Chapter 4

Stochastic Volatility

Empirical observations of time series of returns indicate that the amplitude of returns is positively autocorrelated in time. Both the local stochastic volatility model and the Merton model are conserving independence of log-returns. To overcome this problem, we will consider a model where the volatility process itself is stochastic and possibly correlated with the stock price; A stochastic volatility model. Several types of stochastic volatility models were developed in the late 1980s and in the beginning of the 1990s. For example the Hull-&-White model (Hull and White, 1987), the Wiggins-model (Wiggins, 1987) and the Stein-&-Stein-model (Stein and Stein, 1991). Neither of these models allow the volatility and spot returns to be correlated. The Heston model (Heston, 1993) was therefore a major breakthrough. The model assumptions imply a semi-closed solution for the arbitrage-free price of $\mathcal{X}$ written on an asset with stochastic volatility. In addition, the model allows for a non-zero correlation between volatility and the asset. As a starting point, we will define the model and derive Heston's partial differential equation used for pricing purposes. The PDE as well as the bivariate version of Itŏ's formula is used to derive the Fundamental Theorem of Derivative Trading. Next, a semi-closed valuation formula for $\mathcal{X}$ is determined using the conditional characteristic function of $\log S_T$. This formula will lead to simple expressions for the Greeks and in particular simple formulas for both the Δ-hedge and the locally risk-minimizing hedge. Simulation studies will illustrate the performance of the two hedging strategies. The chapter is completed with a derivation of a pricing formula as well as hedging strategies in Bates' stochastic volatility model allowing for random jumps to occur in the stock price.

4.1 The Heston Model

Assume that the underlying stock $(S_t)_{t\geq 0}$ satisfies the stochastic differential equation

$$dS_t = \mu S_t dt + f(\nu_t) S_t d\overline{W}_t$$

where $(\overline{W}_t)_{t\geq 0}$ is a standard Brownian motion, while the volatility process $(f(\nu_t))_{t\geq 0}$ is positive for all $t \geq 0$. Different choices of $f(\nu_t)$ lead to different stochastic volatility models. The reader should note that the process $(S_t)_{t\geq 0}$ is no longer Markovian; the evolution of the stock price depends on both its current value and its level of volatility. However, the bivariate process $(S_t, f(\nu_t))_{t\geq 0}$ is a Markovian diffusion and the extra dimension allows us to model a correlation between movements in the volatility and movements in the stock price. To be more specific we will assume that $f(x) = \sqrt{x}$ and that $(\nu_t)_{t\geq 0}$ satisfies the stochastic differential equation

$$d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t}dW_t^2$$

where κ, θ and σ are positive constants, while $(W_t^2)_{t\geq 0}$ is a standard Brownian motion with $\mathrm{Cov}(d\overline{W}_t, dW_t^2) = \rho dt$ for $\rho \in [-1, 1]$. The process $(\nu_t)_{t\geq 0}$ is often referred to as a square root process or a CIR process (after Cox, Ingersoll and Ross). Hence, we consider the model where $(S_t, \nu_t)_{t\geq 0}$ has $\mathbb{P}$-dynamics

$$dS_t = \mu S_t dt + \sqrt{\nu_t} S_t \left(\sqrt{1-\rho^2} dW_t^1 + \rho dW_t^2\right) \tag{4.1}$$

$$d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t}dW_t^2, \tag{4.2}$$

and both processes $(W_t^1)_{t\geq 0}$ and $(W_t^2)_{t\geq 0}$ are standard Brownian motions assumed independent of each other. The following section will give a brief description of the variance process.

4.1.1 The CIR process

The process, $(\nu_t)_{t\geq 0}$, is defined (integral representation) as the solution to stochastic differential equation

$$\nu_t = \nu_0 + \kappa \int_0^t (\theta - \nu_s)ds + \sigma \int_0^t \sqrt{\nu_s} dW_s^2,$$

where $\nu_0 \geq 0$. The process is continuous and furthermore positive under the assumption that the constants κ, θ and σ are positive. When/if the process touches zero, the drift term pushes the process in the positive direction. We will the make the technical assumption

$$\sigma^2 \leq 2\kappa\theta \tag{4.3}$$

implying that $\nu_t > 0$ for $t \in [0, T]$. Taking the expectation of ν_t implies

$$E[\nu_t] = \nu_0 + \kappa \int_0^t (\theta - E[\nu_s])\, ds.$$

The ordinary differential equation $\partial_t m(t) = \kappa\theta - \kappa m(t)$ with boundary condition $m(0) = \nu_0$ is obtained by differentiating $m(t) = E[\nu_t]$ with respect to t. The solution to the differential equation is given by

$$m(t) = E[\nu_t] = \theta + (\nu_0 - \theta)e^{-\kappa t},$$

and converges to θ for $t \to \infty$. Thus, it is natural to call ν_t a mean-reversion process with long-term mean, θ, and rate of mean-reversion κ. Furthermore, we will refer to σ as the volatility-of-volatility. The variance process can be used to simulate the Heston model: Introduce the log-price process $(X_t)_{t\geq 0}$ with dynamics

$$dX_t = \left(\mu - \frac{1}{2}\nu_t\right) dt + \sqrt{\nu_t}\left(\sqrt{1-\rho^2}dW_t^1 + \rho dW_t^2\right),$$

which does not depend on its own state. Define the time-grid $0 = t_1 < t_2 < ... < t_n = T$ and simulate $(\nu_t)_{t\in[0,T]}$ by the recursive formula[1] $\nu_{t_{i+1}} = \nu_{t_i} + \kappa(\theta - \nu_{t_i})(t_{i+1} - t_i) + \sigma\sqrt{\max(\nu_{t_i}, 0)}\sqrt{t_{i+1} - t_i}W_{i+1}$ for $i = 0, ..., n-1$ with $W_1, ..., W_n$ being IID standard normally distributed variables and $\nu_{t_1} = \nu_0$. Then simulate $X_{t_{i+1}}$ and $S_{t_{i+1}}$ through ν_{t_i}. Figure 4.1 illustrates simulations of the dynamics 4.1 and 4.2.

[1] This is known as an Euler discretization of the process. The method may lead to negative values of ν_t due the discretization error. To avoid negative values see the algorithm in (Glasserman, 2004, Page 124)

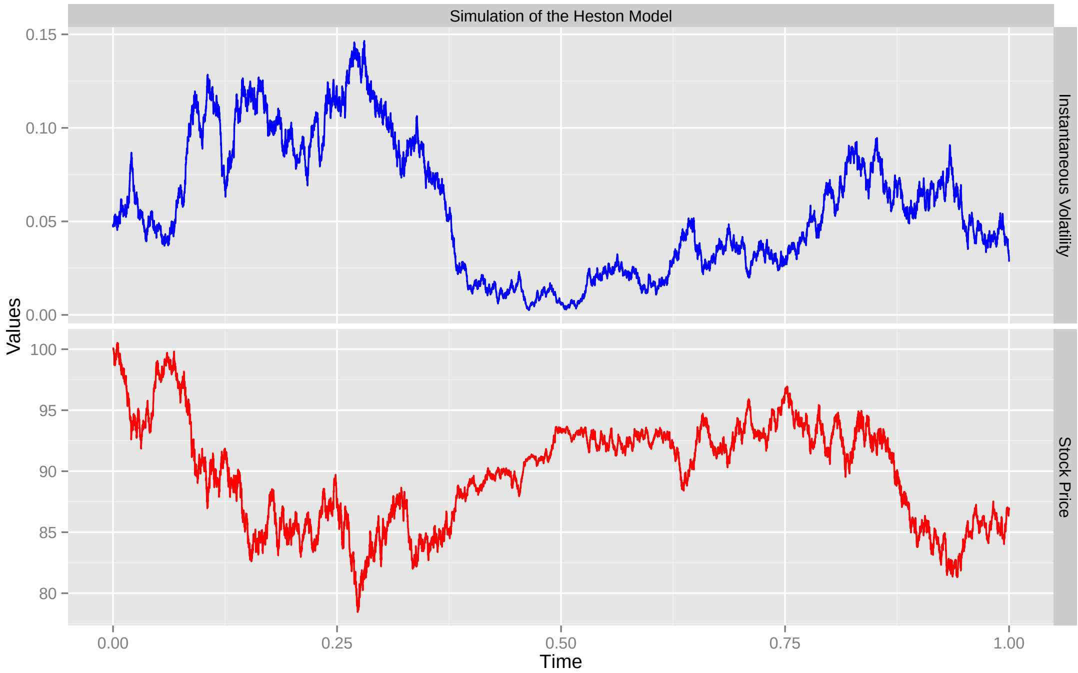

Figure 4.1: *A simulation of the variance process and the stock price process in Heston's stochastic volatility model. The two paths are simulated over the time-grid* $0 = t_1 < t_2 < ... < t_n = T$ *with* $T = 1$ *and* $n = 10000$*. The model paramters are chosen in accordance with benchmark settings:* $S_0 = 100$, $\mu = 0.1$, $\theta = 0.0483$, $\kappa = 4.75$, $\sigma = 0.55$, $\rho = -0.569$ *and* $\nu_0 = \theta$*. The negative correlation between the stock price and the variance process is clearly seen.*

4.1.2 The Heston PDE

We want do derive a partial differential equation for pricing purposes that accounts for random changes in the volatility. It remains to be seen that the derivation depends on the market price of volatility risk. We will therefore explore the class of equivalent martingale measures in Heston's model. Consider the usual Girsanov theorem (Björk, 2009, Theorem 11.3): Let $(\gamma(t))_{t\geq 0}$ be any 2-dimensional column process adapted to the filtration and introduce the Likelihood process

$$L_t = e^{\int_0^t -\left(\gamma_1(s)dW_s^1 + \int_0^t \gamma_2(s)dW_s^2\right) - \frac{1}{2}\left(\int_0^t \gamma_1^2(s)ds + \int_0^t \gamma_2^2(s)ds\right)}$$

restricted to any time $0 \leq t \leq T$. Under the assumption $E[L_T] = 1$, we can define an equivalent probability measure, $\mathbb{Q}$, by $L_T = d\mathbb{Q}/d\mathbb{P}$ on $\mathcal{F}_T$. If $W_t = (W_t^1, W_t^2)^\top$, we get

$$dW_t = -\gamma(t)dt + dW_t^{\mathbb{Q}}$$

where $(W_t^{\mathbb{Q}})_{t\geq 0}$ is a 2-dimensional standard Brownian motion under $\mathbb{Q}$. The discounted process $(\hat{S}_t)_{t\geq 0}$ must be a martingale under the pricing measure $\mathbb{Q}$. Hence we have the necessary condition

$$\frac{\mu - r}{\sqrt{\nu_t}} = \sqrt{1-\rho^2}\gamma_1(t) + \rho\gamma_2(t) \tag{4.4}$$

in order for an equivalent martingale measure to exists. The process $(\gamma(t))_{t\geq 0}$ is called *the risk premium processes*, since $(\gamma_i(t))_{t\geq 0}$ is interpreted as the market price of risk related to $(W_t^i)_{t\geq 0}$ for $i = 1, 2$. Note that condition (4.4) is satisfied for

$$\gamma(t) = \frac{\mu - r}{\sqrt{\nu_t}} \cdot \begin{pmatrix} \sqrt{1-\rho^2} \\ \rho \end{pmatrix},$$

which is the process corresponding to the minimal martingale measure (Poulsen, Schenk-Hoppè, and Ewald, 2009, Proposition 1). Heston proposed (Heston, 1993) the extra condition $\gamma_2(t) = \lambda\sqrt{\nu_t}$ for some constant λ, which implies: 1) the risk premium process

$$\gamma_1(t) = \frac{1}{\sqrt{1-\rho^2}} \left(\frac{\mu - r}{\sqrt{\nu_t}} - \lambda\rho\sqrt{\nu_t} \right)$$

related to W_t^1, and 2) the $\mathbb{Q}$-dynamics of $(S_t)_{t\geq 0}$ and $(\nu_t)_{t\geq 0}$ satisfy

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t \left(\sqrt{1-\rho^2} dW_t^{1,\mathbb{Q}} + \rho dW_t^{2,\mathbb{Q}} \right) \tag{4.5}$$

$$d\nu_t = \kappa^{\mathbb{Q}} \left(\theta^{\mathbb{Q}} - \nu_t \right) dt + \sigma\sqrt{\nu_t} dW_t^{1,\mathbb{Q}} \tag{4.6}$$

with risk-neutral parameters $\kappa^{\mathbb{Q}} = \kappa + \lambda\sigma$ and $\theta^{\mathbb{Q}} = \frac{\kappa\theta}{\kappa+\lambda\sigma}$. Here $(W_t^{1,\mathbb{Q}})_{t\geq 0}$ and $(W_t^{2,\mathbb{Q}})_{t\geq 0}$ are independent standard Brownian motions under $\mathbb{Q}$.

Remark 4.1 It this proved in (Wong and Heyde, 2006) that the proposed choice of $\gamma_2(t)$ implies that $E[L(T)] = 1$. Hence the equivalent martingale measure $\mathbb{Q}$ exists. ○

Remark 4.2 According to (Heston, 1993), the parameter λ could determined by a volatility-dependent asset. ○

Remark 4.3 Assuming that the risk-premium process $(\gamma(t))_{t\geq 0}$ satisfies

$$\gamma(t) = \frac{\mu - r}{\sqrt{\nu_t}} \cdot \begin{pmatrix} \sqrt{1-\rho^2} \\ \rho \end{pmatrix}$$

will imply that the $\mathbb{Q}$-dynamics of the bivariate process $(S_t, \nu_t)_{t\geq}$ are given by

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{\nu_t} S_t \left(\sqrt{1-\rho^2} dW_t^{1,\mathbb{Q}} + \rho dW_t^{2,\mathbb{Q}} \right) \\ d\nu_t &= \kappa^{\mathbb{Q}} \left(\theta^{\mathbb{Q}} - \nu_t \right) dt + \sigma \sqrt{\nu_t} dW_t^{1,\mathbb{Q}} \end{aligned}$$

with $\kappa^{\mathbb{Q}} = \kappa$ and $\theta^{\mathbb{Q}} = \theta - \frac{(\mu - r)\rho\sigma}{\kappa}$. ∘

It is important to emphasize that all relevant change-of-measures in this thesis imply dynamics of $(S_t)_{t\geq 0}$ and $(\nu_t)_{t\geq 0}$ with same parametric form as under $\mathbb{P}$. At risk of repeating ourselves, we will give a formal definition of the dynamics under consideration.

Definition 4.1 (The Heston dynamics under $\mathbb{P}$ and $\mathbb{Q}$) *Assume that $(S_t)_{t\geq 0}$ and $(\nu_t)_{t\geq 0}$ are solutions to the stochastic differential equations*

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{\nu_t} S_t \left(\sqrt{1-\rho^2} dW_t^1 + \rho dW_t^2 \right) \\ d\nu_t &= \kappa(\theta - \nu_t) dt + \sigma \sqrt{\nu_t} dW_t^2, \end{aligned}$$

where $(W^1)_{t\geq 0}$ and $(W_t^2)_{t\geq 0}$ are independent standard Brownian motions, $\rho \in [-1, 1]$ determines the correlation between $(S_t)_{t\geq 0}$ and $(\nu_t)_{t\geq 0}$, θ is the long-term mean of $(\nu_t)_{t\geq 0}$, κ is the speed of mean-reversion to θ and σ is the volatility-of-volatility. Moreover, the dynamics under the (allowed) pricing measure $\mathbb{Q}$ satisfy

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t \left(\sqrt{1-\rho^2} dW_t^{1,\mathbb{Q}} + \rho dW_t^{2,\mathbb{Q}} \right) \tag{4.7}$$

$$d\nu_t = \kappa^{\mathbb{Q}} (\theta^{\mathbb{Q}} - \nu_t) dt + \sigma \sqrt{\nu_t} dW_t^{2,\mathbb{Q}}, \tag{4.8}$$

where $(W_t^{1,\mathbb{Q}})_{t\geq 0}$ and $(W_t^{2,\mathbb{Q}})_{t\geq 0}$ are independent standard Brownian motions, r is the short risk-free rate and $\kappa^{\mathbb{Q}}$ and $\theta^{\mathbb{Q}}$ are risk-adjusted parameters.

A rigorous definition of the dynamics in Heston's model appears important in the derivation of the partial differential equation for pricing purposes.

Proposition 4.2 (Heston's partial differential equation) *Consider the financial market consisting of a risky asset $(S_t)_{t\geq 0}$ and the money account $(B_t)_{t\geq 0}$. Assume the bivariate Markovian process $(S_t, \nu_t)_{t\geq 0}$ is solution to the system of stochastic differential equations* (4.7) *and* (4.8) *under some pricing measure $\mathbb{Q}$. The arbitrage-free price of the European option $\mathcal{X}$ is determined by the function $C : [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ solving the partial differential equation*

$$\begin{aligned} rC(t,s,\nu) =& \partial_t C(t,s,\nu) + \frac{1}{2}\nu \left(s^2 \partial_{ss} C(t,s,\nu) + \sigma^2 \partial_{\nu\nu} C(t,s,\nu)\right) \\ &+ \rho\sigma\nu s \partial_{s\nu} C(t,s,\nu) + rs\partial_s C(t,s,\nu) \\ &+ \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - \nu)\partial_\nu C(t,s,\nu) \end{aligned} \tag{4.9}$$

with boundary condition $C(T,s,\nu) = (s-K)^+$.

Proof. Let $(\Pi_t)_{t\geq 0}$ denote the value process of a portfolio long $\mathcal{X}$, short h_t^1 units of the stock and short h_t^2 units of another risky asset with value process $X(t, S_t, \nu_t)$:

$$\Pi_t = C(t, S_t, \nu_t) - h_t^1 S_t - h_t^2 X(t, S_t, \nu_t).$$

Applying the bivariate version of Itô's formula yields

$$\begin{aligned} d\Pi_t =& dC(t,S_t,\nu_t) - h_t^1 dS_t - h_t^2 dX(t,S_t,\nu_t) \\ =& \left\{\partial_t C(t,S_t,\nu_t) + \frac{1}{2}\nu_t \left(S_t^2 \partial_{ss} C(t,S_t,\nu_t) + \sigma^2 \partial_{\nu\nu} C(t,S_t,\nu_t)\right) + \rho\sigma\nu_t S_t \partial_{\nu s} C(t,S_t,\nu_t)\right\} dt \\ &- h_t^2 \left\{\partial_t X(t,S_t,\nu_t) + \frac{1}{2}\nu_t \left(S_t^2 \partial_{ss} X(t,S_t,\nu_t) + \sigma^2 \partial_{\nu\nu} X(t,S_t,\nu_t)\right) + \rho\sigma\nu_t S_t \partial_{\nu s} X(t,S_t,\nu_t)\right\} dt \\ &+ \left\{\partial_s C(t,S_t,\nu_t) - h_t^2 \partial_s X(t,S_t,\nu_t) - h_t^1\right\} dS_t + \left\{\partial_\nu C(t,S_t,\nu_t) - h_t^2 \partial_\nu C(t,S_t,\nu_t)\right\} d\nu_t. \end{aligned}$$

The following conditions should be satisfied in order to obtain a risk-neutral portfolio

$$\partial_s C(t,S_t,\nu_t) = h_t^1 + h_t^2 \partial_s X(t,S_t,\nu_t) \tag{4.10}$$

$$\partial_\nu C(t,S_t,\nu_t) = h_t^2 \partial_\nu X(t,S_t,\nu_t), \tag{4.11}$$

since these choices of $h_1(t)$ and $h_2(t)$ will eliminate the dS_t-term and the $d\nu_t$-term. To avoid arbitrage the return of the risk-free portfolio must equal the risk-free rate r:

$$d\Pi_t = r\Pi_t dt = r\left(C(t,S_t,\nu_t) - h_t^1 S_t - h_t^2 X(t,S_t,\nu_t)\right) dt.$$

Equating the two expressions for $d\Pi_t$ combined with the conditions (4.10) and (4.11) leads to the equation (where the arguments in C and X are omitted)

$$\begin{aligned}&\frac{\partial_t C + \frac{1}{2}\nu_t\left(S_t^2\partial_{ss}C + \sigma^2\partial_{\nu\nu}C\right) + \rho\sigma\nu_t S_t\partial_{\nu s}C + +rS_t\partial_s C - rC}{\partial_\nu C}\\ =&\frac{\partial_t X + \frac{1}{2}\nu_t\left(S_t^2\partial_{ss}X + \sigma^2\partial_{\nu\nu}X\right) + \rho\sigma\nu_t S_t\partial_{\nu s}X + +rS_t\partial_s X - rX}{\partial_\nu X}.\end{aligned} \tag{4.12}$$

The left-hand side only depends on function $C(t, S_t, \nu_t)$ while the right-hand only depends on the function $X(t, S_t, \nu_t)$. Thus, the equality 4.12 can only be satisfied for all t if both sides equal some arbitrary function g of the independent variables t, S_t and ν_t. Without loss of generality we can choose

$$g(t, S_t, \nu_t) = -\left(\kappa(\theta - \nu_t) - \tilde{\gamma}(t, S_t, \nu_t)\right)$$

where $\tilde{\gamma}(t, S_t, \nu_t)$ determines how much of return of expected return of $C(t, S_t, \nu_t)$ is explained by the risk of ${\nu_t}$[2]. Thus, $\tilde{\gamma}(t, S_t, \nu_t)$ is the market price of volatility risk. By assumption $g(t, S_t, \nu_t) = -\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - \nu_t)$ for all allowed choices of $\mathbb{Q}$ by definition 4.1. Replace the right-hand side in equation (4.12) with $g(t, S_t, \nu_t) = -\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - \nu_t)$ and rearrange the terms to obtain the equation

$$\begin{aligned} rC(t, S_t, \nu_t) =& \partial_t C(t, S_t, \nu_t) + \frac{1}{2}\nu_t\left(S_t^2\partial_{ss}C(t, S_t, \nu_t) + \sigma^2\partial_{\nu\nu}C(t, S_t, \nu_t)\right)\\ &+ \rho\sigma\nu_t S_t\partial_{\nu s}C(t, S_t, \nu_t) + rS_t\partial_s C(t, S_t, \nu_t)\\ &+ \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - \nu_t)\partial_\nu C(t, S_t, \nu_t) \end{aligned} \tag{4.13}$$

with boundary condition $C(T, S_T, \nu_T) = (S_T - K)^+$. Finally, we conclude that equation (4.13) must hold for all $S_t \in \mathbb{R}_+$ and $\nu_t \in \mathbb{R}_+$ for any fixed $t \in [0, T]$. The proposition is proved. □

[2] Equate the left-hand side in (4.12) and reorder terms such $-\left(\kappa(\theta - \nu_t) - \gamma(t, S_t, \nu_t)\right)\partial_\nu C(t, S_t, \nu_t)$ is isolated on one of the sides.

Remark 4.4 For technical reasons it is often convenient to express the PDE in terms of the log price. Introduce the substitution $X_t = \log S_t$ to obtain the equation

$$\begin{aligned} rC(t, X_t, \nu_t) =& \partial_t C(t, X_t, \nu_t) + \frac{1}{2}\nu_t \left\{\partial_{xx} C(t, X_t, \nu_t) + \sigma^2 \partial_{\nu\nu} C(t, X_t, \nu_t)\right\} \\ &+ \rho\sigma\nu_t \partial_{\nu x} C(t, X_t, \nu_t) + \left(r - \frac{1}{2}\nu_t\right) \partial_x C(t, X_t, \nu_t) \\ &+ \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - \nu_t)\partial_\nu C(t, X_t, \nu_t). \end{aligned} \tag{4.14}$$

Note that the coefficients of the partial derivatives in (4.14) do not contain X_t. ○

Remark 4.5 From now on we will drop the "$\mathbb{Q}$" in $\kappa^{\mathbb{Q}}$ and $\theta^{\mathbb{Q}}$. Thus, κ and θ will be interpreted as risk-adjusted parameters. ○

4.2 FTODT: The Heston Model

With the partial differential equation deduced, we can derive the Fundamental Theorem of Derivative Trading under Heston's model. Pay attention to the issue that the model includes two types of volatility; The instantaneous volatility and the volatility-of-volatility. Thus, we could derive two different versions of FTODT corresponding to the scenarios:

1. Full knowledge of parameter values and the states of $(S_t)_{t\geq 0}$, while $(\nu_t)_{t\geq 0}$ is an unobservable quantity: Choose $(\nu_t^h)_{t\geq 0}$ as the hedging volatility.
2. Full knowledge of state variables and the parameters κ, θ and ρ, while σ is is an unobservable quantity: Choose $\sigma^h(t)$ as the hedging volatility-of-volatility.

We prefer scenario 1: It is not realistic to assume both full knowledge of state variables and σ as an unobservable quantity. Indeed, trading in real life is highly based on forecasting and estimating the actual volatility. We observe the implied Black-Scholes volatility but not the actual volatility determining the sign of the volatility risk-premium. The following theorem states the FTODT under Heston's stochastic volatility model.

Theorem 4.3 (FTODT: Heston's model) *Let* $C_t = C(t, S_t, \nu_t) \in \mathcal{C}^{1,2,2}([0,\infty)\times\mathbb{R}\times\mathbb{R})$ *be the value process of the European call* $\mathcal{X}$. *Assume we at time* $t = 0$ *acquire such an option for the market-price* $C(0, S_0, \nu_0^i)$ *corresponding to the implied volatility* $\sqrt{\nu_0^i}$. *If we* Δ*-hedge this position under the assumption that* $\sqrt{\nu_0^h}$ *is the correct volatility leading to a fair price* $C(0, S_0, \nu_0^h)$, *the present value we obtain by hold holding this portfolio over the interval* $[0, T]$ *is*

$$\begin{aligned}
P\&L_T^h =& C(0, S_0, \nu_0^h) - C(0, S_0, \nu_0^i) \\
&+ \int_0^T e^{-rt} \Bigg\{ \left(\nu_t^r - \nu_t^h\right) \left\{ \frac{1}{2}\left(S_t^2 \partial_{ss} C(t, S_t, \nu_t^h) + \sigma^2 \partial_{\nu\nu} C(t, S_t, \nu_t^h)\right)\right. \\
&\left. + \sigma\rho S_t \partial_{s\nu} C(t, S_t, \nu_t^h) - \kappa \partial_\nu C(t, S_t, \nu_t^h)\right\} dt + \sigma\sqrt{\nu_t} dW_t^{2,\mathbb{Q}} \Bigg\}.
\end{aligned}$$

Proof. We consider the portfolio long $\mathcal{X}$, short $\partial_s C^h(t, S_t, \nu_t)$ in the underlying and choose the money account, B_t, such that the portfolio is self-financing:

$$\Pi_t^h = C(t, S_t, \nu_t^i) - \partial_s C^h(t, S_t, \nu_t^h) S_t + B_t = 0.$$

The self-financing condition implies that

$$\begin{aligned}
d\Pi_t^h &= dC^i(t, S_t, \nu_t) - \partial_s C^h(t, S_t, \nu_t) dS_t + dB_t \\
&= dC^i(t, S_t, \nu_t) - \partial_s C^h(t, S_t, \nu_t)\left(dS_t - rS_t dt\right) - rC^i(t, S_t, \nu_t) dt. \qquad (4.15)
\end{aligned}$$

The bivariate version of Itŏ's formula yields that

$$\begin{aligned}
dC(t, S_t, \nu_t^h) =& \partial_t C(t, S_t, \nu_t^h) dt + \partial_s C(t, S_t, \nu_t^h) dS_t + \partial_\nu C(t, S_t, \nu_t^h) d\nu_t \\
&+ \frac{1}{2}\left(\partial_{ss}(dS_t)^2 + \partial_{\nu\nu} C(t, S_t, \nu_t^h)(d\nu_t)^2\right) + \partial_{s\nu} C(t, S_t, \nu_t^h) dS_t d\nu_t \\
=& \partial_t C(t, S_t, \nu_t^h) dt + \partial_s C(t, S_t, \nu_t^h) dS_t + \kappa\theta \partial_\nu C(t, S_t, \nu_t^h) dt \\
&+ \nu_t^r \left\{ \frac{1}{2}\left(S_t^2 \partial_{ss} C(t, S_t, \nu_t^h) + \sigma^2 \partial_{\nu\nu} C(t, S_t, \nu_t^h)\right)\right. \\
&\left. + \sigma\rho S_t \partial_{s\nu} C(t, S_t, \nu_t^h) - \kappa \partial_\nu C(t, S_t, \nu_t^h)\right\} dt + \sigma\sqrt{\nu_t^r} dW_t^{2,\mathbb{Q}}. \qquad (4.16)
\end{aligned}$$

On the other hand, Heston's PDE with $\nu_t = \nu_t^h$ can rewritten to

$$
\begin{aligned}
rC(t, S_t, \nu_t^h) =& \partial_t C(t, S_t, \nu_t^h) + rS_t \partial_s C(t, S_t, \nu_t^h) + \kappa\theta \partial_\nu C(t, S_t, \nu_t^h) \\
& + \nu_t^h \Bigg\{ \frac{1}{2} \left(S_t^2 \partial_{ss} C(t, S_t, \nu_t^h) + \sigma^2 \partial_{\nu\nu} C(t, S_t, \nu_t^h) \right) \\
& + \rho\sigma S_t \partial_{s\nu} C(t, S_t, \nu_t^h) - \kappa \partial_\nu C(t, S_t, \nu_t^h) \Bigg\}.
\end{aligned} \tag{4.17}
$$

Now, substitute the PDE (4.17) into the dynamics of $C^h(t, S_t, \nu_t)$ (4.16)

$$
\begin{aligned}
0 =& - dC(t, S_t, \nu_t^h) + rC(t, S_t, \nu_t^h)dt - \partial_s C(t, S_t, \nu_t^h) r S_t dt \\
& + \left(\nu_t^r - \nu_t^h \right) \Bigg\{ \frac{1}{2} \left(S_t^2 \partial_{ss} C(t, S_t, \nu_t^h) + \sigma^2 \partial_{\nu\nu} C(t, S_t, \nu_t^h) \right) \\
& + \sigma\rho S_t \partial_{s\nu} C(t, S_t, \nu_t^h) - \kappa \partial_\nu C(t, S_t, \nu_t^h) \Bigg\} dt + \sigma \sqrt{\nu_t^r} dW_t^{2,\mathbb{Q}}.
\end{aligned} \tag{4.18}
$$

Finally, add equation (4.18) to the dynamics of the hedging portfolio (4.15) to obtain the equation

$$
\begin{aligned}
d\Pi_t^h =& dC(t, S_t, \nu_t^i) - dC(t, S_t, \nu_t^h) - r \left(C(t, S_t, \nu_t^i) - C(t, S_t, \nu_t^h) \right) dt \\
& + \left(\nu_t^r - \nu_t^h \right) \Bigg\{ \frac{1}{2} \left(S_t^2 \partial_{ss} C(t, S_t, \nu_t^h) + \sigma^2 \partial_{\nu\nu} C(t, S_t, \nu_t^h) \sigma^2 \right) \\
& + \sigma\rho S_t \partial_{s\nu} C(t, S_t, \nu_t^h) - \partial_\nu C(t, S_t, \nu_t^h) \kappa \Bigg\} dt + \sigma \sqrt{\nu_t^r} dW_t^2 \\
=& e^{rt} d \left(e^{-rt} \left(C(t, S_t, \nu_t^i) - dC(t, S_t, \nu_t^h) \right) \right) \\
& + \left(\nu_t^r - \nu_t^h \right) \Bigg\{ \frac{1}{2} \left(S_t^2 \partial_{ss} C(t, S_t, \nu_t^h) + \sigma^2 \partial_{\nu\nu} C(t, S_t, \nu_t^h) \right) \\
& + \sigma\rho S_t \partial_{s\nu} C(t, S_t, \nu_t^h) - \kappa \partial_\nu C(t, S_t, \nu_t^h) \Bigg\} dt + \sigma \sqrt{\nu_t^r} dW_t^{2,\mathbb{Q}}.
\end{aligned}
$$

In particular, the present value of the portfolio equals $P\&L_T^h = \int_0^T e^{-rt} d\Pi_t^h$ and is given by

$$
\begin{aligned}
P\&L_T^h =& C(0, S_0, \nu_0^h) - C(0, S_0, \nu_0^i) \\
&+ \int_0^T e^{-rt} \Bigg\{ \left(\nu_t^r - \nu_t^h\right) \left\{ \frac{1}{2} \left(S_t^2 \partial_{ss} C(t, S_t, \nu_t^h) + \sigma^2 \partial_{\nu\nu} C(t, S_t, \nu_t^h) \right) \right. \\
&\left. + \sigma\rho S_t \partial_{s\nu} C(t, S_t, \nu_t^h) - \kappa \partial_\nu C(t, S_t, \nu_t^h) \right\} dt + \sigma \sqrt{\nu_t^r} dW_t^{2,\mathbb{Q}} \Bigg\}
\end{aligned}
$$

as promised. □

Obviously, perfect hedging is impossible as expected since the market is incomplete. Choosing $\nu_t^h = \nu_t^r$ for all t yields a hedging error given by the stochastic integral

$$
\sigma \int_0^T e^{-rt} \sqrt{\nu_t^r} dW_t^{2,\mathbb{Q}}.
$$

The hedging error can both be positive and negative dependent on the path of the Brownian motion as well as the actual volatility process. Furthermore, if we hedge our position with implied volatility, the final hedging error will also depends on the option's Gamma, Vomma, Vanna and Vega, which can be computed from the pricing function derived in the next section.

To summarize and compare our findings, we have derived the FTODT for three different financial models: The generalized Black-Scholes model, the jump-diffusion model and finally Heston's model. The generalized Black-Scholes model is complete, and we can eliminate all risk choosing $\sigma^h(t, S_t) = \sigma^r(t, S_t)$. This is not possible for the jump-diffusion model respectively the Heston model. Incompleteness in the former is a consequence of the jump-risk represented by the dN_T-terms in theorem 3.10. The Heston model is incomplete due to the presence of volatility risk represented by the $dW_t^{2,\mathbb{Q}}$-term.

4.3 Pricing in Heston

Consider the arbitrage-free price of $\mathcal{X}$ at time t given by

$$\begin{aligned}C(t,S_t,\nu_t) &= e^{-r\tau}E_t^{\mathbb{Q}}[\mathcal{X}]\\ &= e^{-r\tau}E_t^{\mathbb{Q}}[S_T 1_{S_T>K}] - e^{-r\tau}KE_t^{\mathbb{Q}}[1_{S_T>K}]\end{aligned}$$

under a pricing measure $\mathbb{Q}$, which implies risk-neutral dynamics as in definition 4.1. The price is seen as a decomposition of two digital calls with pay-off spot-or-noting respectively cash-or-nothing. The above expression can be rewritten as

$$\begin{aligned}C(t,S_t,\nu_t) &= e^{-r\tau}E_t^{\mathbb{Q}}[S_T 1_{S_T>K}] - e^{-r\tau}KE_t^{\mathbb{Q}}[1_{S_T>K}]\\ &= S_t E_t^{\mathbb{Q}}\left[1_{S_T>K}\frac{B_t/B_T}{S_t/S_T}\right] - e^{-r\tau}KE_t^{\mathbb{Q}}[1_{S_T>K}]\\ &= S_t E_t^{\tilde{\mathbb{Q}}}[1_{S_T>K}] - e^{-r\tau}KE_t^{\mathbb{Q}}[1_{S_T>K}],\end{aligned}$$

where $\tilde{\mathbb{Q}}$ is the equivalent martingale measure that uses the underlying as numeraire (see section B.3), i.e.

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \frac{B_t/B_T}{S_t/S_T}.$$

Define the log-price $X_t = \log S_t$. The price can then be expressed as

$$C(t,S_t,\nu_t) = S_t P_1(t,X_t,\nu_t) - e^{-r\tau}KP_2(t,X_t,\nu_t),$$

where $P_1(t,X_t,\nu_t)$ and $P_2(t,X_t,\nu_t)$ are probabilities that the option expires in-the-money at a fixed time T under $\tilde{\mathbb{Q}}$ respectively $\mathbb{Q}$ conditioned on the states of X_t and ν_t at time t. Both the left-term and the right-term should satisfy Heston's partial differential equation[3] written in log-terms (4.14). For instance, substitute $e^{-r\tau}KP_2(t,X_t,\nu_t)$ into equation (4.14) to obtain

[3]The left-term denotes the value of a digital spot-or-nothing while the right-term is the value of a digital cash-or-nothing

$$
\begin{aligned}
0 =& \partial_t P_2(t, X_t, \nu_t) + \frac{1}{2}\nu_t \left\{\partial_{xx} P_2(t, X_t, \nu_t) + \sigma^2 \partial_{\nu\nu} P_2(t, x_t, \nu_t)\right\} \\
&+ \rho\sigma\nu_t \partial_{\nu x} P_2(t, X_t, \nu_t) + \left(r - \frac{1}{2}\nu_t\right) \partial_x P_2(t, X_t, \nu_t) \\
&+ \kappa(\theta - \nu_t)\partial_\nu P_2(t, X_t, \nu_t)
\end{aligned}
$$

subject to the terminal condition $P_2(T, X_T, \nu_T) = 1_{\{X_T > \log K\}}$. The conditional probability are not immediately available in closed-form. Assume that the random variable $X_T = \log S_T$ has conditional density $q(x)$ with respect to the Lebesgue measure on $\mathbb{R}$ under $\mathbb{Q}$ and note

$$
\begin{aligned}
P_2(t, X_t, \nu_t) &= \mathbb{Q}_t(X_T > \log K) \\
&= \int_{\log K}^{\infty} q(x)dx.
\end{aligned}
$$

The conditional characteristic function of X_T under $\mathbb{Q}$ is defined by

$$
f(\phi) = E_t^{\mathbb{Q}}\left[e^{i\phi X_T}\right] = \underbrace{\int_{-\infty}^{\infty} \cos(\phi x) q(x) dx}_{\mathrm{Re}[f(\phi)]} + i \underbrace{\int_{-\infty}^{\infty} \sin(\phi x) q(x) dx}_{\mathrm{Im}[f(\phi)]}
$$

for $\phi \in \mathbb{R}$ where $\mathrm{Re}[z]$ and $\mathrm{Im}[z]$ denotes the real part respectively the imaginary part of the complex number $z \in \mathbb{C}$. Moreover, the conditional cumulative distribution function $\mathbb{Q}_t(X_T \leq x)$ can now be expressed using the Gil-Pelaez inversion formula

$$
\mathbb{Q}_t(X_T \leq x) = \frac{1}{2} - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\phi x} f(\phi)}{i\phi} d\phi \tag{4.19}
$$

for $x \in \mathbb{R}$. Classic Fourier inversion implies that

$$
\begin{aligned}
q(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\phi x} f(\phi) d\phi \\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos(-\phi x)\mathrm{Re}[f(\phi)] d\phi - i\int_{-\infty}^{\infty} \sin(-\phi x)\mathrm{Im}[f(\phi)] d\phi\right) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Re}\left[e^{-i\phi x} f(\phi)\right] d\phi \\
&= \frac{1}{\pi}\int_{0}^{\infty} \mathrm{Re}\left[e^{-i\phi x} f(\phi)\right] d\phi,
\end{aligned}
$$

where we have used the fact that $q(x)$ is real and hence $\mathrm{Im}[e^{-i\phi x} f(\phi)]$ must vanish as well as the fact that the real part of the characteristic function is even. A similar argument implies that equation (4.19) with $x = \log K$ can be expressed as

$$\begin{aligned}
\mathbb{Q}_t(X_T \leq \log K) &= \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \log K} f(\phi)}{i\phi}\right] d\phi \qquad \Leftrightarrow \\
\mathbb{Q}_t(X_T > \log K) &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \log K} f(\phi)}{i\phi}\right] d\phi.
\end{aligned}$$

Computing the conditional probability, $P_2(T, X_t, \nu_t)$, thereby reduces to finding an expression for the characteristic function $f(\phi)$. By a similar argument, the conditional probability that the option expires in-the-money under $\tilde{\mathbb{Q}}$ is given by

$$\begin{aligned}
P_1(t, X_t, \nu_t) &= \tilde{\mathbb{Q}}_t(X_T > \log K) \\
&= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \log K} \tilde{f}(\phi)}{i\phi}\right] d\phi,
\end{aligned}$$

where $\tilde{f}$ is the conditional characteristic function of X_T under $\tilde{\mathbb{Q}}$. We will express $\tilde{f}$ through f. The Radon-Nikodym derivative of $\tilde{\mathbb{Q}}$ with respect to $\mathbb{Q}$ on $\mathcal{F}_T$ is given by

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \frac{B_t/B_T}{S_t/S_T} = \frac{e^{X_T}}{E_t^{\mathbb{Q}}\left[S_T \frac{B_t}{B_T} \frac{B_T}{B_t}\right]} = \frac{e^{X_T}}{E_t^{\mathbb{Q}}[e^{X_T}]}$$

using that $(\hat{S}_t)_{t\geq 0}$ is $\mathbb{Q}$-martingale. Applying Radon-Nikodým's theorem yields that the (conditional) characteristic function of X_T under $\tilde{\mathbb{Q}}$ can be expressed as

$$\begin{aligned}
\tilde{f}(\phi) &= E_t^{\tilde{\mathbb{Q}}}\left[e^{i\phi X_T}\right] \\
&= E_t^{\mathbb{Q}}\left[e^{i\phi X_T} \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}\right] \\
&= \frac{1}{E_t^{\mathbb{Q}}[e^{X_T}]} \int_{-\infty}^\infty e^{i\phi x} e^x q(x) dx \\
&= \frac{1}{f(-i)} \int_{-\infty}^\infty e^{i(\phi - i)x} q(x) dx. \\
&= \frac{f(\phi - i)}{f(-i)},
\end{aligned}$$

where f is the characteristic function of X_T under $\mathbb{Q}$. In particular

$$\begin{aligned}
P_1(t, X_t, \nu_t) &= \tilde{\mathbb{Q}}_t(X_T > \log K) \\
&= \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \operatorname{Re}\left[\frac{e^{-i\phi \log K} \tilde{f}(\phi)}{i\phi}\right] d\phi \\
&= \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \operatorname{Re}\left[\frac{e^{-i\phi \log K} f(\phi - i)}{i\phi f(-i)}\right] d\phi.
\end{aligned}$$

The problem deriving the pricing function $C(t, S_t, \nu_t)$ is reduced to determining f.

Proposition 4.4 *Consider the bivariate markov process $(S_t, \nu_t)_{t \geq 0}$ with $\mathbb{Q}$-dynamics given as in definition 4.1. The characteristic function of $X_T = \log S_T$ conditioned on X_t and ν_t under $\mathbb{Q}$ is given by*

$$f(\phi) = e^{C(\tau) + D(\tau)\nu_t + i\phi X_t}, \tag{4.20}$$

where C and D can be found by the equations

$$\begin{aligned}
C(\tau) &= ri\phi\tau + \frac{\kappa\theta}{\sigma^2}\left((\kappa - \rho\sigma i\phi + d)\tau - 2\log\left(\frac{1 - ge^{d\tau}}{1 - g}\right)\right) \\
D(\tau) &= \frac{\kappa - \rho\sigma i\phi + d}{\sigma^2}\left(\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}\right)
\end{aligned}$$

with $d = \sqrt{(\rho\sigma i\phi - \kappa)^2 - \sigma^2(-\phi^2 - i\phi)}$ and

$$g = \frac{\kappa - \rho\sigma i\phi + d}{\kappa - \rho\sigma i\phi - d}.$$

Proof. Consider a twice twice-differentiable function h equal to the conditional expectation under $\mathbb{Q}$ of some function g of X_t and ν_t at the expiry T:

$$h(t, X_t, \nu_t) = E^{\mathbb{Q}}\left[g(X_T, \nu_T) \mid \ X_t = x, \nu_t = \nu\right].$$

Note a change in notation is used to emphasize the form of h. The bivariate version of Itǒ's formula implies that $h(t, x_t, \nu_t)$ has dynamics

$$
\begin{aligned}
dh(t, X_t, \nu_t) =& \Big\{ \partial_t h(t, X_t, \nu_t) + \frac{1}{2}\nu_t \left\{ \partial_{xx} h(t, X_t, \nu_t) + \sigma^2 \partial_{\nu\nu} h(t, X_t, \nu_t) \right\} \\
&+ \left(r - \frac{1}{2}\nu_t \right) \partial_x h(t, X_t, \nu_t) + \rho\sigma\nu_t \partial_{\nu x} h(t, X_t, \nu_t) + \left(\kappa(\theta - \nu_t)\right) \partial_\nu h(t, X_t, \nu_t) \Big\} dt \\
&+ \sqrt{\nu_t} \partial_x h(t, X_t, \nu_t) \left(\sqrt{1-\rho^2} dW_t^{1,\mathbb{Q}} + \rho dW_t^{2,\mathbb{Q}} \right) + \sigma \sqrt{\nu_t} \partial_\nu h(t, X_t, \nu_t) dW_t^{2,\mathbb{Q}}.
\end{aligned}
$$

The function h is a martingale by iterated expectations. Thus, $h(t, X_t, \nu_t)$ must satisfy the following partial differential equation

$$
\begin{aligned}
0 =& \partial_t h(t, X_t, \nu_t) + \frac{1}{2}\nu_t \left\{ \partial_{xx} h(t, X_t, \nu_t) + \sigma^2 \partial_{\nu\nu} h(t, X_t, \nu_t) \right\} \\
&+ \rho\sigma\nu_t \partial_{\nu x} h(t, X_t, \nu_t) + \left(r - \frac{1}{2}\nu_t \right) \partial_x h(t, X_t, \nu_t) + \left(\kappa(\theta - \nu_t)\right) \partial_\nu h(t, X_t, \nu_t)
\end{aligned}
$$

with terminal condition $h(T, X_T, \nu_T) = g(X_T, \nu_T)$. If we choose $g(x, \nu) = e^{i\phi x}$ for $\phi \in \mathbb{R}$ the solution is the conditional characteristic function. Inverting the time direction, $\tau(t) = T - t$, implies the partial differential equation

$$
\begin{aligned}
0 =& - \partial_t h(\tau, X_t, \nu_t) + \frac{1}{2}\nu_t \left\{ \partial_{xx} h(\tau, X_t, \nu_t) + \sigma^2 \partial_{\nu\nu} h(\tau, X_t, \nu_t) \right\} \\
&+ \rho\sigma\nu_t \partial_{\nu x} h(\tau, X_t, \nu_t) + \left(r - \frac{1}{2}\nu_t \right) \partial_x h(\tau, X_t, \nu_t) \\
&+ \left(\kappa(\theta - \nu_t)\right) \partial_\nu h(\tau, X_t, \nu_t)
\end{aligned} \tag{4.21}
$$

subject to the initial condition $h(0, X_T, \nu_T) = e^{i\phi X_T}$. Due to the linearity of the coefficients in (4.21), we guess a solution of the form

$$
h(\tau, X_t, \nu_t) = e^{C(\tau) + D(\tau)\nu_t + i\phi X_t}
$$

with restrictions $C(0) = D(0) = 0$. Substituting the guess into (4.21) yields

$$
\begin{aligned}
0 = h(\tau, X_t, \nu_t) \Big\{ & - \left(C'(\tau) + D'(\tau)\nu_t \right) + \frac{1}{2}\nu_t \left\{ -\phi^2 + \sigma^2 D^2(\tau) \right\} \\
&+ \rho\sigma\nu_t i\phi D(\tau) + \left(r - \frac{1}{2}\nu_t \right) i\phi + \left(\kappa(\theta - \nu_t)\right) D(\tau) \Big\}
\end{aligned}
$$

implying the equation

$$
\begin{aligned}
0 =& \nu_t \left(-D'(\tau) + \frac{1}{2}\nu_t \left\{ -\phi^2 + \sigma^2 D^2(\tau) \right\} + \rho\sigma i\phi D(\tau) - \frac{1}{2} i\phi - \kappa D(\tau) \right) \\
&+ ri\phi + \kappa\theta D(\tau) - C'(\tau),
\end{aligned}
$$

which must hold for all t. Determining $C(\tau)$ and $D(\tau)$ reduces to solving the two differential equations

$$
D'(\tau) = \frac{1}{2}\sigma^2 D^2(\tau) + \rho\sigma i\phi D(\tau) - \frac{1}{2}\phi^2 - \frac{1}{2} i\phi - \kappa D(\tau) \tag{4.22}
$$

$$
C'(\tau) = ri\phi + \kappa\theta D(\tau), \tag{4.23}
$$

where we recognize (4.22) as a Ricatti equation[4]. The substitution $D(\tau) = -2y'(\tau)/(\sigma^2 y(\tau))$ leads to the following homogeneous second-order linear differential equation

$$
y''(\tau) - (\rho\sigma i\phi - \kappa)\, y'(\tau) + \frac{\sigma^2}{4}(-\phi^2 - i\phi) y(\tau) = 0. \tag{4.24}
$$

Thus, we can establish the characteristic equation of degree 2

$$
z^2 - (\rho\sigma i\phi - \kappa)\, z + \frac{\sigma^2}{4}(-\phi^2 - i\phi) = 0 \tag{4.25}
$$

on which the general solution to (4.24) depends. The two roots in (4.25) are given

$$
\begin{aligned}
z_- &= \frac{\rho\sigma i\phi - \kappa - d}{2} \\
z_+ &= \frac{\rho\sigma i\phi - \kappa + d}{2}
\end{aligned}
$$

with $d = \sqrt{(\rho\sigma i\phi - \kappa)^2 - \sigma^2(-\phi^2 - i\phi)}$. By the superposition principle[5], the general solution will be of the form

$$
y(\tau) = \alpha_1 e^{z_- \tau} + \alpha_2 e^{z_+ \tau}
$$

[4] A first-order ordinary differential equation quadratic in the unknown function

[5] Let $x_1, ..., x_n$ be n linear independent solutions to a differential equation. Then $\sum_{i=1}^{n} x_i c_i$ is also a solution to the differential equation for all values $c_1, ..., c_n$.

with boundary conditions

$$\begin{aligned}\alpha_1 + \alpha_2 &= y(0)\\ \alpha_1 z_- + \alpha_2 z_+ &= 0.\end{aligned}$$

The boundary conditions are obtained trough the relation between $D(\tau)$ and $y'(\tau)$ as well as the condition $D(0) = 0$. Isolating the coefficients yields

$$\begin{aligned}\alpha_1 &= \frac{\frac{z_+}{z_-} y(0)}{z_+/z_- - 1}\\ \alpha_2 &= -\frac{y(0)}{z_+/z_- - 1},\end{aligned}$$

which are substituted into the general solution implying

$$\begin{aligned}y(\tau) &= \frac{y(0)}{z_+/z_- - 1}\left(\frac{z_+}{z_-}e^{z_-\tau} - e^{z_+\tau}\right)\\ y'(\tau) &= \frac{y(0)}{z_+/z_- - 1}\left(z_+ e^{z_-\tau} - z_+ e^{z_+\tau}\right).\end{aligned}$$

Now, remember that we can express $D(\tau)$ by y' and y

$$\begin{aligned}D(\tau) &= -\frac{2}{\sigma^2}\frac{y'(\tau)}{y(\tau)}\\ &= -\frac{2}{\sigma^2}\frac{z_+\left(e^{z_-\tau} - e^{z_+\tau}\right)}{\left(\frac{z_+}{z_-}e^{z_-\tau} - e^{z_+\tau}\right)}\\ &= -\frac{2}{\sigma^2}z_-\frac{z_+\left(1 - e^{d\tau}\right)}{z_+ - z_- e^{d\tau}}\\ &= \frac{\kappa - \rho\sigma i\phi + d}{\sigma^2}\left(\frac{1 - e^{d\tau}}{1 - g e^{d\tau}}\right),\end{aligned}$$

where

$$g \equiv \frac{z_-}{z_+} = \frac{\kappa - \rho\sigma i\phi + d}{\kappa - \rho\sigma i\phi - d}.$$

Finally equation (4.23) can be solved by mere integration

$$\begin{aligned}
C(\tau) &= \int_0^\tau (ri\phi + \kappa\theta D(s))\, ds \\
&= ri\phi\tau - \frac{2}{\sigma^2}\kappa\theta \int_0^\tau \frac{y'(s)}{y(s)} ds \\
&= ri\phi\tau - \frac{2}{\sigma^2}\kappa\theta \left[\log y(s)\right]_0^\tau \\
&= ri\phi\tau - \frac{2}{\sigma^2}\kappa\theta \log\left(\frac{y(\tau)}{y(0)}\right) \\
&= ri\phi\tau + \frac{\kappa\theta}{\sigma^2}\left((\kappa - \rho\sigma i\phi + d)\tau - 2\log\left(\frac{1-ge^{d\tau}}{1-g}\right)\right).
\end{aligned}$$

The proposition is proved. □

Remark 4.6 It is important to consider the scenario $f(-i)$. Especially if we want to use the pricing function for numerical computations. Indeed, computing the arbitrage-free price of $\mathcal{X}$ at time t requires an expression for $f(-i)$ which is only defined in the limits: Note that $\phi \to -i$ implies $d \to \kappa - \rho\sigma$, $g \to \infty$ and $D(\tau) \to 0$, while the limit of $C(\tau)$ is a more "complex" issue. L'Hôpital yields

$$\log\left(\frac{1-ge^{d\tau}}{1-g}\right) \to e^{d\tau}$$

for $g \to \infty$ and in particular $C(\tau) \to r\tau$ as ϕ tends to $-i$. We conclude that

$$f(\phi) \to e^{r\tau + x_t} = S_t e^{r\tau}$$

for $\phi \to -i$. This is actually a sanity check since we already knew

$$f(-i) = E^{\mathbb{Q}}\left[e^{X_T}\right] = e^{r\tau} S_t \tag{4.26}$$

using that the process $(\hat{S}_t)_{t\geq 0}$ is a $\mathbb{Q}$-martingale. ∘

Due to our thorough work, we are finally ready to compute the price of $\mathcal{X}$ in Heston's model. The main outcome in our analysis of the Heston model is summarized in the following theorem.

Theorem 4.5 *Consider the European Call $\mathcal{X}$ in Heston's model. The arbitrage-free price at time t is determined by the formula*

$$C(S_t, \nu_t, t) = S_t P_1(t, X_t, \nu_t) - e^{-r\tau} K P_2(t, X_t, \nu_t). \tag{4.27}$$

The expiry-in-money probabilities are given by

$$P_1(t, X_t, \nu_t) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \log K} \tilde{f}(\phi)}{i\phi}\right] d\phi \tag{4.28}$$

$$P_2(t, X_t, \nu_t) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \log K} f(\phi)}{i\phi}\right] d\phi \tag{4.29}$$

with $f(\phi)$ and $\tilde{f}(\phi) = f(\phi - i)/f(-i)$ being the conditional characteristic functions of X_T under the pricing measure $\mathbb{Q}$ respectively under the equivalent martingale measure $\tilde{\mathbb{Q}}$ that uses the stock as numeraire. Furthermore,

$$f(\phi) = e^{C(\tau) + D(\tau)\nu_t + i\phi X_t}, \tag{4.30}$$

where the functions C and D can be found by the equations

$$C(\tau) = ri\phi\tau + \frac{\kappa\theta}{\sigma^2}\left((\kappa - \rho\sigma i\phi + d)\tau - 2\log\left(\frac{1 - ge^{d\tau}}{1 - g}\right)\right) \tag{4.31}$$

$$D(\tau) = \frac{\kappa - \rho\sigma i\phi + d}{\sigma^2}\left(\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}\right) \tag{4.32}$$

with

$$d = \sqrt{(\rho\sigma i\phi - \kappa)^2 - \sigma^2(-\phi^2 - i\phi)} \tag{4.33}$$

and

$$g = \frac{\kappa - \rho\sigma i\phi + d}{\kappa - \rho\sigma i\phi - d}. \tag{4.34}$$

Remark 4.7 The result is stated in many different versions across the literature of mathematical finance. We will of course mention the original article Heston (1993) in which the solution was first derived. The main difference between our result and the result in Heston (1993) is that we express $\tilde{f}(\phi)$ through $f(\phi)$. Note that $C(\tau)$ and $D(\tau)$ also depends on the parameter ϕ. ◦

4.4 Hedging in Heston

This section provides formulas for the Δ-hedge and the locally risk-minimizing hedge $\Delta^{\min}$ related to the minimal martingale measure $\mathbb{Q}^{\min}$. In order to compute these hedges, derivatives of $C(t, S_t, \nu_t)$ with respect to S_t and ν_t are required. Fortunately, theorem 4.5 shows that this should not cause any problems. The pricing function is of the form

$$C(t, S_t, \nu_t) = S_t P_1(t, X_t, \nu_t) - e^{-r\tau} K P_2(t, X_t, \nu_t),$$

where P_1 and P_2 are given in theorem 4.5. Note that we can express $\partial_S C(t, S_t, \nu_t)$ using $\partial_K C(t, S_t, \nu_t)$: Let $\alpha > 0$ and consider a European call with strike αK written on the underlying $(\alpha S_t)_{t\in[0,T]}$. Trivial computations yields that the price of an European call with these scaled arguments is given by

$$C(t, \alpha S_t, \nu_t) = \alpha S_t P_1(t, X_t, \nu_t) - \alpha e^{-r\tau} K P_2(t, X_t, \nu_t) = \alpha C(t, S_t, \nu_t).$$

We conclude that the price function is homogeneous of degree one with respect to S_t and K. As a consequence,

$$\begin{aligned} C(t, S_t, \nu_t) &= S_t \partial_s C(t, S_t, \nu_t) + K \partial_K C(t, S_t, \nu_t) \qquad \Leftrightarrow \\ \partial_s C(t, S_t, \nu_t) &= \frac{C(t, S_t, \nu_t) - K \partial_K C(t, S_t, \nu_t)}{S_t} \end{aligned}$$

using Euler's Homogeneous Function Theorem. We claim that it is easier to compute $\partial_K C(t, S_t, \nu_t)$ rather than $\partial_s C(t, S_t, \nu_t)$. Using Leibniz's Integral Rule yields

$$\begin{aligned} \partial_K C(t, S_t, \nu_t) &= \partial_K e^{-r\tau} E^{\mathbb{Q}} \left[(S_T - K)^+) \right] \\ &= e^{-r\tau} \int_K^\infty \partial_K (y - K) q(y) dy \\ &= -e^{-r\tau} P_2(t, X_t, \nu_t). \end{aligned}$$

Especially the Δ-hedge is given by the expression

$$\Delta_t = \partial_s C(t, S_t, \nu_t) = P_1(t, X_t, \nu_t). \tag{4.35}$$

Proposition 4.6 (Heston's $\boldsymbol{\Delta}$-hedge) *The Δ-hedge in Heston's stochastic volatility model is given by*

$$\Delta_t = P_1(t, X_t, \nu_t),$$

where the probability P_1 is found in theorem 4.5.

The interpretation is that a Δ-hedging trader should invest an amount of the underlying equal to the $\tilde{\mathbb{Q}}$-probability that the option is in-the-money at expiry. This strategy is not optimal in a risk-minimizing context. We want a strategy that includes the volatility risk. In parallel with chapter 3, we will investigate the locally risk-minimizing hedge.

4.4.1 The Locally Risk-Minimizing Delta

We consider the problem stated in subsection 3.11. Find a strategy $h(t) = (h_1(t), h_2(t))$ that replicates $\mathcal{X}$ exactly and minimizes the conditional expectation

$$R_t^h \equiv E\left[\left(\hat{L}_T^h - \hat{L}_t^h\right)^2 \mid \mathcal{F}_t\right] \tag{4.36}$$

over all continuations of h from t to T, where $(\hat{L}_t^h)_{t\geq 0}$ is the discounted cost process given in equation (3.29) related to the value process $V^h(t) = h_1(t)B_t + h_2(t)S_t$. A solution to this problem is given in theorem 4.7 in terms of the locally risk-minimizing hedge. Consider the hedging portfolio with a long option position and short $h_2(t)$ units of the stock. Itǒ's formula yields that

$$dC(t, S_t, \nu_t) = (...)dt + \partial_s C(t, S_t, \nu_t)dS_t + \partial_\nu C(t, S_t, \nu_t)d\nu_t$$

ignoring the dt-term due to the definition of the quadratic variation. Thus, the value of the hedging portfolio, $\Pi_t = C(t, S_t, \nu_t) - h_2(t)S_t$, has dynamics

$$d\Pi_t = (...)dt + \left(\partial_s C(t, S_t, \nu_t) - h_2(t)\right)dS_t + \partial_\nu C(t, S_t, \nu_t)d\nu_t.$$

The locally risk-minimizing hedge, $\Delta_t^{\min}$, minimizes the conditional variance of $d\Pi_t$

$$\begin{aligned}\mathrm{Var}_t[d\Pi_t] = & \left(\partial_s C(t,S_t,\nu_t) - h_2(t)\right)^2 \mathrm{Var}_t[dS_t] + \left(\partial_\nu C(t,S_t,\nu_t)\right)^2 \mathrm{Var}_t[d\nu_t] \\ & + 2\left(\partial_s C(t,S_t,\nu_t) - h_2(t)\right) \partial_\nu C(t,S_t,\nu_t) \mathrm{Cov}_t[dS_t, d\nu_t],\end{aligned}$$

which is convex in $h_2(t)$. Note that

$$\begin{aligned}\mathrm{Var}_t(dS_t) &= S_t^2 \nu_t dt \\ \mathrm{Var}_t(d\nu_t) &= \sigma^2 \nu_t dt \\ \mathrm{Cov}_t(dS_t, d\nu_t) &= S_t \sigma \rho \nu_t dt.\end{aligned}$$

Now, plug these values into the expression for the condition variance of $d\Pi_t$ to obtain

$$\frac{\partial \mathrm{Var}_t(d\Pi_t)}{\partial h_2(t)} = 2\left\{S_t^2 \nu_t \left(h_2(t) - \partial_s C(t,S_t,\nu_t)\right) - S_t \sigma \rho \nu_t \partial_\nu C(t,S_t,\nu_t)\right\} dt.$$

Due to the convexity in $h_2(t)$, the risk-minimizing strategy is given by

$$\begin{aligned}\Delta_t^{\min} &= \partial_s C(t,S_t,\nu_t) + \frac{\rho\sigma}{S_t} \partial_\nu C(t,S_t,\nu_t) \\ &= \Delta_t + \frac{\rho\sigma}{S_t} \partial_\nu C(t,S_t,\nu_t).\end{aligned} \tag{4.37}$$

Equation (4.37) **is** the expression that we will use. However, the formula depends on the pricing function C, that depends on a non-defined martingale measure, which the hedge depends on. The answer begs the question. Instead consider the three step procedure proposed in (Poulsen, Schenk-Hoppè, and Ewald, 2009, Page 6-7):

Complete Consider the martingale measure $\mathbb{Q}^{\min}$ corresponding to the risk premium process

$$\gamma^{\min}(t) = \frac{\mu - r}{\sqrt{\nu_t}} \cdot \begin{pmatrix} \sqrt{1-\rho^2} \\ \rho \end{pmatrix} \tag{4.38}$$

and introduce a volatility-dependent asset $(\tilde{S}_t)_{t\geq 0}$ completing the market under $\mathbb{Q}^{\min}$: Let $(\tilde{S}_t)_{t\geq 0}$ satisfy the stochastic differential equation

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \left(r + \frac{\sigma\rho}{\nu_t}(\mu - r)\right) dt + \frac{\sigma}{\sqrt{\nu_t}} dW_t^2.$$

The choice of $\gamma^{\min}(t)$ implies

$$\begin{aligned}\frac{dS_t}{S_t} &= rdt + \sqrt{\nu_t}\left(\sqrt{1-\rho^2}dW_t^{1,\mathbb{Q}^{\min}} + \rho dW_t^{2,\mathbb{Q}^{\min}}\right)\\ \frac{d\tilde{S}_t}{\tilde{S}_t} &= rdt + \frac{\sigma}{\sqrt{\nu_t}}dW_t^{2,\mathbb{Q}^{\min}}.\end{aligned}$$

Surely, we can replicate $\mathcal{X}$ with a self-financing strategy. The market is completed.

Compute The optimal hedge - which replicates $\mathcal{X}$ exactly - is found in same way as in the simple Black-Scholes universe. Let $h_1(t)$ be the amount of holdings in S_t, $h_2(t)$ the amount of holdings in $\tilde{S}_t$ and choose the money such that the strategy is self-financing. We will use the self-financing condition to express the dynamics, $d\Pi_t$, of the value process Π_t corresponding to the hedging strategy. Moreover, the value of $\mathcal{X}$ at time t is given by

$$C(t, S_t, \nu_t) = e^{-r\tau}E_t^{\mathbb{Q}^{\min}}[\mathcal{X}].$$

Since Π_t replicates $\mathcal{X}$, we must have $d\Pi_t = dC(t, S_t, \nu_t)$ for $t \leq T$. Using the bivariate version of Itǒ's formula and equating the two dynamics yields that the strategy which eliminates all risk is given by

$$\begin{aligned}h_1(t) &= \partial_s C(t, S_t, \nu_t)\\ h_2(t) &= \frac{\nu_t}{\tilde{S}_t}\partial_\nu C(t, S_t, \nu_t).\end{aligned}$$

Project Using the proposition (El Karoui et al., 1997, Proposition 1.1) leads to the locally risk-minimizing hedge in Heston's model

$$\Delta^{\min}(t) = \partial_s C(t, S_t, \nu_t) + \frac{\rho\sigma}{S_t}\partial_\nu C(t, S_t, \nu_t).$$

The result is summarized in the following theorem (Poulsen, Schenk-Hoppè, and Ewald, 2009, Proposition 1).

Theorem 4.7 *Consider the stochastic volatility model 4.1. The locally risk-minimizing* *strategy,* $\Delta^{\min}(t)$*, holds*

$$\Delta^{\min}(t) = \partial_s C(t, S_t, \nu_t) + \rho \frac{\nu_t g(\nu_t)}{S_t^{1+\gamma} f(\nu_t)} \partial_\nu C(t, S_t, \nu_t) \tag{4.39}$$

units of the stock at time t*. The price of the option,* $C(t, S_t, \nu_t)$*, is given by*

$$C(t, S_t, \nu_t) = e^{-r\tau} E_t^{\mathbb{Q}^{\min}} \left[(S_T - K)^+ \right],$$

where $\mathbb{Q}^{\min}$ *is the minimal martingale measure corresponding to the risk premium process* $(\gamma^{\min}(t))_{t \geq 0})$ *given in equation* (4.38)*. In addition,* S_t *and* ν_t *has* $\mathbb{Q}^{\min}$*-dynamics given by*

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t \left(\sqrt{1-\rho^2} dW_t^{\mathbb{Q}_{\min},1} + \rho dW_t^{\mathbb{Q}_{\min},2} \right) \tag{4.40}$$

$$d\nu_t = \left(\kappa(\theta - \nu_t) - \rho\sigma(\mu - r) \right) dt + \sigma \sqrt{\nu_t} dW_t^{\mathbb{Q}_{\min},2}. \tag{4.41}$$

Here $(W_t^{\mathbb{Q}^{\min},i})_{t \geq 0}$ *is a* $\mathbb{Q}^{\min}$*-Brownian motion for* $i = 1, 2$*. Finally the investment in the risk-free asset is* $C(t, S_t, \nu_t) - \Delta^{\min}(t) S_t$*.*

Remark 4.8 If $\rho = 0$ the locally risk-minimizing hedge coincides with the Δ-hedge. The two processes S_t and ν_t are uncorrelated for $\rho = 0$; hence we can only hedge the risk related to the underlying stock . Typically, we will see a negative correlation between the underlying stock and the stochastic variance. ○

Remark 4.9 Assume negative correlation between the underlying and the stochastic variance. The contract function for the European Call is convex implying that the Vega $\partial_\nu C(t, S_t, \nu_t)$ is positive. Especially a delta hedger will invest too much in the underlying. ○

Remark 4.10 Let $\kappa^{\mathbb{Q}^{\min}} = \kappa$ and $\theta^{\mathbb{Q}^{\min}} = \theta - \frac{\rho\sigma}{\kappa}(\mu - r)$. The $\mathbb{Q}^{\min}$-dynamics of $(\nu_t)_{t \geq 0}$ can be expressed as

$$d\nu_t = \kappa^{\mathbb{Q}^{\min}} (\theta^{\mathbb{Q}^{\min}} - \nu_t) dt + \sigma \sqrt{\nu_t} dW_t^{\mathbb{Q}_{\min},2},$$

and the parametric form is the same as under $\mathbb{P}$. ○

In order to determine the locally risk-minimizing strategy, we first note

$$\partial_\nu C(t, S_t, \nu_t) = S_t \partial_\nu P_1(t, X_t, \nu_t) - e^{-r\tau} K \partial_\nu P_2(t, X_t, \nu_t). \tag{4.42}$$

Combining Leibniz's Integral Rule with the fact the derivative of the real part of a complex function equals the real part of the derivative of the complex function (Cauchy-Riemann equations Berg (2012)) yields that

$$\begin{aligned}\partial_\nu P_1(t, X_t, \nu_t) &= \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[D(\tau;\ \phi - i)\frac{e^{-i\phi \log K}\tilde{f}(\phi)}{i\phi}\right] d\phi \\ \partial_\nu P_2(t, X_t, \nu_t) &= \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[D(\tau;\phi)\frac{e^{-i\phi \log K}f(\phi)}{i\phi}\right] d\phi.\end{aligned}$$

Note that $\tilde{f}(\phi) = f(\phi - i)/f(-i)$. Now, plug the derivatives of the conditional probabilities into equation (4.42) to obtain a semi-closed expression for the locally risk-minimizing hedge

$$\Delta_t^{\min} = \partial_s C(t, S_t, \nu_t) + \frac{\rho\sigma}{S_t}\partial_\nu C(t, S_t, \nu_t). \tag{4.43}$$

4.5 Numerical Examples

As for the Merton model (subsection 3.6.1), we will consider two simulation experiments. The first study will analyse the performance of hedging strategies under Heston's model based on terminal hedge errors. The second study will conduct a version of Wilmott's hedge experiment under the Heston model. The first experiment is almost identical to the corresponding example in chapter 3, while Wilmott's hedge experiment is designed differently due to the stochastic variance process. Simulations of the underlying and the instantaneous volatility are based on the Euler discretization. This is adequate for our purpose - see (Andersen, 2007) for a more efficient simulation of the model.

4.5.1 Performance of the Strategies: Hedging Error

Based on benchmark settings for Heston's model, market parameters as well as model parameters under $\mathbb{P}$ are specified as in table 4.1.

Parameters	S_0	K	r	q	T	μ	ν_0	κ	θ	σ	ρ
Values	100	100	0.04	0	1/2	0.1	0.0483	4.75	0.0483	0.55	-0.569

Table 4.1: Specification of the market and model parameters in Heston's stochastic volatility model.

We assume the long-run variance to equal the initial variance. Moreover, we assume a negative correlation between the stock price and the volatility. Thus, we will expect that the Δ-trader invests too heavily in the stock. Hedging and pricing are done under the pricing measures $\mathbb{Q}^{\text{min}}$ and $\mathbb{Q}^{\text{mg}}$, where the former corresponds to the bivariate risk-premium process, $\gamma^{\text{min}}(t)$, in theorem 4.7, while the latter assumes that $(\hat{S}_t)_{t\geq 0}$ is a $\mathbb{P}$-martingale, i.e. $\mu = r$ and $\mathbb{P}$ is a martingale measure. Risk-minimizing hedges are carried out under both $\mathbb{Q}^{\text{min}}$ and $\mathbb{Q}^{\text{mg}}$. As in subsection 3.6.1, we will use the term *misconceived risk-minimizing hedge* for the latter. For Δ-hedges we stick to $\mathbb{Q}^{\text{mg}}$ although it would not be wrong to use $\mathbb{Q}^{\text{min}}$. Risk-adjusted parameters under the two measures are given by $\theta^{\mathbb{Q}^{\text{min}}} = 0.054$, $\theta^{\mathbb{Q}^{\text{mg}}} = 0.0483$ and $\kappa^{\mathbb{Q}^{\text{min}}} = \kappa^{Q^{\text{mg}}} = 4.75$. As usual we set up the portfolio long one European option, short in the underlying and choose the money account in accordance with the self-financing condition. Figure 4.2 illustrates the terminal hedge errors for all three strategies.

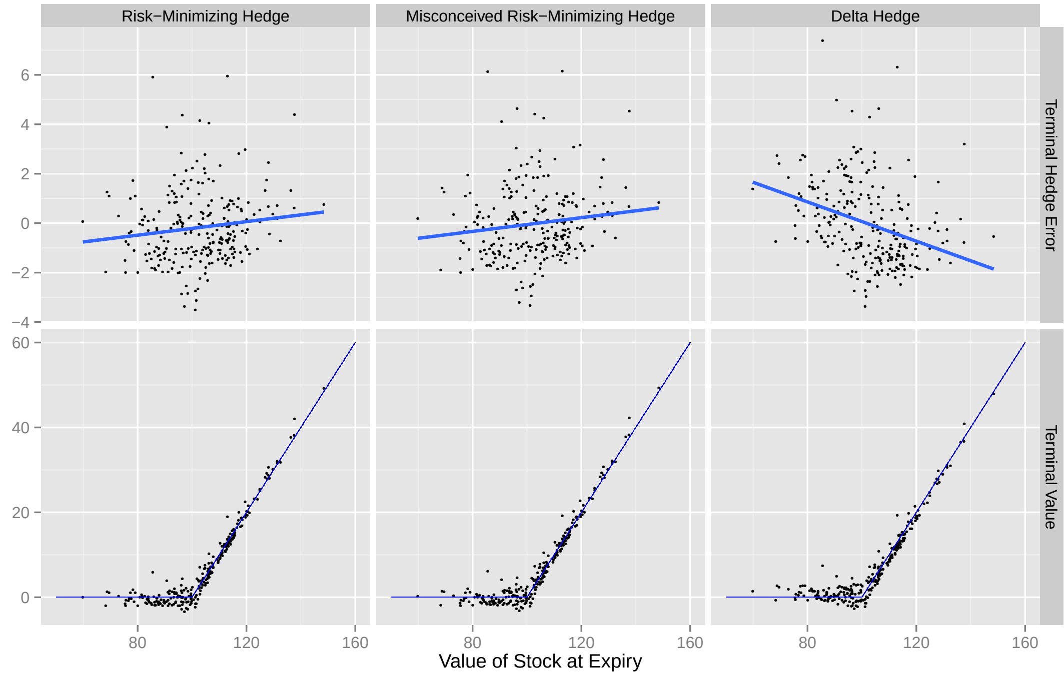

Figure 4.2: *We have simulated* 250 *paths of the stock price assuming Heston's stochastic volatility model with parameters specified in table 4.1. For each path, the terminal hedge error is computed with the* Δ*-hedge, the locally risk-minimizing hedge and the misconceived risk-minimizing hedge. The upper plots show terminal hedge errors as a function of* S_T*. Based on these values, a simple linear regression model is fitted and the blue line illustrates the fitted regression line. The lower plots show the terminal values of the hedging portfolios as a function of* S_T*. The blue line represents the pay-off* $(S_T - 100)^+$ *for the European call.*

According to figure 4.2, the upper plots indicate 1) terminal hedge errors under $\mathbb{Q}^{\min}$ respectively $\mathbb{Q}^{\text{mg}}$ do not differ significantly and 2) terminal hedge errors for the Δ-hedge depend more on S_T than the terminal hedge errors related to risk-minimizing strategies. As in subsection 3.6.1, we compute the standard deviation of profit-&-loss at maturity relative to the initial value of the option in percentages:

$$\xi = 100 \cdot \frac{\sqrt{\text{Var}^{\mathbb{P}}[\Pi_T]}}{e^{-rT} E^{\mathbb{Q}}[\mathcal{X}]}. \tag{4.44}$$

Table 4.2 presents the estimates for ξ.

Option \ Strategy	$\mathbb{Q}^{\min}$	$\mathbb{Q}^{\mathrm{mg}}$	Δ	$\mathbb{Q}^{\min}$-gain	$\mathbb{Q}^{\mathrm{mg}}$-gain
Call	20.51	21.17	23.95	3.44	2.78
Put	28.33	29.49	33.36	5.03	3.87

Table 4.2: The hedging error ξ for both call and put options using the locally risk-minimizing delta hedge ($\mathbb{Q}^{\min}$), the misconceived risk-minimizing hedge ($\mathbb{Q}^{\mathrm{mg}}$) and finally the Δ-hedge. The two last columns show the error-difference between the locally risk-minimizing hedge and the Δ-hedge respectively the misconceived risk-minimizing hedge and the Δ-hedge.

An improvement for the risk-minimizing strategies compared with the Δ-hedge is clearly seen. Using the locally risk-minimizing strategy rather than the Δ-hedge will reduce the hedge errors by a factor of 16%. Moreover, the locally risk-minimizing strategy will also outperform the misconceived risk-minimizing strategy due to the estimates of ξ; the reduction in hedge errors is given by the factor of $3-4\%$.

4.5.2 Wilmott's Hedge Experiment: The Heston Model

In contrast with both Wilmott's hedge experiment in the Black-Scholes model and in Merton's jump-diffusion model, we will not assume the actual volatility to be constant. Obviously, assuming $\sqrt{\nu_t^r} = \sqrt{\nu^r}$ for some positive constant ν^r corresponds to assume the Black-Scholes model. Instead we could design the experiment within the same framework in a stochastic sense, e.g. assume constant implied volatility, an initial volatility that exceeds this implied volatility and also a long-run volatility exceeding the implied volatility. Another approach is to assume that the actual volatility always exceeds implied volatility - for example by an ad-hoc assumption $\sqrt{\nu_t^r} = \sqrt{2\nu_t^i}$. Our experiment is based on the former approach. Table 4.3 shows specifications of market and model parameters.

Parameters	S_0	K	r	q	T	μ	ν_0	κ	θ	σ	ρ	ν^i
Values	100	100	0.05	0	1	0.1	0.05	4.75	0.05	0.55	-0.569	0.15^2

Table 4.3: Specification of the market and model parameters in Willmot's hedge example assuming that the stock price and the volatilitly is a realisation of Heston's stochastic volatility model. Note that $\sqrt{\nu^i}$ defines the implied volatility determined by the market.

At first, we will explore the terminal hedge errors under the Δ-hedge, where the hedging volatility is chosen as the actual volatility respectively the implied volatility. According to theorem 4.3, neither the implied or the actual volatility can prevent a hedging error due to the dW_t^2-term. Figure 4.3 illustrates our results.

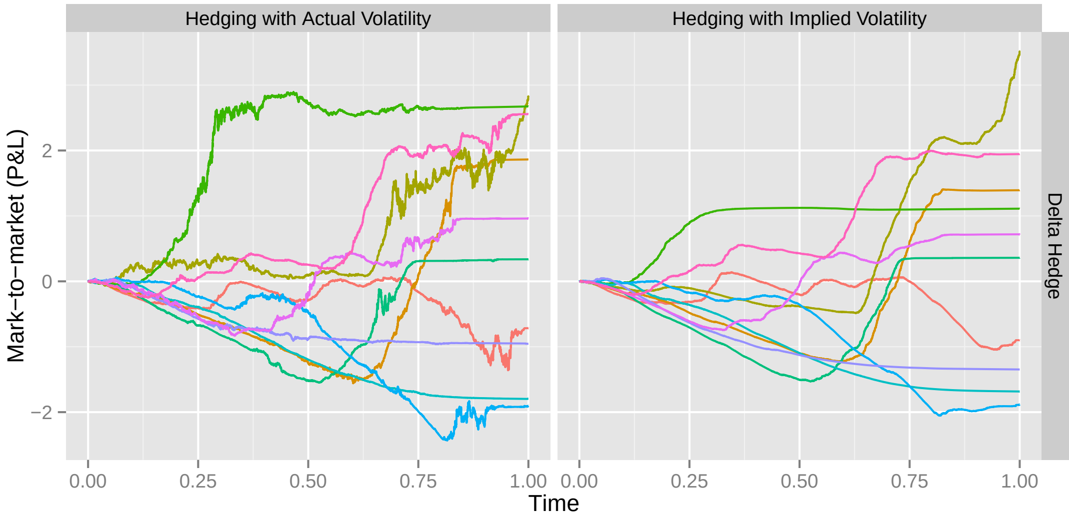

Figure 4.3: *We have simulated* 10 *paths of the stock price process assuming Heston's model with parameters specified in table 4.3. For each simulation we compute the* $P\&L$ *on a day-to-day basis over the time period* $[0, 1]$ *using the* Δ*-hedge. The left diagram corresponds to hedging with actual volatility, while implied volatility is used in the right diagram.*

Obviously, we cannot make profit with probability one. Four expiry values are negative. As a second remark, pay attention to the smoothness of the paths. As in the Black-Scholes example, hedging with the implied volatility rather than the actual volatility will increase the smoothness. Moreover, the $P\&L$'s related to the implied volatility seems to fluctuate less than the $P\&L$'s corresponding to the actual volatility. The experiment is completed with an investigation of the locally risk-minimizing hedge. Figure 4.4 illustrates our findings.

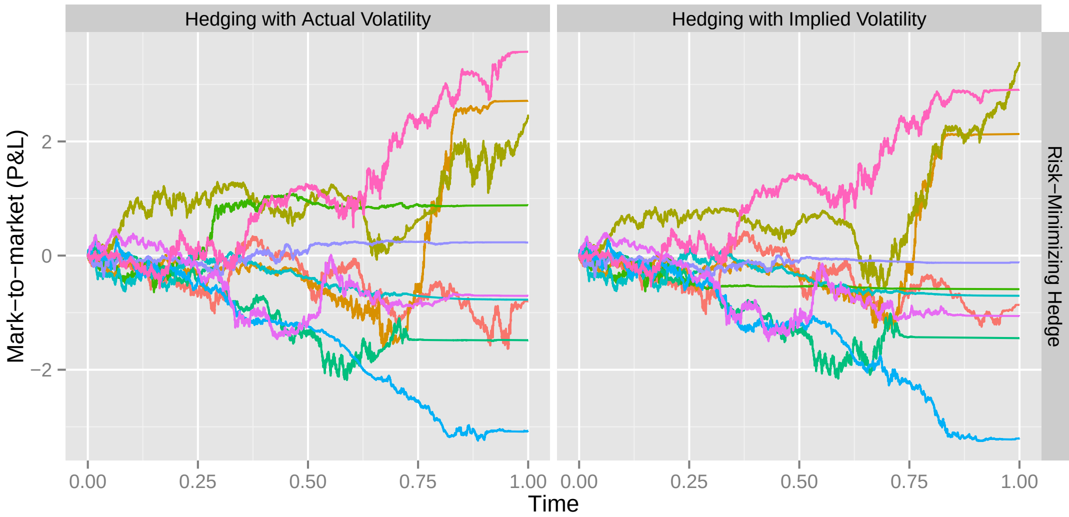

Figure 4.4: *We have simulated* 10 *paths of the stock price assuming Heston's stochastic volatility model with parameters specified in table 4.3. For each simulation we compute the* $P\&L$ *on a day-to-day basis over the time period* $[0, 1]$ *using the locally risk-minimizing hedge. The paths in the diagram to the left corresponds to hedging with actual volatility, while the hedge volatility equals the implied volatility in the diagram to the right.*

From the theoretical perspective, we cannot use theorem 4.3 directly for an interpretation of the results. The Fundamental Theorem of Derivative Trading is derived by the Δ-hedge, while the risk-minimizing hedge can be seen as a risk-adjusted Δ-hedge. Based on the variation of the paths as well as the terminal hedge errors, it seems that hedging with the implied volatility is again less painful than hedging with actual volatility. However, this is not as obvious as for the Δ-hedge scenario. In particular, the smoothness of the paths disappears hedging with the implied volatility. The reason for this is a more complex strategy including several parameters. A Comparison between the Δ-hedge and the locally risk-minimizing hedge is difficult. Immediately we would prefer the Δ-hedge with the hedging volatility chosen as the implied volatility due to the smoothness of the $P\&L$. On the other hand, the simulation study in the previous subsection (see table 4.2) showed that using the risk-minimizing hedge rather than the Δ-hedge would reduce terminal hedge errors. We conclude that the results are in accordance with theorem 4.3; volatility arbitrage is not present when the volatility is stochastic. Moreover, we will presume that the best trading strategy is obtained with the locally risk-minimizing hedge, where the implied volatility is used as hedging volatility.