

# Risk Minimization in Stochastic Volatility Models: Model Risk and Empirical Performance\*

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## Abstract

In this paper the performance of locally risk-minimizing delta hedge strategies for European options in stochastic volatility models is studied from an experimental as well as from an empirical perspective. These hedge strategies are derived for a large class of diffusion-type stochastic volatility models, and they are as easy to implement as usual delta hedges. Our simulation results on model risk show that these risk-minimizing hedges are robust with respect to uncertainty and misconceptions about the underlying data generating process. The empirical study, which includes the U.S. sub-prime crisis period, documents that in equity markets risk-minimizing delta hedges consistently outperform usual delta hedges by approximately halving the standard deviation of the profit-and-loss ratio.

**Key words:** Locally risk-minimizing delta hedge, stochastic volatility, model risk, empirical hedge performance.

**JEL classification:** C90, G13.

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# 1 Introduction

In this paper we study hedging in stochastic volatility models. Such models are incomplete; the seller of a contingent claim cannot eliminate all risk by trading the primary assets. The main objective is therefore to deal efficiently with this risk. Schweizer (1991) proposes the concept of locally risk-minimizing delta hedges which aims at minimizing the variance of the cost process of non-self-financing hedges. We derive an explicit formula for the locally risk-minimizing delta hedge in a general class of stochastic volatility models in a rigorous and novel fashion using results from El Karoui, Peng & Quenez (1997). The formula shows that the hedge can be decomposed as a sum of the usual delta hedge and a volatility/correlation-risk term. We also give an intuitive derivation illustrating the mechanics behind local risk-minimization; and we explain why this proof has a “loose end.” Applying some form of risk-minimization to stochastic volatility models is natural, so the formula appears in various guises in the literature, e.g. Frey (1997, Prop. 6.5), Bakshi, Cao & Chen (1997, Eq. (21)), Ahn & Wilmott (2003, Eq. (4)), Bouchaud & Potters (2003, Chapter 15), Alexander & Nogueira (2007*a*, Eq. (9)) and Alexander & Nogueira (2007*b*, Eq. (5)).

The locally risk-minimizing delta hedge *is* model- and parameter-dependent. The first issue analyzed in this paper is “But by how much?” To this end we conduct a number of controlled experiments that test the Heston model’s locally risk-minimizing delta hedge against various types of model risk. We study parameter uncertainty (small effect), misspecification of the functional form of the stochastic volatility (small effect if the models are calibrated to same data), and “plain ignorance” (using vega from a constant volatility model hurts hedge performance markedly).

The empirical performance of locally risk-minimizing delta hedges is tested using U.S. and European stock indices and currency option markets over a four-and-half-year period that includes the U.S. sub-prime crisis and the credit crunch that followed.<sup>1</sup> In the stock markets there is a strong negative correlation between returns on the underlying and volatility which manifests itself as a skew in implied volatilities across strikes, and here we find a significant improvement in hedge performance when using locally risk-minimizing delta hedges based on a stochastic volatility model. Compared to usual delta hedging, the standard deviations of daily profit-and-loss ratios are reduced by 50%. The USD/EURO currency market displays a higher degree of symmetry (close to zero correlation between underlying and volatility; smile rather than skew in implied volatilities) and there is no gain from using the locally risk-minimizing delta hedge strategy—as also suggested by the formula for the hedge. However, there is no loss of out-of-sample performance either. Results are robust to the use of the Heston (1993) stochastic volatility model or the SABR model from Hagan, Kumar, Lesniewski & Woodward (2002).

The paper is organized as follows. Section 2 briefly reviews local risk-minimization

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<sup>1</sup>Murphy (2008) gives an account of the credit crunch.

and introduces a class of stochastic volatility models considered here. Section 3 studies model risk. The empirical performance of local risk-minimizing delta hedges is analyzed in Section 4. Section 5 concludes and suggests some ideas for future research.

## 2 Local Risk-Minimization and Stochastic Volatility

Hedge strategies in incomplete markets generally are not replicating self-financing portfolios. Depending on the particular restrictions on trading in the risk-free asset, hedges either do not replicate the payoff perfectly or are not self-financing. In the latter case a hedge is associated with a cost process that aggregates any additional investments. Traders describe this as “bleeding.”<sup>2</sup> Consider a market with a risk-free and a risky asset with prices  $B(t)$  resp.  $S(t)$ . Then the cost process associated to a trading strategy  $\varphi(t) = (\varphi^0(t), \varphi^1(t))$  (where the components are the holdings—in number of units—of the risk-free and the risky asset) is given by

$$\text{Cost}_\varphi(t) = V_\varphi(t) - \int_0^t \varphi^0(s)dB(s) - \int_0^t \varphi^1(s)dS(s)$$

where  $V_\varphi(t) = \varphi^0(t)B(t) + \varphi^1(t)S(t)$  denotes the value of the trading strategy at time  $t$ . The cost of a trading strategy is simply the difference between the value of the holdings and the cumulative gains or losses up to the current point in time.

**Locally risk-minimizing delta hedges.** Constant cost processes are associated with self-financing trading strategies. The availability of a self-financing hedge  $\varphi$  for a contingent claim  $H$  means that the seller of  $H$  can guarantee payment of his obligation (at the time  $H$  expires) simply by investing the (known) initial amount  $V_\varphi(0)$  to buy  $\varphi$ . Using a non-self-financing hedge to meet a seller’s obligation at expiry, carries the risk associated to the cost process.<sup>3</sup> The total cost of the hedge becomes uncertain; something that the seller might not appreciate and, therefore, seeks to reduce. One criterion, proposed by Föllmer & Schweizer (1990), is the minimization of the conditional variance process of the cost process which is defined as

$$R_\varphi(t) := \mathbb{E}((\text{Cost}_\varphi(T) - \text{Cost}_\varphi(t))^2 | \mathcal{F}_t). \quad (1)$$

However, this dynamic optimization problem may have no solution. This existence problem is overcome by a localized version of the risk-minimization criterion introduced Schweizer (1991). The solution is called a *locally risk-minimizing delta hedge*.

A more formal discussion is given of (local) risk-minimizing delta hedges for models with a money market account  $B(t)$  and a single stock  $S(t)$  whose dynamics are of the

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<sup>2</sup>For an insightful account see Jesper Andreasen’s talk “Derivatives—The View From The Trenches,” available at <http://www.math.ku.dk/~rolf/jandreasen.pdf>.

<sup>3</sup>A contingent claim  $H$  can always be hedged with a trading strategy that is not self-financing. For instance let  $\varphi^0(s) = \varphi^1(s) = 0$  for all  $s \in [0, T)$ ,  $\varphi^0(T) = H$  and  $\varphi^1(T) = 0$ .

type  $dS(t)/S(t) = b(t)dt + \sum_j \sigma_j(t)dW^j(t)$ , where  $b(\cdot)$  and  $\sigma_j(\cdot)$  are stochastic processes and  $W^j(\cdot)$  are independent standard Brownian motions on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

A trading strategy  $(\psi^0, \psi^1)$  is said to be an admissible continuation of a trading strategy  $(\varphi^0, \varphi^1)$  from time  $t \in [0, T]$  on, if

$$\psi^0(s) = \varphi^0(s), \quad s < t; \quad \psi^1(s) = \varphi^1(s), \quad s \leq t; \quad \text{and} \quad V_\psi(T) = V_\varphi(T) \quad \mathbb{P}\text{-a.s.}$$

Föllmer & Schweizer (1990) define the trading strategy  $\varphi$  to be *risk-minimizing* ( $R$ -minimizing) if, for any  $t \in [0, T]$  and for any admissible continuation  $\psi$  of  $\varphi$  from  $t$  on,

$$R_\psi(t) \geq R_\varphi(t) \quad \mathbb{P}\text{-a.s. for all } t \in [0, T].$$

When the stock price  $S$  is a  $\mathbb{P}$ -martingale this criterion guarantees that the cost process is a  $\mathbb{P}$ -martingale as well, i.e., the hedge is self-financing on average. Otherwise a risk-minimizing hedge for  $H$  may not exist, see Föllmer & Schweizer (1990).

The local version of the risk-minimization criterion is quite technical, see Schweizer (1991) for further details. Call a trading strategy  $(\delta^0, \delta^1)$  a small perturbation if both  $\delta^1$  and  $\int_0^T |\delta^1(t)S(t)b(t)| dt$  are bounded and  $\delta^0(T) = \delta^1(T) = 0$ . Given a small perturbation  $(\delta^0, \delta^1)$  and a subinterval  $(s, t] \subset [0, T]$ , define the small perturbation  $\delta_{|(s, t]} := (\delta_{|[s, t]}^0, \delta_{|[s, t]}^1)$  with  $\delta_{|[s, t]}^0(u, \omega) := \delta^0(u, \omega) \cdot \mathbf{1}_{[s, t]}(u)$  and  $\delta_{|[s, t]}^1(u, \omega) := \delta^1(u, \omega) \cdot \mathbf{1}_{(s, t]}(u)$ . For a partition  $\tau$  of the interval  $[0, T]$  and a small perturbation  $\delta$  we finally define

$$r^\tau(t, \varphi, \delta) := \sum_{t_i \in \tau} \frac{R_{\varphi + \delta_{|(t_i, t_{i+1}]}}(t_i) - R_\varphi(t_i)}{\mathbb{E} \left( \int_{t_i}^{t_{i+1}} S(t)^2 \|\sigma(t)\|^2 dt \middle| \mathcal{F}_{t_i} \right)} \cdot \mathbf{1}_{(t_i, t_{i+1}]}(t).$$

The strategy  $\varphi$  is called *locally risk-minimizing* if, for every small perturbation  $\delta$ ,

$$\liminf_{|\tau| \rightarrow 0} r^\tau(t, \varphi, \delta) \geq 0 \quad \mathbb{P}\text{-a.s. for all } t \in [0, T].$$

**A general class of stochastic volatility models.** In the remainder of this section we consider the class of stochastic volatility models of the form

$$\begin{aligned} dS(t)/S(t) &= \mu dt + S(t)^\gamma f(V(t)) \left[ \sqrt{1 - \rho^2} dW^1(t) + \rho dW^2(t) \right] \\ dV(t)/V(t) &= \beta(V(t))dt + g(V(t))dW^2(t) \end{aligned} \tag{2}$$

with independent standard Brownian motions  $W^1(\cdot)$  and  $W^2(\cdot)$ . The probability space is denoted, as earlier, by  $(\Omega, \mathcal{F}, \mathbb{P})$ .  $S(t) > 0$  denotes the price of the (traded) asset and  $V(t) > 0$  is the (non-traded) stochastic local return variance. The risk-free asset  $B$  pays the constant interest rate  $r$ . These models allow for level (also known as scale) dependence ( $\gamma \neq 0$ ) and correlation between returns and variance ( $\rho \neq 0$ ). This class of stochastic volatility models contains most of those (without jump component) that are commonly used in research as well as in practice. Table 1 provides an overview.

Author(s) & year	Specification	Remarks
Hull/White 1987	$f(v) = v,$ $\beta(v) = 0,$ $g(v) = \sigma,$ $\rho = 0, \gamma = 0$	Instantaneous variance: Geometric Brownian motion. Options priced by mixing.
Wiggins 1987	$f(v) = e^{v/2},$ $\beta(v) = \kappa(\theta - v)/v,$ $g(v) = \sigma,$ $\rho = 0, \gamma = 0$	Instantaneous volatility: Ornstein-Uhlenbeck in logarithms.
Stein/Stein 1991	$f(v) =  v ,$ $\beta(v) = \kappa(\theta - v)/v,$ $g(v) = \sigma/v,$ $\rho = 0, \gamma = 0$	Instantaneous volatility: Reflected Ornstein-Uhlenbeck.
Heston 1993	$f(v) = \sqrt{v},$ $\beta(v) = \kappa(\theta - v)/v,$ $g(v) = \sigma/\sqrt{v},$ $\rho \in [-1, 1], \gamma = 0$	Instantaneous variance: CIR process. First model with correlation. Options priced by Fourier inversion.
Romano/Touzi 1997	$f, \beta,$ and $g$ free, $\rho \in [-1, 1], \gamma = 0$	Extension of mixing to correlation.
Schöbel/Zhu 1999	$f(v) =  v ,$ $\beta(v) = \kappa(\theta - v)/v,$ $g(v) = \sigma/v,$ $\rho \in [-1; 1], \gamma = 0$	Stein/Stein with correlation. Options priced by Fourier inversion.
Hagan et al. 2002	$f(v) = v$ $\beta(v) = 0,$ $g(v) = \sigma,$ $\rho \in [-1, 1], \gamma \in [-1, 0]$	Level dependence in returns. Options priced by perturbation technique. Acronym'ed as "SABR."

Table 1: Specification of stochastic volatility models for Eq. (2).

**A three step procedure.** El Karoui et al. (1997, Proposition 1.1) introduces a technique to determine the locally risk-minimizing delta hedge. In the following we describe in detail a three step procedure to implement their technique for the class of models (2).<sup>4</sup> The steps are: complete the market, compute the hedging strategy in the completed market, and finally project onto the original market. The method works for general contingent claims in possibly non-Markovian models, path dependent options in particular. But, of course, the practical usefulness of the method hinges on finding efficient ways of calculating prices and hedge portfolios in the completed market. For

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<sup>4</sup>El Karoui et al. (1997) assume boundedness of the volatility matrix and its inverse. No commonly used stochastic volatility model satisfies this condition. However, their Proposition 1.1 can still be used as long as the Doleans-Dade exponential of the risk premium process corresponding to the minimal martingale measure is a true martingale. This property must be verified on a case-by-case basis. For the Heston model, which is employed in the empirical study, Theorem 1 in Cheridito, Filipovic & Kimmel (2007) shows that the martingale property holds if the Feller condition is satisfied under both the original measure and the minimal martingale measure.

more on risk-minimizing delta hedges for exotic options we refer the reader to Alexander & Nogueira (2007a). For the stochastic volatility model, the steps work out like this.

**Complete.** The risk premium (process) corresponding to the minimal martingale measure is

$$\lambda^{\min}(t) = \frac{\mu - r}{S(t)^\gamma f(V(t))} \cdot \begin{pmatrix} \sqrt{1 - \rho^2} \\ \rho \end{pmatrix}.$$

The stochastic volatility model can be completed by introducing a second, volatility-dependent, tradable asset; El Karoui et al. (1997, Proposition 1.1) tells us to do that in such a way that the risk premium in the completed market is equal to  $\lambda^{\min}$ . This is achieved by introducing the new asset  $\hat{S}$  with dynamics

$$d\hat{S}(t)/\hat{S}(t) = \left( r + \rho \frac{g(V(t))}{S(t)^\gamma f(V(t))} (\mu - r) \right) dt + g(V(t)) dW^2(t).$$

Other choices of assets that complete the market are possible (for instance a specific option), but there is not *that* much flexibility because of the constraint on the market price of risk. With the above choice,  $\hat{S}(t)$  can be interpreted as the price of a pure volatility derivative.

**Compute.** In the completed market the hedge strategy for a European contingent claim with payoff  $H(S(T))$  is found in the same way as in the Black-Scholes model. Let  $X(t)$  denote the value process of a (perfect) hedge for  $H$  with positions  $\Delta(t)$  and  $\hat{\Delta}(t)$  in the assets  $S$  and  $\hat{S}$ , respectively. The position in the risk-free asset account is adjusted so as to make the hedge self-financing. Then, on the one hand,

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + \hat{\Delta}(t)d\hat{S}(t) + r \left[ X(t) - \Delta(t)S(t) - \hat{\Delta}(t)\hat{S}(t) \right] dt \\ &= (\dots) dt + \Delta(t)S(t)^{1+\gamma}f(V(t)) \left[ \sqrt{1 - \rho^2}dW^1(t) + \rho dW^2(t) \right] \\ &\quad + \hat{\Delta}(t)\hat{S}(t)g(V(t))dW^2(t). \end{aligned}$$

On the other hand, the original market model is Markovian, which allows to write the price of the claim  $H$  as

$$e^{-r(T-t)} \mathbb{E}_t^{\min}(H(S(T))) = C(t, S(t), V(t))$$

for some function  $C(\cdot, \cdot, \cdot)$ . The conditional expected value  $\mathbb{E}_t^{\min}$  is calculated under the minimal martingale measure. Since  $X(t) = C(t, S(t), V(t))$ , the application of Itô's formula to describe the dynamics of  $dC$  and a comparison of the resulting diffusion terms with the above equation for  $dX$  yields

$$\Delta(t) = C_S \text{ and } \hat{\Delta}(t) = \frac{C_V}{\hat{S}(t)} V(t)$$

where  $C_S$  and  $C_V$  denote partial derivatives.

**Project.** Finally, El Karoui et al. (1997, Proposition 1.1) says that the investment in asset  $S$  under the locally risk-minimizing delta hedge of the original market (2) is given by

$$S(t)\varphi_{\min}^1(t) = C_S S(t) + \rho \frac{V(t)g(V(t))}{S(t)^\gamma f(V(t))} C_V.$$

These findings can be summarized as follows.

**Proposition 1** *Consider the stochastic volatility model (2). The locally risk-minimizing delta hedge of a European contingent claim with payoff  $H(S(T))$  holds*

$$\varphi_{\min}^1(t) = C_S + \rho \frac{V(t)g(V(t))}{S(t)^{1+\gamma} f(V(t))} C_V \quad (3)$$

units of the stock, where

$$C(t, S(t), V(t)) = e^{-r(T-t)} \mathbb{E}_t^{\min}(H(S(T))).$$

$\mathbb{E}_t^{\min}$  denotes the conditional expectation with respect to the minimal martingale measure  $\mathbb{Q}_{\min}$  under which the dynamics are given by

$$\begin{aligned} dS(t)/S(t) &= rdt + S(t)^\gamma f(V(t)) \left[ \sqrt{1 - \rho^2} dW_t^{\min,1} + \rho dW_t^{\min,2} \right] \\ dV(t)/V(t) &= \left[ \beta(V(t)) - \rho \frac{g(V(t))}{S(t)^\gamma f(V(t))} (\mu - r) \right] dt + g(V(t)) dW_t^{\min,2} \end{aligned} \quad (4)$$

where  $dW^{\min} = dW + \lambda_t^{\min} dt$  defines a  $\mathbb{Q}_{\min}$ -Brownian motion. The investment in the risk-free asset is  $C(t, S(t), V(t)) - \varphi_{\min}^1(t)S(t)$ .

If the changes in the underlying and the instantaneous variance are correlated ( $\rho \neq 0$ ), the locally risk-minimizing delta hedge and the standard delta hedge do not coincide. Suppose  $\rho$  is negative (as is typical in stock markets) and the payoff function  $H$  is convex. Then  $C_V$  is positive (see e.g. Romano & Touzi (1997, Proposition 4.2)), and (3) tells us that a delta hedger invests too heavily in the stock.

The minimal martingale measure is often described loosely as “the one that changes as little as possible.” Proposition 1 highlights that when return and volatility are correlated and there is an equity risk premium ( $\mu \neq r$ ), the minimal martingale measure does *not* merely change the drift rate of the stock to  $r$  while leaving the volatility dynamics unaltered. In the presence of correlation a change in the stock price dynamics (when switching to the minimal martingale measure) entails a change in the volatility dynamics.

**A (deceptively) simple derivation of locally risk-minimizing strategies.** Intuition for the locally risk-minimizing delta hedge (3) (e.g. “where is something actually being minimized?”) can be gained by using the direct approach from Bakshi et al. (1997,

pp. 2033-4). Suppose at some point in time  $t$ , a trader takes a position that is (a) long one unit of the European contingent claim with payoff  $H(S(T))$ , which is valued at  $C(t, S(t), V(t))$ , and (b) short  $\Delta$  units of the stock, where  $\Delta$  is to be determined. Itô's formula yields  $dC = \dots dt + C_S dS + C_V dV$  which implies that the change in value of the hedge over a small time-interval  $[t, t + dt]$  (i.e. locally) is given by

$$dX = dC - \Delta dS = \dots dt + (C_S - \Delta)dS + C_V dV.$$

For the conditional variance the  $dt$ -term does not matter, and thus

$$\begin{aligned} \text{var}_t(dX) &= (C_S - \Delta)^2 \text{var}_t(dS) + C_V^2 \text{var}_t(dV) + 2(C_S - \Delta)C_V \text{cov}_t(dS, dV) \\ &= [(C_S - \Delta)^2 S^{2(1+\gamma)} f^2(V) + C_V^2 V^2 g^2(V) + 2(C_S - \Delta)C_V S^{1+\gamma} f(V) V g(V) \rho] dt. \end{aligned}$$

From the trader's perspective a sensible choice of  $\Delta$  is the one that minimizes this variance. The first-order condition

$$-2(C_S - \Delta^{\min}) S^{2(1+\gamma)} f^2(V) - 2C_V S^{1+\gamma} f(V) V g(V) \rho = 0$$

yields  $\Delta^{\min} = \varphi_{\min}^1$  which coincides with the above result.

However, the shortcoming of this derivation is its inability to tie down the price  $C$  of the contingent claim. Implementation of this hedge requires taking an expectation when calculating the function  $C$ , but the derivation gives no indication as to *which* of the many martingale measures to use. It does not help to “close the model” by assuming that agents use risk-minimizing delta hedge strategies (and do not care about residuals) and setting the price of the claim equal to the price of this particular hedge. There is a Catch-22: the hedge depends on the pricing function which, in turn, depends on the hedge. The approach of El Karoui et al. (1997) does not encounter this problem because it derives the price as well as the hedge by considering trading in primary assets only.

### 3 Model Risk

In this section we analyze experimentally to what extent the performance of risk-minimizing delta hedge is sensitive to model risk, i.e., what happens if you get things a little bit wrong? In particular we are asking whether (and to what extent) the risk-minimizing delta hedge is robust. As stressed by Cont (2006) this is a highly relevant practical issue. In the study of this question we use the popular Heston (1993) model as the benchmark. Four likely sources of error are considered and their effects are quantified: “Wrong” martingale measure (little effect), parameter uncertainty (detectable effect, but not nearly strong enough to outweigh the benefits), wrong Greeks (considerable negative effect) and wrong data-generating process (surprisingly small effect).

The Heston model has the dynamics (see Table 1)

$$\begin{aligned} dS(t) &= S(t) \left( \mu dt + \sqrt{V(t)} [\sqrt{1 - \rho^2} dW^1(t) + \rho dW^2(t)] \right) \\ dV(t) &= \kappa(\theta - V(t))dt + \sigma \sqrt{V(t)} dW^2(t). \end{aligned} \tag{5}$$



Our simulation experiments use the parameter estimates from the comprehensive study in Eraker (2004) which can be seen as reflecting the consensus in the literature. Table 2 summarizes the values (annualized and in non-percentage terms) and the interpretation of these parameters. The table also includes (a) standard errors of the estimated parameters and (b) option-based parameter estimates of the pricing measure used in the market. These latter estimates, which can also be called risk-adjusted, were obtained by Eraker through a joint time-series and cross-sectional estimation of spot and option prices. The values of the estimated parameters reflect the empirical fact that the conditional standard deviation of returns (“historical volatility”) is typically lower than the implied volatility of at-the-money options.

Symbol	Text reference/interpretation	Numerical value
$r$	risk-free rate	0.04
$\mu$	expected stock return	0.10 [0.022]
$\theta$	long term variance	0.0483 [0.0012] ( $\approx 0.220^2$ )
$\kappa$	speed of mean reversion; deviations from $\theta$ have a half-life of $(\ln 2)/\kappa$ years	4.75 [1.8]
$\sigma$	volatility of volatility	0.550 [0.018]
$\rho$	correlation	-0.569 [0.014]
$S(0)$	initial stock price	100
$V(0)$	initial variance	$\theta$
$T$	expiry date of option	<i>varies</i> ; often 1 year
$K$	strike of (call) option	<i>varies</i> ; often forward at-the-money: $S(0)e^{rT}$
$\theta^{\text{option}}$	typical at-the-money implied volatility (squared)	0.0834 ( $\approx 0.289^2$ )
$\kappa^{\text{option}}$	option market implied speed of mean reversion	2.75

Table 2: Benchmark settings for the parameters of the Heston model. The numbers in square brackets are the standard errors of the estimates.

Proposition 1 states that the position in the stock of the locally risk-minimizing delta hedge is given by

$$\varphi_{\min}^1(t) = C_S + \rho\sigma \frac{C_V}{S(t)}. \quad (6)$$

The option pricing formula  $C$  and the related Greeks (which are as easy—or as hard—to calculate as the option price itself) are implemented using the Lipton-Lewis reformulation of Heston’s original expression to increase computational stability, see Lipton (2002).<sup>5</sup>

<sup>5</sup>When using Lipton (2002) one must correct a typo by changing the sign of  $\beta^{(SV)}$  in either equation

We report hedge errors as the standard deviation of the cost process at expiry divided by the initial option value (in percentage terms),

$$\text{hedge error} = 100 \times \frac{\sqrt{\text{var}^{\mathbb{P}}(\text{Cost}(T))}}{e^{-rT} \mathbb{E}^{\min}([S(T) - K]^+)}. \quad (1)$$

This error measurement tracks the hedge all the way until the date of expiry; it is thus related to the global variance in the sense of Eq. (1).

Hedge errors are estimated as follows: (1) simulate paths of stock prices and volatilities; (2) apply the above hedge strategy for a particular option (a forward-at-the-money call unless otherwise said) at a particular frequency (daily unless otherwise said) along each path; (3) record the path-specific terminal cost; and (4) compute sample moments from many paths.

For the given parameters, a small simulation study (not reported) shows that using risk-minimizing delta hedges rather than usual delta hedges reduces hedge errors by a factor of 10-15% for the typical range of liquid options. The longer the time-to-expiry and the more the option is out-of-the-money, the larger the reduction.

**Picking different martingale measures.** As mentioned above, part of the result in Proposition 1 is that the  $C$ -function in Eq. (6) is a conditional expected value under the minimal martingale measure. But what if the hedger uses another measure? In the context of the Heston model different martingale measures can be obtained by different choices of  $\kappa$  and  $\theta$  in the instantaneous variance dynamics of (5).<sup>6</sup> The choice of measure affects the parameters in the  $C$ -function, and hence the hedger's positions. Our analysis looks at three different martingale measures<sup>7</sup>: (1) the minimal martingale measure (the one to use if you want to minimize local variance); (2) the one obtained by just replacing  $\mu$  by  $r$  (a not uncommon misconception of the minimal martingale measure); and (3) the “market measure” as estimated from option data in Eraker (2004) (the parameters are given in Table 2).

The results of this exercise are summarized in Table 3. One finds that the choice of measure has little effect. That is comforting because the minimal martingale measure dynamics depend on  $\mu$  (the expected stock return) which is notoriously hard to estimate. This finding also means that the loose end in the deceptively simple derivation of the risk-minimizing strategy (in the previous section) is not of major practical importance. Moreover the risk-minimizing strategy based on local considerations does have stable (actually close-to-optimal) global behavior.

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(6) or (7).

<sup>6</sup>Cheridito et al. (2007) show that this does indeed give absolutely continuous measure changes as long as the Feller conditions hold under both measures, which is true for our parameter choice in the simulation experiments.

<sup>7</sup>There are numerous stories and analyses as to which martingale measure “the market picks.” Functions corresponding to other criteria than the variance in Eq. (1) give rise to different valuation measures, see Henderson, Hobson, Howison & Kluge (2005).

Martingale measure; $\mathbb{Q}$	Minimal	Misconceived minimal	Market
$\mathbb{Q}$ -parameters $\theta$	0.229 <sup>2</sup>	0.220 <sup>2</sup>	0.289 <sup>2</sup>
$\kappa$	4.75	4.75	2.75
Hedge error	19.7	19.7	20.3

Table 3: Hedge errors under different conceptions about the Heston model’s martingale measures. Hedge portfolios are adjusted daily and the target option is a 1-year forward-at-the-money call.

**Parameter estimation risk.** Rather than addressing the question of picking the right or the wrong measure, the previous analysis can be seen as an investigation of (a very particular form of) parameter uncertainty. This analysis can be extended to studying the effects of the uncertainty associated to the use of estimated parameters. Eraker (2004) reports standard errors<sup>8</sup> of his estimates, see Table 2. With these at hand, we quantify the estimation risk by the following experiment. Suppose that the true parameters  $(\mu, \theta, \kappa, \sigma, \rho)$  are given by the estimates in Table 2, but that the hedger (along a path) uses parameters drawn from the (asymptotically normal) distribution of the estimator. Repeat this simulation over many paths (each time drawing a new hedge parameter, but keeping it fixed along the path).

Expiry	Moneyness	Hedge frequency		
		monthly	weekly	daily
3M	At-the-money	0.1%	0.2%	0.3%
	10% Out-of-the-money	1.1%	1.5%	1.7%
1Y	At-the-money	1.2%	2.9 %	3.5%
	10% Out-of-the-money	2.8%	3.8%	4.0%

Table 4: Effects of Heston-parameter uncertainty on locally risk-minimizing delta hedges. The table shows the relative increase in the hedge errors when the hedger uses parameters drawn from the distribution of Eraker’s estimator rather than the true parameter.

Table 4 compares the performance of the “random-parameter hedger” to that of someone who uses the true parameter. Results are reported in terms of the relative increase in the hedge errors. As one would expect, the hedge quality deteriorates when the true parameter value is not known. Indeed, the more frequently one hedges, the bigger the effect. (With infrequent hedging, the differences “drown.”) The main message from Table 4 is that the adverse effects (in the range of 0-4% in relative terms) of parameter estimation risk in a stochastic volatility model are small compared to what is

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<sup>8</sup>Only standard errors, not correlations, are reported. We treat things as independent, which should give conservative estimates of the effects.

gained from using risk-minimizing delta hedges (in the range of 10-15%). This parameter uncertainty analysis applies to a hedger who bases his inference solely on observations of the underlying. In practice, hedgers are likely to calibrate at least some of the parameters to option prices observed in the market. That situation raises different questions which are analyzed at the end of this section and, in more detail, in the empirical section.

**Using Black-Scholes' Greeks.** The partial derivatives  $C_S$  and  $C_V$  in Eq. (3) could be computed within a Black-Scholes model. This might indeed be tempting for a trader who generally has these functions readily available.<sup>9</sup> To investigate this case, we run a simulation experiment in the Heston model with parameters as in Table 2. Three hypothetical (but realistic types of) hedgers are looked at:

- A Heston hedger who has full knowledge of parameter values and state variables and uses the risk-minimizing strategy from Eq. (6).
- A simple Black-Scholes hedger who uses the delta from the Black-Scholes model with the role of volatility played by  $\sqrt{V(t)}$ , i.e.

$$h^{SBM}(t) = \Phi \left( \underbrace{\frac{\ln(S(t)/K) + (r + V(t)/2)(T - t)}{\sqrt{V(t)(T - t)}}}_{=:d_+(V(t))} \right),$$

where  $\Phi$  is the standard normal distribution function.

- A risk-minimizing Black-Scholes hedger who has full knowledge of parameters and state variables and uses

$$\begin{aligned} h^{RMBS}(t) &= \left. \frac{\partial BS}{\partial S} \right|_{\sigma_{BS}=\sqrt{V(t)}} + \frac{\rho\sigma}{S(t)} \left. \frac{\partial BS}{\partial \sigma_{BS}^2} \right|_{\sigma_{BS}=\sqrt{V(t)}} \\ &= \Phi(d_+(V(t))) + \frac{\rho\sigma\sqrt{T-t}}{2\sqrt{V(t)}} \phi(d_+(V(t))), \end{aligned}$$

where  $\phi$  denotes the standard normal density function.

Figure 1 shows the hedge errors for different values of the correlation parameter  $\rho$ . For  $\rho = 0$  using Black-Scholes's Greeks (or Greek, as only the delta enters the formula) does little harm as all hedges show fairly identical performance. But, as  $|\rho|$  increases, the quality of the Black-Scholes hedges deteriorate (more pronounced for longer-dated options) relative to the risk-minimizing Heston hedge. Surprisingly, perhaps, for 1-year options the Black-Scholes hedge with attempted risk-minimization does far worse than the simple Black-Scholes hedge. The correction works the right way (lowering the number

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<sup>9</sup>Note that  $C_V$  is the derivative with respect to the variance. As the usual vega is a derivative with respect to the standard deviation, the chain rule is needed.

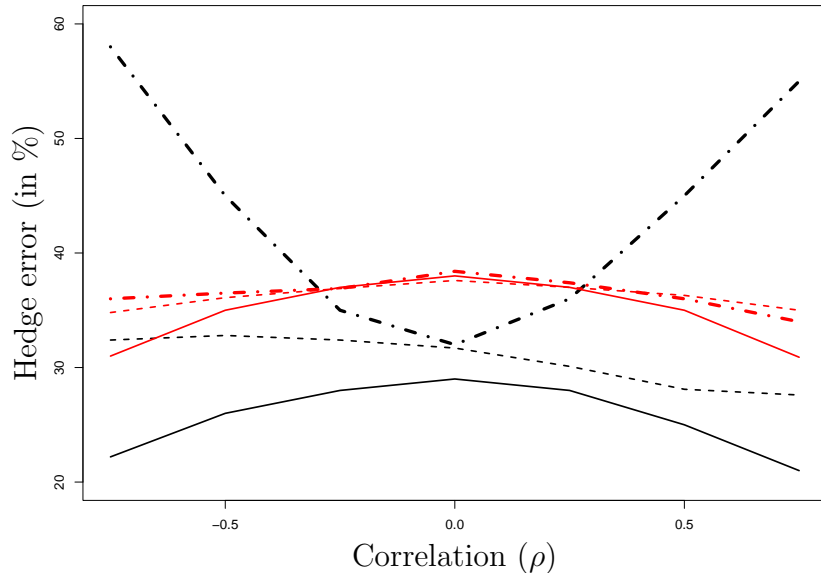


Figure 1: Hedge errors when using Black-Scholes resp. Heston Greeks. The target option are forward-at-the-money calls (black for 1-year, red for 3 month). The solid lines correspond to Heston, the dashed lines are simple Black-Scholes, and the dash-dotted lines show risk-minimizing Black-Scholes.

of units of the underlying in the case of call), but it overshoots because the variance sensitivity (in the sense of a partial derivative) in the Black-Scholes model is much larger than in the Heston model (100 vs. 20 as initial numbers in this case). The intuition for this property is straightforward: if you start out thinking that the (instantaneous) volatility is and always will be, say, 20%, then a 21% volatility next week (which you then believe in forever) is going to create far larger option price changes than if your option price formulas actually reflect that volatility can change. In other words, our finding illustrate the importance of using a genuine stochastic volatility model. One cannot just use sensitivities from a static model to hedge successfully in the dynamic model. We like to think of this as a finance analogy of the Lucas critique in economics.

**SABR as the data-generating process.** What if the Heston model is *not* the correct stochastic volatility model? (After all, alternatives abound.) Hagan et al. (2002) suggest the SABR<sup>10</sup> model for maturity- $T$  forward prices<sup>11</sup> given by

$$\begin{aligned} dF(t; T) &= \alpha(t; T) F^\beta(t; T) dW^1(t) \\ d\alpha(t; T) &= \nu \alpha(t; T) dW^2(t) \end{aligned}$$

where the pricing measure Brownian motions  $W^1$  and  $W^2$  have correlation  $\rho$ . Hagan

<sup>10</sup>SABR is anglicized acronym for “stochastic  $\alpha$ - $\beta$ - $\rho$ ”, i.e. mostly the names of the parameters.

<sup>11</sup>With constant interest rate  $r$  and dividend yield  $\delta$ , one has  $F(t; T) = S(t) \exp((r - \delta)(T - t))$ .

et al. (2002) derive closed-form approximations to option prices in the SABR model by perturbation techniques.<sup>12</sup> The model is qualitatively different from the Heston model in several ways. Instantaneous return variance ( $\alpha/F^{2(b-1)}$ ) is log-normal rather than non-central  $\chi^2$ , and does not mean-revert. Moreover, if  $\beta \neq 1$  then the return variance depends on the level of the underlying; in the language of Alexander & Nogueira (2007b) the model is not scale-invariant.<sup>13</sup> If the parameter  $\beta < 1$ , a negative relation (sometimes referred to as leverage or back-bone) is created between returns and their volatility and, thus, one observes (elements of) a skew in implied volatilities. In fact, as Figure 2 shows, scale-invariance and negative correlation can give the same (to the naked eye) option prices as scale-dependence and zero correlation. Dependence of volatility on the (absolute) level of the process is quite reasonable when modeling quantities that are thought of as exhibiting “more stationarity” than stock prices (such as interest rates or commodity prices).

With parameters as given in Figure 2, let us assume the SABR model is the true data-generating process and look at hedgers who use a Heston model. This investigation of the performance of the locally risk-minimizing delta hedge and the delta hedge in the (wrong) Heston model is carried out as follows: (1) simulate stock prices and volatilities from the SABR model; (2) for each path implement the Heston-based locally risk-minimizing strategy (using the initially calibrated parameters and the simulated Heston-sense local variance along each path) as well as a delta hedge; and (3) implement the SABR model’s delta hedge (which, because of zero correlation of the Brownian motions, coincides with the locally risk-minimizing delta hedge) using the pricing formula given in Hagan et al. (2002).

Hedge method	SABR RiskMin	Heston RiskMin	Heston Delta
Hedge error	13.3	13.9	18.4

Table 5: Hedge error under a misspecified data-generating process (SABR). The target option is a 1-year forward-at-the-money call.

Table 5 presents the results. The locally risk-minimizing delta hedge from the (incorrect) Heston model is almost as good as the one from the (correct) SABR model. It is important, however, to use the calibrated Heston model’s (spurious) correlation; using a Heston-based ordinary delta hedge increases the error by about one third. In essence

<sup>12</sup>Obloj (2008) gives corrections to the formulas in Hagan et al. (2002).

<sup>13</sup>A model is called (spatially) scale-invariant if the conditional distribution of the ratio  $S(u)/S(t)$  is independent of  $S(t)$  for all  $u \geq t$ , i.e. if log-return distributions do not depend on the current level of the underlying. (Some authors, Joshi (2003) for example, call this homogeneity and—as pointed out in Footnote 4 in Alexander & Nogueira (2007b)—there exist many subtle variations in the definition of scale-invariance.) Both the Black-Scholes model and the Heston model are scale-invariant. Alexander & Nogueira (2007b) investigate the properties of scale-invariant models, one of which is that call-option prices are homogenous of degree one in spot and strike.

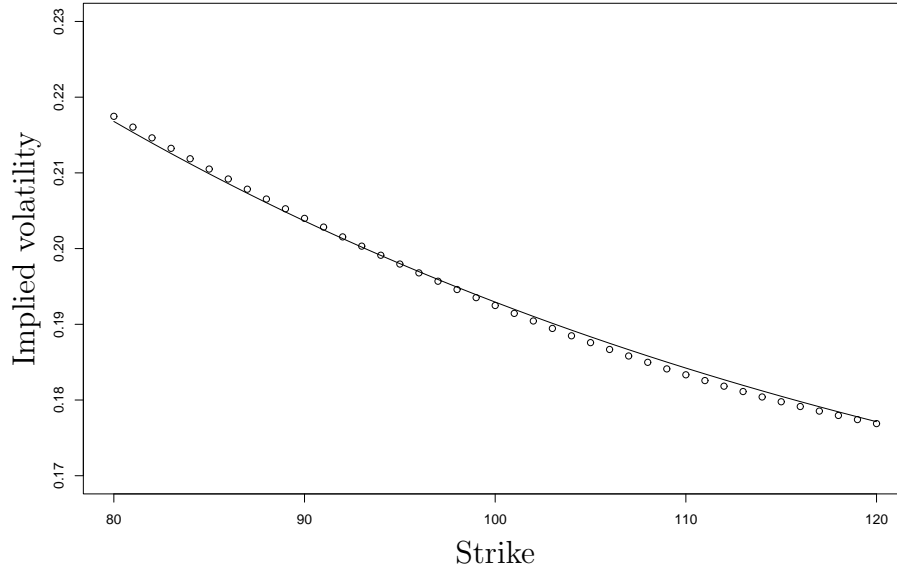


Figure 2: Correlation and scale-dependence can both explain a skew. The graph shows 1-year implied volatilities in the Heston model (circles) and in the SABR model (solid line). Parameters for the Heston model are as specified in Table 2 (except for  $r = \mu = 0$ ), and the SABR settings are  $\alpha(0) = 1.92$ ,  $\nu = 0.2$ ,  $\beta = 0$ , and  $\rho = 0$ , with the two latter being fixed and the two former then calibrated.

the result of this controlled experiment can be interpreted as follows: if the stochastic volatility model is reasonably calibrated, it does not matter much which particular model is used. As explained below, the same is true for real market data.

## 4 Empirical Performance

In this section we present an empirical test of the performance of locally risk-minimizing delta hedge strategies.

**Data.** We collected<sup>14</sup> times series for spot and option prices for three different markets: The U.S. S&P 500 index, the European EUROSTOXX 50 index and the USD/EURO exchange rate. The data, which cover the period from early 2004 to early 2008, are summarized in Figure 3. The markets display both differences and similarities. Volatilities rise markedly after the beginning of the U.S. sub-prime crisis in July 2007 and

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<sup>14</sup>Time series of foreign exchange rate options can be found at British Bankers' Association home page <http://www.bba.org.uk/bba/jsp/polopoly.jsp?d=129&a=799>. (This series was discontinued in January 2008.) The weekly data on stock index options were kindly provided by two investment banks, and can be found on the corresponding author's home page <http://www.math.ku.dk/~rolf>.

the credit crunch that ensued. The varying implied volatilities of at-the-money options would be hard to explain without a stochastic volatility model. In both the U.S. and the European stock markets there is a strong negative correlation between returns and at-the-money implied volatility; for both markets it is around -0.85 over the full sample. In the exchange rate market there is low correlation between changes in the underlying and in the implied volatility; -0.07 in the sample. Moreover implied volatilities across strikes display a smile, rather than a skew.

**Experimental design.** At a point in time  $t$  the hedger sets up a position that holds  $h(t)$  (prescribed by some model and strategy) units of the underlying  $S$  and  $b(t) = \pi(t) - S(t)h(t)$  in a money market account, where  $\pi(t)$  denotes the observed market price of the option to be hedged. This position then requires an investment of  $\pi(t)$ . As common among traders, we measure the quality (or riskiness) of a hedge portfolio by its weekly (i.e.  $dt = 1/52$ ) profit-and-loss ratio (P&L ratio)

$$\text{P\&L}_{t+dt} = \frac{h(t)S(t+dt)e^{\delta dt} + b(t)e^{r dt} - \pi(t+dt)}{\pi(t)},$$

where  $r$  and  $\delta$  denote interest rate and dividend yield.<sup>15</sup> This reflects the profit or loss (or alternatively viewed: the instantaneous cost) relative to the size of the transaction of someone who has sold an option and is trying to cover the position by the underlying and the money market. We always consider hedgers who sell at- or out-of-the-money options. We consider hedges based on the Heston and SABR models, and as a benchmark or “sanity check” we also include a simple Black-Scholes hedge, i.e. the hedger holds a number of units of the underlying that is equal to the Black-Scholes delta evaluated at the implied volatility of the target option.

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<sup>15</sup>Interest rates and dividend yields (= the foreign interest rates in the case of currencies) were estimated from bond and forward/futures prices along with the put/call parity.



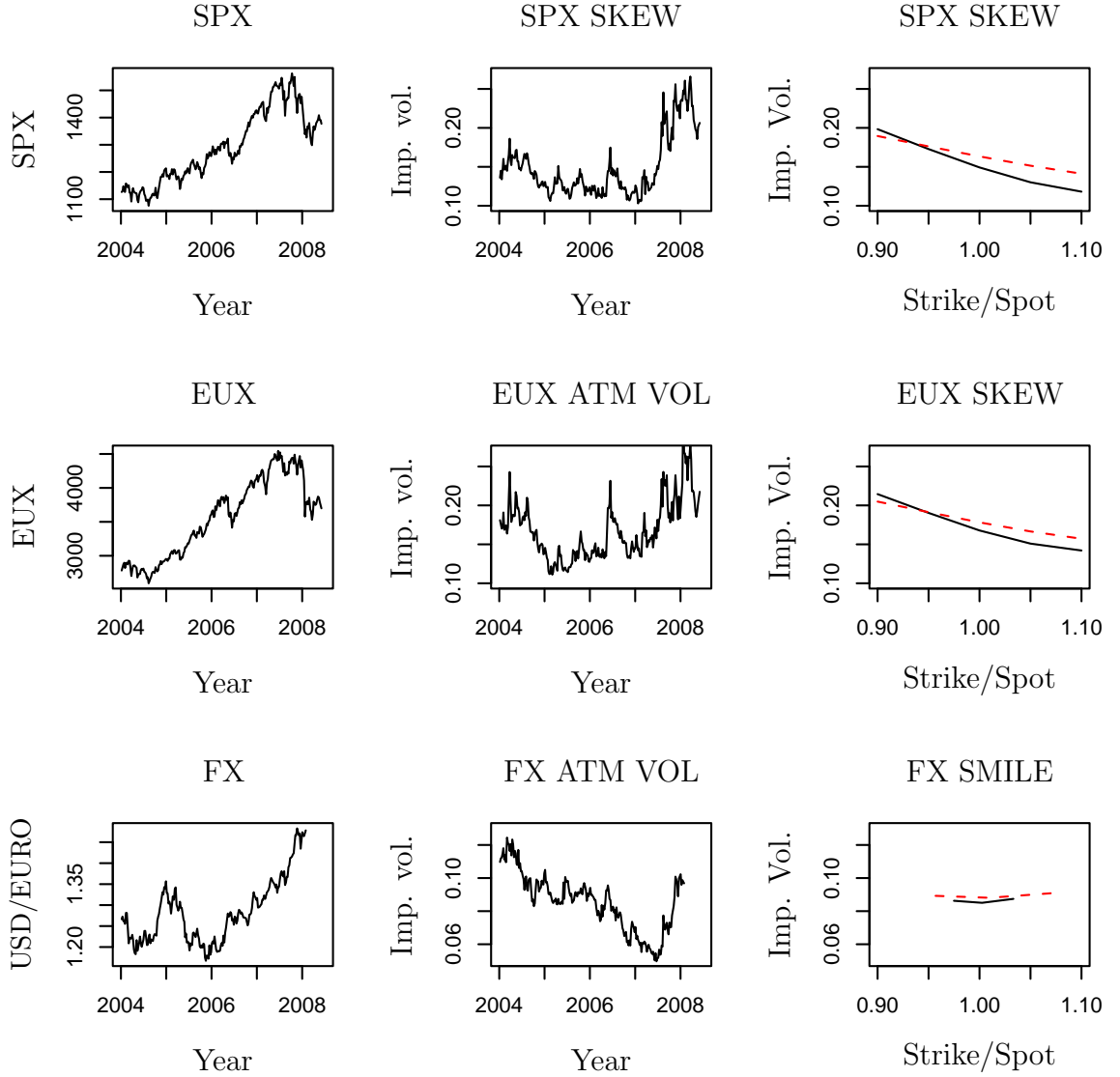


Figure 3: The data used in the study. Each row corresponds to a different market: S&P500 (top), EUROSTOXX 50 (middle) and USD/EURO exchange rate (bottom). The columns correspond to the following data. Left: spot prices over the observation period 2004-2008; Middle: time series of implied volatilities of 3-month at-the-money options; and Right: time-series averages of implied volatilities across strikes for 3-month (black, solid line), 1-year (red, dashed line).

				OTM put			ATM			OTM call		
				SPX	EUX	FX	SPX	EUX	FX	SPX	EUX	FX
Expiry 1M												
Average												
Black-Scholes Delta				2.1			1.3			3.2		
Heston Delta				1.9			1.3			3.5		
Heston RiskMin				1.7			1.2			3.4		
SABR Delta				2.8			1.4			2.9		
SABR RiskMin				2.8			1.3			2.8		
Std. deviation												
Black-Scholes Delta				41.0			18.5			37.6		
Heston Delta				40.7			18.5			37.6		
Heston RiskMin				41.2			18.6			37.9		
SABR Delta				42.5			18.6			37.9		
SABR RiskMin				42.4			18.6			38.0		
Expiry 3M												
Average												
Black-Scholes Delta				2.6 <sup>†</sup>	1.1	1.0	1.3 <sup>‡</sup>	0.9	0.6	3.3 <sup>‡</sup>	2.2	1.5
Heston Delta				2.9 <sup>†</sup>	1.6	1.0	1.5 <sup>†</sup>	1.2	0.6	4.0 <sup>‡</sup>	2.8	1.3
Heston RiskMin				2.3	0.7	1.2	1.1	0.6	0.6	2.6	1.1	1.2
SABR Delta				3.1 <sup>†</sup>	1.8	1.4	1.5 <sup>†</sup>	1.2	0.6	3.9 <sup>‡</sup>	2.7	1.2
SABR RiskMin				3.2 <sup>†</sup>	1.7	1.4	1.1 <sup>‡</sup>	0.5	0.5	2.4 <sup>‡</sup>	1.0	1.1
Std. deviation												
Black-Scholes Delta				17.5 <sup>‡</sup>	19.7 <sup>‡</sup>	16.3	7.3 <sup>‡</sup>	10.5 <sup>‡</sup>	7.6	18.2 <sup>†</sup>	21.1 <sup>†</sup>	15.5
Heston Delta				24.1 <sup>‡</sup>	27.0 <sup>‡</sup>	17.3	11.1 <sup>‡</sup>	15.2 <sup>‡</sup>	7.4	25.7 <sup>‡</sup>	26.6 <sup>‡</sup>	15.8
Heston RiskMin				12.3	14.4	17.8	4.8	7.1	7.5	13.6	14.6	16.1
SABR Delta				24.9 <sup>‡</sup>	25.2 <sup>‡</sup>	17.4	10.9 <sup>‡</sup>	14.8 <sup>‡</sup>	7.5	26.1 <sup>‡</sup>	27.7 <sup>‡</sup>	16.1
SABR RiskMin				12.7	13.0	17.1	4.9	7.4	7.5	12.9	15.6	16.3
SABR Delta ( $\beta = 1/2$ )				23.8 <sup>‡</sup>	25.8 <sup>‡</sup>	17.1	10.2 <sup>‡</sup>	13.9 <sup>‡</sup>	7.5	24.7 <sup>‡</sup>	27.7 <sup>‡</sup>	16.4
SABR RiskMin ( $\beta = 1/2$ )				12.6	13.1	17.2	4.9	7.4	7.5	12.8	15.6	16.2
Expiry 1Y												
Average												
Black-Scholes Delta				0.4	0.3	0.4	0.2	0.2	0.4	0.4	0.4	0.8
Heston Delta				0.5	0.6	0.6	0.3	0.4	0.4	0.7	0.7	0.7
Heston RiskMin				0.5	0.1	0.5	0.1	0.0	0.4	0.0	-0.2	0.7
SABR Delta				0.8	0.6	0.6	0.3	0.4	0.4	0.7	0.9	0.7
SABR RiskMin				0.3	-0.1	0.5	0.1	-0.1	0.4	0.1	-0.4	0.6
Std. deviation												
Black-Scholes Delta				8.3 <sup>‡</sup>	8.8 <sup>‡</sup>	6.7	3.5 <sup>‡</sup>	4.9 <sup>‡</sup>	3.4	8.1 <sup>‡</sup>	10.1 <sup>‡</sup>	6.9
Heston Delta				12.1 <sup>‡</sup>	12.9 <sup>‡</sup>	7.3	5.4 <sup>‡</sup>	7.4 <sup>‡</sup>	3.3	11.3 <sup>‡</sup>	13.6 <sup>‡</sup>	7.4
Heston RiskMin				5.5	6.2	7.6	2.4	3.2	3.4	5.9	6.7	7.7
SABR Delta				11.8 <sup>‡</sup>	12.7 <sup>‡</sup>	7.2	5.2 <sup>‡</sup>	7.3 <sup>‡</sup>	3.4	11.7 <sup>‡</sup>	14.2 <sup>‡</sup>	7.4
SABR RiskMin				5.7	6.0	6.9	2.5	3.2	3.3	6.0	6.8	7.5
SABR Delta ( $\beta = 1/2$ )				10.6 <sup>‡</sup>	11.2 <sup>‡</sup>	7.0	4.6 <sup>‡</sup>	6.3 <sup>‡</sup>	3.4	10.4 <sup>‡</sup>	12.4 <sup>‡</sup>	7.8
SABR RiskMin ( $\beta = 1/2$ )				5.6	5.8	7.0	2.5	3.1	3.3	6.0	6.0	7.5

Table 6: Sample averages and standard deviations (in %) over the period early 2004 to early 2008 for weekly P&L ratios across different models and hedge strategies for options on the S&P 500 index (SPX), the EUROSTOXX50 index (EUX), and the USD/EURO exchange rate (FX). 1-month expiry options were only available for FX. Means are tested equal to zero, standard deviations equal to the Heston RiskMin standard deviation, and <sup>†</sup> resp. <sup>‡</sup> indicate that differences are significant at the 5% resp. 1% level.

**Calibration.** We assume that, at a given date  $t$ , the hedger calibrates his model’s parameters to the option prices observed at this date in the following way:

**Heston** The speed of mean-reversion  $\kappa$  is estimated from a first-order auto-regression of the implied 3-month at-the-money (squared) volatility. On any given day  $V_t$ ,  $\theta$ ,  $\sigma$ , and  $\rho$  are then chosen to minimize the sum of squared differences between observed and model-based implied volatilities for 3-month and 1-year options across a range of strikes.

**SABR** We fix  $\beta$  at some value (1 unless stated) and choose  $\alpha$ ,  $\nu$  and  $\rho$  to minimize differences between the model’s and the market’s implied volatilities. Following the spirit of how the SABR model is used, this is done separately for each expiry.

**Results.** Averages and standard deviations of the profit-and-loss (P&L) ratio across markets, models, moneyness and option expiries are given in Table 6. The main message is that the risk-minimizing delta hedges offer a clear benefit in markets in which changes in the underlying and the instantaneous variance are correlated. The standard deviation of the risk-minimizing delta hedges’ profit-and-loss ratios are approximately half of the models’ usual delta hedges. Put differently: based on recent, large and varied data-sets, our results give a strongly affirmative answer to the questions-in-title posed by authors from Nandi (1998) to Doran, Peterson & Tarrant (2007). Some papers, e.g. Carr, Geman, Madan & Yor (2007) and Eberlein & Madan (2009), advocate the use of pure (“infinite intensity”) jump models. Our results show that there is a (return,volatility)-correlation component that can be treated, i.e. better hedged or risk-managed, by a diffusion-type stochastic volatility model. The skew, so to say, can be “tamed.” If the correlation is close to zero (as in exchange markets), there is (as one would expect) no gain from the suggested risk-minimization. But nothing is lost either: one could fear that a complicated model and frequent re-calibrations might lead to “over-fitting the data” and, thus, deteriorate the out-of-sample hedge performance. This is not the case. The results reported in Table 6 also reveal that long-term options are easier to hedge than short-term ones and that risk-minimization is comparatively most effective for out-of-the-money options. Both of these results coincide with the findings obtained in the simulation studies.

At first sight it might be puzzling that the simple Black-Scholes delta hedges (i.e. the ones where everything is just calculated at implied volatility) perform better than standard delta hedges based on genuine stochastic volatility models. To see why this is so, consider a call option. In a scale-invariant model—such as Black-Scholes or Heston—the call-price, say  $C$ , is homogeneous of degree one in spot and strike. This means that by Euler’s Theorem,  $C = SC_S + KC_K$ , i.e.

$$C_S = \frac{1}{S}(C - KC_K)$$

cf. Lee (2001).

In the Heston model with negative correlation, the risk-minimizing delta hedge holds fewer than  $C_S^{Heston}$  units of the underlying asset. The negative correlation entails the property that implied volatilities are—except at extremely high strikes—decreasing in the strike. This means that  $C_K^{BS} < C_K^{Heston} < 0$ . If the two were equal, implied volatilities would be (locally) flat by definition. So  $C_S^{BS}$  is smaller than  $C_S^{Heston}$  and thus more likely to be closer to the risk-minimizing position.

Some researchers<sup>16</sup> argue that local variance is closer to log-normal than to non-central  $\chi^2$ . An investigation of the stability of the calibrated parameters (not reported) lends some support to this. For the weekly hedge performance however there is no significance or systematic pattern when comparing Heston to SABR. The scale-dependence parameter  $\beta$  matters little; one may as well use a scale-invariant model. This lends empirical support to the suggestions made in Ayache, Henrotte, Nassar & Wang (2004) and the methodology described in Cont & da Fonseca (2002).

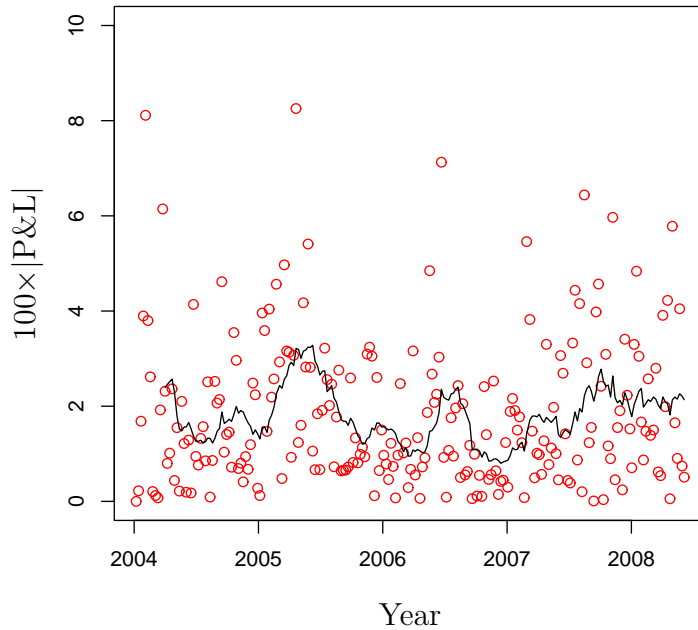


Figure 4: Absolute profit-and-loss (P&L) over time for Heston-based risk-minimizing delta hedges of 1-year, at-the-money call options on the S&P 500. The solid line denotes the 13-week moving average.

<sup>16</sup>See for instance Paul Wilmott's view on this issue:  
<http://www.wilmott.com/images/246/VolForecastingOpTradingCM.wmv>

Average P&L ratios are small, but not zero; this reflects a volatility risk-premium. In our sample there are typically positive expected profits to option sellers who hedge their positions with the underlying. Traders will see this as reward for “short volatility (or gamma) exposure,” while closer inspection, see Branger & Schlag (2008), reveals that one should be careful about what to read into the signs of average hedge errors.<sup>17</sup>

Finally, the natural question to be asked at this time in history: “How do stochastic volatility model hedge performance cope with the credit crunch?” can be answered: “Quite well.” A typical plot of the time-series behavior of profit-and-loss rates is shown in Figure 4. There is no deterioration of the hedge quality towards the end of the time period (which corresponds to the credit crunch).

## 5 Conclusion and Future Research

In this paper we calculated locally risk-minimizing delta hedge strategies for a general class of stochastic volatility models. Our empirical tests (across different markets, time and option-types) showed that the risk-minimizing delta hedges offer what an economist would call a Pareto-improvement over usual delta hedges: Risk-minimizing delta hedges are as easy and reliable to implement as usual delta hedges and one is never worse off when using risk-minimizing delta hedges but sometimes one is better off by quite a margin. We presented experimental and empirical evidence on the importance of model risk (or the lack of it). Our findings reveal that when volatility is stochastic, it is important to model it as such; short-cuts will not do. However, as long as the modeling is done sensibly, the exact model seems to matter little for hedging plain vanilla options.

An interesting topic for future research is the application of locally risk-minimizing delta strategies to exotic options. For example, the form of the risk-minimizing delta hedge in Proposition 1 carries over verbatim to barrier options; we just don’t know of any truly closed-form expressions for the relevant conditional expectations and partial derivatives. For exotic options one should use static positions in the plain vanilla options to create Vega (and/or other Greeks such as “Gamma”, “Volga” or “Vanna”) neutral positions, and then dynamically hedge the residual with the underlying in a risk-minimizing way. Because the frictions (transaction costs, bid/ask spreads) related to trading options relative to trading the underlying are typically orders of magnitude apart, this entails a non-trivial trade-off; see Siven & Poulsen (2008) for some preliminary investigations. Moreover, there could be a stronger element of model risk because, as pointed out by Ayache et al. (2004) among others, models that produce similar prices of plain vanilla options (by calibration) may give markedly different prices of barrier (and other exotic) options.

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<sup>17</sup>In the authors’ personal experience, calculating average P&L’s is a good way of debugging one’s code.

## References

- Ahn, H. & Wilmott, P. (2003), ‘Stochastic volatility and mean-variance analysis’, *Wilmott Magazine* **8**, 84–90.
- Alexander, C. & Nogueira, L. (2007*a*), ‘Model-free hedge ratios and scale-invariant models’, *Journal of Banking and Finance* **31**, 1839–1861.
- Alexander, C. & Nogueira, L. (2007*b*), ‘Model-free hedge ratios for homogeneous claims on tradable assets’, *Quantitative Finance* **7**, 473–479.
- Ayache, E., Henrotte, P., Nassar, S. & Wang, X. (2004), ‘Can anyone solve the smile problem?’, *Wilmott Magazine* **9**, 78–96.
- Bakshi, G., Cao, C. & Chen, Z. (1997), ‘Empirical performance of alternative option pricing models’, *Journal of Finance* **52**, 2003–2049.
- Bouchaud, J.-P. & Potters, M. (2003), *Theory of Financial Risks and Derivative Pricing*, Cambridge.
- Branger, N. & Schlag, C. (2008), ‘Can tests based on option hedging errors correctly identify volatility risk premia?’, *Journal of Financial and Quantitative Analysis* **32**, 213–247.
- Carr, P., Geman, H., Madan, D. & Yor, M. (2007), ‘Self-decomposability and option pricing’, *Mathematical Finance* **17**, 31–57.
- Cheridito, P., Filipovic, D. & Kimmel, R. L. (2007), ‘Market price of risk specifications for affine models: Theory and evidence’, *Journal of Financial Economics* **83**, 123–170.
- Cont, R. (2006), ‘Model uncertainty and its impact on the pricing of derivative instruments’, *Mathematical Finance* **16**, 519–542.
- Cont, R. & da Fonseca, J. (2002), ‘Dynamics of implied volatility surfaces’, *Quantitative Finance* **2**, 45–60.
- Doran, J. S., Peterson, D. R. & Tarrant, B. C. (2007), ‘Is there information in the volatility skew?’, *Journal of Futures Markets* **27**, 921–959.
- Eberlein, E. & Madan, D. B. (2009), ‘Sato processes and the valuation of structured products’, *Quantitative Finance* **9**, 27–42.
- El Karoui, N., Peng, S. & Quenez, M. (1997), ‘Backward stochastic differential equations in finance’, *Mathematical Finance* **7**, 1–77.

- Eraker, B. (2004), ‘Do stock prices and volatility jump? Reconciling evidence from spot and option prices’, *Journal of Finance* **59**, 1367–1403.
- Föllmer, H. & Schweizer, M. (1990), Hedging of contingent claims under incomplete information, in M. Davis & R. Elliott, eds, ‘Applied Stochastic Analysis’, Gordon and Breach, London, pp. 389–414.
- Frey, R. (1997), ‘Derivative asset analysis in models with level-dependent and stochastic volatility’, *CWI Quarterly* **10**, 1–34.
- Hagan, P. S., Kumar, D., Lesniewski, A. S. & Woodward, D. E. (2002), ‘Managing smile risk’, *Wilmott Magazine* **1**, 84–108.
- Henderson, V., Hobson, D., Howison, S. & Kluge, T. (2005), ‘A comparison of option prices under different pricing measures in a stochastic volatility model with correlation’, *Review of Derivatives Research* **8**, 5–25.
- Heston, S. (1993), ‘A closed-form solution for options with stochastic volatility with applications to bond and currency options’, *Review of Financial Studies* **6**, 327–344.
- Joshi, M. S. (2003), *The Concepts and Practice of Mathematical Finance*, Cambridge.
- Lee, R. (2001), ‘Implied and local volatilities under stochastic volatility’, *International Journal of Theoretical and Applied Finance* **4**, 45–89.
- Lipton, A. (2002), ‘The vol smile problem’, *Risk Magazine* **15**, 61–65.
- Murphy, D. (2008), ‘A preliminary enquiry into the causes of the Credit Crunch’, *Quantitative Finance* **8**, 435–451.
- Nandi, S. (1998), ‘How important is the correlation between returns and volatility in a stochastic volatility model? Empirical evidence from pricing and hedging in the S&P 500 index options market’, *Journal of Banking and Finance* **22**, 589–610.
- Obloj, J. (2008), ‘Fine-tune your smile: Correction to Hagan et al.’, *Wilmott Magazine* **35**, 102–104.
- Romano, M. & Touzi, N. (1997), ‘Contingent claims and market completeness in a stochastic volatility model’, *Mathematical Finance* **7**, 399–412.
- Schweizer, M. (1991), ‘Option hedging for semimartingales’, *Stochastic Processes and their Applications* **37**, 339–363.
- Siven, J. & Poulsen, R. (2008), ‘The long and short of static hedging with frictions’, *Wilmott Magazine* **38**, 62–67.