Fundamental Theorem of Derivative Trading

Meeting 6

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Model Settings (Generalized Black-Scholes)

- Two assets with P-dynamics

$$
\begin{aligned}
dB_t &= rB_t dt \\
dS_t &= \mu_t S_t dt + \sigma_t S_t dW_t
\end{aligned}
$$

- Dynamics under the EMM Q

$$
\begin{aligned}
dB_t &= rB_t dt \\
dS_t &= rS_t dt + \sigma_t S_t dW_t^Q
\end{aligned}
$$

- The pricing measure Q is unique $\sim$ the model is free of arbitrage and complete

- Let $\mathcal{X}$ be the European call with maturity T and strike K, e.g., the contract function is given by $\Phi(s) = (s - K)^+$.

- Moreover, let $\pi(t)$ denote the fair price of $\mathcal{X}$. Recall that $\pi(t)$ is on the form

$$
\pi(t) = C(t, S_t)
$$

where the pricing function $C(t, s)$ is given by the Black-Scholes formula.

Δ-Hedging

- Let $h_t = (h_t^0, h_t^1)$ be a self-financing portfolio replicating $\mathcal{X}$, that is

$$V_T^h = \mathcal{X} \qquad P - a.s.$$

- Recall that holding the replicating portfolio is equivalent to holding $\mathcal{X}$, i.e.,

$$C(t, S_t) = V_t^h$$

- Δ-hedging: Let $h_t^1 = \Delta_t := C_s(t, S_t)$ — which implicitly depends on σ — and choose h_t^0 to fund this investment. Result: This strategy will replicate/hedge $\mathcal{X}$.

- It may be convenient considering the *adjusted portfolio*, $h_t = (h_t^0, \Delta_t, -1)$, with value process

$$V_t^h = h_t^0 B_t + \Delta_t S_t - C(t, S_t)$$

Three Volatilities...

- True model volatility at time t, σ_t
- The market implied volatility at time t, σ_t^i, given by

$$C(t, S_t; \sigma_t^i) = \hat{C}_t,$$

  where $\hat{C}_t$ denotes the observed market price at time t
- Note that $\sigma_t^i \neq \sigma_t$ may be interpreted as *mispricing* at time t. We are mainly interested in the 'normal' scenario: $\sigma_t^i > \sigma_t$
- Finally, the Δ-hedging trader has to pick a *hedging volatility* when computing Δ_t. For this choice of volatility, we will use the notation σ_t^h.
- As FTODT will show — supported by Wilmott's hedge example — two obvious choices for the chosen hedging volatility appear:
  1. Hedging with implied volatility: $\sigma_t^h = \sigma_t^i$
  2. Hedging with true model volatility: $\sigma_t^h = \sigma_t$
- If reality is described by the simple BS-model, both choices will lead to *volatility arbitrage*

Theorem: FTODT

- Assume that we sell a European option with strike K and maturity T for the market price $C(0, S_0\ \sigma_0^i)$ and want to Δ-hedging this position. Let σ_t^h denote the hedging volatility chosen by the Δ-hedging trader. The **present value** of the hedging error (profit-and-loss) when holding the portfolio over $[0, T]$ is given by

$$P\&L_T = C(0, S_0; \sigma_0^i) - C(0, S_0; \sigma_0^h) + \int_0^T e^{-rt} \frac{1}{2}((\sigma_t^h)^2 - \sigma_t^2) S_t^2 \Gamma(t, S_t; \sigma_t^h) dt \tag{1}$$

- Thus the hedging error is decomposed by the upfront-premium and a path-dependent (random) integral

- Pick $\sigma_t^h = \sigma_t$ and $P\&L_T$ becomes deterministic

$$P\&L_T = C(0, S_0; \sigma_0^i) - C(0, S_0; \sigma_0)$$

- Pick $\sigma_t^h = \sigma_t^i$ and $P\&L_T$ becomes stochastic

$$P\&L_T = \int_0^T \frac{1}{2} e^{-rt} ((\sigma_t^i)^2 - \sigma_t^2) S_t^2 \Gamma(t, S_t; \sigma_t^i) dt$$

- Note: Sign of hedging error depends on whether $\sigma_t^i < \sigma_t$ or $\sigma_t^i > \sigma_t$

Wilmott's Hedge Example

- Assume simple BS-model: $\sigma_t = \sigma$ for all t. Also assume that $\sigma_t^i = \sigma^i$ for all t (market implied volatility is constant).
- Suppose that $\sigma_i > \sigma$, i.e., the market price exceeds the theoretical price
- $\sigma_i - \sigma$ is called the *volatility risk premium* measuring the magnitude of mispricing
- We want to reap this premium
- Intuition: The option is overpriced, so we may sell it and receive the market price $C(0, S_0, \sigma_i)$. Finally, we eliminate the risk (we have just sold an option) by Δ-hedging this position.
- According to FTODT hedging with true volatility yields the positive and deterministic profit-and-loss

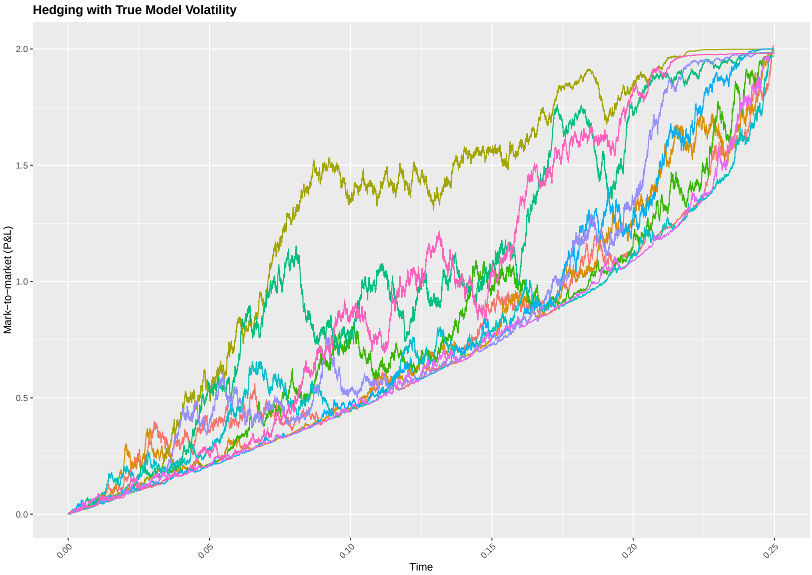
  $$P\&L_T = C(0, S_0; \sigma^i) - C(0, S_0; \sigma) > 0$$

  since $\sigma^i > \sigma$
- ...whereas hedging with implied volatility yields the positive and stochastic profit-and-loss

  $$P\&L_T = \frac{1}{2} e^{-rT} ((\sigma^i)^2 - \sigma^2) \int_0^T S_t^2 \Gamma(t, S_t; \sigma^i) dt$$

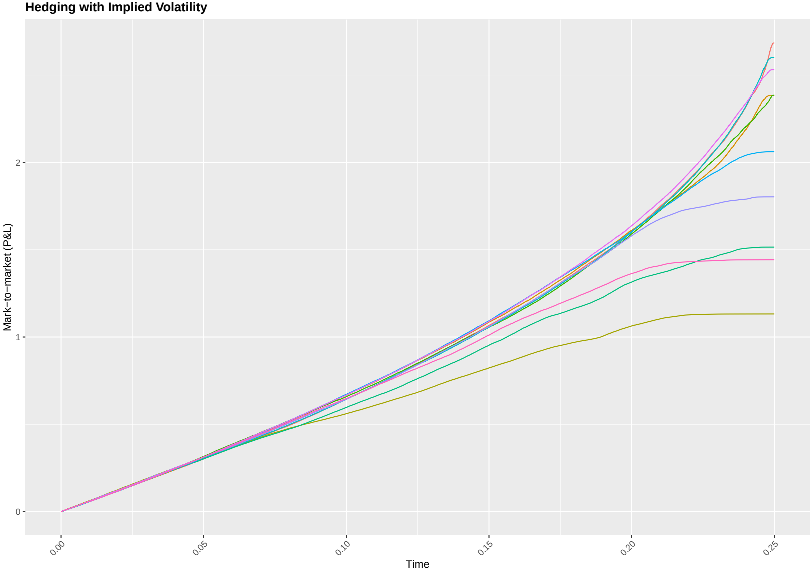
  since $\sigma^i > \sigma$ and $\Gamma(t, S_t; \sigma^i) > 0$

Case I: Hedging with true volatility



Model parameters: $S_0 = 100$, $\mu = 0.1$ and $\sigma = 0.1$. Market parameters: $r = 0.02$, $\sigma^i = 0.2$, $K = 100$ and $T = 1/4$. As a result $e^{rT}(C(0, S_0; \sigma^i) - C(0, S_0; \sigma)) = 1.99$.

Case II: Hedging with Implied Volatility



Terminal hedging error is now random — but paths appear less erratic!

Proof for FTODT: Step 1

- Assume that we sell one European option and want to Δ-hedge this position. The value of the adjusted portfolio $h_t = (B_t, C_s(t, S_t; \sigma_t^h), -1)$ at time t is given by

$$V_t^h = B_t + C_s(t, S_t; \sigma_t^h) S_t - C(t, S_t; \sigma_t^i)$$

- Now, we choose to B_t such that the net position is zero, i.e.,

$$V_t^h = 0 \qquad \Leftrightarrow \qquad B_t = C(t, S_t; \sigma_t^i) - C_s(t, S_t; \sigma_t^h) S_t$$

- This strategy is by construction self-financing which implies

$$\begin{aligned} dV_t^h &= dB_t + C_s(t, S_t; \sigma_t^h) dS_t - dC(t, S_t; \sigma_t^i) \\ &= C_s(t, S_t; \sigma_t^h)(dS_t - rS_t dt) + rC(t, S_t; \sigma_t^i) dt - dC(t, S_t; \sigma_t^i) \end{aligned}$$

Proof for FTODT: Step 2

- Firstly Ito's formula yields

$$dC(t, S_t; \sigma_t^h) = C_t(t, S_t; \sigma_t^h)dt + C_s(t, S_t; \sigma_t^h)dS_t + \frac{1}{2}C_{ss}(t, S_t; \sigma_t^h)\sigma_t^2 S_t^2 dt \tag{2}$$

- Furthermore, the Black-Scholes PDE implies that

$$C_t(t, S_t; \sigma_t^h) = rC(t, S_t; \sigma_t^h) - rS_t C_s(t, S_t; \sigma_t^h) - \frac{1}{2}(\sigma_t^h)^2 S_t^2 C_{ss}(t, S_t; \sigma_t^h) \tag{3}$$

- Now substitute this guy into equation (2) to obtain

$$\begin{aligned} 0 &= -dC(t, S_t; \sigma_t^h) + \left(rC(t, S_t; \sigma_t^h) - rS_t C_s(t, S_t; \sigma_t^h) - \frac{1}{2}(\sigma_t^h)^2 S_t^2 C_{ss}(t, S_t; \sigma_t^h) \right) dt \\ &\qquad + C_s(t, S_t; \sigma_t^h)dS_t + \frac{1}{2}C_{ss}(t, S_t; \sigma_t^h)\sigma_t^2 S_t^2 dt \\ &= -dC(t, S_t; \sigma_t^h) + C_s(t, S_t; \sigma_t^h)dS_t \\ &+ \left(rC(t, S_t; \sigma_t^h) - rS_t C_s(t, S_t; \sigma_t^h) + \frac{1}{2}(\sigma_t^2 - (\sigma_t^h)^2) S_t^2 C_{ss}(t, S_t; \sigma_t^h) \right) dt \end{aligned} \tag{4}$$

Proof for FTODT: Step 3

- Now we subtract equation (4) from dV_t^h:

$$\begin{aligned} dV_t^h &= dC(t, S_t; \sigma_t^h) - dC(t, S_t; S\sigma_t^i) - r(C(t, S_t; \sigma_t^h) - C(t, S_t; \sigma_t^i))dt \\ &\quad + \frac{1}{2}\left((\sigma_t^h)^2 - \sigma_t^2)S_t^2 C_{ss}(t, S_t; \sigma_t^h)\right) dt \\ &= e^{rt}\left(d\left(e^{-rt}(C(t, S_t; \sigma_t^h) - C(t, S_t; \sigma_t^i)\right)\right) \\ &\quad + \frac{1}{2}\left((\sigma_t^h)^2 - \sigma_t^2)S_t^2 C_{ss}(t, S_t; \sigma_t^h)\right) dt \end{aligned}$$

- At last the profit-and-loss from holding this portfolio is defined by $P\&L_T := \int_0^T e^{-rt} dV_t^h$:

$$P\&L_T = C(t, S_t; \sigma_t^i) - C(t, S_t; \sigma_t^h) + \frac{1}{2}\int_0^T e^{-rt}((\sigma_t^h)^2 - \sigma_t^2)S_t^2 \Gamma(t, S_t; \sigma_t^h)dt$$

  using that $C(T, S_T; \sigma_T^i) = C(T, S_T; \sigma_T^h) = (S_T - K)^+$.