UNIVERSITY OF COPENHAGEN

Mandatory Bachelor Project

Math-Econ Programme

Financial Modeling with Continuous-Time Models

Theory and Applications

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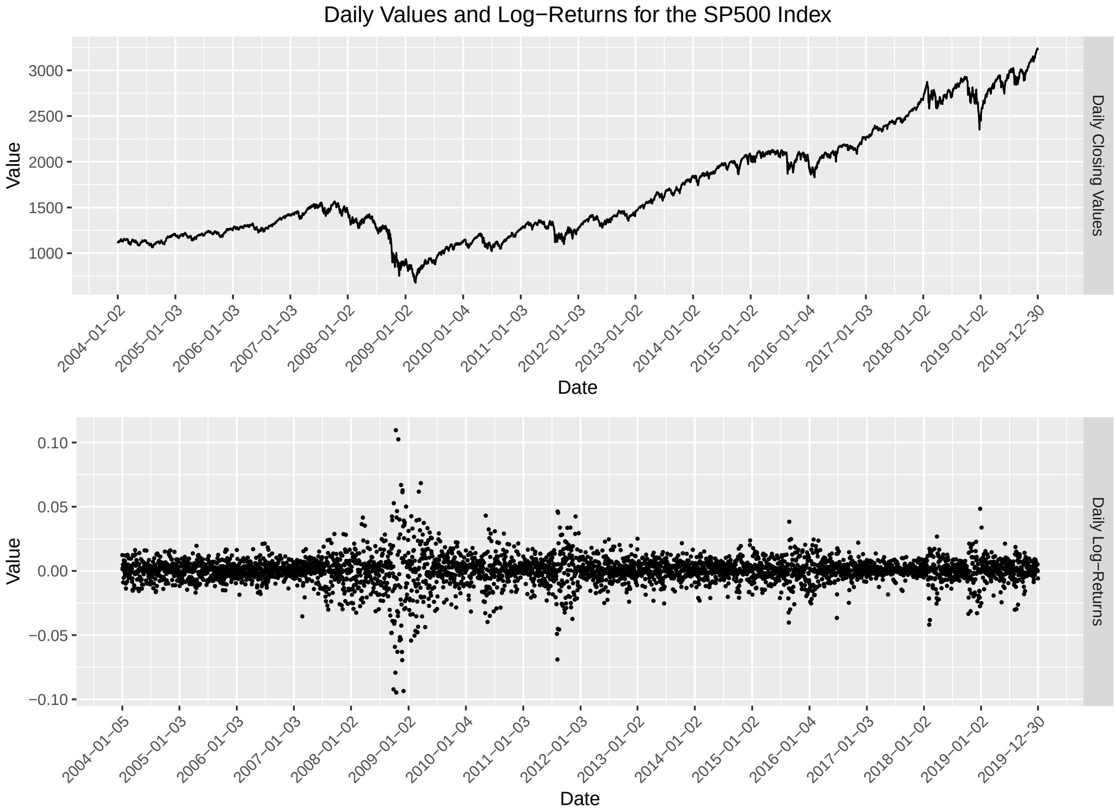

Figure 1: The top plot shows daily closing values for the SP500 index between 2004/01/02-2019/12/30. Corresponding log-returns are represented by the black points in the lower plot. Source: https://finance.yahoo.com/quote/%5EGSPC/.

1 Introduction

The financial market consists of a risky asset (a stock) and a risk-free asset (e.g., zero-coupon bond), both traded continuously up to some fixed time horizon T. As usual, the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{0 \le t \le T}, P\right)$ is introduced to capture the information flow: Ω contains all possible states of the market, $\mathcal{F}$ is the corresponding σ-algebra, P is the physical measure, and $(\mathcal{F}_t^W)$ is the σ-field generated by the Brownian motion $(W_t)_{0 \le t \le T}$, i.e.,

$$\mathcal{F}_t^W = \sigma \left\{ W_s \mid 0 \le s \le t \right\}. \tag{1}$$

In the remainder of this project, S_t denotes the value of the risky asset at time t, and the process is assumed to solve the stochastic differential equation (SDE)

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t \tag{2}$$

$$S_0 = s > 0 \tag{3}$$

where $\mu(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are deterministic functions possibly depending on both time (t) and the state of the underlying (S_t). Moreover, $\sigma(t,s) > 0$ for all $(t,s) \in [0,T] \times \mathbb{R}^+$, and both $\mu(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are sufficiently well-behaved such that a solution to the above SDE exists. We will use following terminology: $\mu(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are called the drift and volatility, respectively, of (S_t), whereas dS_t is referred to as the dynamics of the process.

The risk-free asset's value through time is given by the process $(B_t)_{0\leq t\leq T}$ with dynamics

$$dB_t = rB_t dt \tag{4}$$
$$B_0 = 1 \tag{5}$$

where r is constant. The risk-free asset is interpreted as the money account with short rate of interest r.

Questions: Model Assumptions

The economic interpretation of the above setting is crucial for working with continuous-time models in practice. Answering the questions below will provide some intuition behind the model assumptions. I would recommend you to read chapters 2 and 4 in Björk (2009).

1.1

Discuss the term *risk*. What is the difference between a risk-free and a risky asset? What would a locally risk-free asset be? How many sources of risk appear in this model setting? Is $\mathcal{F}_t^W = \mathcal{F}_t^S$? What is the financial interpretation of the $\mathcal{F}_t^S$.

1.2

Explain the economic interpretation of $D_t := B_t/B_T$. Is $S_0 > 0$ a strong assumption? Illustrate B_t, D_t, S_t and $D_t S_T$. You can use the following pseudo-code for generating (S_t) (for fun: adjust n as well as the *seed*):

```
# Parameters
S <-              # Starting value of process
mu <- 0.05        # drift
sigma <- 0.2      # volatility

# Time grid
T <- 1            # End time
n <-              # Number of evaluations
dt <- T/(n)       # Equidistant time step

S_vec <- numeric(n)

for (i in 1:n){

  Z <-            # Generate value from normal distribution with mean=0 and var=1
```

```
  dS <-          # Compute S_i - S_{i-1}

  S <- S + dS # Update S

  S_vec[i] <- S
}
```

1.3

Is it reasonable to assume that stock prices are represented by continuous processes? Discuss the (dis)advantage of using continuous-time models for representing stock prices (e.g., discuss the trade off between numerical tractability and modeling reality/financial markets).

2 Geometric Brownian Motion

Assume that

$$S_t = S_0 e^{\left(\mu - \sigma^2/2\right)t + \sigma W_t} \tag{6}$$

where $S_0 = s > 0$ is the (time-0) starting value of the process. The process in equation (6) is called a Geometric Brownian Motion (GBM) and is closely linked to equations (2)-(3), although we can't prove that until we get Ito's formula in our toolbox (next topic). The Geometric Brownian Motion is defined and discussed by Björk (2009) in chapter 5.

Questions: Geometric Brownian Motion

2.1

Is (S_t) continuous? Illustrate (S_t) using pseudo-code from the box. Also compare this simulation technique with Euler's method and build-in function from sde-package respectively.

```
for (i in 1:n){
  Z <- # Generate value from normal dist. with mean=0 and var=1
  X <- # log(S_i/S_{i-1})
  S <- # Update S
  S_vec[i] <- S
}

# Or without the for-loop....
Z <- # Generate n values from multi dim. normal dist.
X <- cumsum(...) # Argument should be a n-dim vector
S <- # Generate GBM
```

2.2

Compute $E(S_t)$ and $V(S_t)$. What is the distribution of S_t? Also compute $E(S_t \,|\, \mathcal{F}_s^S)$ for $s < t$ (hint: realize that $S_t = S_s \times S_t/S_s$). Is (S_t) a martingale under P?

2.3

Let $R_t := (S_t - S_{t-1})/S_t$ denote the *return* over one period. Determine the distribution of the associated log-return $r_t := \log(S_t/S_{t-1})$. How is R_t linked with r_t? Explain the convenience of using log-returns when working with financial time-series.

2.4

Would you prefer holding the risk-free asset or the risky-asset?

2.5

Produce figure 1. Are data a realization of a Geometric Brownian Motion (i.e., discuss whether model assumptions for GBM are violated)?

2.6

Derive the maximum likelihood estimator of μ and σ. Estimate μ and σ for the sample used to generate figure 1. Finally, discuss a significant shortfall for the maximum likelihood estimator of the drift μ (e.g., you may support your arguments with data).

3 Itǒ Calculus

Itǒ calculus and in particular Itǒ's formula play a crucial role in mathematical finance. The following results are from chapter 4 in Björk (2009) (p. 49-62, read it!).

Theorem 3.1 (Itǒ's formula) *Assume that the process $X = (X_t)_{t\geq 0}$ has dynamics given by*

$$dX_t = \mu_t dt + \sigma_t dW_t, \tag{7}$$

where μ_t and σ_t are adapted processes, and let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be $C^{1,2}$-function. Introduce a new process by $Z_t = f(t, X_t)$. Then $Z = (Z_t)_{t\geq 0}$ has dynamics given by

$$dZ_t = df(t, X_t) = \left(f_t(t, X_t) + \mu_t f_x(t, X_t) + \frac{1}{2}\sigma_t^2 f_{xx}(t, X_t) \right) dt + \sigma_t f_x(t, X_t) dW_t, \tag{8}$$

where f_t, f_x and f_{xx} denote the derivative with respect to time, the derivative with respective to space and the second-derivative with respective to space, respectively.

Another formulation of Ito's formula is stated in the following proposition — you may find this version useful!

Proposition 3.2 *With the assumptions as in 3.1, dZ_t is given by*

$$dZ_t = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, X_t)(dX_t)^2, \tag{9}$$

where

$$(dt)^2 = 0, \qquad dt \cdot dW_t = 0 \qquad \textit{and} \qquad (dW_t)^2 = dt.$$

Finally, the multidimensional version of Ito's formula is given in definition 4.16 in Björk (2009).

Questions: Ito Calculus

Itŏ

3.1

State a condition such that the process $(X_t)_{t\geq 0}$ with dynamics given by equation (7) becomes a P-martingale. What do we call X when $\mu_t \geq 0$ P-almost surely? Finally, derive/state the condition such that $Z_t = f(t, X_t)$ becomes a P-martingale.

3.2

Compute the dynamics of $(S_t)_{t\geq 0}$ where S_t is given by equation (6). Moreover, apply Ito's formula to compute $E(S_t)$.

From now on, we will specify $(S_t)_{t\geq 0}$ by its stochastic differential and not equation (6).

3.3

Compute the dynamics of $(Z_t)_{t\geq 0}$ where $Z_t = S_t/B_t$. Hint: Use the multidimensional Ito formula from Björk (2009). Is the process a martingale?

4 Black-Scholes Model and Q-dynamics

The Black-Scholes model consists of two assets with dynamics given by

$$dB_t = rB_t dt, \tag{10}$$
$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{11}$$

where $B_0 = 1$ and $S_0 = s > 0$, and r, μ and $\sigma > 0$ are constants. This famous model was (and to some degree still is) used for pricing T-claims. Following definition 7.1 in Björk (2009), the stochastic variable $\mathcal{X}$ is called a T-claim if it's $\mathcal{F}_T^S$-measurable. Here T is the date of maturity (or exercise date). A simple T-claim is on the form $\mathcal{X} = \Phi(S_T)$, where $\Phi(\cdot)$ is the contract function.

The focal point is to price $\mathcal{X}$ at any point in time $t < T$, i.e. determine the pricing function $\pi(t;\mathcal{X})$ for $t < T$ with terminal condition $\pi(T;\mathcal{X}) = \mathcal{X}$. The Black-Scholes model is free of arbitrage as well as complete (we will discuss both concepts next time), and we will also assume that

$$\pi(t;\mathcal{X}) = F(t, S_t), \tag{12}$$

where F is a $C^{1,2}$-function.

Pricing in this type of models is conducted under a *risk-neutral* probability measure (also called equivalent martingale measure (EMM), see definition) Q. In the Black-Scholes model, the pricing measure Q is unique — this follows directly from no-arbitrage and completeness of the model. Moreover, the discounted process (S_t/B_t) is a martingale under this measure. Theorem 10.5 and theorem 10.9 yield in Björk (2009) yield the existence of Q, whereas theorem 10.17 implies uniqueness of the pricing measure. Definition 10.11 and theorem 10.14 tell us that (S_t/B_t) is a martingale under this pricing measure.

In this section, we will show that the process the discounted process (S_t/B_t) has Q-dynamics given by

$$dS_t = rS_t dt + \sigma S_t dW_t^Q, \tag{13}$$

where W_t^Q is a Brownian motion under Q (not P!).

Questions: Black-Scholes Model

4.1

According to theorem 11.3 (Girsanov) in Björk (2009), we can define a new probability Q on $\mathcal{F}_T$ implying that

$$dW_t = \phi_t dt + dW_t^Q,$$

where W_t^Q is Q-Brownian motion. The process ϕ_t is called the Girsanov kernel. Compute the Q-dynamics of (S_t/B_t). Hint: Replace W_t with $\phi_t dt + dW_t^Q$ and apply Itǒ's formula.

4.2

Choose ϕ_t implying that the discounted process is a Q-martingale.

4.3

Verify that are the solution satisfies the assumptions in Girsavno's theorem. Conclude that Q (generated from ϕ_t) is the risk-neutral pricing measure.

4.4

Compute the Q-dynamics of (S_t).

4.5 (Black-Scholes Equation)

Compute the Q-dynamics of $F(t, S_t)/B_t$ where $F(t, S_t)$ is the pricing funcion of the T-claim $\mathcal{X}$. Derive the condition (PDE) such that $F(t, S_t)/B_t$ becomes a Q-martingale. Finally, state the time-T boundary equation for the PDE.

5 Monte Carlo Methods

Monte Carlo simulation is a useful tool for estimating the expectation of a random variable. For instance, consider the random variable X with mean μ and variance σ^2. The objective — in this simple setting — is to estimate the mean

$$\mu = EX = \int xf(x)fx.$$

Suppose now we draw/simulate n points $X_1, ..., X_n$ independently from this distribution. Since $X_1, ..., X_n$ is a sequence IID random variables, it immediately follows from the (strong) law of large numbers that

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i \overset{a.s.}{\to} EX = \mu$$

for $n \to \infty$. We say that $\hat{\mu}$ is the Monte Carlo estimator of μ.

Questions: Monte Carlo Methods

5.1

Let $\hat{\mu}_n - \mu$ denote the error. Compute the expectation and standard deviation of the error. Interpret your findings for large n. Determine the limit distribution of $\sqrt{n}(\hat{\mu}_n - \mu)$ for $n \to \infty$.

5.2

Consider the Geometric Brownian Motion (S_t) on $[0, T]$ with drift μ and volatility σ. Furthermore, for simulation of (S_t), let $0 = t_0 < ... < t_m = T$ be a equidistant time-grid with $t_k - t_{k-1} = 1/m$ for $k = 1, ..., m$. Using techniques from the previous weeks, we are now able to simulate many (n) paths of (S_t). To avoid notational clutter let $\mathbb{S}_i = (S_{0,i}, ..., S_{m,i})$ denote the i'th simulated sample path.

Estimate the drift and volatility using Monte Carlo simulation:

1. Simulate n independent paths of (S_t).
2. For each sample path compute the maximum likelihood estimator of (μ, σ) (see slides from week 2).
3. Finally, compute the Monte Carlo estimator of μ and σ respectively (the average of the n maximum likelihood estimates).

As a sanity check you should compare your findings/estimates with the true values.

5.3

Discuss the difference between sampling error (arising from the Monte Carlo simulation) and discretization error (discreteness of time).

5.4

Explain how we can use Monte Carlo techniques to compute the arbitrage-free price of the T-claim $\mathcal{X} = \Phi(S_T)$.

5.5

Consider the cash-or-nothing claim from week 3 (this time with $c = K$)

$$\mathcal{X} = K \cdot 1_{S_T > K}.$$

Is this a simple T-claim? Determine the contract function $\Phi(x)$ and compute the risk-neutral price at time 0, $\pi(0; \mathcal{X})$, using Monte Carlo simulation:

1. Simulate n independent paths of (S_t).
2. For each sample path compute the pay-off $K \cdot 1_{S_T > K}$.
3. Finally, compute the Monte Carlo estimator of arbitrage-free price respectively (remember discounting).

6 Pricing Options in the Black-Scholes Model

We are finally ready to price T-claims in the Black-Scholes universe. Fundamentally, there are three approaches to value $\mathcal{X}$ at time $t < T$: numerically, analytically or with Monte Carlo techniques, respectively.

6.1

State an equation (depending on time and space) that $F(t, S_t)$ from equation (12) must satisfy.

6.2

Use proposition 5.6 from Björk (2009) to express $F(t, S_t)$ as a conditional expectation. Interpret/discuss the expression for the arbitrage-free price of $\mathcal{X}$ at time $t < T$.

6.3

Determine the contract function for the European call with maturity T and strike K. Do the same for the European put option (also with maturity T and strike K. Draw both payoffs as a function of the terminal stock value S_T.

6.4

Demonstrate (hint: illustrate) how we can use the underlying stock, the money account and the put option to hedge the European call at maturity T. Use your hedge to state the put-call parity in proposition 9.2 in Björk (2009) (hint: use proposition 9.1 from Björk (2009)).

6.5

Use Monte Carlo simulation to compute the price of the European call. Compare with the closed-form solution in Björk (2009).

7 Greeks, Δ-hedging and Implied Volatility

Let $C(t, s)$ and $P(t, s)$ denote the pricing function of European call and European put, respectively (with maturity T and strike K). Recall that the two functions also depend on r and σ.

Questions: Greeks

Greeks are — in finance — the quantities representing the sensitivity of the price of derivatives such as options to a change in underlying parameters on which the value of an instrument or portfolio of financial instruments is dependent. In particular, Greeks are used to measure and hedge (eliminate) risk when trading options. See chapter 9 in Björk (2009).

7.1

State the Delta, Gamma, Vega, Theta and Rho for a European call with maturity T and strike K (see chapter 9 in Björk (2009). Provide an intuition/interpretation for each of the five expressions (hint: look at the signs).

7.2

Compute similar Greeks for the put option (hint: use the put call parity). Again, interpret your findings.

Questions: Δ-Hedging

Let $h(t) = (h_t^0, h_t^1)$ be a self-financing portfolio with value process $(V_t^h)_{0 \leq t \leq T}$. In this setting h_t^0 denotes the amount held in the money account whereas h_t^1 denotes number of assets at time t, i.e.

$$V_t^h = h_t^0 B_t + h_t^1 S_t \tag{14}$$

for $t \in [0, T]$. See chapter 6, 8 and 9 in Björk (2009).

7.3

Compute the dynamics of V_t^h under P and Q, respectively.

7.4

Compute the dynamics of the discounted value process, V_t^h/B_t under P and Q, respectively. Discuss the results.

7.5

Provide a numerical example (simulation study) for the Δ-hedging strategy (hint: see the note on Δ-hedging).

Questions: Implied Volatility

Market participants buying and selling European options observe option prices (C), strike levels (K), (time-to-)maturities $(T - t)$ and risk-free rates (r). However, the underlying *true* volatility — which is assumed constant in the BS model — is not observable. See p. 110 in Björk (2009).

7.6

Explain how we can use observed quantities to compute the *market implied volatility* (hint: use the pricing function for the European call option).

7.7

Assume that we at time $t < T$ observe prices $\hat{C}_1$, $\hat{C}_2$ and $\hat{C}_3$ for three European options written on the same underlying (S_t) but with different strike prices K_1, K_2 and K_3, respectively. In the Black-Scholes model, what should be expect for the three corresponding implied volatilities. The relationship between implied volatility and the strike price is called the *volatility smile*.

7.8

Assume that we at times t_1, t_2 and t_3 observe prices $\hat{C}_1$, $\hat{C}_2$ and $\hat{C}_3$ for a European options written on the underlying (S_t) with strike price K. In the Black-Scholes model, what should be expect for the three corresponding implied volatilities. The relationship between implied volatility and time-to-maturity is called the *term structure of volatility*.

7.9

Finally, assume that the true model volatility σ is equal to 0.15, while the implied volatility observed at time t, σ_t^i, is equal to 0.2. Assuming the Black-Scholes model, is the corresponding option priced correctly?

8 The Fundamental Theorem of Derivative Trading

Let $h_t = (h_t^0, h_t^1)$ be a self-financing portfolio with value process (V_t^h). Moreover, let $\mathcal{X}$ denote the European call with strike K and maturity T.

8.1

Assume that the trader sells one European option at time 0 for the market price associated with the implied volatility σ_0^i. Suppose now that the same trader wants to Δ-hedge his position using the money account and the underlying stock S_t. Show/prove that the profit-and-loss of holding this portfolio over $[0, T]$ is given by

$$P\&L_T = C(0, S_0; \sigma_0^i) - C(0, S_0; \sigma_0^h) + \int_0^T e^{-rt}\frac{1}{2}((\sigma_t^h)^2 - \sigma_t^2)S_t^2\Gamma(t, S_t; \sigma_t^h)dt, \qquad (15)$$

where σ_t^h is the hedging-volatility chosen by the trader. Also show the 'opposite' result where the buys one option and wants to Δ-hedge his position.

8.2

Interpret expression (15). For instance, how are the volatilities related to (volatility) arbitrage? When is the expression deterministic? The impact from the option's Gamma? Also propose a trading strategy depending on the relationship between initial implied volatility σ_0^i and initial true volatility σ_0 (this is related to volatility arbitrage).

8.3

Conduct Wilmott's hedge experiment.

B) Stochastic Volatility

References

Björk, T. (2009). *Arbitrage Theory in Continuous Time* (3 ed.). Oxford University Press.