UNIVERSITY OF COPENHAGEN

Mandatory Bachelor Project

Math-Econ Programme

Financial Modeling with Continuous-Time Models

Theory and Applications

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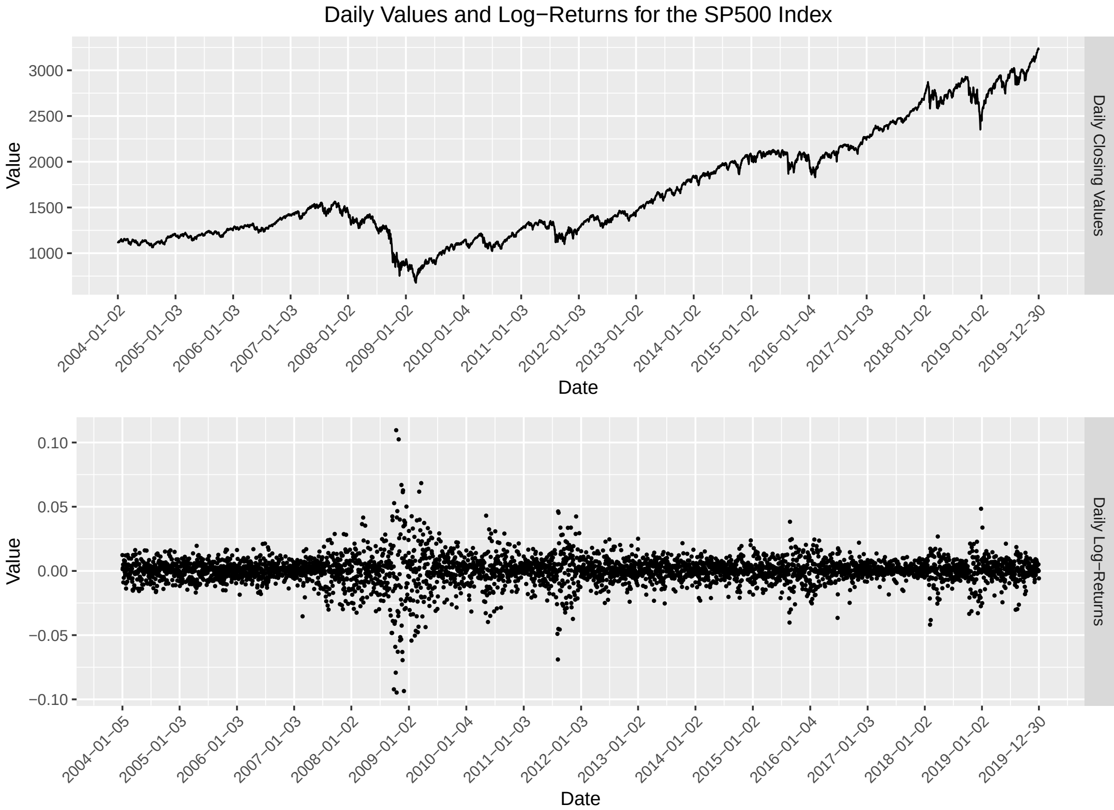

Figure 1: The top plot shows daily closing values for the SP500 index between 2004/01/02-2019/12/30. Corresponding log-returns are represented by the black points in the lower plot. Source: https://finance.yahoo.com/quote/%5EGSPC/.

1 Introduction

The financial market consists of a risky asset (a stock) and a risk-free asset (e.g., zero-coupon bond), both traded continuously up to some fixed time horizon T. As usual, the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{0\le t\le T}, P\right)$ is introduced to capture the information flow: Ω contains all possible states of the market, $\mathcal{F}$ is the corresponding σ-algebra, P is the physical measure, and $(\mathcal{F}_t^W)$ is the σ-field generated by the Brownian motion $(W_t)_{0\le t\le T}$, i.e.,

$$\mathcal{F}_t^W = \sigma\left\{W_s \mid 0 \le s \le t\right\}. \tag{1}$$

In the remainder of this project, S_t denotes the value of the risky asset at time t, and the process is assumed to solve the stochastic differential equation (SDE)

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t \tag{2}$$

$$S_0 = s > 0 \tag{3}$$

where $\mu(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are deterministic functions possibly depending on both time (t) and the state of the underlying (S_t). Moreover, $\sigma(t,s) > 0$ for all $(t,s) \in [0,T] \times \mathbb{R}^+$, and both $\mu(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are sufficiently well-behaved such that a solution to the above SDE exists. We will use following terminology: $\mu(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are called the drift and volatility, respectively, of (S_t), whereas dS_t is referred to as the dynamics of the process.

The risk-free asset's value through time is given by the process $(B_t)_{0\leq t\leq T}$ with dynamics

$$dB_t = rB_t dt \tag{4}$$

$$B_0 = 1 \tag{5}$$

where r is constant. The risk-free asset is interpreted as the money account with short rate of interest r.

Questions: Model Assumptions

The economic interpretation of the above setting is crucial for working with continuous-time models in practice. Answering the questions below will provide some intuition behind the model assumptions. I would recommend you to read chapters 2 and 4 in Björk (2009).

1.1

Discuss the term *risk*. What is the difference between a risk-free and a risky asset? What would a locally risk-free asset be? How many sources of risk appear in this model setting? Is $\mathcal{F}_t^W = \mathcal{F}_t^S$? What is the financial interpretation of the $\mathcal{F}_t^S$.

1.2

Explain the economic interpretation of $D_t := B_t/B_T$. Is $S_0 > 0$ a strong assumption? Illustrate B_t, D_t, S_t and D_tS_T. You can use the following pseudo-code for generating (S_t):

```
# Parameters
S <-            # Starting value of process
mu <- 0.05      # drift
sigma <- 0.2    # volatility

# Time grid
T <- 1          # End time
n <-            # Number of evaluations
dt <- T/(n)     # Equidistant time step

S_vec <- numeric(n)

for (i in 1:n){

  Z <-          # Generate value from normal distribution with mean=0 and var=1
  dS <-         # Compute S_i - S_{i-1}

```

```
  S <- S + dS # Update S

  S_vec[i] <- S
}
```

1.3

Is it reasonable to assume that stock prices are represented by continuous processes? Discuss the (dis)advantage of using continuous-time models for representing stock prices (e.g., discuss the trade off between numerical tractability and modeling reality/financial markets).

2 Geometric Brownian Motion

Assume that

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t} \tag{6}$$

where $S_0 = s > 0$ is the (time-0) starting value of the process. The process in equation (6) is called a Geometric Brownian Motion (GBM) and is closely linked to equations (2)-(3), although we can't prove that until we get Ito's formula in our toolbox (next topic). The Geometric Brownian Motion is defined and discussed by Björk (2009) in chapter 5.

Questions Geometric Brownian Motion

2.1

Is (S_t) continuous? Illustrate (S_t) using the following pseudo-code:

```
for (i in 1:n){
  Z <- # Generate value from normal dist. with mean=0 and var=1
  X <- # log(S_i/S_{i-1})
  S <- # Update S
  S_vec[i] <- S
}

# Or without the for-loop....
Z <- # Generate n values from multi dim. normal dist.
X <- cumsum(...) # Argument should be n-dim vector
S <- # Generate GBM
```

2.2

Compute $E(S_t)$ and $V(S_t)$. What is the distribution of S_t? Also compute $E(S_t \,|\, \mathcal{F}_s^S)$ for $s < t$ (hint: realize that $S_t = S_s \times S_t/S_s$). Is (S_t) a martingale under P?

2.3

Let $R_t := (S_t - S_{t-1})/S_t$ denote the *return* over one period. Determine the distribution of the associated log-return $r_t := \log(S_t/S_{t-1})$. How is R_t linked with r_t? Explain the convenience of using log-returns when working with financial time-series.

2.4

Would you prefer holding the risk-free asset or the risky-asset?

2.5

Produce figure 1. Are data a realization of a Geometric Brownian Motion (i.e., discuss whether model assumptions for GBM are violated)?

2.6

Derive the maximum likelihood estimator of μ and σ. Estimate μ and σ for the sample used to generate figure 1. Finally, discuss a significant shortfall for the maximum likelihood estimator of the drift μ (e.g., you may support your arguments with data).

3 Ito Calculus, Hedging Portfolios and the Black-Scholes model

...

4 Option Pricing and Monte Carlo Methods

5 tba

6 tba

References

Björk, T. (2009). *Arbitrage Theory in Continuous Time* (3 ed.). Oxford University Press.