



Mandatory Bachelor Project

Math-Econ Programme

Financial Modeling with Continuous-Time Models

Theory and Applications

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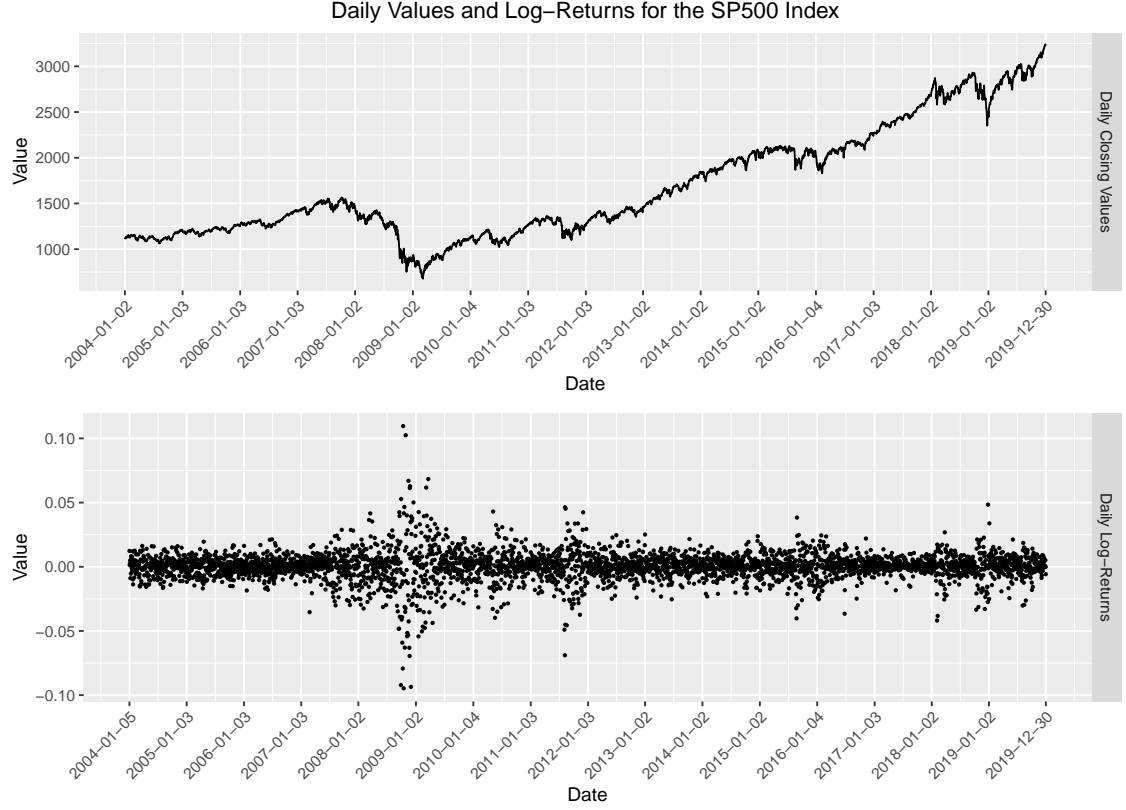


Figure 1: The top plot shows daily closing values for the SP500 index between 2004/01/02-2019/12/30. Corresponding log-returns are represented by the black points in the lower plot. Source: <https://finance.yahoo.com/quote/%5EGSPC/>.

1 Introduction

The financial market consists of a risky asset (a stock) and a risk-free asset (e.g., zero-coupon bond), both traded continuously up to some fixed time horizon T . As usual, the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{0 \leq t \leq T}, P)$ is introduced to capture the information flow: Ω contains all possible states of the market, \mathcal{F} is the corresponding σ -algebra, P is the physical measure, and (\mathcal{F}_t^W) is the σ -field generated by the Brownian motion $(W_t)_{0 \leq t \leq T}$, i.e.,

$$\mathcal{F}_t^W = \sigma \{W_s \mid 0 \leq s \leq t\}. \quad (1)$$

In the remainder of this project, S_t denotes the value of the risky asset at time t , and the process is assumed to solve the stochastic differential equation (SDE)

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t \quad (2)$$

$$S_0 = s > 0 \quad (3)$$

where $\mu(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ are deterministic functions possibly depending on both time (t) and the state of the underlying (S_t). Moreover, $\sigma(t, s) > 0$ for all $(t, s) \in [0, T] \times \mathbb{R}^+$, and both $\mu(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ are sufficiently well-behaved such that a solution to the above SDE exists. We will use following terminology: $\mu(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ are called the drift and volatility, respectively, of (S_t) , whereas dS_t is referred to as the dynamics of the process.

The risk-free asset's value through time is given by the process $(B_t)_{0 \leq t \leq T}$ with dynamics

$$dB_t = rB_t dt \quad (4)$$

$$B_0 = 1 \quad (5)$$

where r is constant. The risk-free asset is interpreted as the money account with short rate of interest r .

Questions: Model Assumptions

The economic interpretation of the above setting is crucial for working with continuous-time models in practice. Answering the questions below will provide some intuition behind the model assumptions. I would recommend you to read chapters 2 and 4 in [Björk \(2009\)](#).

1.1

Discuss the term *risk*. What is the difference between a risk-free and a risky asset? What would a locally risk-free asset be? How many sources of risk appear in this model setting? Is $\mathcal{F}_t^W = \mathcal{F}_t^S$? What is the financial interpretation of the \mathcal{F}_t^S .

1.2

Explain the economic interpretation of $D_t := B_t/B_T$. Is $S_0 > 0$ a strong assumption? Illustrate B_t , D_t , S_t and $D_t S_t$. You can use the following pseudo-code for generating (S_t) (for fun: adjust n as well as the *seed*):

```

1 # Parameters
2 S <-          # Starting value of process
3 mu <- 0.05    # drift
4 sigma <- 0.2   # volatility
5
6 # Time grid
7 T <- 1        # End time
8 n <-          # Number of evaluations
9 dt <- T/(n)   # Equidistant time step
10
11 S_vec <- numeric(n)
12
13 for (i in 1:n){
14
15   Z <-          # Generate value from normal distribution with mean=0 and var=1

```

```

16 dS <-          # Compute S_i - S_{i-1}
17
18 S <- S + dS # Update S
19
20 S_vec[i] <- S
21 }

```

1.3

Is it reasonable to assume that stock prices are represented by continuous processes? Discuss the (dis)advantage of using continuous-time models for representing stock prices (e.g., discuss the trade off between numerical tractability and modeling reality/financial markets).

2 Geometric Brownian Motion

Assume that

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t} \quad (6)$$

where $S_0 = s > 0$ is the (time-0) starting value of the process. The process in equation (6) is called a Geometric Brownian Motion (GBM) and is closely linked to equations (2)-(3), although we can't prove that until we get Ito's formula in our toolbox (next topic). The Geometric Brownian Motion is defined and discussed by Björk (2009) in chapter 5.

Questions: Geometric Brownian Motion

2.1

Is (S_t) continuous? Illustrate (S_t) using pseudo-code from the box. Also compare this simulation technique with Euler's method and build-in function from **sde**-package respectively.

```

1 for (i in 1:n){
2   Z <- # Generate value from normal dist. with mean=0 and var=1
3   X <- # log(S_i/S_{i-1})
4   S <- # Update S
5   S_vec[i] <- S
6 }
7
8 # Or without the for-loop....
9 Z <- # Generate n values from multi dim. normal dist.
10 X <- cumsum(...) # Argument should be a n-dim vector
11 S <- # Generate GBM

```

2.2

Compute $E(S_t)$ and $V(S_t)$. What is the distribution of S_t ? Also compute $E(S_t | \mathcal{F}_s^S)$ for $s < t$ (hint: realize that $S_t = S_s \times S_t/S_s$). Is (S_t) a martingale under P ?

2.3

Let $R_t := (S_t - S_{t-1})/S_t$ denote the *return* over one period. Determine the distribution of the associated log-return $r_t := \log(S_t/S_{t-1})$. How is R_t linked with r_t ? Explain the convenience of using log-returns when working with financial time-series.

2.4

Would you prefer holding the risk-free asset or the risky-asset?

2.5

Produce figure 1. Are data a realization of a Geometric Brownian Motion (i.e., discuss whether model assumptions for GBM are violated)?

2.6

Derive the maximum likelihood estimator of μ and σ . Estimate μ and σ for the sample used to generate figure 1. Finally, discuss a significant shortfall for the maximum likelihood estimator of the drift μ (e.g., you may support your arguments with data).

3 Itô Calculus

Itô calculus and in particular Itô's formula play a crucial role in mathematical finance. The following results are from chapter 4 in Björk (2009) (p. 49-62, read it!).

Theorem 3.1 (Itô's formula) Assume that the process $X = (X_t)_{t \geq 0}$ has dynamics given by

$$dX_t = \mu_t dt + \sigma_t dW_t, \quad (7)$$

where μ_t and σ_t are adapted processes, and let $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ be $C^{1,2}$ -function. Introduce a new process by $Z_t = f(t, X_t)$. Then $Z = (Z_t)_{t \geq 0}$ has dynamics given by

$$dZ_t = df(t, X_t) = \left(f_t(t, X_t) + \mu_t f_x(t, X_t) + \frac{1}{2} \sigma_t^2 f_{xx}(t, X_t) \right) dt + \sigma_t f_x(t, X_t) dW_t, \quad (8)$$

where f_t , f_x and f_{xx} denote the derivative with respect to time, the derivative with respect to space and the second-derivative with respect to space, respectively.

Another formulation of Ito's formula is stated in the following proposition — you may find this version useful!

Proposition 3.2 *With the assumptions as in 3.1, dZ_t is given by*

$$dZ_t = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, X_t)(dX_t)^2, \quad (9)$$

where

$$(dt)^2 = 0, \quad dt \cdot dW_t = 0 \quad \text{and} \quad (dW_t)^2 = dt.$$

Finally, the multidimensional version of Ito's formula is given in definition 4.16 in Björk (2009).

Questions: Ito Calculus

Itô

3.1

State a condition such that the process $(X_t)_{t \geq 0}$ with dynamics given by equation (7) becomes a P -martingale. What do we call X when $\mu_t \geq 0$ P -almost surely? Finally, derive/state the condition such that $Z_t = f(t, X_t)$ becomes a P -martingale.

3.2

Compute the dynamics of $(S_t)_{t \geq 0}$ where S_t is given by equation (6). Moreover, apply Ito's formula to compute $E(S_t)$.

From now on, we will specify $(S_t)_{t \geq 0}$ by its stochastic differential and not equation (6).

3.3

Compute the dynamics of $(Z_t)_{t \geq 0}$ where $Z_t = S_t/B_t$. Hint: Use the multidimensional Ito formula from Björk (2009). Is the process a martingale?

4 Black-Scholes Model and Q -dynamics

The Black-Scholes model consists of two assets with dynamics given by

$$dB_t = rB_t dt, \quad (10)$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (11)$$

where $B_0 = 1$ and $S_0 = s > 0$, and r, μ and $\sigma > 0$ are constants. This famous model was (and to some degree still is) used for pricing T -claims. Following definition 7.1 in Björk (2009), the stochastic variable \mathcal{X} is called a T -claim if it's \mathcal{F}_T^S -measurable. Here T is the date of maturity (or exercise date). A simple T -claim is on the form $\mathcal{X} = \Phi(S_T)$, where $\Phi(\cdot)$ is the contract function.

The focal point is to price \mathcal{X} at any point in time $t < T$, i.e. determine the pricing function $\pi(t; \mathcal{X})$ for $t < T$ with terminal condition $\pi(T; \mathcal{X}) = \mathcal{X}$. The Black-Scholes model is free of arbitrage as well as complete (we will discuss both concepts next time), and we will also assume that

$$\pi(t; \mathcal{X}) = F(t, S_t), \quad (12)$$

where F is a $C^{1,2}$ -function.

Pricing in this type of models is conducted under a *risk-neutral* probability measure (also called equivalent martingale measure (EMM), see definition) Q . In the Black-Scholes model, the pricing measure Q is unique — this follows directly from no-arbitrage and completeness of the model. Moreover, the discounted process (S_t/B_t) is a martingale under this measure. Theorem 10.5 and theorem 10.9 yield in Björk (2009) yield the existence of Q , whereas theorem 10.17 implies uniqueness of the pricing measure. Definition 10.11 and theorem 10.14 tell us that (S_t/B_t) is a martingale under this pricing measure.

In this section, we will show that the process the discounted process (S_t/B_t) has Q -dynamics given by

$$dS_t = rS_t dt + \sigma S_t dW_t^Q, \quad (13)$$

where W_t^Q is a Brownian motion under Q (not P !).

Questions: Black-Scholes Model

4.1

According to theorem 11.3 (Girsanov) in Björk (2009), we can define a new probability Q on \mathcal{F}_T implying that

$$dW_t = \phi_t dt + dW_t^Q,$$

where W_t^Q is Q -Brownian motion. The process ϕ_t is called the Girsanov kernel. Compute the Q -dynamics of (S_t/B_t) . Hint: Replace W_t with $\phi_t dt + dW_t^Q$ and apply Itô's formula.

4.2

Choose ϕ_t implying that the discounted process is a Q -martingale.

4.3

Verify that are the solution satisfies the assumptions in Girsavno's theorem. Conclude that Q (generated from ϕ_t) is the risk-neutral pricing measure.

4.4

Compute the Q -dynamics of (S_t) .

4.5 (Black-Scholes Equation)

Compute the Q -dynamics of $F(t, S_t)/B_t$ where $F(t, S_t)$ is the pricing function of the T -claim \mathcal{X} . Derive the condition (PDE) such that $F(t, S_t)/B_t$ becomes a Q -martingale. Finally, state the time- T boundary equation for the PDE.

5 Pricing Options in the Black-Scholes Model

We are finally ready to price T -claims in the Black-Scholes universe. Fundamentally, there are three approaches to value \mathcal{X} at time $t < T$: numerically, analytically or with Monte Carlo techniques, respectively.

5.1

State an equation (depending on time and space) that $F(t, S_t)$ from equation (12) must satisfy.

5.2

Use proposition 5.6 from Björk (2009) to express $F(t, S_t)$ as a conditional expectation. Interpret/discuss the expression for the arbitrage-free price of \mathcal{X} at time $t < T$.

5.3

Explain how we can use Monte Carlo techniques to compute the arbitrage-free price of the T -claim \mathcal{X} .

5.4

Determine the contract function for the European call with maturity T and strike K .

5.5

Use Monte Carlo simulation to compute the price of the European call.

6 TBA

A) The Fundamental Theorem of Derivative Trading

B) Stochastic Volatility

References

Björk, T. (2009). *Arbitrage Theory in Continuous Time* (3 ed.). Oxford University Press.