Greeks, Hedging and Implied Volatility

Vejledningsmøde 5

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Greeks (European Options)

- The definition of Greeks are given in definition 9.4, while proposition 9.5 states the quantities for a European call in the Black-Scholes model:

$$\begin{aligned}\Delta_t &= C_s(t,s) = N(d_1(t,s)) > 0\\ \Gamma_t &= C_{ss}(t,s) = \frac{\phi(d_1(t,s))}{s\sigma\sqrt{T-t}} > 0\\ \rho_t &= C_r(t,s) > 0\\ \theta_t &= C_t(t,s) < 0\\ \mathcal{V}_t &= C_\sigma(t,s) = s\phi(d_1(t,s))\sqrt{T-t}\end{aligned}$$

  with $\phi(\cdot)$ being the density for the standard normal distribution

- We may utilize the put-call parity to compute Greeks for the European put. Recall that

$$P(t,s) = Ke^{-r(T-t)} + C(t,s) - s$$

- ....for instance

$$P_s(t,s) = C_s(t,s) - 1, \qquad P_{ss}(t,s) = \qquad C_{ss}(t,s) \qquad \text{and} \qquad P_\sigma = C_\sigma.$$

Self-Financing Portfolios

- Let $h(t) = (h_t^0, h_t^1)$ be a self-financing portfolio with value process $(V_t^h)_{0 \leq t \leq T}$:

$$V_t^h = h_t^0 B_t + h_t^1 S_t$$

- We will focus on *self-financing* portfolios, that is

$$dV_t^h = h_t^0 dB_t + h_t^1 dS_t$$

- Dynamics of V_t^h under P and Q, respectively,

$$\begin{aligned} P: \qquad & dV_t^h = ? \\ Q: \qquad & dV_t^h = ? \end{aligned}$$

- Dynamics of V_t^h / B_t under P and Q, respectively,

$$\begin{aligned} P: \qquad & d(e^{-rt} V_t^h) = ? \\ Q: \qquad & d(e^{-rt} V_t^h) = ? \end{aligned}$$

Self-Financing Portfolios — (No-)Arbitrage

- Definition 7.5: Markedet indeholder arbitrage-muligheder, hvis der eksiterer en selv-finansierende portefølje, hvor

$$V_0^h = 0, \tag{1}$$

$$P(V_T^h \geq 0) = 1, \tag{2}$$

$$P(V_T^h > 0) > 0 \tag{3}$$

- Proposition 7.6: Lad h_t være en selv-finansierende portefølje med

$$dV_t^h = k_t V_t^h dt,$$

hvor k_t er en tilpasset proces. Så er $k_t = r$ — ellers eksisterer der arbitrage-muligheder.

Self-Financing Portfolios — Completeness

- Definition 8.1: Lad $\mathcal{X}$ være et T-claim. Den selvf-finansierende portefølje h_t hedger $\mathcal{X}$, hvis

$$V_T^h = \mathcal{X}$$

P-næsten-sikkert.

- Sætning 8.3: Black-Scholes modellen er *complete* (alle T-claims kan hedges)

- Intuition: Antag at porteføljen (h_t) hedger $\mathcal{X}$
  1. Antag at vi til tid t har beløbet V_t^h
  2. For V_t^h køber vi portefølen h_t
  3. Mellem $[t, T]$ holder vi nu h_t, hvilket ikke koster os noget, da porteføljen er selv-finansierende. Bemærk h_t stadig ændrer sig dynamiks over tid
  4. Til tid T har vi, at $V_T^h = \mathcal{X}$, da h_t hedger $\mathcal{X}$

- Specielt gælder der, at $\pi(t\ \mathcal{X}) = V_t^h$ — at holde hedging porteføljen er ækvivalent med at holde $\mathcal{X}$!

Δ-Neutrality

- Lad h_t være hedging portefølje for $\mathcal{X}$ med
$$V_t^h = h_t^0 B_t + h_1^t S_t$$
- Ofte er det belejligt at betragte den *justerede* portefølje, hvor vi sælger $\mathcal{X}$, $h_t = (h_t^0, h_t^1, -1)$, med
$$V_t^h = h_t^0 B_t + h_1^t S_t - F(t, S_t).$$
Her er $F(t,s)$ prisfunktionen for $\mathcal{X}$. Bemærk at $V_T^h = 0$.
- Den justerede portefølje siges at være Δ-neutral, hvis
$$\frac{\partial V_t^h}{\partial s} = 0 \qquad \Leftrightarrow \qquad h_t^1 = F_s(t, S_t),$$
hvor $F_s(t,s)$ er optionens Δ til tid t

Δ-Hedging (se note)

- For hvert t beregnes Δ_t (antal aktier), mens h_t^0 vælges til at finansiere $\Delta_t S_t$. Størrelsen h_0^t er både givet i github-noten samt sætning 8.5

- I praksis hedger vi i diskret tid (fx daglig basis)

- Lad $0 = t_0 < t_1 < ... < t_n = T$ være et time-grid over $[0, T]$ med $dt = 1/n$
  1. $t_0 = 0$: Vi sælger $\mathcal{X}$, modtager $F(0, S_0)$ og køber $\Delta_0 = F_s(0, S_0)$ aktier, mens
  $$h_0^0 = V_0^h + F(0, S_0) - \Delta_0 S_0$$
  med $V_0^h = 0$
  2. Porteføljens (justeret) værdi (hedging error) næste dag (t_1) er da
  $$V_{t_1}^h = h_0^0 e^{r dt} + \Delta_0 S_{t_1} - F(t_1, S_{t_1}),$$
  3. Nu rebalanceres porteføljen: $\Delta_1 = F_s(t_1, S_{t_1})$ og
  $$h_{t_1}^0 = V_{t_1}^h + F(t_1, S_{t_1}) - \Delta_1 S_{t_1}$$
  4. Dette gentages for $t_2, t_3, ..., t_{n-1}$: Beregn Δ_t, h_t^0 og hedging fejlen V_t^h. Strategiens peformance/hedging error er til tid T, da givet ved
  $$V_T^h = h_{t_{n-1}}^0 e^{r dt} + \Delta_{t_{n-1}} S_T - F(T, S_T)$$

Mere om Δ-Hedging...

- Bemærk i diskret tid: $dV^h_{t_i} = V^h_{t_i} - V^h_{t_{i-1}}$ er hedging fejlen ved at holde $h_{t_{i-1}}$ mellem t_{i-1}
- Hvis vi summer daglige hedging errors fås samlet hedring error:

$$\sum_{i=1}^{n} dV^h_{t_i} = V^h_T$$

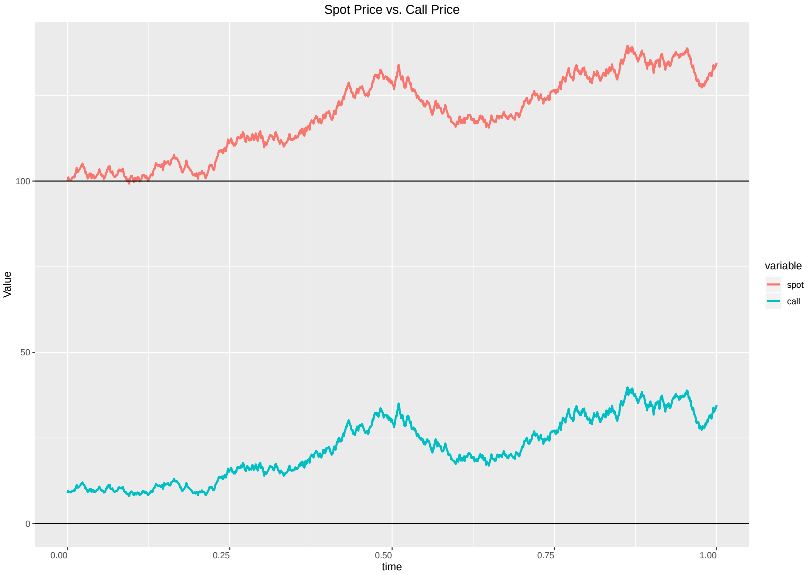
- Proposition 9.7: I kontinuert tid $(dt \to 0)$ vil værdien af (h^0_t, Δ_t) hedge $\mathcal{X}$, altså

$$V^h_T = 0$$

, hvor (V^h_t) er værdiprocessen for den justerede portefølje

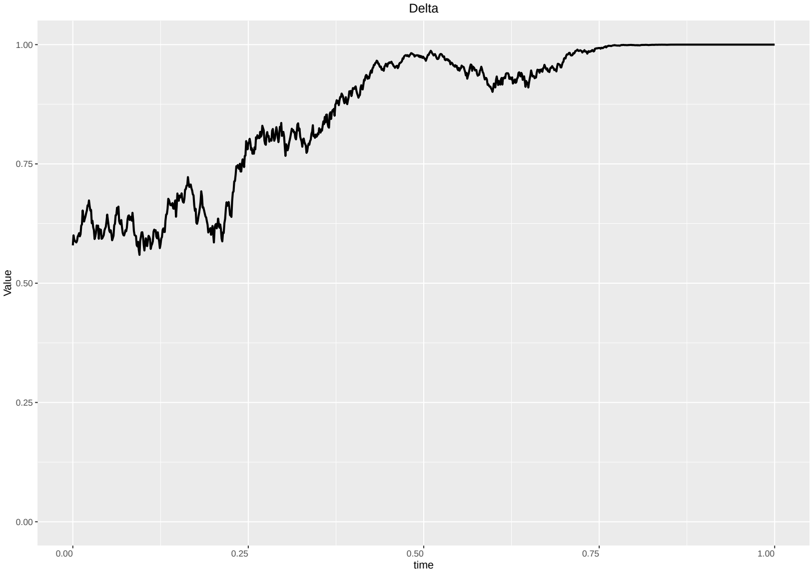
- Overvejelser/problemer:
  - Vi hedger ikke i kontinuert tid (men måske på daglig basis)
  - Vi har antaget, at transaktionsomkostninger er 0
  - Vi har antaget, at BS-modellen beskriver virkeligheden
  - Vi har antaget, at vi kender alle BS-parametre (vi kender ikke σ)

Example: Spot Price and Call Price

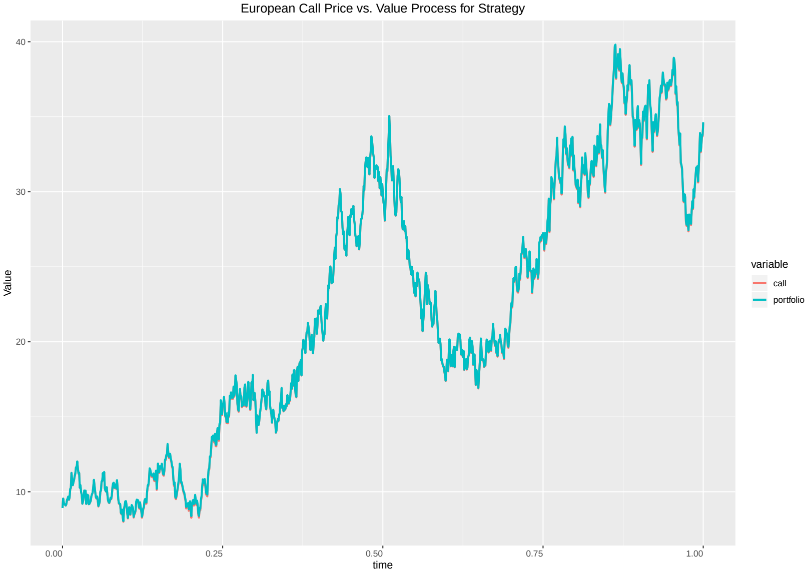


Model parameters: $S_0 = 100$, $\mu = 0.07$ and $\sigma = 2$. Market parameters: $r = 0.02$, $K = 100$ and $T = 1$. Finally $dt = 1/1000$.

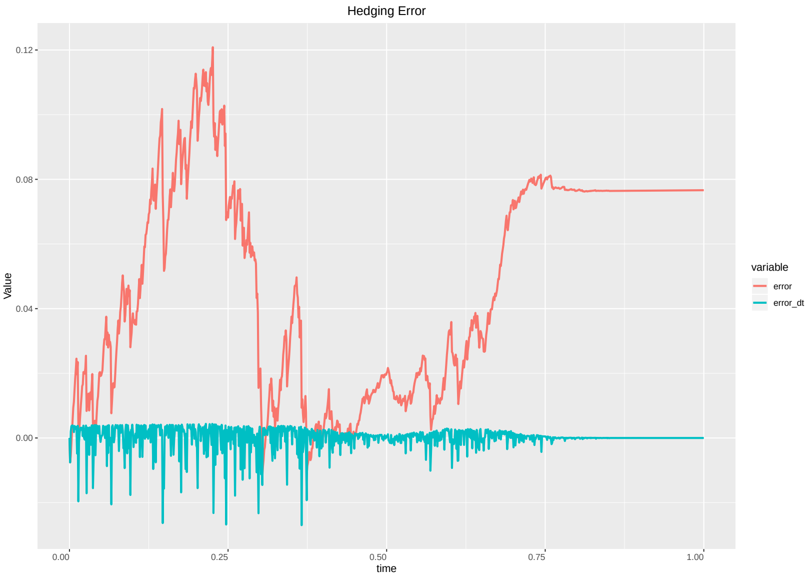
....Δ_t of the European Option



Call Price vs. Value of Δ-hedging Portfolio



Hedging Errors



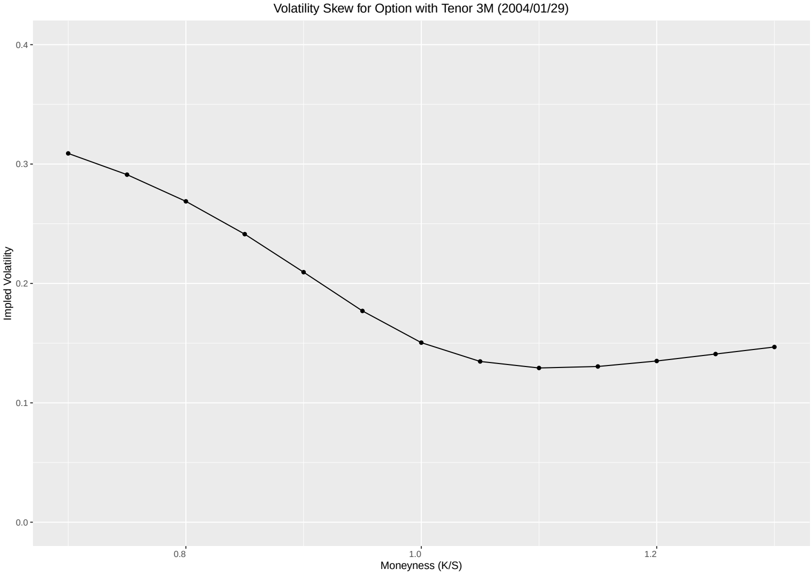
Implied Volatility

- On $0 = t_0 < t_1 < ... < t_n = T$ assume that we observe *market prices* $\hat{C}_0, ..., \hat{C}_{n-1}$ for a European call with strike K and maturity T
- The market implied volatility at time t_j, σ_j^i, is then expressed through the BS-formula and solves

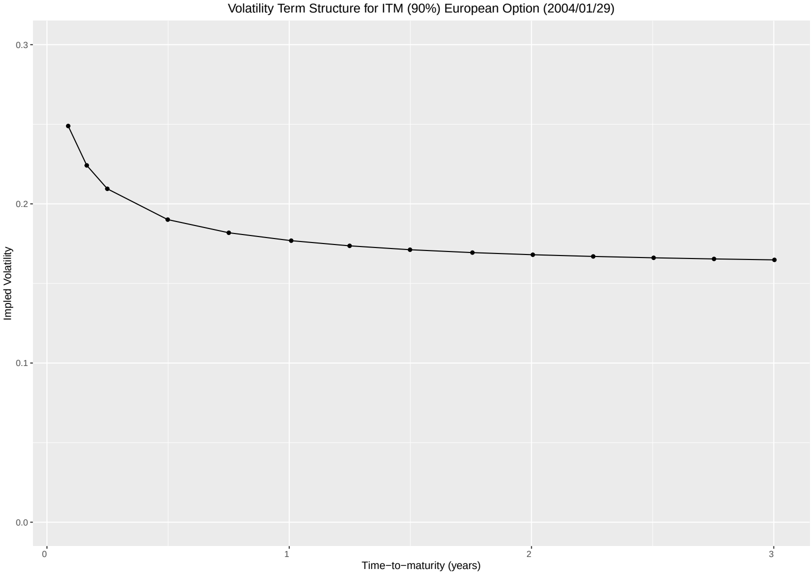
$$C(t_i, S_{t_i}, r, \sigma_j^i, K) = \hat{C}_j$$

- Recall that $\mathcal{V}_t = C_\sigma(t, s) > 0$: the solution to the above expression is unique
- The volatility skew: The market implied volatility depends on the strike level (or moneyness $M_t = K/S_t$). Typically decreasing in K.
- The volatility term structure: The market implied volatility depends on time-to-maturity $T - t$
- Combining the two observations yields the *implied volatility surface*

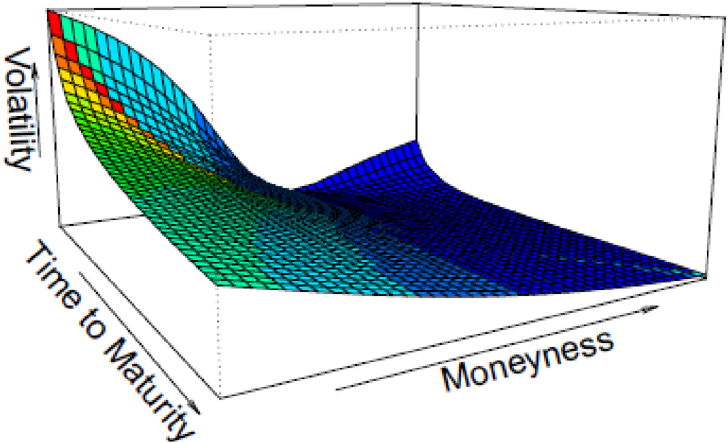
The volatility Skew (3-months European Option)



The volatility Term Structure



The volatility Surface



Summary

- If BS-model describes reality, market implied volatilities shouldn't depend on the option's moneyness and time-to-maturity
- Observed market prices show that this is clearly not the case (volatility surface)
- Focal point for option pricing models: Describe the observed volatility surface (for instance by including stochastic volatility)
- Regarding hedging European Option, we can't expect a perfect hedge the option because
  - Hedging is not conducted in continuous time
  - Option prices contradict the BS-model $\sim$ even in continuous time, the BS Δ-hedge wouldn't replicate the option. However, this may imply that we can make profit when hedging the option (volatility arbitrage)
- Practical considerations: We need to pick a hedging volatility
  - True model volatility (σ) — requires estimation
  - Market implied volatility (σ^i) — observed from market prices