymbol{q}_t = (q_1(t, X_{1t}), q_2(t, X_{2t}), ..., q_n(t, X_{nt}))^\intercal$ codifies the continuous dividend yields and $\circ$ is the Hadamard (entry-wise) product[5]. Jointly, the two previous equations entail that

$$d\Pi_t^h = dV_t^i - \nabla_{\boldsymbol{x}} V_t^h \bullet (d\boldsymbol{X}_t - (r_t \boldsymbol{\iota} - \boldsymbol{q}_t) \circ \boldsymbol{X}_t dt) - r_t V_t^i dt, \tag{5}$$

where $\boldsymbol{\iota} = (1, 1, ..., 1)^\intercal \in \mathbb{R}^n$.

Now consider the option valued under $\boldsymbol{\sigma}_h(t, \boldsymbol{X}_t)$; from the multi-dimensional Itō formula (see for instance Björk (2009), p. 65.) we have that

$$dV_t^h = \{\partial_t V_t^h + \tfrac{1}{2}\mathrm{tr}[\boldsymbol{\sigma}_r^\intercal(t, \widetilde{\boldsymbol{X}}_t)\mathbf{D}_{\boldsymbol{X}} \nabla_{\boldsymbol{xx}}^2 V_t^h \mathbf{D}_{\boldsymbol{X}} \boldsymbol{\sigma}_r(t, \widetilde{\boldsymbol{X}}_t)]\}dt + \nabla_{\boldsymbol{x}} V_t^h \bullet d\boldsymbol{X}_t, \tag{6}$$

where we have used the fact that $\boldsymbol{X}_t$ is governed by (2). Meanwhile, V_t^h satisfies the multi-dimensional Black Scholes equation for dividend paying underlyings (see for instance Björk, Theorem 13.1 and Proposition 16.7),

$$r_t V_t^h = \partial_t V_t^h + \nabla_{\boldsymbol{x}} V_t^h \bullet ((r_t \boldsymbol{\iota} - \boldsymbol{q}_t) \circ \boldsymbol{X}_t) + \tfrac{1}{2}\mathrm{tr}[\boldsymbol{\sigma}_h^\intercal(t, \boldsymbol{X}_t)\mathbf{D}_{\boldsymbol{X}} \nabla_{\boldsymbol{xx}}^2 V_t^h \mathbf{D}_{\boldsymbol{X}} \boldsymbol{\sigma}_h(t, \boldsymbol{X}_t)]. \tag{7}$$

Combining this expression with the Itō expansion we obtain,

$$\begin{aligned} 0 = &\ - dV_t^h + r_t V_t^h dt + \nabla_{\boldsymbol{x}} V_t^h \bullet (d\boldsymbol{X}_t - (r_t \boldsymbol{\iota} - \boldsymbol{q}_t) \circ \boldsymbol{X}_t dt) \\ &+ \tfrac{1}{2}\mathrm{tr}[\mathbf{D}_{\boldsymbol{X}}\big(\boldsymbol{\sigma}_r(t, \widetilde{\boldsymbol{X}}_t)\boldsymbol{\sigma}_r^\intercal(t, \widetilde{\boldsymbol{X}}_t) - \boldsymbol{\sigma}_h(t, \boldsymbol{X}_t)\boldsymbol{\sigma}_h^\intercal(t, \boldsymbol{X}_t)\big)\mathbf{D}_{\boldsymbol{X}} \nabla_{\boldsymbol{xx}}^2 V_t^h]dt \end{aligned} \tag{8}$$

where we have used the fact that the trace is invariant under cyclic permutations of its constituent matrices. Finally, defining $\boldsymbol{\Sigma}_{rh}(t, \widetilde{\boldsymbol{X}}_t)$ as in (4), and adding (8) to (5) we obtain

$$\begin{aligned} d\Pi_t^h = &\ dV_t^i - dV_t^h - r_t(V_t^i - V_t^h)dt + \tfrac{1}{2}\mathrm{tr}[\mathbf{D}_{\boldsymbol{X}} \boldsymbol{\Sigma}_{rh}(t, \widetilde{\boldsymbol{X}}_t)\mathbf{D}_{\boldsymbol{X}} \nabla_{\boldsymbol{xx}}^2 V_t^h]dt \\ = &\ e^{\int_0^t r_u du} d(e^{-\int_0^t r_u du}(V_t^i - V_t^h)) + \tfrac{1}{2}\mathrm{tr}[\mathbf{D}_{\boldsymbol{X}} \boldsymbol{\Sigma}_{rh}(t, \widetilde{\boldsymbol{X}}_t)\mathbf{D}_{\boldsymbol{X}} \nabla_{\boldsymbol{xx}}^2 V_t^h]dt. \end{aligned} \tag{9}$$

Whilst a perfect hedge would render this infinitesimal value-change in the portfolio *zero*, this is clearly not the case here. In fact, upon discounting (9) back to the present ($t = 0$) and integrating up the infinitesimal components, we find that net profit-&-loss incurred over the life-time of the portfolio is

$$\begin{aligned} P\&L_{\mathbb{T}}^h &= \int_0^T d(e^{-\int_0^t r_u du}(V_t^i - V_t^h)) + \int_0^T e^{-\int_0^t r_u du} \tfrac{1}{2}\mathrm{tr}[\mathbf{D}_{\boldsymbol{X}} \boldsymbol{\Sigma}_{rh}(t, \widetilde{\boldsymbol{X}}_t)\mathbf{D}_{\boldsymbol{X}} \nabla_{\boldsymbol{xx}}^2 V_t^h]dt \\ &= V_0^h - V_0^i + \tfrac{1}{2}\int_0^T e^{-\int_0^t r_u du}\ \mathrm{tr}[\mathbf{D}_{\boldsymbol{X}} \boldsymbol{\Sigma}_{rh}(t, \widetilde{\boldsymbol{X}}_t)\mathbf{D}_{\boldsymbol{X}} \nabla_{\boldsymbol{xx}}^2 V_t^h]dt. \end{aligned}$$

where $P\&L_{\mathbb{T}}^h \equiv \int_0^T e^{-\int_0^t r_u du} d\Pi_t^h$ and the last line makes use of the fact that $V_T^i = V_T^h = g(\boldsymbol{X}_T)$. This is the desired result. □

[5]Per definition, if A and B are matrices of equal dimensions, then $(A \circ B)_{ij} = A_{ij}B_{ij}$.

Remark 2. A few observations on this proof are in order: first, the relative simplicity of (3) clearly boils down to the assumption that the market is perceived to be driven by a local volatility model. If this assumption is dropped equation (7) no longer holds. Secondly, it should be clear that the value of the *P&L* changes sign if we are short on the derivative and long the underlying. Thirdly, the market price of the derivative enters only though the initial price V_0. That is because we look at the profit-&-loss accrued over the entire life-time of the portfolio. The case of marking-to-market requires further analysis and/or assumption. We will elaborate on this in the following subsection.

Remark 3. From a generalist's perspective, theorem 1 suffers from a number of glaring limitations: for instance, the governing asset price dynamics only considers Brownian stochasticity, the hedge is assumed to be a workaday Δ-hedge, and the option type is vanilla European in the sense that the terminal pay-off is determined by the instantaneous price of the underlying assets. Fortunately, the Fundamental Theorem can readily be extended in various directions: e.g. it can be shown that if $V_t = V(t, X_t, A_t)$ is an Asian option written on the continuous average A_t of the underlying process X_t, then the Fundamental Theorem remains form invariant. In section three we consider one particularly topical dynamical modification viz. the incorporation of possible market crashes through jump diffusion.

2.2 The Implications for Δ-Hedging.

From a first inspection, the Fundamental Theorem quite clearly demonstrates that reasonably successful hedging is possible *even* under significant model uncertainty. Indeed, as Davis (2010) puts it "without some robustness property of this kind, it is hard to imagine that the derivatives industry could exist at all". In this section, we dive further into the implications of what happens to our portfolio, by considering the case where we hedge with (a) the real volatility, and (b) the implied volatility.

Hedging With the Real Volatility. Suppose we happen to be bang-on our estimate of the real volatility matrix in our Δ-hedge, i.e. let $\boldsymbol{\sigma}_h(t, \boldsymbol{X}_t) = \boldsymbol{\sigma}_r(t, \widetilde{\boldsymbol{X}}_t)$ a.s. $\forall t \in [0, T]$, then $\boldsymbol{\Sigma}_{rr}(t, \widetilde{\boldsymbol{X}}_t) = \mathbf{0}$ and the present valued profit-&-loss amounts to

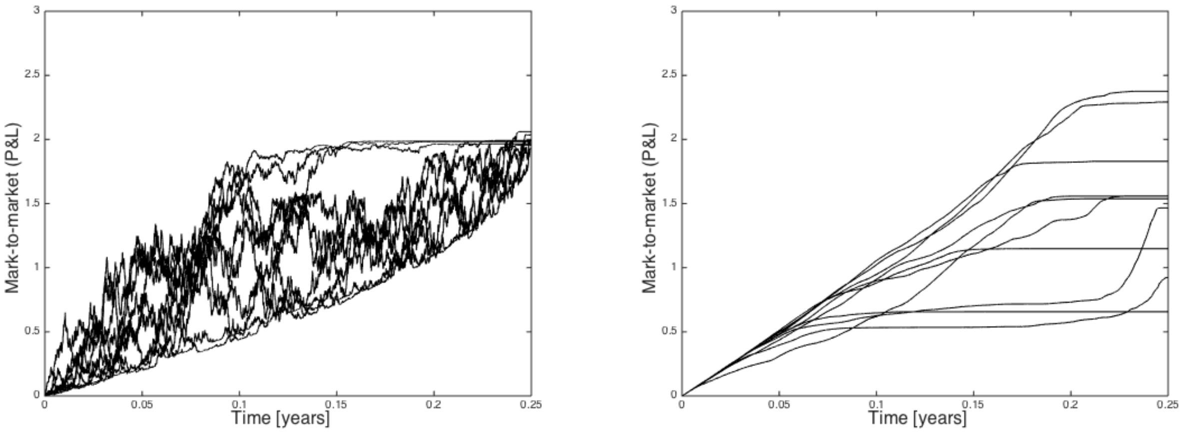
$$P\&L^r_{\mathbb{T}} = V^r_0 - V^i_0,$$

which is manifestly deterministic. However, we observe that this relies crucially on us holding the portfolio until expiry of the option. Day-to-day fluctuations of the profit-&-loss still vary stochastically (erratically) as it is vividly demonstrated by combining equation (8) (where $h = i$) with equation (5) (where $h = r$):

$$\begin{aligned} d\Pi^r_t &= \tfrac{1}{2}\,\mathrm{tr}[\mathbf{D}_{\boldsymbol{X}}\boldsymbol{\Sigma}_{ri}(t, \widetilde{\boldsymbol{X}}_t)\mathbf{D}_{\boldsymbol{X}}\nabla^2_{\boldsymbol{xx}}V^i_t]dt \\ &\quad + \nabla_{\boldsymbol{x}}\left(V^i_t - V^r_t\right) \bullet \left\{(\boldsymbol{\mu}^r_t - r_t\boldsymbol{\iota} + \boldsymbol{q}_t) \circ \boldsymbol{X}_t dt + \mathbf{D}_{\boldsymbol{X}}\boldsymbol{\sigma}_r(t, \widetilde{\boldsymbol{X}}_t)d\boldsymbol{W}_t\right\}, \end{aligned}$$

cf. the explicit dependence of the Brownian increment. As for the *profitability* of the Δ-hedging strategy, this is a complex issue which ultimately must be studied on a case-by-case basis. However, for options with positive *vega*[6], it suffices to require that the real volatility everywhere exceeds the implied volatility.

[6] A clear example of vega being manifestly positive would be European calls and puts, which satisfy the assumptions needed to derive the Black-Scholes formula. Explicitly, $\nu \equiv \frac{\partial V}{\partial \sigma} = S_t e^{-\delta(T-t)}\phi(d_1)\sqrt{T-t} > 0$ where ϕ is the standard normal pdf and d_1 has the usual definition.



(a) P&L paths with the real volatility

(b) P&L paths with the implied volatility

Figure 1: (a) Delta hedging a portfolio assuming that $\sigma_h = \sigma_r$. The parameter specifications are: $r = 0.05$, $\mu = 0.1$, $\sigma_i = 0.2$, $\sigma_r = 0.3$, $S_0 = 100$, $K = 100$, $q = 0$ and $T = 0.25$. The portfolio is rebalanced 5000 times during the lifetime of the option. Observe that while the P&L fluctuates randomly along the path of S_t due to the presence of dW_t, the accumulated P&L at the maturity of the option is the deterministic quantity $\Pi_T = e^{rT}(V_0^r - V_0^i)$. From the Black-Scholes formula it follows that $V_0^r = 6.583$ and $V_0^i = 4.615$ so $\Pi_{T=1} = 1.993$. The fact that our ten paths only approximately hit this terminal value is attributable to the discretisation of the hedging which should be done in continuous time. (b) Delta hedging a portfolio assuming that $\sigma_h = \sigma_i$.The parameter specifications are as before. Evidently, the accumulated P&L stays highly path dependent for the *entire* duration of the option. However, the curves per se are smooth, which highlights that $d\Pi_t^i$ does not depend explicitly on the Brownian increment.

Hedging With the Implied Volatility. Suppose instead we hedge the portfolio using the implied volatility matrix $\boldsymbol{\sigma}_i(t, \boldsymbol{X}_t)$ $\forall t \in [0, T]$, then the associated present-valued profit-&-loss is of the form

$$P\&L_{\mathbb{T}}^i = \frac{1}{2} \int_0^T e^{-\int_0^t r_u du} \operatorname{tr}[\mathbf{D}_{\boldsymbol{X}} \boldsymbol{\Sigma}_{ri}(t, \widetilde{\boldsymbol{X}}_t) \mathbf{D}_{\boldsymbol{X}} \nabla_{\boldsymbol{xx}}^2 V_t^i] dt.$$

As we find ourselves integrating over the stochastic process $\boldsymbol{X}_t$, this profit-&-loss is manifestly stochastic. Notice though that $d\Pi_t^i$ here does *not* depend explicitly on the Brownian increment (the daily profit-and-loss is $\mathcal{O}(dt)$) which gives rise to point that "bad models cause bleeding - not blow-ups". As for the *profitability* of the strategy, again this is a complex issue: however, insofar as $\boldsymbol{\Sigma}_{ri}(t, \widetilde{\boldsymbol{X}}_t) \circ \nabla_{\boldsymbol{xx}}^2 V_t^i$ is *positive definite* a.s. for all $t \in [0, T]$, then we're making a profit with probability one. To see this, recall that the trace can be written as[7]

$$\operatorname{tr}[\mathbf{D}_{\boldsymbol{X}} \boldsymbol{\Sigma}_{ri}(t, \widetilde{\boldsymbol{X}}_t) \mathbf{D}_{\boldsymbol{X}} \nabla_{\boldsymbol{xx}}^2 V_t^i] = \boldsymbol{X}_t^\intercal (\boldsymbol{\Sigma}_{ri}(t, \widetilde{\boldsymbol{X}}_t) \circ \nabla_{\boldsymbol{xx}}^2 V_t^i) \boldsymbol{X}_t,$$

In particular, if $\boldsymbol{\Sigma}_{ri}(t, \widetilde{\boldsymbol{X}}_t) \circ \nabla_{\boldsymbol{xx}}^2 V_t^i$ is positive definite at all times, i.e.

$$\forall t \in [0, T] \;\; \forall \boldsymbol{X}_t \in \mathbb{R}^n : \;\; \boldsymbol{X}_t^\intercal (\boldsymbol{\Sigma}_{ri}(t, \widetilde{\boldsymbol{X}}_t) \circ \nabla_{\boldsymbol{xx}}^2 V_t^i) \boldsymbol{X}_t > 0,$$

[7] This follows from the general identity for matrices $\mathbf{A}$ and $\mathbf{B}$ of corresponding dimensions: $\mathbf{x}^\intercal(\mathbf{A} \circ \mathbf{B})\mathbf{y} = \operatorname{tr}[\mathbf{D_x} \mathbf{A} \mathbf{D_y} \mathbf{B}^\intercal]$ where $\mathbf{x}$ and $\mathbf{y}$ are vectors.

then $P\&L^i_{\mathbb{T}} > 0$. A sufficient condition for this to be the case is that $\mathbf{\Sigma}_{ri}(t, \widetilde{\boldsymbol{X}}_t)$ and $\nabla^2_{\boldsymbol{xx}} V^i_t$ individually are positive definite $\forall t$, as demonstrated by the *Schur Product Theorem*.

Wilmott's Hedge Experiment. The points imbued in the previous two paragraphs are forcefully demonstrated in the event that there is only one risky asset in existence, the derivative is a European call option and all volatilities are assumed constant. Based on Wilmott and Ahmad, Figure 1 clearly illustrates the behaviour of the profit-&-loss paths insofar as we hedge with (a) the real volatility, and (b) the implied volatility. Again, the main insights are as follows: hedging V^i_t with the real volatility causes the P&L of the portfolio to fluctuate erratically over time, only to land at a deterministic value at maturity. On the other hand, hedging V^i_t with the implied volatility yields smoother (albeit still stochastic) P&L curves. Nonetheless, here there is no way of telling what the P&L actually amounts to at maturity.

Rather perturbingly, both strategies blatantly suggest the relative ease with which we can make *volatility arbitrage*. Specifically, assuming that the historical volatility is a reasonable proxy for the real volatility, $\sigma_{\text{hist}} \approx \sigma_r$, and that $\sigma_{\text{hist}} > \sigma_i$ ($\sigma_{\text{hist}} < \sigma_i$), it would suffice to go long (short) on the hedge portfolio for $\mathbb{P}(P\&L_{\mathbb{T}} \geq 0) = 1$ and $\mathbb{P}(P\&L_{\mathbb{T}} > 0) > 0$.

Reality, of course, is not always as simple as our abstract idealisations, wherefore we dedicate section four to an empirical investigation of Wilmott's hedge experiment.

2.3 Applications

Due to the presence of the real volatility, the exact nature of which transcends our epistemic domain, one might reasonably ponder whether the Fundamental Theorem conveys any practical points besides those of the preceding subsection. Using two poignant (even if somewhat eccentric) examples, we will argue that the gravity of the Fundamental Theorem propagates well into risk management and volatility surface calibration. Zero rates and dividends will be assumed throughout.

Example 1. Let $V_t(T, K)$ be the price process of a European strike K maturity T call or put option, written on an underlying which obeys Geometric Brownian Motion, $dX_t = \mu_r X_t dt + \sigma_r X_t dW_t$, where μ_r, σ_r are constants. Suppose we Δ-hedge a long position on V_t at the *implied* volatility, $\sigma_h = \sigma_i$, then the Fundamental Theorem implies that

$$P\&L^i_{\mathbb{T}} = \tfrac{1}{2} \int_0^T (\sigma_r^2 - \sigma_i^2) X_t^2 \Gamma^i_t dt,$$

where $\Gamma^i_t = \phi(d^i_1)/(X_t \sigma_i \sqrt{T-t})$ is the option's gamma, $\phi : \mathbb{R} \mapsto \mathbb{R}_+$ is the standard normal pdf and $d^i_1 \equiv \frac{1}{\sigma_i \sqrt{T-t}} \left\{ \ln(X_t/K) + \frac{1}{2}\sigma_i^2 (T-t) \right\}$. Since $\forall t \ \Gamma^i_t > 0$ the strategy is profitable if and only if $\sigma_r^2 > \sigma_i^2$.[8] Furthermore, by maximising the integrand with respect to X_t we find that the $P\&L^i_{\mathbb{T}}$ is maximal when

$$X^*_t = K e^{\frac{1}{2}\sigma_i^2 (T-t)},$$

Specifically, upon evaluating the integral explicitly we find that

$$\max_{X_t} P\&L^i_{\mathbb{T}} = \sqrt{\frac{T}{2\pi}} \frac{K}{\sigma_i} (\sigma_r^2 - \sigma_i^2).$$

[8]In the real market one often finds $\sigma_r < \sigma_i$. In other words, we should have shorted the portfolio.

Using elementary statistics we can compute a confidence interval for the real volatility based on historical observations. Hence, we can compute a confidence interval for the maximal profit-&-loss we might face upon holding the hedge portfolio till expiry.

Example 2. Let $V_t = C_t(T, K)$ be the price process of a European strike K maturity T call option written on an underlying price process X. As in (1) we assume the fundamental dynamics to be of the form $dX_t = \mu_r(t, \widetilde{X}_t)X_t dt + \sigma_r(t, \widetilde{X}_t)X_t dW_t$, where $\widetilde{X}_t$ is defined as the $(m+1)$-dimensional vector $(X_t, \boldsymbol{\chi}_t)^\intercal$ where $\boldsymbol{\chi}$ is a state variable. Since r is assumed zero, it follows that X_t is a local martingale under the risk neutral measure. Specifically, defining

$$d\mathbb{Q} \equiv \exp\left\{-\int_0^T \theta_t dW_t - \tfrac{1}{2}\int_0^T \theta_t^2 dt\right\} d\mathbb{P}, \quad \text{where} \;\; \theta_t \equiv \frac{\mu_r(t, \widetilde{X}_t)}{\sigma_r(t, \widetilde{X}_t)},$$

we have the $\mathbb{Q}$-dynamics $dX_t = \sigma_r(t, \widetilde{X}_t)X_t dW_t^{\mathbb{Q}}$.

Now consider the (admittedly somewhat contrived) scenario of a Δ-hedged portfolio, long one unit of the call, for which σ_h and $\sigma_i = 0$ are both zero[9]. The associates value process is

$$\Pi_t^i = C_t^i(T, K) + B_t - \partial_x C_t^h(T, K) \cdot X_t = (X_t - K)^+ + B_t - \mathbb{1}_{\{X_t > K\}}X_t, \tag{10}$$

where $\mathbb{1}_{\{X_t > K\}}$ is the indicator function. The important point here is that $(X_t - K)^+$ may be reinterpreted as the terminal pay-off of a strike K maturity t call option (obviously, the specification $\sigma_h = \sigma_i = 0$ is paramount here). Substituting (10) into the infinitesimal form of the Fundamental Theorem, $d\Pi_t^i = \frac{1}{2}(\sigma_r^2(t, \widetilde{X}_t) - \sigma_i^2)X_t^2 \partial_{xx}^2 C_t^i(T, K)dt$, we find that

$$d((X_t - K)^+ + B_t - \mathbb{1}_{\{X_t > K\}}X_t) = \tfrac{1}{2}\sigma_r^2(t, \widetilde{X}_t)X_t^2 \delta(X_t - K)dt, \tag{11}$$

where we once again have made use of $\sigma_i = 0$, alongside the fact that $\partial_x \mathbb{1}_{\{X_t > K\}}$ is the Dirac delta-function $\delta(X_t - K)$. Taking the risk neutral expectation of (11), conditional on $\mathscr{F}_0$, the left-hand side reduces to

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[LHS] &= \mathbb{E}^{\mathbb{Q}}[d(X_t - K)^+] + \mathbb{E}^{\mathbb{Q}}[dB_t - \mathbb{1}_{\{X_t > K\}}dX_t] \\
&= d\mathbb{E}^{\mathbb{Q}}[(X_t - K)^+] - \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{X_t > K\}}\sigma_r(t, \widetilde{X}_t)X_t dW_t^{\mathbb{Q}}] \\
&= dC_0^r(t, K) - \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{X_t > K\}}\sigma_r(t, \widetilde{X}_t)X_t dW_t^{\mathbb{Q}}|\mathscr{F}_t]] \\
&= dC_0^r(t, K) - \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{X_t > K\}}\sigma_r(t, \widetilde{X}_t)X_t \mathbb{E}^{\mathbb{Q}}[dW_t^{\mathbb{Q}}|\mathscr{F}_t]] \\
&= dC_0^r(t, K),
\end{aligned} \tag{12}$$

where the second line uses $r = 0$ (whence $dB_t = 0$) and the fact risk neutral dynamics obeys the real dynamics $dX_t = \sigma_r(t, \widetilde{X}_t)X_t dW_t^{\mathbb{Q}}$, whilst the third line uses the law of iterated expectations and the fact that $\mathbb{E}^{\mathbb{Q}}[(X_t - K)^+]$ is the time zero price of a strike K maturity t call option. Finally, the fourth line follows from the $\mathscr{F}_t$-measurability of $\mathbb{1}_{\{X_t > K\}}\sigma_r(t, \widetilde{X}_t)X_t$, whilst the fifth line exploits $\mathbb{E}^{\mathbb{Q}}[dW_t^{\mathbb{Q}}] = 0$.

[9] To be precise, the contrived part is the assumption that the call trades at zero volatility; less so that we hedge it at zero volatility. The latter corresponds to a so-called *stop-loss strategy*, see Carr & Jarrow (1990).

As for the right-hand side, let $f^{\mathbb{Q}}_{\sigma_r^2, X_t}(\sigma^2, x)$ be the joint probability density of X_t and $\sigma_r^2 \equiv \sigma_r^2(t, \widetilde{X}_t)$ under $\mathbb{Q}$, then

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[RHS] &= \tfrac{1}{2} \iint_{\mathbb{R}_+^2} \sigma^2 x^2 \delta(x-K) f^{\mathbb{Q}}_{\sigma_r^2, X_t}(\sigma^2, x) d\sigma^2 dx dt \\
&= \tfrac{1}{2} \iint_{\mathbb{R}_+^2} \sigma^2 x^2 \delta(x-K) f^{\mathbb{Q}}_{\sigma_r^2}(\sigma^2 | X_t = x) f^{\mathbb{Q}}_{X_t}(x) d\sigma^2 dx dt \\
&= \tfrac{1}{2} \int_{\mathbb{R}_+} x^2 \delta(x-K) f^{\mathbb{Q}}_{X_t}(x) \left\{ \int_{\mathbb{R}_+} \sigma^2 f^{\mathbb{Q}}_{\sigma_r^2}(\sigma^2 | X_t = x) d\sigma^2 \right\} dx dt \\
&\equiv \tfrac{1}{2} \int_{\mathbb{R}_+} x^2 \delta(x-K) f^{\mathbb{Q}}_{X_t}(x) \mathbb{E}^{\mathbb{Q}}[\sigma_r^2(t, \widetilde{X}_t) | X_t = x] dx dt \\
&= \tfrac{1}{2} K^2 f^{\mathbb{Q}}_{X_t}(K) \mathbb{E}^{\mathbb{Q}}[\sigma_r^2(t, \widetilde{X}_t) | X_t = K] dt
\end{aligned} \tag{13}$$

Now, it is easy to verify that $f^{\mathbb{Q}}_{X_t}(K) = \partial^2_{KK} C_0^r(t, K)$: indeed, we have that $\partial_K \mathbb{E}^{\mathbb{Q}}[(X_t - K)\mathbb{1}_{\{X_t > K\}}] = -\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{X_t > K\}}]$ and $-\partial_K \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_{\{X_t > K\}}] = \mathbb{E}^{\mathbb{Q}}[\delta(X_t - K)]$ (this is the so-called Breeden-Litzenberger formula). Therefore (12) and (13) jointly imply that

$$\frac{dC_0^r}{dt}(t, K) = \tfrac{1}{2} \partial^2_{KK} C_0^r(t, K) K^2 \mathbb{E}^{\mathbb{Q}}[\sigma_r^2(t, \widetilde{X}_t) | X_t = K].$$

Using the change of notation $t = T$ to emphasise that t is the *maturity* of the option (not its value at time t), this expression may be recast in the following more familiar form

$$\mathbb{E}^{\mathbb{Q}}[\sigma_r^2(T, \widetilde{X}_T) | X_T = K] = \frac{\partial_T C_0^r(T, K)}{\tfrac{1}{2} K^2 \partial^2_{KK} C_0^r(T, K)}, \tag{14}$$

which in known as the Dupire-Gyöngy-Derman formula.[10] Using some amount of extrapolation,[11] the righthand side is empirically measurable, hence (14) provides a way of calibrating the volatility surface to observed call option prices in the market[12].

Remark 4. The above derivation is arguably unconventional and neither rigorous nor the quickest way to demonstrate (14). In fact, the entire point of setting $\sigma_i = 0$ is essentially to extract the Itō-(Tanaka) formula applied to $(X_t - K)^+$, from which Derman et al.'s derivation takes its starting point. We keep the derivation here, as it provides a curious glimpse into how two philosophically quite distinct theorems can be interconnected.

3 The Gospel of the Jump

Following remark 3, it is worthwhile exploring how the Fundamental Theorem can be adapted to new terrain. For instance, it is well known that Brownian motion in itself does not adequately capture the sporadic discontinuities that emerge in stock price processes. Hence, it is

[10]See for instance Dupire (1994) or Derman & Kani (1998).

[11]Exactly how to do this extrapolation has turned out to be sufficiently non-trivial to spurn numerous papers and successive quant-of-the-year awards a-decade-and-a-half later, see Andreasen & Huge (2011) (pure local volatility) Guyon & Henry-Labordère (2012) (decorated stochastic volatility models).

[12]In Wittgensteinian terms we must "throw away the ladder" to arrive at this final conclusion. Hitherto, we have assumed that the real parameters (r) are fundamentally unobservable, whilst the implied parameters (i) are those we are exposed to in the market. Yet, no such distinction exists in the works of Dupire et al., whence the r superscript in (14) really ought to be dropped.

opportune to scrutinise the effect of a jump diffusion process, which in turn will give rise to another valuable lesson on the profitability of imperfect hedging.

Already, it is a well-known fact that exact hedges generally do not exist in a jump economy where the true dynamics of the underlying is perfectly disseminated (see e.g. Shreve (2008) or Privault (2013)). It is thus of some theoretical interest to see how this preexisting hedge error is further complicated under the model error framework of the Fundamental Theorem. We note that this problem has been treated (with various degrees of rigour) in Andreasen (2003) and Davis (2010) when the hedge volatility is implied. Our main contribution is to generalise the result to an arbitrary hedge volatility.

Suppose the real dynamics of the underlying price process obeys

$$dX_t = X_{t-}[\mu_r(t, \widetilde{X}_t)dt + \sigma_r(t, \widetilde{X}_t)dW_t + dY_t], \tag{15}$$

where $\{Y_t\}_{t\geq 0}$ is the compound Poisson process $Y_t \equiv \sum_{k=1}^{N_t} J_k$, such that $\{N_t\}_{t\geq 0}$ is an intensity-λ Poisson process, and $\{J_k\}_{k\geq 1}$ is a sequence of relative jump-sizes, assumed to be i.i.d. square-integrable random variables with density function $f : \mathbb{R} \mapsto \mathbb{R}_+$. Oblivious to the true nature of (15), we imagine that pricing and hedging should be performed under the tuple $\langle \sigma_h(t, X_t), \mathbb{Q}\rangle$, where $\sigma_h(\cdot, \cdot)$ is a local volatility function and $\mathbb{Q}$ is the risk neutral measure

$$d\mathbb{Q}_{\theta,\lambda^{\mathbb{Q}},f^{\mathbb{Q}}} \equiv \exp\left\{-(\lambda^{\mathbb{Q}} - \lambda)T - \int_0^T \theta_t dW_t - \tfrac{1}{2}\int_0^T \theta_t^2 dt\right\} \prod_{k=1}^{N_T} \frac{\lambda^{\mathbb{Q}} f^{\mathbb{Q}}(J_k)}{\lambda f(J_k)} d\mathbb{P},$$

where $\{\theta_t\}_{t\geq 0}$ is a bounded adapted process, and $\lambda^{\mathbb{Q}}, f^{\mathbb{Q}}$ respesctively represent the jump intensity and jump-size distribution under $\mathbb{Q}$. Specifically, the price of an option with terminal pay-off $g(X_T)$ is determined as $V_t^h = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_u du} g(X_T)]$, where the underlying is driven by

$$dX_t = X_{t-}[r_t dt + \sigma_h(t, X_t)dW_t^{\mathbb{Q}} + dY_t - \lambda^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}[J_k]dt],$$

and $\mathbb{Q}_{\theta,\lambda^{\mathbb{Q}},f^{\mathbb{Q}}}$ has been specified such that

$$\mu_h(t, X_t) - \sigma_h(t, X_t)\theta_t + \lambda^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}[J_k] = r_t, \tag{16}$$

is satisfied.[13]

Theorem 2. ***The Fundamental Theorem of Derivative Trading with Jumps.*** *Let $V_t = V(t, X_t) \in \mathcal{C}^{1,2}([0,\infty) \times \mathbb{R})$ be the price process of a European option with terminal pay-off $V_T = g(X_T)$. Assume we at time $t = 0$ acquire such an option for the market-price V_0^i, with the associated implied volatility $\sigma_i(0, X_0)$. Furthermore, suppose we set out to Δ-hedge our position, but remain under the impression that the correct volatility ought, in fact, to be $\sigma_h(0, X_0)$, leading to the fair price V_0^h. Then the present value of the profit-&-loss we incur from holding such a portfolio over the interval $\mathbb{T} = [0, T]$ is*

$$\begin{aligned} P\&L_{\mathbb{T}}^h = V_0^h - V_0^i + \int_0^T e^{-\int_0^t r_u du}\Big\{&\tfrac{1}{2}\big(\sigma_r^2(t, \widetilde{X}_t) - \sigma_h^2(t, X_t)\big)X_t^2 \partial_{xx}^2 V_t^h dt \\ &+ \big(V^h(t, X_{t-}(1 + J_{N_t})) - V^h(t, X_{t-}) - X_{t-}J_{N_t}\partial_x V_t^h\big)dN_t \\ &- \lambda^{\mathbb{Q}}\big(\mathbb{E}^{\mathbb{Q}}[V^h(t, x(1 + J_k)) - V^h(t, x)]|_{x=X_t} - X_{t-}\mathbb{E}^{\mathbb{Q}}[J_k]\partial_x V_t^h\big)dt\Big\}, \end{aligned} \tag{17}$$

[13]It should be clear the $\mathbb{Q}$ is not uniquely determined. In fact, for (16) to admit only one solution, we would require that either (i) $\lambda = \lambda^{\mathbb{Q}} = 0$ (there are *no* jumps), in which case we recover the standard Girsanov theorem with $\theta_t = (\mu_h - r)/\sigma_h$, or (ii) when $\sigma_h = 0$ and $Y_t = \alpha_t N_t$ (there are *only* jumps (of constant size α)) in which case $\mu_h - r_t = -\alpha\lambda^{\mathbb{Q}}$.

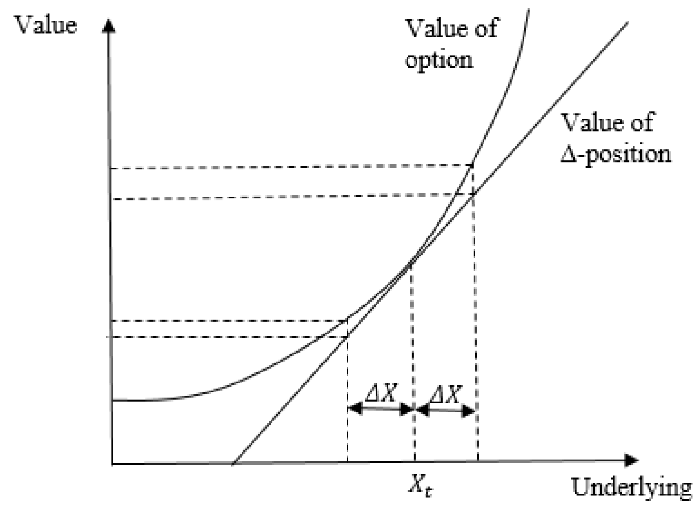


Figure 2: Suppose we Δ-hedge a long position in an option with a convex pricing function. Insofar as a jump in the underlying occurs, $X_t \mapsto X_t \pm \Delta X_t$, it follows that the value of the option will exceed the value of the Δ-position. Hence, our net *P&L* benefits from such an occurrence. Obviously, the converse will be true if we hold a short position in the option.

where $V^h(t, X_{t-}(1+J_t)) - V^h(t, X_{t-})$ *represents the change in value of the option when the underlying jumps.* ■

Sketch Proof: The proof runs in parallel with that of theorem 1. Specifically, the analogue of expression (5) is

$$d\Pi_t^h = dV_t^i - \partial_x V_t^h (dX_t^c - r_t X_t dt) - X_{t-} J_{N_t} \partial_x V_t^h dN_t - r_t V_t^i dt,$$

where dX_t^c is the continuous part of (15) i.e. $dX_t^c = X_{t-}[\mu_r(t, \widetilde{X}_t)dt + \sigma_r(t, \widetilde{X}_t)dW_t]$. Furthermore, in analogy with (6) and (7) we have the Itō formula (see for instance Privault (2013, Chapter 15))

$$\begin{aligned} dV_t^h = &\partial_t V_t^h dt + \partial_x V_t^h dX_t^c + \tfrac{1}{2}\sigma_r^2(t, \widetilde{X}_t) X_t^2 \partial_{xx}^2 V_t^h dt \\ &+ \big(V^h(t, X_{t-}(1+J_{N_t})) - V^h(t, X_{t-})\big) dN_t, \end{aligned}$$

and the partial integro-differential equation for pricing purposes

$$\begin{aligned} rV_t^h = &\partial_t V_t^h + r_t X_t \partial_x V_t^h + \tfrac{1}{2}\sigma_h^2(t, X_t) X_t^2 \partial_{xx}^2 V_t^h - \lambda^{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}[J_k] X_t \partial_x V_t^h \\ &+ \lambda^{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}[V^h(t, x(1+J_k)) - V^h(t, x)]|_{x=X_t}. \end{aligned}$$

Combining these three expressions as above yields the desired result. □

Remark 5. The last two lines in (17) (which we denote by $P\&L_J$) represent the present-valued profit-&-loss brought about by our inability to hedge the jump risk completely. In the simplest case where all jump sizes are constant and of equal magnitude, i.e. $Y_t = \alpha N_t \ \alpha \in \mathbb{R}$, these terms simplify to

$$P\&L_J = \int_0^T e^{-\int_0^t r_u du} \big\{ V^h(t, X_{t-}(1+\alpha)) - V^h(t, X_{t-}) - \alpha X_{t-} \partial_x V_t^h \big\} (dN_t - \lambda^{\mathbb{Q}} dt). \quad (18)$$

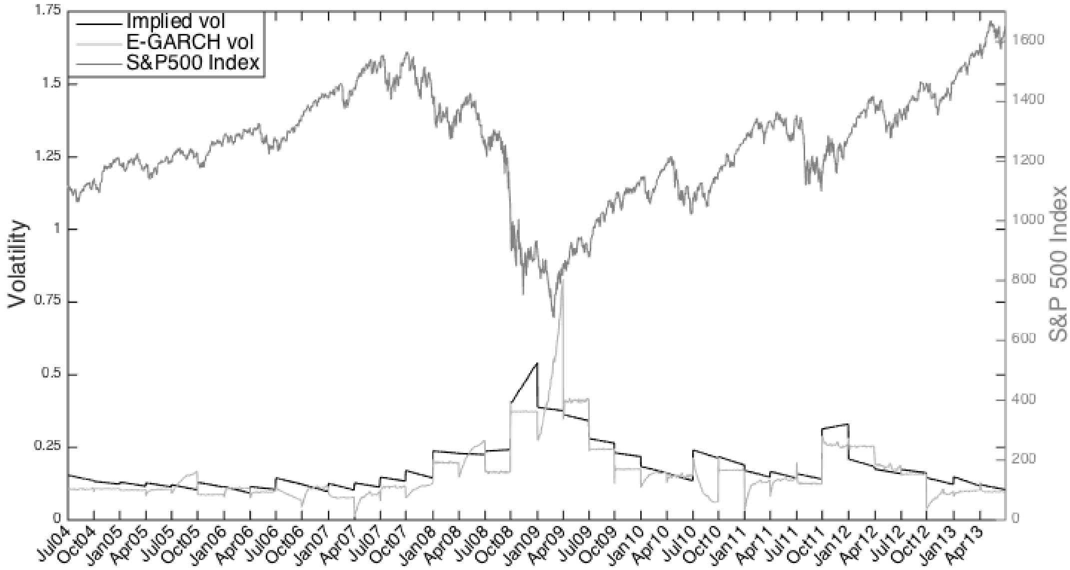


Figure 3: The top **grey curve** is the S&P500 Index plotted from July 2004 to July 2013 [units on right hand axis]. The tic-dates on the time axis have deliberately been chosen to match the purchasing dates $\{t_i\}_{i=1}^{36}$ of the 36 delta-hedged portfolios under investigation (each of which is of three months' duration). The **light grey curve** is the actual (stochastic) volatility estimated from a lognormal volatility model. Specifically, every time segment between purchasing dates $[t_i, t_{i+1})$ reflects a Monte Carlo simulated forecast based upon an EGARCH(1,1) fitted to market data from the previous time segment $[t_{i-1}, t_i)$. Finally, the **black curve** is the three-month ATM implied volatility. Specifically, every time segment between purchasing dates $[t_i, t_{i+1})$ is a static forecast based upon ATM implied volatility data from the purchasing date t_i. Both volatility curves have their units on the left hand axis.

If V is convex (a property it will inherit from the payoff function under mild conditions) then $\Delta V > \partial_x V \Delta X$ whence the integrand in $P\&L_J$ is positive. Thus, our hedge portfolio actually benefits from jumps (in either direction) of the underlying price process. Conversely, if we had shorted the option, the hedge profit would obviously take a hit in the event of a jump (in traders' terms, holding hedge portfolios with short option positions correspond to "picking up pennies in front of a steam roller"). A vivid illustration of this point is provided in figure 2.

4 Insights From Empirics: On Arbitrage and Erraticism

Inspired by Wilmott's theoretical hedge experiment, we now look into the empirical performance of Δ-hedging strategies based on (I) forecasted implied volatilities and (II) forecasted actual volatilities. Specifically, we are interested in the properties of the accumulated P&L, insofar as we Δ-hedge, till expiry, a three-month call-option on the S&P500 index, initially purchased at-the-money. We investigate a totality of 36 such portfolios over disjoint intervals between July 2004 and July 2013. This involves market data on both the underlying index and on options. Daily data on the S&P500 index is readily and freely available. For option data, we combine a 2004-2009 data set from a major commercial bank[14] with more recent prices from OptionMetrics obtained via the Wharton Financial Database.

Whilst ATM call option prices straightforwardly are obtained from the data set, the (forecasted) implied and actual volatilities require a bit of manipulation. In case of the former, we define the daily implied volatility, over the life-time of the portfolio, as the ATM implied volatility of corresponding tenor obtained at the portfolio purchasing date (the resulting volatility process is illustrated by the black curve in Figure 3). In case of the latter, we require a suitable volatility model fitted to historical data in order to predict the "actual" volatility process. Specifically, we define the daily actual volatility, over the life-time of the portfolio, as the conditional expectation of a volatility model which has been fitted to market data from the previous portfolio period. In this context, we observe that models with lognormal volatility dynamics generally have more empirical support than, say, Heston's model (see Gatheral, Jaisson & Rosenbaum (2014) and their references). The Exponential General Autoregressive Conditional Heteroskedasticity model (EGARCH(1,1)) has proven particularly felicitous in the context of S&P 500 forecasting (see Awartani & Corradi (2005)) - a result we assume applies universally for each of the 36 portfolios investigated. Thus, we hold it to be the case that daily log returns, r_t, can be modelled as $r_t = \mu + \varepsilon_t$, where μ is the mean return, and ε_t has the interpretation of a hetereoskedastic error. In particular, ε_t is construed to be the product between a white noise process, $z_t \sim N(0,1)$, and a daily standard deviation, σ_t, which obeys the relation

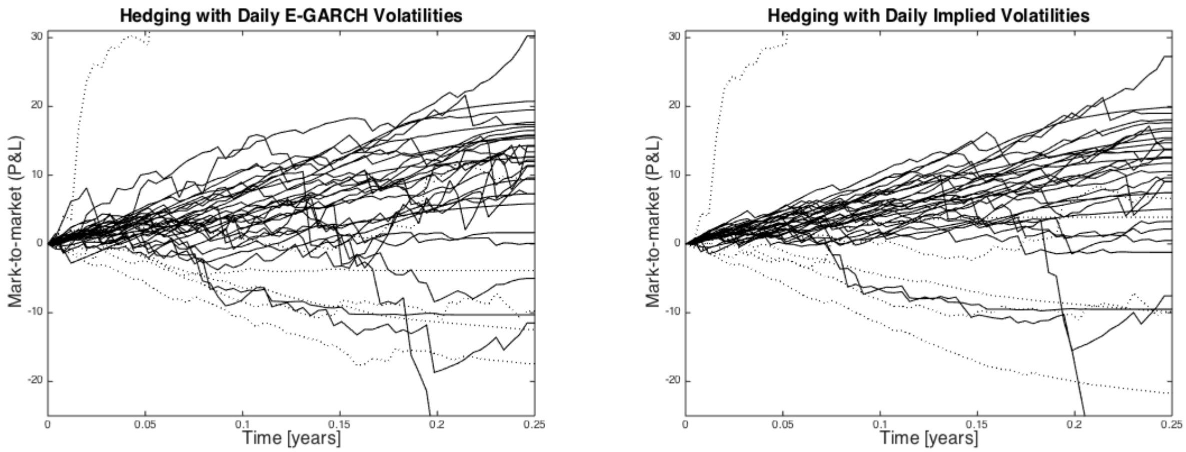
$$\log \sigma_t^2 = \alpha_0 + \alpha_1 \log \sigma_{t-1}^2 + \alpha_2 \left[\frac{|\varepsilon_{t-1}|}{\sigma_{t-1}} - \sqrt{\frac{2}{\pi}} \right] + \alpha_3 \frac{\varepsilon_{t-1}}{\sigma_{t-1}}, \tag{19}$$

where $\alpha_0, \alpha_1, \alpha_2$ and α_3 are constants. The resulting volatility process is illustrated by the light grey curve in Figure 3.

A few remarks on the estimated volatility processes are in order. First, we clearly see that volatility can change dramatically during the life-time of a portfolio. We also see that implied volatility typically is higher than actual volatility. This oft-reported result can be explained theoretically by the stochastic volatility having a market price of risk attached, see for instance Henderson, Hobson, Howison & Kluge (2005). Finally, there is a clear negative correlation between stock returns and volatility during the financial turmoil which followed the Lehman default in September 2008. All in all, reality (unsurprisingly) turns out to be a bit more complicated than the set-up in Wilmott's experiment. Still and all, does its main messages carry over? To test this, we perform a hedge experiment with the following design:

- For any given portfolio, we compute the daily implied volatilities $\{\sigma_t^{\text{imp}}\}_{t=1}^{63}$ and the daily actual volatilities $\{\sigma_t^{\text{act}}\}_{t=1}^{63}$ as outlined above. We assume there are 63 trading days over a three months period (labelled by $t = 1, 2, ..., 63$) and let S_t, r_t and q_t denote the time t value of the index, interest rate and dividend yield.
- For each of the two hedging strategies $x \in \{\sigma^{\text{imp}}, \sigma^{\text{act}}\}$ we do the following: If $\sigma_1^{\text{act}} < \sigma_1^{\text{imp}}$ we short the call ($\gamma = -1$); otherwise, we go long the call ($\gamma = +1$). Then, we set up the delta neutral portfolio $\Pi_1 = B_1 - \gamma \Delta_1^{\text{BS}}(x_1) S_1 + \gamma C_1^{\text{BS}}(\sigma_1^{\text{imp}})$ s.t. $\Pi_1 = 0$.

[14] The bank shall remain nameless, but the data can be downloaded from http://www.math.ku.dk/~rolf/Svend/



(a) Hedge error paths with EGARCH vol.

(b) Hedge error paths with implied vol.

Figure 4: Panels (a) (actual) and (b) (implied) show the path-for-path hedge error behaviour for the 36 non-overlapping three-month hedges. Dotted paths correspond to cases where we initially take a long position in the option.

- For $t = 2, 3, ..., 63$ we do the following: compute the time t value of the portfolio set up the previous day: $\tilde{\Pi}_t = B_{t-1}e^{r_{t-1}\Delta t} - \gamma\Delta_t^{\text{BS}}(x_t)S_{t+1}e^{q_{t-1}\Delta t} + \gamma C_t^{\text{BS}}(\sigma_t^{\text{imp}})$. The quantity $dP\&L_t = \tilde{\Pi}_t - \Pi_{t-1}$ defines the profit-&-loss accrued over the interval $[t-1, t]$. Next, we rebalance the portfolio such that it, once again, is delta-neutral, $\Pi_t = B_t - \gamma\Delta_t^{\text{BS}}(x_t)S_1 + \gamma C_t^{\text{BS}}(\sigma_t^{\text{imp}})$, where B_t is chosen in accordance with the self-financing condition: $\tilde{\Pi}_t = \Pi_t$.
- Finally, at maturity $t = 63$, we compute the terminal P&L, $\tilde{\Pi}_{63}$, as well as its lifetime quadratic variation, $\sum_{t=1}^{63} |dP\&L_t|^2/63$.

The 36 hedge error (or P&L) paths and the distributions of the quadratic variation of the two methods are shown in Figure 4. Table 1 reports descriptive statistics and a statistical tests of various hypotheses.

Quantity	**Mean** (m)	**Std. Dev.** (sd)	**Hypothesis Tests**
Hedge error, actual volatility	7.7	17.3	**Q:** $m = 0$? **A:** No; p-value = 1%.
Hedge error, implied volatility	7.7	15.6	**Q**: $m = 0$? **A**: No; p-value = 1%. **Q**: $sd_{act} = sd_{imp}$? **A**: Yes; p-value = 55%
Quadratic variation, actual volatility	1.2	2.1	**Q**: $m_{QV_{act}} = m_{QV_{imp}}$? **A**: No; p-val. = 1.4%.
Quadratic variation, implied volatility	0.81	2.0	

Table 1: Summary statistics and hypothesis tests for different hedge strategies.

First, we note (top panels figure 4) that even though implied volatility typically is above actual volatility, this far from creates volatility arbitrage. Hedge errors for the two methods readily become negative. A primary explanation for this is the randomness of volatility. Our Δ-hedged strategy only makes us a profit if realised volatility ends up "on the right side" of initial implied volatility. And that we don't know for sure until after the hedging period is over; we have to base our decisions on forecasts; initial forecasts even, for the fundamental theorem to apply. Notice though that the averages for both hedge errors are significantly positive. This shows that there is a risk premium that can be picked up, most often by selling options and Δ-hedging them. Because the hedge is not perfect, this compensation is anticipated. The question is, is it financially significant? In theory the hedged portfolio has an initial cost of zero, so it is not obvious how define a rate of return, but the initial option price would seem a reasonable (possibly conservative) benchmark for the collateral that would need to be posted on a hedged short call option position. From column three in Table 2 the average option price is \$ 49.2. Comparing this to the means (~ 7.7; remember this is over a three-month horizon) and standard deviations (~ 15.5; ditto) of the hedge errors in Table 1 shows that the gains are also significant in economic terms. Put differently, a crude calculation $((4 \cdot 7.7/49.2 - 0.02)/(\sqrt{4} \cdot 15.5/49.2))$ gives annualised Sharpe-ratios around 1.

If we look just at the terminal hedge errors, then the difference in riskiness (as measured by standard deviation) between hedging with actual and hedging with implied volatility is in no way statistically significant (the p-value for equality of variances is 55%). Also, the correlation between the terminal hedge error from the two approaches is 0.97. However, if we consider the quadratic variations as the measure of riskiness, then the picture changes. The average quadratic variation of the implied hedge error (0.81) is only two-thirds of the average quadratic variation of the actual hedge error (1.2) (a paired t-test for equality yields a p-value of 1.4%).

All in all this shows that volatility arbitrage is difficult, but the following insight from Wilmott's experiment stands: If you are in the business of hedging, then the use of implied volatility should make you sleep better at night.

5 Conclusion

In the world of finance, no issue is more pressing than that of hedging our risks, yet remarkably little attention has been paid to the risk brought about by the possibility that our models might be wrong. To remedy this deplorable situation, we have in this paper derived a meta-theorem that quantifies the P&L of a Δ-hedged portfolio with an erroneous volatility specification. *Meta-* to the extent that one of the constituent parameters (the real volatility) is transcendental; yet, also a theorem with some very concrete "real world" corollaries. For instance, it was shown that hedging with the implied volatility gives rise to smooth (i.e. $\mathcal{O}(dt)$) P&L-paths, whilst any other hedge volatility yields erratic (i.e. $\mathcal{O}(dW_t)$) P&L paths. In a somewhat quirkier context, the Dupire-Gyöngy-Derman formula for volatility surface calibration was shown to be a corollary.

Whilst the theorem proved in section one is more general than the versions typically found in the literature, it does not go *far enough*. Extensive empirical support has been added to the case of discontinuities in the stock price process: thus, in the Gospel of the Jump we extended the Fundamental Theorem to include compound Poisson processes, which came with the revelation that jumps unambiguously hurt you when you try to hedge short put and call option positions.

One of the most conspicuous implications of the Fundamental Theorem is undoubtedly the apparent ease with which arbitrage can be made: e.g. in the constant parameter framework of Wilmott's experiment, a free lunch is guaranteed insofar as we can establish $\max\{\sigma_{\text{hist}}, \sigma_i\}$ (in case of the former, we go long on the option - otherwise, we short it). Studying this strategy empirically, we find that the mean P&L indeed is in the positive; nonetheless, qua a significant dispersion the profit readily turns negative: the statistical arbitrage accordingly relies on us being willing to take so some significant hits along the way. Indeed, this is without even factoring in the non-negligible role of transaction costs. On the other hand, there is strong evidence that hedging at the implied volatility does yield smoother P&L paths.

A Data Used for Table 1

Contract	ATM strike	Option price	P&L$_T^{\text{act.}}$	P&L$_T^{\text{imp.}}$	Q.V.$^{\text{act.}}$	Q.V.$^{\text{imp.}}$
07-Jul-2004	1118.3	36.5852	12.2045	15.1591	0.5269	0.2615
05-Oct-2004	1134.5	33.0392	5.8372	5.0520	0.1683	0.1386
05-Jan-2005	1183.7	34.7050	11.4080	13.6705	0.1975	0.1759
06-Apr-2005	1184.1	34.9985	7.3072	9.0917	0.3162	0.1693
06-Jul-2005	1194.9	34.4864	11.9818	10.5282	0.2974	0.0894
04-Oct-2005	1214.5	37.8141	7.2779	7.4261	0.5384	0.1670
06-Jan-2006	1285.4	37.1621	12.6952	12.4934	0.1539	0.1406
07-Apr-2006	1295.5	38.2703	0.0765	0.5022	0.2827	0.2444
07-Jul-2006	1265.5	45.5356	15.3714	13.7452	0.3655	0.1974
05-Oct-2006	1353.2	42.7682	12.6179	12.6400	0.0904	0.0945
08-Jan-2007	1412.8	45.4682	-5.0096	2.1741	2.4476	1.0569
09-Apr-2007	1444.6	47.0689	19.4885	7.4564	0.7699	0.0865
09-Jul-2007	1531.8	55.8378	-11.4976	-7.5524	1.8603	1.1396
05-Oct-2007	1557.6	63.1625	1.6451	-1.2115	1.2330	0.4542
09-Jan-2008	1409.1	74.2874	9.6117	9.6158	1.1975	0.6555
09-Apr-2008	1354.5	66.2276	17.3617	19.0049	0.8019	0.6270
09-Jul-2008	1244.7	62.8179	-56.9636	-47.0345	8.0872	10.4193
07-Oct-2008	996.2	83.8510	55.3847	51.8900	9.7721	6.4129
09-Jan-2009	890.3	69.9489	14.1892	3.2637	3.0947	0.4083
10-Apr-2009	856.6	62.9702	30.2400	27.2551	0.5701	0.4336
09-Jul-2009	882.7	49.8464	-12.4499	-9.8467	0.1245	0.1039
07-Oct-2009	1057.6	49.0640	17.0496	18.0507	0.2944	0.2135
07-Jan-2010	1141.7	42.2410	16.4989	16.4106	0.2595	0.1990
09-Apr-2010	1194.4	36.6784	-10.3121	-9.5031	0.5463	0.5578
09-Jul-2010	1078.0	52.2001	15.6833	17.6455	3.0501	0.3326
07-Oct-2010	1158.1	50.6050	20.7394	19.8607	0.1926	0.2166
06-Jan-2011	1273.8	43.6970	9.4015	11.7384	0.3762	0.2400
07-Apr-2011	1333.5	44.7866	13.3942	13.8116	0.3490	0.3055
08-Jul-2011	1343.8	43.0900	-3.8722	3.8883	0.1692	0.3196
06-Oct-2011	1165.0	73.4417	14.3245	16.8015	0.8601	0.7112
06-Jan-2012	1277.8	53.9770	-17.4158	-21.6739	0.3472	0.1853
09-Apr-2012	1382.2	48.9735	-9.7517	-9.9641	0.4760	0.4018
09-Jul-2012	1352.5	47.5814	15.8417	15.4475	0.3181	0.3184
05-Oct-2012	1460.9	42.9608	11.2648	9.0422	3.0925	0.8156
09-Jan-2013	1461.0	43.7355	17.6935	14.7094	0.2747	0.1360
11-Apr-2013	1593.4	39.2535	9.4000	6.6261	1.0037	0.7546

Table 2: The first column lists the purchasing dates of the 36 contracts. Column two shows the ATM strikes at which the contracts are purchased and column three show the prices at which this happens. The fourth column gives the terminal P&L for each contract, when the hedge is performed with an "actual" (EGARCH(1,1)) volatility forecast. Column five likewise, but when the hedge is with the implied volatilities. Finally, columns six and seven give the quadratic variation, defined as $\sum_{i=1}^{N} |dP\&L_i|^2/N$, where $N = 63$ is the number of trading days, for the entire actual and implied paths respectively.

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