Course: Theory of Probability II
Term: Spring 2015
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Lecture 20
Itô's formula

Itô's formula

Itô's formula is for stochastic calculus what the Newton-Leibnitz formula is for (the classical) calculus. Not only does it relate differentiation and integration, it also provides a practical method for computation of stochastic integrals. There is an added benefit in the stochastic case. It shows that the class of continuous semimartingales is closed under composition with C^2 functions. We start with the simplest version:

Theorem 20.1. *Let X be a continous semimartingale taking values in a segment $[a,b] \subseteq \mathbb{R}$, and let $f : [a,b] \to \mathbb{R}$ be a twice continuously differentiable function ($f \in C^2[a,b]$). Then the process $f(X)$ is a continuous semimartingale and*

$$f(X_t) - f(X_0) = \int_0^t f'(X_u)\, dM_u + \int_0^t f'(X_u)\, dA_u + \tfrac{1}{2} \int_0^t f''(X_u)\, d\langle M \rangle_u. \tag{20.1}$$

Remark 20.2. The Leibnitz notation $\int_0^t f'(X_u)\, dM_u$ (as opposed to the "semimartingale notation" $f'(X) \cdot M$) is more common in the context of the Itô formula. We'll continue to use both.

Before we proceed with the proof, let us state and prove two useful related results. In the first one we compute a simple stochastic integral explicitly. You will immediately see how it differs from the classical Stieltjes integral of the same form.

Lemma 20.3. *For $M \in \mathcal{M}_0^{2,c}$, we have*

$$M \cdot M = \tfrac{1}{2} M^2 - \tfrac{1}{2} \langle M \rangle.$$

Proof. By stopping, we may assume that M and $\langle M \rangle$ are bounded. For a partition $\Delta = \{0 = t_0 < t_1 < \dots\}$, let $\bar{M}^\Delta$ denote the "left-continuous simple approximation"

$$\bar{M}^\Delta_t = M_{t_{k^\Delta(t)}}, \text{ where } k^\Delta(t) = \sup_{k \in \mathbb{N}_0} t_k \leq t.$$

Nothing but rearrangement of terms yields that

$$(\bar{M}^{\Delta} \cdot M)_t = \tfrac{1}{2}(M_t^2 - \langle M \rangle_t^{\Delta}) \text{ for all } t \geq 0, \tag{20.2}$$

and all partitions $\Delta \in P_{[0,\infty)}$. Indeed, assuming for notational simplicity that Δ is such that $t_n = t$, we have

$$\begin{aligned}(\bar{M}^{\Delta} \cdot M)_t &= \sum_{k=1}^{n} M_{t_{k-1}}(M_{t_k} - M_{t_{k-1}}) = \sum_{k=1}^{n} \left(\tfrac{1}{2}(M_{t_k} + M_{t_{k-1}}) - \tfrac{1}{2}(M_{t_k} - M_{t_{k-1}})\right)(M_{t_k} - M_{t_{k-1}}) \\ &= \sum_{k=1}^{n} \tfrac{1}{2}(M_{t_k}^2 - M_{t_{k-1}}^2) - \sum_{k=1}^{n} \tfrac{1}{2}(M_{t_k} - M_{t_{k-1}})^2 = \tfrac{1}{2}M_t^2 - \tfrac{1}{2}\langle M \rangle_t^{\Delta}.\end{aligned} \tag{20.3}$$

By Theorem 18.10, the right-hand side of (20.3) converges to $\frac{1}{2}(M_t^2 - \langle M \rangle_t)$ in $\mathbb{L}^2$, as soon as $\Delta \to \mathrm{Id}$. To show that the limit of the left-hand side converges in $M \cdot M$, it is enough to use the stochastic dominated-convergence theorem (Proposition 19.10). Indeed, the integrands are all, unifromly, bounded by a constant. □

The second preparatory result is the stochastic analogue of the **integration-by-parts formula**. We remind the reader that for two semimartingales $X = M + A$ and $Y = N + C$, we have $\langle X, Y \rangle = \langle M, N \rangle$.

Proposition 20.4. *Let $X = M + A, Y = N + C$ be semimartingale decompositions of two continuous semimartingales. Then XY is a continuous semimartingale and*

$$X_t Y_t = X_0 Y_0 + \int_0^t Y_u \, dX_u + \int_0^t X_u \, dY_u + \langle X, Y \rangle_t. \tag{20.4}$$

Proof. By stopping, we can assume that the processes M, N, A and C are bounded (say, by $K \geq 0$). Moreover, we assume that $X_0 = Y_0 = 0$ - otherwise, just consider $X - X_0$ and $Y - Y_0$. We write XY as $(M + A)(N + C)$ and analyze each term. Using the polarization identity, the product MN can be written as $MN = \frac{1}{2}\left((M+N)^2 - M^2 - N^2\right)$, and Lemma 20.3 implies that

$$M_t N_t = \int_0^t M_u \, dN_u + \int_0^t N_u dM_u + \langle M, N \rangle_t, \tag{20.5}$$

holds for all $t \geq 0$. As far as the FV terms A and C are concerned, the equality

$$A_t C_t = \int_0^t A_u \, dC_u + \int_0^t C_u \, dA_u \tag{20.6}$$

follows by a representation of both sides as a limit of Riemann-Stieltjes sums. Alternatively, you can view the left-hand side of the above equality as the area (under the product measure $dA \times dC$) of the square $[0, t] \times [0, t] \subseteq \mathbb{R}^2$. The right-hand side can also be interpreted as the area of $[0,t] \times [0,t]$ - the two terms corresponding to the areas above and below the diagonal $\{(s,s) \in \mathbb{R}^2 : s \in [0,t]\}$ (we leave it up to the reader to supply the details).

Let us focus now on the mixed term MC. Take a sequence $\{\Delta_n\}_{n\in\mathbb{N}}$ in $P_{[0,\infty)}$ with $\Delta_n = \{t^n_k\}_{k\in\mathbb{N}}$ and $\Delta_n \to \mathrm{Id}$ and write

$$\begin{aligned}
M_t C_t &= \sum_{k=0}^{\infty} (M_{t\wedge t^n_{k+1}} C_{t\wedge t^n_{k+1}} - M_{t\wedge t^n_k} C_{t\wedge t^n_k}) = I^1_n + I^2_n + I^3_n, \text{ where} \\
I^1_n &= \sum_{k=0}^{\infty} M_{t\wedge t^n_k} (C_{t\wedge t^n_{k+1}} - C_{t\wedge t^n_k}) \\
I^2_n &= \sum_{k=0}^{\infty} C_{t\wedge t^n_k} (M_{t\wedge t^n_{k+1}} - M_{t\wedge t^n_k}) \\
I^3_n &= \sum_{k=0}^{\infty} (M_{t\wedge t^n_{k+1}} - M_{t\wedge t^n_k})(C_{t\wedge t^n_{k+1}} - C_{t\wedge t^n_k})
\end{aligned}$$

By the properties of the Stieltjes integral (basically, the dominated convergence theorem), we have $I^1_n \to \int_0^t M_u \, dC_u$, a.s. Also, by the uniform continuity of the paths of M on compact intervals, we have $I^3_n \to 0$, a.s. Finally, we note that

$$I^2_n = \int_0^t \bar{C}^{\Delta_n}_u \, dM_u,$$

where $\bar{C}^{\Delta_n}_u = C_{t^{\Delta_n}(u)}$. By uniform continuity of the paths of C on $[0,t]$, $\bar{C}^{\Delta_n} \to C$, uniformly on $[0,t]$, a.s., and $\left| \bar{C}^{\Delta_n}_u - C_u \right| \leq 2C^*_u$, which is an adapted and continuous process. Therefore, we can use the stochastic dominated convergence theorem (Proposition 19.10) to conclude that $I^2_n \to \int_0^t C_u \, dM_u$ in probability, so that

$$M_t C_t = \int_0^t M_u \, dC_u + \int_0^t C_u \, dM_u \tag{20.7}$$

and, in the same way,

$$A_t N_t = \int_0^t A_u \, dN_u + \int_0^t N_u \, dA_u \tag{20.8}$$

The required equality (20.4) is nothing but the sum of (20.5), (20.6), (20.7) and (20.8). $\square$

Remark 20.5. By formally differentiating (20.4) with respect to t, we write

$$d(XY)_t = X_t \, dY_t + Y_t \, dX_t + dX_t dY_t.$$

While meaningless in the strict sense, the "differential" representation above serves as a good mnemonic device for various formulas in stochastic analysis. One simply has to multiply out all the terms, disregard all terms of order larger than 2 (such as $(dX_t)^3$ or $(dX_t)^2 dY_t$), and use the following multiplication table:

	dM	dA
dN	$d\langle M,N\rangle$	0
dC	0	0

where $X = M + A$, $Y = N + C$ are semimartingale decompositions of X and Y. When $M = N = B$, where B is the Brownian motion, the multiplication table simplifies to

	dB	dA
dB	dt	0
dC	0	0

Proof of Theorem 20.1. Let $\mathcal{A}$ be the family of all functions $f \in C^2[a,b]$ such that the formula (20.1) holds for all $t \geq 0$ and all continuous semimartingales X which take values in $[a,b]$. It is clear that $\mathcal{A}$ is a linear space which contains all constant functions. Moreover, it is also closed under multiplication, i.e., $fg \in \mathcal{A}$ if $f, g \in \mathcal{A}$. Indeed, we need to use the integration-by-parts formula (20.4) and the associativity of stochastic integration (Problem 19.5, (2)) applied to $f(X)$ and $g(X)$. Next, the identity is clearly in $\mathcal{A}$, and so, $P \in \mathcal{A}$ for each polynomial P. It remains to show that $\mathcal{A} = C^2[a,b]$. For $f \in C^2[a,b]$, let $\{P''_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials with the property that $P''_n \to f''$, uniformly on $[a,b]$. This can be achieved by the Weierstrass-Stone theorem, because polynomials are dense in $C[a,b]$. It is easy to show that the polynomials $\{P''_n\}_{n\in\mathbb{N}}$ can be taken to be second derivatives of a sequence $\{P_n\}_{n\in\mathbb{N}}$ of polynomials with the property that $P_n \to f$, $P'_n \to f'$ and $P''_n \to f''$, uniformly on $[a,b]$. Indeed, just use $P'_n(x) = f'(x) + \int_a^x P''_n(\xi)\,d\xi$ and $P_n(x) = f(x) + \int_a^x P'_n(\xi)\,d\xi$. Then, as the reader will easily check, the stochastic dominated convergence theorem 19.10 will imply that all terms in (20.1) for P_n converge in probability to the corresponding terms for f. This shows that $C^2[a,b] = \mathcal{A}$. $\square$

Remark 20.6. The proof of the Itô's formula given above is slick, but it does not help much in terms of intuition. One of the best ways of understanding the Itô formula is the following non-rigorous, heuristic, derivation, where $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = t$ is a partiton of $[0,t]$. The main insight is that the second-order term in the Taylor's expansion $f(t) - f(s) = f'(s)(t-s) + \frac{1}{2}f''(s)(t-s)^2 + o((t-s)^2)$ has to be kept and cannot be discarded because - in contast to the classical case - the second (quadratic) variation does not vanish:

$$\begin{aligned} f(X_t) - f(X_0) &= \sum_{k=0}^{n} \big(f(X_{t_{k+1}}) - f(X_{t_k}) \big) \\ &\approx \sum_{k=0}^{n} \Big(f'(X_{t_k})(X_{t_{k+1}} - X_{t_k}) + \tfrac{1}{2} f''(X_{t_k})(X_{t_{k+1}} - X_{t_{k-1}})^2 \Big) \\ &= \sum_{k=0}^{n} f'(X_{t_k})(M_{t_{k+1}} - M_{t_k}) + \sum_{k=0}^{n} f'(X_{t_k})(A_{t_{k+1}} - A_{t_k}) \\ &\quad + \tfrac{1}{2} \sum_{k=0}^{n} f''(X_{t_k})(X_{t_{k+1}} - X_{t_k})^2 \\ &\approx \int_0^t f'(X_u)\, dM_u + \int_0^t f'(X_u)\, dA_u + \tfrac{1}{2} \int_0^t f''(X_u)\, d\langle X \rangle_u. \end{aligned}$$

Finally, one simply needs to remember that $\langle X \rangle = \langle M \rangle$.

The requirements that the semimartingale is one-dimensional, or that it takes values in a compact set $[a, b]$, are not necessary in general. Here is a general version of the Itô formula for continuous semimartingales. We do not give a proof as it is very similar to the one-dimensional case. The requirement that the process takes values in a compact set is relaxed by stopping.

Theorem 20.7. *Let $X^1, X^2, \dots, X^d$ be d continuous semimartingales and let $D \subseteq \mathbb{R}^d$ be an open set, such that $X_t = (X^1_t, \dots, X^d_t) \in D$, for all $t \geq 0$, a.s. Moreover, let $f : D \to \mathbb{R}$ be a twice continuously differentiable function ($f \in C^2(D)$). Then the process $\{f(X_t)\}_{t \in [0,\infty)}$ is a continuous semimartingale and*

$$f(X_t) - f(X_0) = \sum_{k=1}^{d} \int_0^t \frac{\partial}{\partial x_k} f(X_u)\, dX^k_u + \tfrac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(X_u)\, d\langle X^i, X^j \rangle_u. \tag{20.9}$$

Remark 20.8.

1. When the local martingale parts of some of the components of the process $X = (X^1, \dots, X^d)$ vanish, one does not need the full C^2 differentiability in those coordinates; C^1 will be enough.
2. Using the differential notation (and the multiplication table) of Remark 20.5, we can write the Itô formula in the more compact form:

$$df(X_t) = \sum_{k=1}^{n} \frac{\partial}{\partial x_k} f(X_u)\, dX^k_u + \tfrac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} f(X_u)\, dX^i_u dX^j_u.$$

Lévy's characterization of the Brownian motion

We start with a deep (but easy to prove) result which loosely states that if a continuous local martingale has the speed of the Brownian motion, then it is a Brownian motion.

Problem 20.1. Let $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a function in $C^{1,2}([0, \infty) \times \mathbb{R})$ (the space functions continuously differentiable in t and twice continuously differentiable in x). Such a function f is said to be **space-time harmonic** if it satisfies $f_t + \frac{1}{2} f_{xx} = 0$. For $f \in C^{1,2}([0, \infty) \times \mathbb{R})$ and a Brownian motion B, show that the process $f(t, B_t)$ is a local martingale if and only if f is space-time harmonic. Show, additionally, that $f(t, B_t)$ is a martingale if f is space-time harmonic and f_x is bounded on each domain of the form $[0, t] \times \mathbb{R}$, $t \geq 0$.

Theorem 20.9 (Lévy's Characterization of Brownian Motion)**.** *Suppose that $M \in \mathcal{M}_0^{loc,c}$ has $\langle M \rangle_t = t$, for all t, a.s. Then M is an $\{\mathcal{F}_t\}_{t \in [0,\infty)}$-Brownian motion.*

Proof. Consider the complex-valued process

$$Z_t^u = \exp(iuM_t + \tfrac{1}{2}u^2 t), \ t \geq 0,$$

for $u \in \mathbb{R}$. We can use Itô's formula (applied separately to its real and imaginary parts) to show that Z^u is a local martingale for each $u \in \mathbb{R}$. Indeed, $Z_t^u = f(t, M_t)$, where $f(t, x) = e^{iux + \frac{1}{2}u^2 t}$, so that, as you can easily check, $f_t(t, x) + \frac{1}{2} f_{xx}(t, x) = 0$. Therefore,

$$\begin{aligned} f(t, M_t) &= f(0, M_0) + \int_0^t f_t(u, M_u)\, du + \int_0^t f_x(u, M_u)\, dM_u + \tfrac{1}{2} \int_0^t f_{xx}(u, M_u)\, d\langle M \rangle_u \\ &= f(0, 0) + \int_0^t (f_t(u, M_u) + \tfrac{1}{2} f_{xx}(u, M_u))\, du + \int_0^t f_x(u, M_u)\, dM_u \\ &= 1 + \int_0^t f_x(u, M_u)\, dM_u. \end{aligned}$$

We can, actually, assert that Z^u is a (true) martingale because it is bounded by $\exp(\frac{1}{2}u^2 t)$ on $[0, t]$, and, therefore, of class (DL). Consequently, we have $\mathbb{E}[Z_t^u | \mathcal{F}_s] = Z_s^u$, for all $u \in \mathbb{R}$, $0 \leq s \leq t < \infty$. With a bit of rearranging, we get

$$\mathbb{E}[e^{iu(M_t - M_s)} | \mathcal{F}_s] = e^{-\frac{1}{2}u^2 (t-s)}.$$

In other words, $e^{-\frac{1}{2}u^2(t-s)}$ is a regular conditional characteristic function of $M_t - M_s$, given $\mathcal{F}_s$. That implies that $M_t - M_s$ is independent of $\mathcal{F}_s$ and that its characteristic function is given by $e^{-\frac{1}{2}u^2(t-s)}$, i.e., that $M_t - M_s$ is normal with mean 0 and variance $t - s$. □

Remark 20.10*.* Continuity is very important in the Lévy's characterization. Note that the process $(N_t - t)^2 - t$ is a martingale (prove it!), so that the deterministic process t can be interpreted as a quadratic variation of the martingale $N_t - t$. Clearly, $N_t - t$ is not a Brownian motion.

Problem 20.2. Remember that a stochastic process $\{X_t\}_{t\in[0,\infty)}$ is said to be **Gaussian** if all of its finite-dimensional distributions are multivariate normal. Let $\{M_t\}_{t\in[0,\infty)}$ be a continuous martingale with $\langle M\rangle_t = f(t)$, for some continuous deterministic non-decreasing function $f : [0,\infty) \to [0,\infty)$. Show that $\{M_t\}_{t\in[0,\infty)}$ is Gaussian. Is it true that that each continuous, Gaussian martingale has deterministic quadratic variation?

Itô processes

Definition 20.11. A continuous semimartingale $\{X_t\}_{t\in[0,\infty)}$ is said to be an **Itô process** if there exist progressively measurable processes $\{\alpha_t\}_{t\in[0,\infty)}$ and $\{\beta_t\}_{t\in[0,\infty)}$ such that $\int_0^t \big(|\alpha_s| + \beta_s^2\big)\, ds < \infty$, a.s., and

$$X_t = X_0 + \int_0^t \alpha_s \, ds + \int_0^t \beta_s \, dB_s. \tag{20.10}$$

Note: This is usually written in differential notation as $dX_t = \alpha_t \, dt + \beta_t \, dB_t$.

Problem 20.3 (Stability of Itô processes)**.** Let X and Y be two Itô processes, and let H be a progressively measurable process in $L(X)$.

1. Show that $H \cdot X$ is an Itô process.
2. Let $F : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ be in the class $C^{1,2,2}$. Show that $Z_t = F(t, X_t, Y_t)$ is also an Itô process.

Definition 20.12. Let X be an Itô process with decomposition (20.10) such that the relation

$$\alpha_t = \mu(t, X_t), \ \beta_t = \sigma(t, X_t),$$

holds for some measurable μ, σ for all $t \geq 0$, a.s. Then, X is said to be an **inhomogeneous diffusion**. If μ and σ do not depend on t, X is said to be a **(homogeneous) diffusion**.

Example 20.13.

1. The Brownian motion $X = B$, the **scaled Brownian motion** $X = \sigma B$ and the **Brownian motion with drift** $X_t = B_t + ct$ (for $c \in \mathbb{R}$) are Itô processes (they are diffusions, as well).
2. The process $|B_t|$ is not an Itô process. Can you prove that? *Hint:* Assume that it is apply Itô's formula on $B_t^2 = |B_t|^2$. Derive an expression for $|B|$ and show that it cannot be non-negative all the time.

   Note: It can be shown, though, that $|B_t|$ is a continuous semimartingale.

3. Let B be a Brownian motion, and let M_t be the process defined by

$$M_t = \begin{cases} 0, & t \leq 1, \\ \mathbf{1}_{\{B_1 > 0\}}, & t > 1. \end{cases}$$

Consider the process X given by

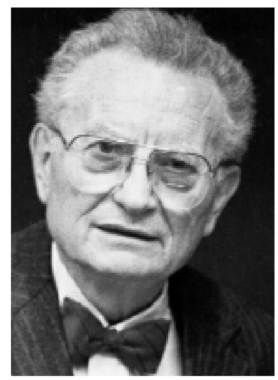
$$X_t = \int_0^t M_u \, du + B_t.$$

In words, X follows a Brownian motion until time 1 and then it gives it an extra drift of speed 1 if and only if $B_1 > 0$. It is clear that X is an Itô process. It is not a diffusion (homogeneous or not). Indeed, by the uniqueness of the semimartingale decomposition, it will be enough to show that M_u cannot be written as a deterministic function $\mu(\cdot,\cdot)$ of t and X_t. If it were possible, there would exist a function $f(x) = \mu(2, x)$ such that $\mathbf{1}_{\{B_1>0\}} = f(B_2 + \mathbf{1}_{\{B_1>0\}})$.

4. **(Geometric Brownian Motion)** Paul Samuelson proposed the Itô process

$$S_t = \exp\Big(\sigma B_t + (\mu - \tfrac{1}{2}\sigma^2)t\Big), \; S_0 = s_0 > 0, \tag{20.11}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are parameters, as the model for the time-evolution of the price of a common stock. This process is sometimes referred to as the **geometric Brownian motion (with drift)**. Itô formula yields the following expression for the process S in the differential notation:

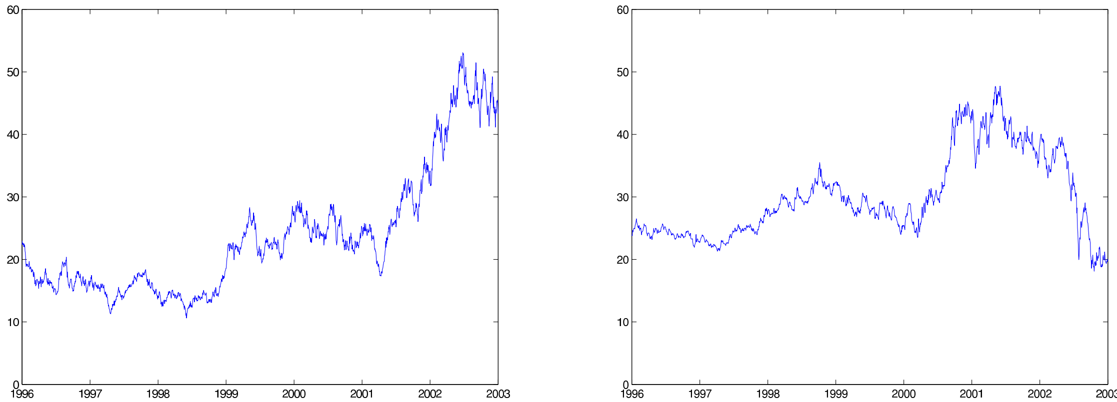


Paul Samuelson

$$\begin{aligned} dS_t &= \exp\Big(\sigma B_t + (\mu - \tfrac{1}{2}\sigma^2)t\Big)(\mu - \tfrac{1}{2}\sigma^2 + \tfrac{1}{2}\sigma^2)\, dt + \exp\Big(\sigma B_t + (\mu - \tfrac{1}{2}\sigma^2)t\Big)\sigma \, dB_t \\ &= S_t \mu \, dt + S_t \sigma dB_t. \end{aligned}$$

It follows that S is a diffusion with $\mu(x) = \mu x$ and $\sigma(x) = \sigma x$ (where we use the Greek letters μ and σ to denote both the functions appearing the definition of the diffusion, and the numerical coefficients in (20.12)).

One of the graphs below shows a simulated path of a geometric Brownian motion (Samuelson's model) and the other shows the actual evolution of a price of a certain stock. Can you guess which is which?



To understand the model a bit better, let us start with the case $\sigma = 0$, first. The relationship

$$dS_t = S_t \mu \, dt,$$

is then an example of a linear homogeneous ODE with constant coefficients and its solution describes the exponential growth:

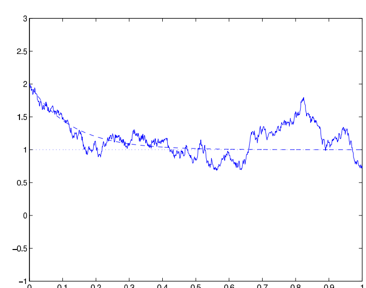
$$S_t = s_0 \exp(\mu t).$$

If we view μ as an interest-rate, S will model the amount of money we will have in the bank after t units of time, starting from s_0 and with the continuously compounded interest at the constant rate of μ.

When $\sigma \neq 0$, we can think of an increment of S as composed of two parts. The first on $S_t \mu \, dt$ models (deterministic) growth, and the second one $S_t \sigma \, dB_t$ the effect of a random fluctuation of size σ. Discretized, it looks like this:

$$\frac{\Delta S_{t_i}}{S_{t_i}} = \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} \approx \mu \Delta t + \sigma \Delta B_t \approx N(\mu dt, (\sigma \sqrt{dt})^2).$$

Therefore, might say that the stock-prices, according to Samuelson, are bank accounts + noise. It makes sense, then, to call the parameter μ the **mean rate of return** and σ the **volatility**.



Ornstein-Uhlenbeck process (solid) and its noise-free (dashed) version, with parameters $m = 1$, $\alpha = 8$, $\sigma = 1$, started at $X_0 = 2$.

5. **(The Ornstein-Uhlenbeck process)** The Ornstein-Uhlenbeck process (often abbreviated to OU-process) is an Itô process with a long history and a number of applications (in physics, related to the **Langevin equation** or in finance, where it drives the **Vasiček model** of interest rates). The OU-process can be defined as the unique process with the property that $X_0 = x_0$ and

$$dX_t = \alpha(m - X_t) \, dt + \sigma dB_t, \qquad (20.12)$$

for some constants $m, \alpha, \sigma \in \mathbb{R}$, $\alpha, \sigma > 0$ (we'll show shortly that such a process exists).

Intuitively, the OU-process models a motion of a particle which drifts toward the level m, perturbed with the Brownian noise of intensity σ. Notice how the drift coefficient is negative when $X_t > m$ and positive when $X_t < m$.

Because of that feature, OU-process is sometimes also called a **mean-reverting process**. Another one of its nice properties is that the OU process is a Gaussian process, making it amenable to empirical and theoretical study. In the remainder of this example, we will show how this process can be constructed by using some methods similar to the ones you might have seen in your ODE class. The solution we get will not be completely explicit, since it will contain a stochastic integral, but as equations of this form go, even that is usually more than we can hope for.

First of all, let us simplify the notation (without any real loss of generality) by setting $m = 0$, $\sigma = 1$. Supposing that a process satisfying (20.12) exists, let us start by defining the process $Y_t = R_t X_t$, with $R_t = \exp(\alpha t)$, inspired by the technique one would use to solve the corresponding noiseless equation

$$d\hat{X}_t = -\alpha \hat{X}_t \, dt.$$

Clearly, the process R_t is a deterministic Itô process with $dR_t = \alpha R_t \, dt$, so the integration-by-parts formula (note that $dR_t dB_t = 0$) gives

$$\begin{aligned} dY_t &= R_t \, dX_t + X_t dR_t + dX_t dR_t = -\alpha X_t R_t \, dt + R_t \, dB_t + X_t \alpha R_t \, dt \\ &= R_t \, dB_t = e^{\alpha t} \, dB_t. \end{aligned}$$

It follows now that

$$X_t = x_0 e^{-\alpha t} + e^{-\alpha t} \int_0^t e^{\alpha s} \, dB_s.$$

Problem 20.4. Let $\{X_t\}_{t \in [0,\infty)}$ be an OU process with $m = 0$, $\sigma = 1$ with $\alpha \in \mathbb{R}$, started at $X_0 = x \in \mathbb{R}$. Show that X_t converges in distribution when $t \to \infty$ and find the limiting distribution.

Additional Problems

Problem 20.5 (Bessel processes)**.** For $d \in \mathbb{N}$, $d \geq 2$, let

$$B_t = (B_t^{(1)}, B_t^{(2)}, \dots, B_t^{(d)}), \ t \geq 0,$$

be a d-dimensional Brownian motion, and let the process X, given by

$$X_t = (X_t^{(1)}, X_t^{(2)}, \dots, X_t^{(d)}) = (1, 0, \dots, 0) + (B_t^{(1)}, B_t^{(2)}, \dots, B_t^{(d)}),$$

be a translation of the d-dimensional Brownian motion, started from the point $(1,0,\dots,0) \in \mathbb{R}^d$. Finally, let R be given by

$$R_t = ||X_t||,\ t \geq 0,$$

where $||\cdot||$ denotes the usual Euclidean norm in $\mathbb{R}^d$.

1. Use Itô's formula to show that R is a continuous semimartingale and compute its semimartingale decomposition (the filtration is the usual completion of the filtration generated by B.)
2. Supposing that $d=3$, show that the process $\{L_t\}_{t\in[0,\infty)}$, given by $L_t = R_t^{-1}$, is well-defined and a local martingale.
3. (Quite computational and, therefore, *optional*) Write $\mathbb{E}[L_t]$ as a triple integral and show that the expression inside the integral is dominated by an integrable function for $t \geq 0$. Conclude from there that $\mathbb{E}[L_t] \to 0$, as $t \to \infty$.
4. Given the result in 3. above, can $\{L_t\}_{t\in[0,\infty)}$ be a (true) martingale?

Problem 20.6 (Stochastic exponentials)**.**

1. Show that, for any continuous semimartingale X with $X_0 = 0$, there exists a non-negative continuous semimartingale $\mathcal{E}(X)$ with the property that

   $$\mathcal{E}(X)_t = 1 + \int_0^t \mathcal{E}(X)_u \, dX_u,\ \text{for all } t \geq 0, \text{ a.s.} \tag{20.13}$$

   $\mathcal{E}(X)$ is called the **stochastic exponential of** X. *Hint:* Look for $\mathcal{E}(X)$ of the form $\exp(X - A)$, where A is a process of finite variation.
2. If X and Y are two continuous semimartingales starting at 0, relate the processes $\mathcal{E}(X)\mathcal{E}(Y)$ and $\mathcal{E}(X+Y)$. When are they equal? *Note:* You can use, without proof, the fact that process satisfying (20.13) is necessarily unique.
3. Let M be a continuous local martingale with $M_0 = 0$. Show that

   $$\{\mathcal{E}(M)_\infty = 0\} = \{\langle M\rangle_\infty = \infty\},\ \text{a.s.}$$

   Hint: $\mathcal{E}(M) = \mathcal{E}(\tfrac{1}{2}M)^2 e^{-\frac{1}{4}\langle M\rangle}$

Problem 20.7 (Linear Stochastic Differential Equations)**.** Let Y and Z be two continuous semimartingales. Solve (in closed form) the following linear stochastic differential equation for X where $x \in \mathbb{R}$:

$$\begin{cases} dX_t = X_t \, dY_t + dZ_t \\ X_0 = x, \end{cases}$$

i.e., find a continuous semimartingale X with $X_0 = x$ such that $X_t = x + Z_t + \int_0^t X_u \, dY_u$, for all $t \geq 0$, a.s.

Problem 20.8 (Preservation of the local martingale property)**.** Let $\{\mathcal{G}_t\}_{t\in[0,\infty)}$ be a right-continuous and complete filtration such that $\mathcal{G}_t \subseteq \mathcal{F}_t$, for all $t \geq 0$, and let $\{M_t\}_{t\in[0,\infty)}$ be a continuous $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-local martingale which is adapted to $\{\mathcal{G}_t\}_{t\in[0,\infty)}$. Show that $\{M_t\}_{t\in[0,\infty)}$ is also a continuous local martingale with respect to $\{\mathcal{G}_t\}_{t\in[0,\infty)}$.

Problem 20.9 (The distribution of a Brownian functional)**.** Let B, where

$$B_t = (B_t^{(1)}, B^{(2)})_t), t \geq 0,$$

be a 2-dimensional Brownian motion, and let $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ be the right-continuous augmentation of the natural filtration $\sigma(B_u^{(1)}, B_u^{(2)}; u \leq t)$. Show that $B^{(1)} \in L(B^{(2)})$ and $B^{(2)} \in L(B^{(1)})$, and identify the distribution of the random variable

$$F = \int_0^1 B_u^{(1)} dB_u^{(2)} + \int_0^1 B_u^{(2)} dB_u^{(1)}.$$

Hint: Compute the characteristic function of F, first. Don't try to build exponential martingales; there is an easier way.

Problem 20.10 (Stochastic area)**.** Let $r = (x, y) : [0,1] \to \mathbb{R}^2$ be a smooth curve such that

- $r(0)$ is on the positive x-axis and $r(1)$ on the positive y-axis,
- r takes values in the first quadrant and never hits 0.
- r moves counterclockwise, i.e., $x'(t) < 0$, for all t.

Let A denote the region in $[0,\infty) \times [0,\infty)$ bounded by the coordinate axes from below, and by the image of r from above. First heuristically, and then formally, show that the area $|A|$ of A is given by

$$|A| = \tfrac{1}{2} \int_0^1 x(t)\, dy(t)\, dt - \tfrac{1}{2} \int_0^1 y(t)\, dx(t).$$

With the setup as in Problem 20.8, we define the random variable A (note the negative sign)

$$A = \int_0^1 B_u^{(1)} dB_u^{(2)} - \int_0^1 B_u^{(2)} dB_u^{(1)}.$$

Thanks to 1. above, we can interpret $A(\omega)$ as the double of the (signed) area that the chord with endpoints $(0,0)$ and $(B_t^{(1)}(\omega), B_t^{(2)}(\omega))$ sweeps during the time interval $[0,1]$. Use Itô's formula to show that

$$\mathbb{E}[e^{itA}] = \tfrac{1}{\cosh(t)}. \tag{20.14}$$

Note: The formula (20.14) if often known as **Lévy's stochastic area formula**. The distribution with the characteristic function apearing in it can be shown to admit a density of the form $f_A(x) = \frac{1}{2}(\cosh(\pi x/2))^{-1}$.

Problem 20.11 (Speed measures)**.** A continuous and strictly increasing function $s : \mathbb{R} \to \mathbb{R}$ is said to be a **scale function** for the diffusion $dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t$ if $s(X_t)$ is a local martingale.

1. Suppose that the function $x \mapsto \frac{\mu(x)}{\sigma^2(x)}$ is locally integrable, i.e., that $\int_a^b \frac{|\mu(x)|}{\sigma^2(x)}\,dx < \infty$, for all $[a,b] \subseteq \mathbb{R}$. Show that a scale function exists.

2. Let $[u,d]$ be an interval with $X_0 \in [u,d]$ and $\mathbb{P}[\tau_u \wedge \tau_d = \infty] = 0$, where $\tau_y = \inf\{t \geq 0 : X_t = y\}$, for $y \in \mathbb{R}$. If a scale function s exists and $X_0 = x \in \mathbb{R}$, show that

$$\mathbb{P}[\tau_u < \tau_d] = \frac{s(x) - s(d)}{s(u) - s(d)}.$$

3. Let $X_t = B_t + ct$ be a Brownian motion with drift. For $u > 0$ an $d = 0$, show that $\mathbb{P}[\tau_u \wedge \tau_d = \infty] = 0$ and compute $\mathbb{P}[\tau_d < \tau_u]$.

4. For $c \in \mathbb{R}$, find the probability that $X_t = B_t + ct$ will never hit the level 1.

Problem 20.12 (One unit of activity)**.** Let $\{B_t\}_{t\in[0,\infty)}$ be a standard Brownian Motion and let $\{H_t\}_{t\in[0,\infty)}$ be a predictable process for the standard augmentation of the natural filtration of B. Given that

$$\forall\, t \geq 0,\ \int_0^t H_u^2\,du < \infty, \text{ and } \int_0^\infty H_u^2\,du = \infty.$$

identify the distribution of $\int_0^\tau H_u dB_u$, where

$$\tau = \inf\{t \geq 0 : \int_0^t H_u^2\,du = 1\}.$$

Problem 20.13 (The Lévy transform)**.** Let $(B_t)_{t\in[0,\infty)}$ be a Brownian motion, and let $(X_t)_{t\geq 0}$ be its *Lévy transform*, i.e., the process defined by

$$X_t = \int_0^t \operatorname{sign}(B_u)\,dB_u, \text{ where } \operatorname{sign}(x) = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0. \end{cases}$$

1. Show that X is a Brownian motion,

2. show that the random variables B_t and X_t are uncorrelated, for each $t \geq 0$, and

3. show that B_t and X_t are not independent for $t > 0$. *Hint:* Compute $\mathbb{E}[X_t B_t^2]$.