CHAPTER 7

POISSON'S AND LAPLACE'S EQUATIONS

A study of the previous chapter shows that several of the analogies used to obtain experimental field maps involved demonstrating that the analogous quantity satisfies Laplace's equation. This is true for small deflections of an elastic membrane, and we might have proved the current analogy by showing that the direct-current density in a conducting medium also satisfies Laplace's equation. It appears that this is a fundamental equation in more than one field of science, and, perhaps without knowing it, we have spent the last chapter obtaining solutions for Laplace's equation by experimental, graphical, and numerical methods. Now we are ready to obtain this equation formally and discuss several methods by which it may be solved analytically.

It may seem that this material properly belongs before that of the previous chapter; as long as we are solving one equation by so many methods, would it not be fitting to see the equation first? The disadvantage of this more logical order lies in the fact that solving Laplace's equation is an exercise in mathematics, and unless we have the physical problem well in mind, we may easily miss the physical significance of what we are doing. A rough curvilinear map can tell us much about a field and then may be used later to check our mathematical solutions for gross errors or to indicate certain peculiar regions in the field which require special treatment.

With this explanation let us finally obtain the equations of Laplace and Poisson.

7.1 POISSON'S AND LAPLACE'S EQUATIONS

Obtaining Poisson's equation is exceedingly simple, for from the point form of Gauss's law,

$$\nabla \cdot \mathbf{D} = \rho_v \tag{1}$$

the definition of **D**,

$$\mathbf{D} = \epsilon \mathbf{E} \tag{2}$$

and the gradient relationship,

$$\mathbf{E} = -\nabla V \tag{3}$$

by substitution we have

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = -\nabla \cdot (\epsilon \nabla V) = \rho_v$$

or

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\epsilon} \tag{4}$$

for a homogeneous region in which ϵ is constant.

Equation (4) is *Poisson's equation*, but the "double ∇" operation must be interpreted and expanded, at least in cartesian coordinates, before the equation can be useful. In cartesian coordinates,

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

and therefore

$$\begin{aligned}\nabla \cdot \nabla V &= \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right) \\ &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \end{aligned} \tag{5}$$

Usually the operation $\nabla \cdot \nabla$ is abbreviated ∇^2 (and pronounced "del squared"), a good reminder of the second-order partial derivatives appearing in (5), and we have

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\epsilon}} \tag{6}$$

in cartesian coordinates.

If $\rho_v = 0$, indicating zero *volume* charge density, but allowing point charges, line charge, and surface charge density to exist at singular locations as sources of the field, then

$$\boxed{\nabla^2 V = 0} \tag{7}$$

which is *Laplace's equation*. The ∇^2 operation is called the *Laplacian of V*.

In cartesian coordinates Laplace's equation is

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{(cartesian)}} \tag{8}$$

and the form of $\nabla^2 V$ in cylindrical and spherical coordinates may be obtained by using the expressions for the divergence and gradient already obtained in those coordinate systems. For reference, the Laplacian in cylindrical coordinates is

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \qquad \text{(cylindrical)}} \tag{9}$$

and in spherical coordinates is

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \qquad \text{(spherical)}} \tag{10}$$

These equations may be expanded by taking the indicated partial derivatives, but it is usually more helpful to have them in the forms given above; furthermore, it is much easier to expand them later if necessary than it is to put the broken pieces back together again.

Laplace's equation is all-embracing, for, applying as it does wherever volume charge density is zero, it states that every conceivable configuration of electrodes or conductors produces a field for which $\nabla^2 V = 0$. All these fields are different, with different potential values and different spatial rates of change, yet for each of them $\nabla^2 V = 0$. Since *every* field (if $\rho_v = 0$) satisfies Laplace's equation, how can we expect to reverse the procedure and use Laplace's equation to find one specific field in which we happen to have an interest? Obviously, more information is required, and we shall find that we must solve Laplace's equation subject to certain *boundary conditions*.

Every physical problem must contain at least one conducting boundary and usually contains two or more. The potentials on these boundaries are assigned values, perhaps V_0, $V_1, \ldots$, or perhaps numerical values. These definite equipotential surfaces will provide the boundary conditions for the type of problem to be solved in this chapter. In other types of problems, the boundary conditions take the form of specified values of E on an enclosing surface, or a mixture of known values of V and E.

Before using Laplace's equation or Poisson's equation in several examples, we must pause to show that if our answer satisfies Laplace's equation and also satisfies the boundary conditions, then it is the only possible answer. It would be very distressing to work a problem by solving Laplace's equation with two different approved methods and then to obtain two different answers. We shall show that the two answers must be identical.

✔ **D7.1.** Calculate numerical values for V and ρ_v at point P in free space if: (*a*) $V = \dfrac{4yz}{x^2+1}$, at $P(1, 2, 3)$; (*b*) $V = 5\rho^2 \cos 2\phi$, at $P(\rho = 3, \phi = \dfrac{\pi}{3}, z = 2)$; (*c*) $V = \dfrac{2\cos\phi}{r^2}$, at $P(r = 0.5, \theta = 45^\circ, \phi = 60^\circ)$.

Ans. 12 V, −106.2 pC/m^3; 22.5 V, 0; 4 V, −141.7 pC/m^3

7.2 UNIQUENESS THEOREM

Let us assume that we have two solutions of Laplace's equation, V_1 and V_2, both general functions of the coordinates used. Therefore

$$\nabla^2 V_1 = 0$$

and

$$\nabla^2 V_2 = 0$$

from which

$$\nabla^2 (V_1 - V_2) = 0$$

Each solution must also satisfy the boundary conditions, and if we represent the given potential values on the boundaries by V_b, then the value of V_1 on the boundary V_{1b} and the value of V_2 on the boundary V_{2b} must both be identical to V_b,

$$V_{1b} = V_{2b} = V_b$$

or

$$V_{1b} - V_{2b} = 0$$

In Sec. 4.8, Eq. (44), we made use of a vector identity,

$$\nabla \cdot (V\mathbf{D}) \equiv V(\nabla \cdot \mathbf{D}) + \mathbf{D} \cdot (\nabla V)$$

which holds for any scalar V and any vector $\mathbf{D}$. For the present application we shall select $V_1 - V_2$ as the scalar and $\nabla(V_1 - V_2)$ as the vector, giving

$$\nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)] \equiv (V_1 - V_2)[\nabla \cdot \nabla(V_1 - V_2)] + \nabla(V_1 - V_2) \cdot \nabla(V_1 - V_2)$$

which we shall integrate throughout the volume *enclosed* by the boundary surfaces specified:

$$\int_{\text{vol}} \nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)]\, dv \equiv \int_{\text{vol}} (V_1 - V_2)[\nabla \cdot \nabla(V_1 - V_2)]dv + \int_{\text{vol}} [\nabla(V_1 - V_2)]^2\, dv \tag{11}$$

The divergence theorem allows us to replace the volume integral on the left side of the equation by the closed surface integral over the surface surrounding the volume. This surface consists of the boundaries already specified on which $V_{1b} = V_{2b}$, and therefore

$$\int_{\text{vol}} \nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)]\, dv = \oint_S [(V_{1b} - V_{2b})\nabla(V_{1b} - V_{2b})] \cdot d\mathbf{S} = 0$$

One of the factors of the first integral on the right side of (11) is $\nabla \cdot \nabla(V_1 - V_2)$, or $\nabla^2(V_1 - V_2)$, which is zero by hypothesis, and therefore that integral is zero. Hence the remaining volume integral must be zero:

$$\int_{\text{vol}} [\nabla(V_1 - V_2)]^2\, dv = 0$$

There are two reasons why an integral may be zero: either the integrand (the quantity under the integral sign) is everywhere zero, or the integrand is positive in some regions and negative in others, and the contributions cancel algebraically. In this case the first reason must hold because $[\nabla(V_1 - V_2)]^2$ cannot be negative. Therefore

$$[\nabla(V_1 - V_2)]^2 = 0$$

and

$$\nabla(V_1 - V_2) = 0$$

Finally, if the gradient of $V_1 - V_2$ is everywhere zero, then $V_1 - V_2$ cannot change with any coordinates and

$$V_1 - V_2 = \text{constant}$$

If we can show that this constant is zero, we shall have accomplished our proof. The constant is easily evaluated by considering a point on the boundary. Here

$V_1 - V_2 = V_{1b} - V_{2b} = 0$, and we see that the constant is indeed zero, and therefore

$$V_1 = V_2$$

giving two identical solutions.

The uniqueness theorem also applies to Poisson's equation, for if $\nabla^2 V_1 = -\rho_v/\epsilon$ and $\nabla^2 V_2 = -\rho_v/\epsilon$, then $\nabla^2(V_1 - V_2) = 0$ as before. Boundary conditions still require that $V_{1b} - V_{2b} = 0$, and the proof is identical from this point.

This constitutes the proof of the uniqueness theorem. Viewed as the answer to a question, "How do two solutions of Laplace's or Poisson's equation compare if they both satisfy the same boundary conditions?" the uniqueness theorem should please us by its ensurance that the answers are identical. Once we can find any method of solving Laplace's or Poisson's equation subject to given boundary conditions, we have solved our problem once and for all. No other method can ever give a different answer.

✔ **D7.2.** Consider the two potential fields $V_1 = y$ and $V_2 = y + e^x \sin y$. (*a*) Is $\nabla^2 V_1 = 0$? (*b*) Is $\nabla^2 V_2 = 0$? (*c*) Is $V_1 = 0$ at $y = 0$? (*d*) Is $V_2 = 0$ at $y = 0$? (*e*) Is $V_1 = \pi$ at $y = \pi$? (*f*) Is $V_2 = \pi$ at $y = \pi$? (*g*) Are V_1 and V_2 identical? (*h*) Why does the uniqueness theorem not apply?

Ans. Yes; yes; yes; yes; yes; yes; no; boundary conditions not given for a *closed* surface

7.3 EXAMPLES OF THE SOLUTION OF LAPLACE'S EQUATION

Several methods have been developed for solving the second-order partial differential equation known as Laplace's equation. The first and simplest method is that of direct integration, and we shall use this technique to work several examples in various coordinate systems in this section. In Sec. 7.5 one other method will be used on a more difficult problem. Additional methods, requiring a more advanced mathematical knowledge, are described in the references given at the end of the chapter.

The method of direct integration is applicable only to problems which are "one-dimensional," or in which the potential field is a function of only one of the three coordinates. Since we are working with only three coordinate systems, it might seem, then, that there are nine problems to be solved, but a little reflection will show that a field which varies only with x is fundamentally the same as a field which varies only with y. Rotating the physical problem a quarter turn is no change. Actually, there are only five problems to be solved, one in cartesian coordinates, two in cylindrical, and two in spherical. We shall enjoy life to the fullest by solving them all.

Example 7.1

Let us assume that V is a function only of x and worry later about which physical problem we are solving when we have a need for boundary conditions. Laplace's equation reduces to

$$\frac{\partial^2 V}{\partial x^2} = 0$$

and the partial derivative may be replaced by an ordinary derivative, since V is not a function of y or z,

$$\frac{d^2 V}{dx^2} = 0$$

We integrate twice, obtaining

$$\frac{dV}{dx} = A$$

and

$$V = Ax + B \tag{12}$$

where A and B are constants of integration. Equation (12) contains two such constants, as we should expect for a second-order differential equation. These constants can be determined only from the boundary conditions.

What boundary conditions should we supply? They are our choice, since no physical problem has yet been specified, with the exception of the original hypothesis that the potential varied only with x. We should now attempt to visualize such a field. Most of us probably already have the answer, but it may be obtained by exact methods.

Since the field varies only with x and is not a function of y and z, then V is a constant if x is a constant or, in other words, the equipotential surfaces are described by setting x constant. These surfaces are parallel planes normal to the x axis. The field is thus that of a parallel-plate capacitor, and as soon as we specify the potential on any two planes, we may evaluate our constants of integration.

To be very general, let $V = V_1$ at $x = x_1$ and $V = V_2$ at $x = x_2$. These values are then substituted into (12), giving

$$V_1 = Ax_1 + B \qquad V_2 = Ax_2 + B$$

$$A = \frac{V_1 - V_2}{x_1 - x_2} \qquad B = \frac{V_2 x_1 - V_1 x_2}{x_1 - x_2}$$

and

$$V = \frac{V_1(x - x_2) - V_2(x - x_1)}{x_1 - x_2}$$

A simpler answer would have been obtained by choosing simpler boundary conditions. If we had fixed $V = 0$ at $x = 0$ and $V = V_0$ at $x = d$, then

$$A = \frac{V_0}{d} \qquad B = 0$$

and

$$\boxed{V = \frac{V_0 x}{d}} \tag{13}$$

Suppose our primary aim is to find the capacitance of a parallel-plate capacitor. We have solved Laplace's equation, obtaining (12) with the two constants A and B. Should they be evaluated or left alone? Presumably we are not interested in the potential field itself, but only in the capacitance, and we may continue successfully with A and B or we may simplify the algebra by a little foresight. Capacitance is given by the ratio of charge to potential difference, so we may choose now the potential *difference* as V_0, which is equivalent to one boundary condition, and then choose whatever second boundary condition seems to help the form of the equation the most. This is the essence of the second set of boundary conditions which produced (13). The potential difference was fixed as V_0 by choosing the potential of one plate zero and the other V_0; the location of these plates was made as simple as possible by letting $V = 0$ at $x = 0$.

Using (13), then, we still need the total charge on either plate before the capacitance can be found. We should remember that when we first solved this capacitor problem in Chap. 5, the sheet of charge provided our starting point. We did not have to work very hard to find the charge, for all the fields were expressed in terms of it. The work then was spent in finding potential difference. Now the problem is reversed (and simplified).

The necessary steps are these, after the choice of boundary conditions has been made:

1. Given V, use $\mathbf{E} = -\nabla V$ to find $\mathbf{E}$.
2. Use $\mathbf{D} = \epsilon\mathbf{E}$ to find $\mathbf{D}$.
3. Evaluate $\mathbf{D}$ at either capacitor plate, $\mathbf{D} = \mathbf{D}_S = D_N\mathbf{a}_N$.
4. Recognize that $\rho_S = D_N$.
5. Find Q by a surface integration over the capacitor plate, $Q = \int_S \rho_S dS$.

Here we have

$$V = V_0 \frac{x}{d}$$

$$\mathbf{E} = -\frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{D} = -\epsilon\frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{D}_S = \mathbf{D}\Big|_{x=0} = -\epsilon \frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{a}_N = \mathbf{a}_x$$

$$D_N = -\epsilon \frac{V_0}{d} = \rho_S$$

$$Q = \int_S \frac{-\epsilon V_0}{d} dS = -\epsilon \frac{V_0 S}{d}$$

and the capacitance is

$$\boxed{C = \frac{|Q|}{V_0} = \frac{\epsilon S}{d}} \tag{14}$$

We shall use this procedure several times in the examples to follow.

Example 7.2

Since no new problems are solved by choosing fields which vary only with y or with z in cartesian coordinates, we pass on to cylindrical coordinates for our next example. Variations with respect to z are again nothing new, and we next assume variation with respect to ρ only. Laplace's equation becomes

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

or

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$$

Noting the ρ in the denominator, we exclude $\rho = 0$ from our solution and then multiply by ρ and integrate,

$$\rho \frac{dV}{d\rho} = A$$

rearrange, and integrate again,

$$V = A \ln \rho + B \tag{15}$$

The equipotential surfaces are given by $\rho =$ constant and are cylinders, and the problem is that of the coaxial capacitor or coaxial transmission line. We choose a potential difference of V_0 by letting $V = V_0$ at $\rho = a$, $V = 0$ at $\rho = b$, $b > a$, and obtain

$$\boxed{V = V_0 \frac{\ln(b/\rho)}{\ln(b/a)}} \tag{16}$$

from which

$$\mathbf{E} = \frac{V_0}{\rho} \frac{1}{\ln(b/a)} \mathbf{a}_\rho$$

$$D_{N(\rho=a)} = \frac{\epsilon V_0}{a \ln(b/a)}$$

$$Q = \frac{\epsilon V_0 2\pi a L}{a \ln(b/a)}$$

$$\boxed{C = \frac{2\pi\epsilon L}{\ln(b/a)}} \tag{17}$$

which agrees with our results in Chap. 5.

Example 7.3

Now let us assume that V is a function only of ϕ in cylindrical coordinates. We might look at the physical problem first for a change and see that equipotential surfaces are given by $\phi =$ constant. These are radial planes. Boundary conditions might be $V = 0$ at $\phi = 0$ and $V = V_0$ at $\phi = \alpha$, leading to the physical problem detailed in Fig. 7.1.

Laplace's equation is now

$$\frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} = 0$$

We exclude $\rho = 0$ and have

$$\frac{d^2 V}{d\phi^2} = 0$$

The solution is

$$V = A\phi + B$$

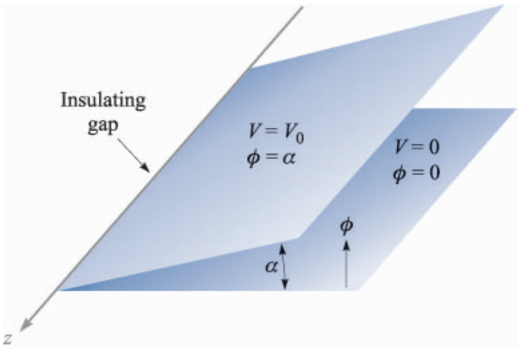

FIGURE 7.1
Two infinite radial planes with an interior angle α. An infinitesimal insulating gap exists at $\rho = 0$. The potential field may be found by applying Laplace's equation in cylindrical coordinates.

The boundary conditions determine A and B, and

$$\boxed{V = V_0 \frac{\phi}{\alpha}} \tag{18}$$

Taking the gradient of (18) produces the electric field intensity,

$$\boxed{\mathbf{E} = -\frac{V_0 \mathbf{a}_\phi}{\alpha\rho}} \tag{19}$$

and it is interesting to note that E is a function of ρ and not of ϕ. This does not contradict our original assumptions, which were restrictions only on the potential field. Note, however, that the *vector* field $\mathbf{E}$ is a function of ϕ.

A problem involving the capacitance of these two radial planes is included at the end of the chapter.

Example 7.4

We now turn to spherical coordinates, dispose immediately of variations with respect to ϕ only as having just been solved, and treat first $V = V(r)$.

The details are left for a problem later, but the final potential field is given by

$$\boxed{V = V_0 \frac{\dfrac{1}{r} - \dfrac{1}{b}}{\dfrac{1}{a} - \dfrac{1}{b}}} \tag{20}$$

where the boundary conditions are evidently $V = 0$ at $r = b$ and $V = V_0$ at $r = a$, $b > a$. The problem is that of concentric spheres. The capacitance was found previously in Sec. 5.10 (by a somewhat different method) and is

$$\boxed{C = \frac{4\pi\epsilon}{\dfrac{1}{a} - \dfrac{1}{b}}} \tag{21}$$

Example 7.5

In spherical coordinates we now restrict the potential function to $V = V(\theta)$, obtaining

$$\frac{1}{r^2 \sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right) = 0$$

We exclude $r = 0$ and $\theta = 0$ or π and have

$$\sin\theta \frac{dV}{d\theta} = A$$

The second integral is then

$$V = \int \frac{A\,d\theta}{\sin\theta} + B$$

which is not as obvious as the previous ones. From integral tables (or a good memory) we have

$$V = A \ln\left(\tan\frac{\theta}{2}\right) + B$$

The equipotential surfaces are cones. Fig. 7.2 illustrates the case where $V = 0$ at $\theta = \pi/2$ and $V = V_0$ at $\theta = \alpha$, $\alpha < \pi/2$. We obtain

$$\boxed{V = V_0 \frac{\ln\left(\tan\frac{\theta}{2}\right)}{\ln\left(\tan\frac{\alpha}{2}\right)}} \tag{22}$$

In order to find the capacitance between a conducting cone with its vertex separated from a conducting plane by an infinitesimal insulating gap and its axis normal to the plane, let us first find the field strength:

$$\mathbf{E} = -\nabla V = \frac{1}{r}\frac{\partial V}{\partial\theta}\mathbf{a}_\theta = -\frac{V_0}{r\sin\theta\ln\left(\tan\frac{\alpha}{2}\right)}\mathbf{a}_\theta$$

The surface charge density on the cone is then

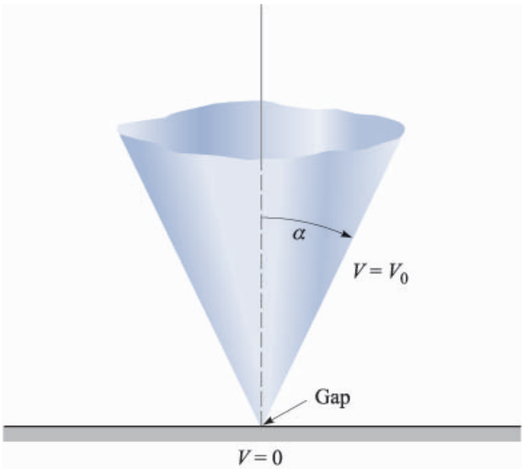

FIGURE 7.2
For the cone $\theta = \alpha$ at V_0 and the plane $\theta = \pi/2$ at $V = 0$, the potential field is given by $V = V_0[\ln(\tan\theta/2)]/[\ln(\tan\alpha/2)]$.

$$\rho_S = \frac{-\epsilon V_0}{r \sin\alpha \ln\left(\tan\frac{\alpha}{2}\right)}$$

producing a total charge Q,

$$Q = \frac{-\epsilon V_0}{\sin\alpha \ln\left(\tan\frac{\alpha}{2}\right)} \int_0^\infty \int_0^{2\pi} \frac{r \sin\alpha \, d\phi \, dr}{r}$$

$$= \frac{-2\pi\epsilon_0 V_0}{\ln\left(\tan\frac{\alpha}{2}\right)} \int_0^\infty dr$$

This leads to an infinite value of charge and capacitance, and it becomes necessary to consider a cone of finite size. Our answer will now be only an approximation, because the theoretical equipotential surface is $\theta = \alpha$, a conical surface extending from $r = 0$ to $r = \infty$, whereas our physical conical surface extends only from $r = 0$ to, say, $r = r_1$. The approximate capacitance is

$$\boxed{C \doteq \frac{2\pi\epsilon r_1}{\ln\left(\cot\frac{\alpha}{2}\right)}} \qquad (23)$$

If we desire a more accurate answer, we may make an estimate of the capacitance of the base of the cone to the zero-potential plane and add this amount to our answer above. Fringing, or nonuniform, fields in this region have been neglected and introduce an additional source of error.

✔ **D7.3.** Find $|\mathbf{E}|$ at $P(3, 1, 2)$ for the field of: (*a*) two coaxial conducting cylinders, $V = 50\,\text{V}$ at $\rho = 2\,\text{m}$, and $V = 20\,\text{V}$ at $\rho = 3\,\text{m}$; (*b*) two radial conducting planes, $V = 50\,\text{V}$ at $\phi = 10^\circ$, and $V = 20\,\text{V}$ at $\phi = 30^\circ$.

Ans. 23.4 V/m; 27.2 V/m

7.4 EXAMPLE OF THE SOLUTION OF POISSON'S EQUATION

To select a reasonably simple problem which might illustrate the application of Poisson's equation, we must assume that the volume charge density is specified. This is not usually the case, however; in fact, it is often the quantity about which we are seeking further information. The type of problem which we might encounter later would begin with a knowledge only of the boundary values of the potential, the electric field intensity, and the current density. From these we would have to apply Poisson's equation, the continuity equation, and some relationship expressing the forces on the charged particles, such as the Lorentz force equation or the diffusion equation, and solve the whole system of equations simultaneously. Such an ordeal is beyond the scope of this text, and we shall therefore assume a reasonably large amount of information.

As an example, let us select a *pn* junction between two halves of a semiconductor bar extending in the x direction. We shall assume that the region for $x < 0$ is doped p type and that the region for $x > 0$ is n type. The degree of doping is identical on each side of the junction. To review qualitatively some of the facts about the semiconductor junction, we note that initially there are excess holes to the left of the junction and excess electrons to the right. Each diffuses across the junction until an electric field is built up in such a direction that the diffusion current drops to zero. Thus, to prevent more holes from moving to the right, the electric field in the neighborhood of the junction must be directed to the left; E_x is negative there. This field must be produced by a net positive charge to the right of the junction and a net negative charge to the left. Note that the layer of positive charge consists of two parts—the holes which have crossed the junction and the positive donor ions from which the electrons have departed. The negative layer of charge is constituted in the opposite manner by electrons and negative acceptor ions.

The type of charge distribution which results is shown in Fig. 7.3*a*, and the negative field which it produces is shown in Fig. 7.3*b*. After looking at these two figures, one might profitably read the previous paragraph again.

A charge distribution of this form may be approximated by many different expressions. One of the simpler expressions is

$$\rho_v = 2\rho_{v0} \operatorname{sech} \frac{x}{a} \tanh \frac{x}{a} \tag{24}$$

which has a maximum charge density $\rho_{v,max} = \rho_{v0}$ that occurs at $x = 0.881a$. The maximum charge density ρ_{v0} is related to the acceptor and donor concentrations N_a and N_d by noting that all the donor and acceptor ions in this region (the *depletion* layer) have been stripped of an electron or a hole, and thus

$$\rho_{v0} = eN_a = eN_d$$

Let us now solve Poisson's equation,

$$\nabla^2 V = -\frac{\rho_v}{\epsilon}$$

subject to the charge distribution assumed above,

$$\frac{d^2V}{dx^2} = -\frac{2\rho_{v0}}{\epsilon} \operatorname{sech}\frac{x}{a} \tanh\frac{x}{a}$$

in this one-dimensional problem in which variations with y and z are not present. We integrate once,

$$\frac{dV}{dx} = \frac{2\rho_{v0}a}{\epsilon} \operatorname{sech}\frac{x}{a} + C_1$$

and obtain the electric field intensity,

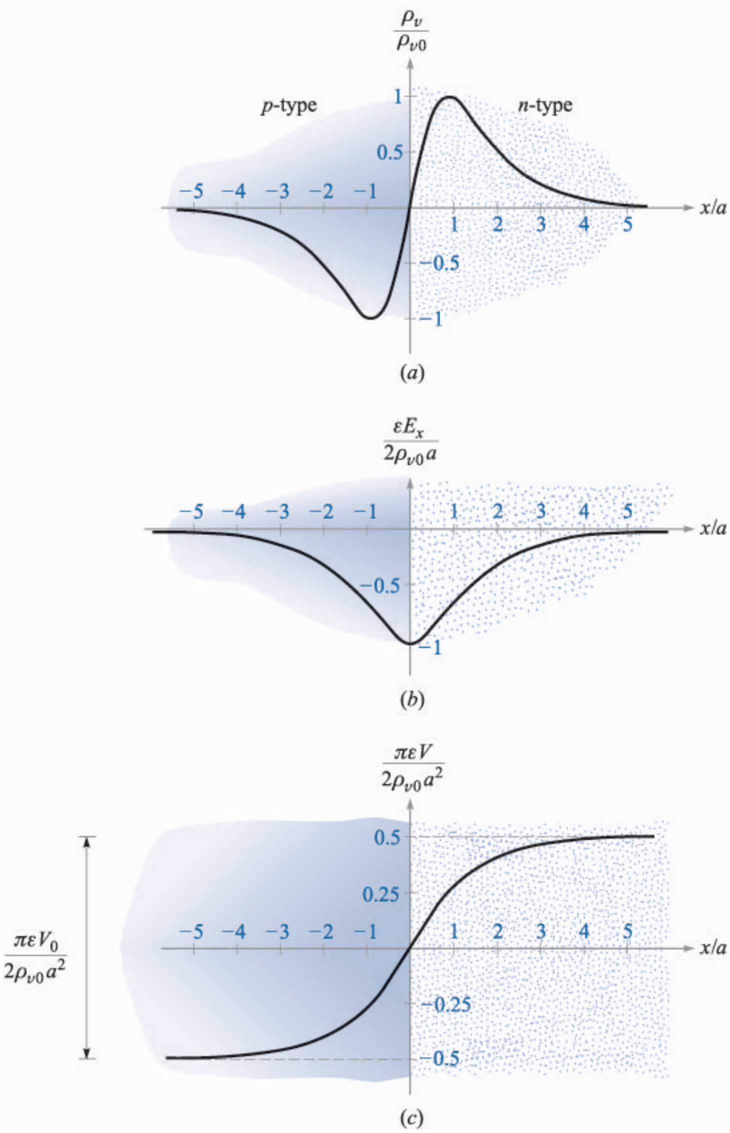

FIGURE 7.3
(*a*) The charge density, (*b*) the electric field intensity, and (*c*) the potential are plotted for a pn junction as functions of distance from the center of the junction. The *p*-type material is on the left, and the *n*-type is on the right.

$$E_x = -\frac{2\rho_{v0}a}{\epsilon}\operatorname{sech}\frac{x}{a} - C_1$$

To evaluate the constant of integration C_1, we note that no net charge density and no fields can exist *far* from the junction. Thus, as $x \to \pm\infty$, E_x must approach zero. Therefore $C_1 = 0$, and

$$E_x = -\frac{2\rho_{v0}a}{\epsilon}\operatorname{sech}\frac{x}{a} \tag{25}$$

Integrating again,

$$V = \frac{4\rho_{v0}a^2}{\epsilon}\tan^{-1} e^{x/a} + C_2$$

Let us arbitrarily select our zero reference of potential at the center of the junction, $x = 0$,

$$0 = \frac{4\rho_{v0}a^2}{\epsilon}\frac{\pi}{4} + C_2$$

and finally,

$$V = \frac{4\rho_{v0}a^2}{\epsilon}\left(\tan^{-1} e^{x/a} - \frac{\pi}{4}\right) \tag{26}$$

Fig. 7.3 shows the charge distribution (a), electric field intensity (b), and the potential (c), as given by (24), (25), and (26), respectively.

The potential is constant once we are a distance of about $4a$ or $5a$ from the junction. The total potential difference V_0 across the junction is obtained from (26),

$$V_0 = V_{x\to\infty} - V_{x\to-\infty} = \frac{2\pi\rho_{v0}a^2}{\epsilon} \tag{27}$$

This expression suggests the possibility of determining the total charge on one side of the junction and then using (27) to find a junction capacitance. The total positive charge is

$$Q = S\int_0^\infty 2\rho_{v0}\operatorname{sech}\frac{x}{a}\tanh\frac{x}{a}\,dx = 2\rho_{v0}aS$$

where S is the area of the junction cross section. If we make use of (27) to eliminate the distance parameter a, the charge becomes

$$Q = S\sqrt{\frac{2\rho_{v0}\epsilon V_0}{\pi}} \tag{28}$$

Since the total charge is a function of the potential difference, we have to be careful in defining a capacitance. Thinking in "circuit" terms for a moment,

$$I = \frac{dQ}{dt} = C\frac{dV_0}{dt}$$

and thus

$$C = \frac{dQ}{dV_0}$$

By differentiating (28) we therefore have the capacitance,

$$C = \sqrt{\frac{\rho_{v0}\epsilon}{2\pi V_0}}S = \frac{\epsilon S}{2\pi a} \tag{29}$$

The first form of (29) shows that the capacitance varies inversely as the square root of the voltage. That is, a higher voltage causes a greater separation of the charge layers and a smaller capacitance. The second form is interesting in that it indicates that we may think of the junction as a parallel-plate capacitor with a "plate" separation of $2\pi a$. In view of the dimensions of the region in which the charge is concentrated, this is a logical result.

Poisson's equation enters into any problem involving volume charge density. Besides semiconductor diode and transistor models, we find that vacuum tubes, magnetohydrodynamic energy conversion, and ion propulsion require its use in constructing satisfactory theories.

✔ **D7.4.** In the neighborhood of a certain semiconductor junction the volume charge density is given by $\rho_v = 750 \text{ sech } 10^6\pi x \tanh \pi x \, \text{C/m}^3$. The dielectric constant of the semiconductor material is 10 and the junction area is $2 \times 10^{-7}\,\text{m}^2$. Find: (*a*) V_0; (*b*) C; (*c*) E at the junction.

Ans. 2.70 V; 8.85 pF; 2.70 MV/m

✔ **D7.5.** Given the volume charge density $\rho_v = -2 \times 10^7 \epsilon_0 \sqrt{x}\, \text{C/m}^3$ in free space, let $V = 0$ at $x = 0$ and $V = 2\,\text{V}$ at $x = 2.5\,\text{mm}$. At $x = 1\,\text{mm}$, find: (*a*) V; (*b*) E_x.

Ans. 0.302 V; -555 V/m

7.5 PRODUCT SOLUTION OF LAPLACE'S EQUATION

In this section we are confronted with the class of potential fields which vary with more than one of the three coordinates. Although our examples are taken in the cartesian coordinate system, the general method is applicable to the other coordinate systems. We shall avoid those applications, however, because the potential fields are given in terms of more advanced mathematical functions, such as Bessel functions and spherical and cylindrical harmonics, and our interest now does not lie with new mathematical functions but with the techniques and methods of solving electrostatic field problems.

We may give ourselves a general class of problems by specifying merely that the potential is a function of x and y alone, so that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{30}$$

We now assume that the potential is expressible as the *product* of a function of x alone and a function of y alone. It might seem that this prohibits too many solutions, such as $V = x + y$, or any sum of a function of x and a function of y, but we should realize that Laplace's equation is linear and the sum of any two solutions is also a solution. We could treat $V = x + y$ as the sum of $V_1 = x$ and $V_2 = y$, where each of these latter potentials is now a (trivial) product solution.

Representing the function of x by X and the function of y by Y, we have

$$V = X\,Y \tag{31}$$

which is substituted into (30),

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

Since X does not involve y and Y does not involve x, ordinary derivatives may be used,

$$Y\frac{d^2 X}{dx^2} + X\frac{d^2 Y}{dy^2} = 0 \tag{32}$$

Equation (32) may be solved by separating the variables through division by XY, giving

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} = 0$$

or

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -\frac{1}{Y}\frac{d^2 Y}{dy^2}$$

Now we need one of the cleverest arguments of mathematics: since $(1/X)d^2X/dx^2$ involves no y and $-(1/Y)d^2Y/dy^2$ involves no x, and since the two quantities are equal, then $(1/X)d^2X/dx^2$ cannot be a function of x either, and similarly, $-(1/Y)d^2Y/dy^2$ cannot be a function of y! In other words, we have shown that each of these terms must be a constant. For convenience, let us call this constant α^2,

$$\frac{1}{X}\frac{d^2 X}{dx^2} = \alpha^2 \tag{33}$$

$$-\frac{1}{Y}\frac{d^2 Y}{dy^2} = \alpha^2 \tag{34}$$

The constant α^2 is called the *separation constant*, because its use results in separating one equation into two simpler equations.

Equation (33) may be written as

$$\frac{d^2X}{dx^2} = \alpha^2 X \tag{35}$$

and must now be solved. There are several methods by which a solution may be obtained. The first method is experience, or recognition, which becomes more powerful with practice. We are just beginning and can barely recognize Laplace's equation itself. The second method might be that of direct integration, when applicable, of course. Applying it here, we should write

$$d\left(\frac{dX}{dx}\right) = \alpha^2 X\, dx$$

$$\frac{dX}{dx} = \alpha^2 \int X\, dx$$

and then pass on to the next method, for X is some unknown function of x, and the method of integration is not applicable here. The third method we might describe as intuition, common sense, or inspection. It involves taking a good look at the equation, perhaps putting the operation into words. This method will work on (35) for some of us if we ask ourselves, "What function has a second derivative which has the same form as the function itself, except for multiplication by a constant?" The answer is the exponential function, of course, and we could go on from here to construct the solution. Instead, let us work with those of us whose intuition is suffering from exposure and apply a very powerful but long method, the infinite-power-series substitution.

We assume hopefully that X may be represented by

$$X = \sum_{a=0}^{\infty} a_n x^n$$

and substitute into (35), giving

$$\frac{d^2X}{dx^2} = \sum_{0}^{\infty} n(n-1)a_n x^{n-2} = \alpha^2 \sum_{0}^{\infty} a_n x^n$$

If these two different infinite series are to be equal for all x, they must be identical, and the coefficients of like powers of x may be equated term by term. Thus

$$2 \times 1 \times a_2 = \alpha^2 a_0$$

$$3 \times 2 \times a_3 = \alpha^2 a_1$$

and in general we have the recurrence relationship

$$(n+2)(n+1)a_{n+2} = \alpha^2 a_n$$

The even coefficients may be expressed in terms of a_0 as

$$a_2 = \frac{\alpha^2}{1 \times 2} a_0$$

$$a_4 = \frac{\alpha^2}{3 \times 4} a_2 = \frac{\alpha^4}{4!} a_0$$

$$a_6 = \frac{\alpha^6}{6!} a_0$$

and, in general, for n even, as

$$a_n = \frac{\alpha^n}{n!} a_0 \qquad (n \text{ even})$$

For odd values of n, we have

$$a_3 = \frac{\alpha^2}{2 \times 3} a_1 = \frac{\alpha^3}{3!} \frac{a_1}{\alpha}$$

$$a_5 = \frac{\alpha^5}{5!} \frac{a_1}{\alpha}$$

and in general, for n odd,

$$a_n = \frac{\alpha^n}{n!} \frac{a_1}{\alpha} \qquad (n \text{ odd})$$

Substituting back into the original power series for X, we obtain

$$X = a_0 \sum_{0,\text{even}}^{\infty} \frac{\alpha^n}{n!} x^n + \frac{a_1}{\alpha} \sum_{1,\text{odd}}^{\infty} \frac{\alpha^n}{n!} x^n$$

or

$$X = a_0 \sum_{0,\text{even}}^{\infty} \frac{(\alpha x)^n}{n!} + \frac{a_1}{\alpha} \sum_{1,\text{odd}}^{\infty} \frac{(\alpha x)^n}{n!}$$

Although the sum of these two infinite series is the solution of the differential equation in x, the form of the solution may be improved immeasurably by recognizing the first series as the hyperbolic cosine,

$$\cosh \alpha x = \sum_{0,\text{even}}^{\infty} \frac{(\alpha x)^n}{n!} = 1 + \frac{(\alpha x)^2}{2!} + \frac{(\alpha x)^4}{4!} + \ldots$$

and the second series as the hyperbolic sine,

$$\sinh \alpha x = \sum_{1,\text{odd}}^{\infty} \frac{(\alpha x)^n}{n!} = \alpha x + \frac{(\alpha x)^3}{3!} + \frac{(\alpha x)^5}{5!} + .$$

The solution may therefore be written as

$$X = a_0 \cosh \alpha x + \frac{a_1}{\alpha} \sinh \alpha x$$

or

$$X = A \cosh \alpha x + B \sinh \alpha x$$

where the slightly simpler terms A and B have replaced a_0 and a_1/α, respectively, and are the two constants which must be evaluated in terms of the boundary conditions. The separation constant is not an arbitrary constant as far as the solution of (35) is concerned, for it appears in that equation.

An alternate form of the solution is obtained by expressing the hyperbolic functions in terms of exponentials, collecting terms, and selecting new arbitrary constants, A' and B',

$$X = A' e^{\alpha x} + B' e^{-\alpha x}$$

Turning our attention now to (34), we see the solution proceeds along similar lines, leading to two power series representing the sine and cosine, and we have

$$Y = C \cos \alpha y + D \sin \alpha y$$

from which the potential is

$$V = XY = (A \cosh \alpha x + B \sinh \alpha x)(C \cos \alpha y + D \sin \alpha y) \tag{36}$$

Before describing a physical problem and forcing the constants appearing in (36) to fit the boundary conditions prescribed, let us consider the physical nature of the potential field given by a simple choice of these constants. Letting $A = 0$, $C = 0$, and $BD = V_1$, we have

$$V = V_1 \sinh \alpha x \sin \alpha y \tag{37}$$

The sinh αx factor is zero at $x = 0$ and increases smoothly with x, soon becoming nearly exponential in form, since

$$\sinh \alpha x = \tfrac{1}{2}(e^{\alpha x} - e^{-\alpha x})$$

The sin αy term causes the potential to be zero at $y = 0$, $y = \pi/\alpha$, $y = 2\pi/\alpha$, and so forth. We therefore may place zero-potential conducting planes at $x = 0$, $y = 0$, and $y = \pi/\alpha$. Finally, we can describe the V_1 equipotential surface by setting $V = V_1$ in (37), obtaining

$$\sinh \alpha x \sin \alpha y = 1$$

or

$$\alpha y = \sin^{-1} \frac{1}{\sinh \alpha x}$$

This is not a familiar equation, but a hand calculator or a set of tables can furnish enough material values to allow us to plot αy as a function of αx. Such a

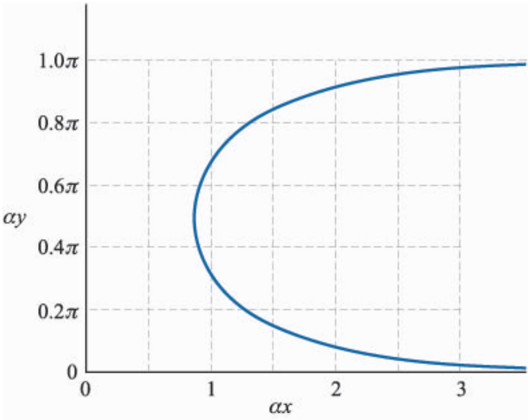

FIGURE 7.4
A graph of the double-valued function $\alpha y = \sin^{-1}(1/\sinh \alpha x)$, $0 < \alpha y < \pi$.

curve is shown in Fig. 7.4. Note that the curve is double-valued and symmetrical about the line $\alpha y = \pi/2$ when αy is restricted to the interval between 0 and π. The information of Fig. 7.4 is transferred directly to the $V = 0$ and $V = V_1$ equipotential conducting surfaces in Fig. 7.5. The surfaces are shown in cross section, since the potential is not a function of z.

It is very unlikely that we shall ever be asked to find the potential field of these peculiarly shaped electrodes, but we should bear in mind the possibility of combining a number of the fields having the form given by (36) or (37) and thus satisfying the boundary conditions of a more practical problem. We close this chapter with such an example.

The problem to be solved is that shown in Fig. 7.6. The boundary conditions shown are $V = 0$ at $x = 0$, $y = 0$, and $y = b$, and $V = V_0$ at $x = d$ for all y

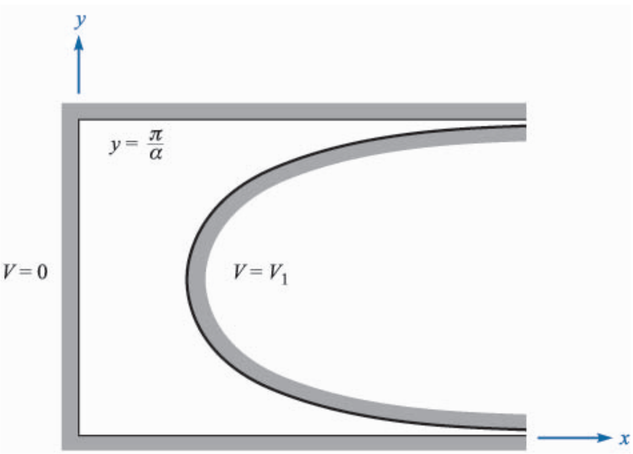

FIGURE 7.5
Cross section of the $V = 0$ and $V = V_1$ equipotential surfaces for the potential field $V = V_1 \sinh \alpha x \sin \alpha y$.

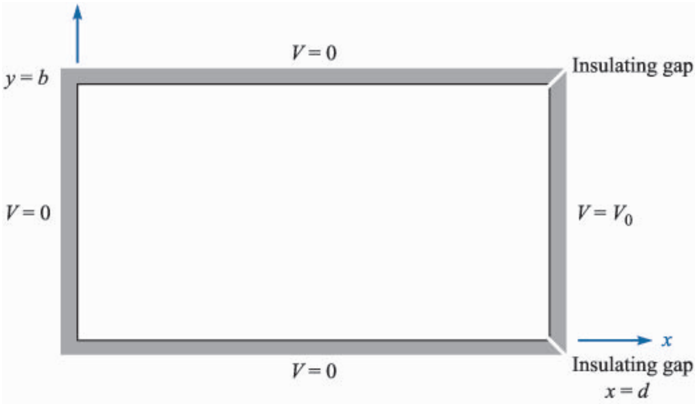

FIGURE 7.6
Potential problem requiring an infinite summation of fields of the form $V = V_1 \sinh \alpha x \sin \alpha y$. A similar configuration was analyzed by the iteration method in Chap. 6.

between 0 and b. It is immediately apparent that the potential field given by(37) and outlined in Fig. 7.5 satisfies two of the four boundary conditions. A third condition, $V = 0$ at $y = b$, may be satisfied by the choice of a, for the substitution of these values of (37) leads to the equation

$$0 = V_1 \sinh \alpha x \sin \alpha b$$

which may be satisfied by setting

$$\alpha b = m\pi \qquad (m = 1, 2, 3, \ldots)$$

or

$$\alpha = \frac{m\pi}{b}$$

The potential function

$$V = V_1 \sinh \frac{m\pi x}{b} \sin \frac{m\pi y}{b} \tag{38}$$

thus produces the correct potential at $x = 0$, $y = 0$, and $y = b$, regardless of the choice of m or the value of V_1. It is impossible to choose m or V_1 in such a way that $V = V_0$ at $x = d$ for each and every value of y between 0 and b. We must combine an infinite number of these fields, each with a different value of m and a corresponding value of V_1,

$$V = \sum_{m=0}^{\infty} V_{1m} \sinh \frac{m\pi x}{b} \sin \frac{m\pi y}{b}$$

The subscript on V_{1m} indicates that this amplitude factor will have a different value for each different value of m. Applying the last boundary condition now,

$$V_0 = \sum_{m=0}^{\infty} V_{1m} \sinh \frac{m\pi d}{d} \sin \frac{m\pi y}{b} \qquad (0 < y < b, m = 1, 2, \ldots)$$

Since $V_{1m} \sinh (m\pi d/b)$ is a function only of m, we may simplify the expression by replacing this factor by c_m,

$$V_0 = \sum_{m=0}^{\infty} c_m \sin \frac{m\pi y}{b} \qquad (0 < y < b, m = 1, 2, \ldots)$$

This is a Fourier sine series, and the c_m coefficients may be determined by the standard Fourier-series methods[1] if we can interpret V_0 as a periodic function of y. Since our physical problem is bounded by conducting planes at $y = 0$ and $y = b$, and our interest in the potential does not extend outside of this region, we may *define* the potential at $x = d$ for y *outside* of the range 0 to b in any manner we choose. Probably the simplest periodic expression is obtained by selecting the interval $0 < y < b$ as the half-period and choosing $V = -V_0$ in the adjacent half-period, or

$$V = V_0 \qquad (x = d, 0 < y < b)$$
$$V = -V_0 \qquad (x = d, b < y < 2b)$$

The c_m coefficients are then

$$c_m = \frac{1}{b}\left[\int_0^b V_0 \sin \frac{m\pi y}{b} dy + \int_b^{2b} (-V_0) \sin \frac{m\pi y}{b} dy\right]$$

leading to

$$c_m = \frac{4V_0}{m\pi} \qquad (m \text{ odd})$$
$$= 0 \qquad (m \text{ even})$$

However, $c_m = V_{1m} \sinh (m\pi d/b)$, and therefore

$$V_{1m} = \frac{4V_0}{m\pi \sinh (m\pi d/b)} \qquad (m \text{ odd only})$$

which may be substituted into (38) to give the desired potential function,

$$V = \frac{4V_0}{\pi} \sum_{1,\text{odd}}^{\infty} \frac{1}{m} \frac{\sinh (m\pi x/b)}{\sinh (m\pi d/b)} \sin \frac{m\pi y}{b} \tag{39}$$

The map of this field may be obtained by evaluating (39) at a number of points and drawing equipotentials by interpolation between these points. If we let $b = d$ and $V_0 = 100$, the problem is identical with that used as the example in

[1] Fourier series are discussed in almost every electrical engineering text on circuit theory. The authors are partial to the Hayt and Kemmerly reference given in the Suggested References at the end of the chapter.

the discussion of the iteration method. Checking one of the grid points in that problem, we let $x = d/4 = b/4$, $y = b/2 = d/2$, and $V_0 = 100$ and obtain

$$
\begin{aligned}
V &= \frac{400}{\pi} \sum_{1,\text{odd}}^{\infty} \frac{1}{m} \frac{\sinh(m\pi/4)}{\sinh m\pi} \sin\frac{m\pi}{2} \\
&= \frac{400}{\pi}\left(\frac{\sinh(\pi/4)}{\sinh \pi} - \frac{1}{3}\frac{\sinh(3\pi/4)}{\sinh 3\pi} + \frac{1}{5}\frac{\sinh(5\pi/4)}{\sinh 5\pi} - \dots\right) \\
&= \frac{400}{\pi}\left(\frac{0.8687}{11.549} - \frac{5.228}{3 \times 6195.8} + \dots\right) \\
&= 9.577 - 0.036 + \dots \\
&= 9.541\ \text{V}
\end{aligned}
$$

The equipotentials are drawn for increments of 10 V in Fig. 7.7, and flux lines have been added graphically to produce a curvilinear map.

The material covered in this discussion of the product solution was more difficult than much of the preceding work, and moreover, it presented three new ideas. The first new technique was the assumption that the potential might be expressed as the product of a function of x and a function of y, and the resultant separation of Laplace's equation into two simpler ordinary differential equations. The second new approach was employed when an infinite-power-series solution was assumed as the solution for one of the ordinary differential equations. Finally, we considered an example which required the combination of an infinite number of simpler product solutions, each having a different amplitude and a different variation in one of the coordinate directions. All these techniques

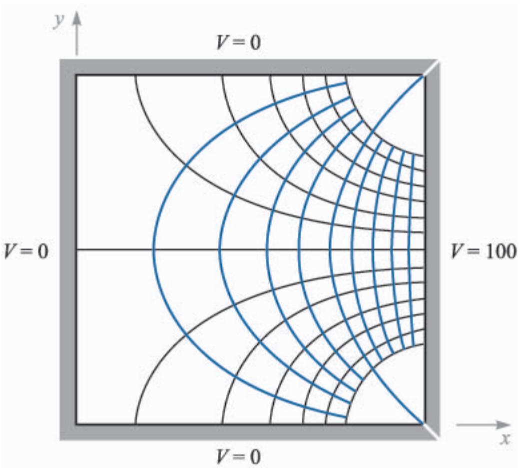

FIGURE 7.7
The field map corresponding to $V = \frac{4V_0}{\pi} \sum_{1,\text{odd}}^{\infty} \frac{1}{m} \frac{\sinh(m\pi x/b)}{\sinh(m\pi d/b)} \sin\frac{m\pi y}{b}$ with $b = d$ and $V_0 = 100$ V.

are very powerful. They are useful in all coordinate systems, and they can be used in problems in which the potential varies with all three coordinates.

We have merely introduced the subject here, and more information can be obtained from the references below, several of which devote hundreds of pages to the solution of Laplace's equation.

SUGGESTED REFERENCES

1. Dekker, A. J.: (see Suggested References for Chap. 5).
2. Hayt, W. H., Jr., and J. E. Kemmerly: "Engineering Circuit Analysis," 5th ed., McGraw-Hill Book Company, New York, 1993.
3. Push, E. M., and E. W. Pugh: "Principles of Electricity and Magnetism," 2d ed., Addison-Wesley Publishing Co., Reading, Mass., 1970. This text provides the physicist's view of electricity and magnetism, but electrical engineering students should find it easy to read. The solution to Laplace's equation by a number of methods is discussed in chap. 4.
4. Ramo, S., J. R. Whinnery, and T. Van Duzer: (see Suggested References for Chap. 6). A more complete and advanced discussion of methods of solving Laplace's equation is given in chap. 7.
5. Seeley, S., and A. D. Poularikas: "Electromagnetics: Classical and Modern Theory and Applications," Marcel Dekker, Inc., New York, 1979. Several examples of the solution of Laplace's equation by separation of variables appear in chap. 4.
6. Smythe, W. R.: "Static and Dynamic Electricity," 3d ed., McGraw-Hill Book Company, New York, 1968. An advanced treatment of potential theory is given in chap. 4.
7. Weber, E.: (see Suggested References for Chap. 6). There are a tremendous number of potential solutions given with the original references.

PROBLEMS

7.1 Let $V = 2xy^2z^3$ and $\epsilon = \epsilon_0$. Given point $P(1, 2, -1)$, find: (*a*) V at P; (*b*) $\mathbf{E}$ at P; (*c*) ρ_v at P; (*d*) the equation of the equipotential surface passing through P; (*e*) the equation of the streamline passing through P. (*f*) Does V satisfy Laplace's equation?

7.2 A potential field V exists in a region where $\epsilon = f(x)$. Find $\nabla^2 V$ if $\rho_v = 0$.

7.3 Let $V(x, y) = 4e^{2x} + f(x) - 3y^2$ in a region of free space where $\rho_v = 0$. It is known that both E_x and V are zero at the origin. Find $f(x)$ and $V(x, y)$.

7.4 Given the potential field $V = A \ln\left(\tan^2 \dfrac{\theta}{2}\right) + B$: (*a*) show that $\nabla^2 V = 0$; (*b*) select A and B so that $V = 100\,\text{V}$ and $E_\theta = 500\,\text{V/m}$ at $P(r = 5, \theta = 60^\circ, \phi = 45^\circ)$.

7.5 Given the potential field $V = (A\rho^4 + B\rho^{-4})\sin 4\phi$: (*a*) show that $\nabla^2 V = 0$; (*b*) select A and B so that $V = 100$ V and $|\mathbf{E}| = 500$ V/m at $P(\rho = 1, \phi = 22.5^\circ, z = 2)$.

7.6 If $V = \dfrac{20\sin\theta}{r^3}$ V in free space, find: (*a*) ρ_v at $P(r = 2, \theta = 30^\circ, \phi = 0)$; (*b*) the total charge within the spherical shell $1 < r < 2$ m.

7.7 Let $V = \dfrac{\cos 2\phi}{\rho}$ V in free space. (*a*) Find the volume charge density at point $A(\frac{1}{2}, 60^\circ, 1)$. (*b*) Find the surface charge density on a conductor surface passing through the point $B(2, 30^\circ, 1)$.

7.8 Let $V_1(r, \theta, \phi) = \dfrac{20}{r}$ and $V_2(r, \theta, \phi) = \dfrac{4}{r} + 4$. (*a*) State whether V_1 and V_2 satisfy Laplace's equation. (*b*) Evaluate V_1 and V_2 on the closed surface $r = 4$. (*c*) Conciliate your results with the uniqueness theorem.

7.9 The functions $V_1(\rho, \phi, z)$ and $V_2(\rho, \phi, z)$ both satisfy Laplace's equation in the region $a < \rho < b$, $0 \le \phi < 2\pi$, $-L < z < L$; each is zero on the surfaces $\rho = b$ for $-L < z < L$; $z = -L$ for $a < \rho < b$; and $z = L$ for $a < \rho < b$; and each is 100 V on the surface $\rho = a$ for $-L < z < L$. (*a*) In the region specified above, is Laplace's equation satisfied by the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? (*b*) On the boundary surfaces specified, are the potential values given above obtained from the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? (*c*) Are the functions V_2, $V_1 + V_2$, $V_1 + 3$, and $V_1 V_2$ identical with V_1?

7.10 Conducting planes at $z = 2$ cm and $z = 8$ cm are held at potentials of -3 V and 9 V, respectively. The region between the plates is filled with a perfect dielectric with $\epsilon = 5\epsilon_0$. Find and sketch: (*a*) $V(z)$; (*b*) $E_z(z)$; (*c*) $D_z(z)$.

7.11 The conducting planes $2x + 3y = 12$ and $2x + 3y = 18$ are at potentials of 100 V and 0, respectively. Let $\epsilon = \epsilon_0$ and find: (*a*) V at $P(5, 2, 6)$; (*b*) $\mathbf{E}$ at P.

7.12 Conducting cylinders at $\rho = 2$ cm and $\rho = 8$ cm in free space are held at potentials of 60 mV and -30 mV, respectively. (*a*) Find $V(\rho)$. (*b*) Find $E_\rho(\rho)$. (*c*) Find the surface on which $V = 30$ mV.

7.13 Coaxial conducting cylinders are located at $\rho = 0.5$ cm and $\rho = 1.2$ cm. The region between the cylinders is filled with a homogeneous perfect dielectric. If the inner cylinder is at 100 V and the outer at 0 V, find: (*a*) the location of the 20-V equipotential surface; (*b*) $E_{\rho\,\max}$; (*c*) ϵ_R if the charge per meter length on the inner cylinder is 20 nC/m.

7.14 Two semi-infinite planes are located at $\phi = -\alpha$ and $\phi = \alpha$, where $\alpha < \pi/2$. A narrow insulating strip separates them along the z axis. The potential at $\phi = -\alpha$ is V_0, while $V = 0$ at $\phi = \alpha$. (*a*) Find $V(\phi)$ in terms of α and V_0. (*b*) Find E_ϕ at $\phi = 20^\circ$, $\rho = 2$ cm, if $V_0 = 100$ V and $\alpha = 30^\circ$.

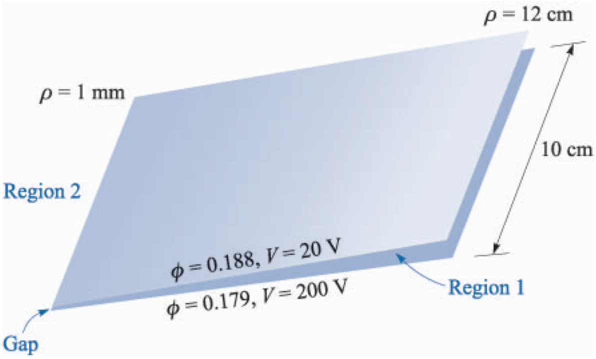

FIGURE 7.8
See Prob. 15.

7.15 The two conducting planes illustrated in Fig. 7.8 are defined by $0.001 < \rho < 0.120$ m, $0 < z < 0.1$ m, $\phi = 0.179$ and 0.188 rad. The medium surrounding the planes is air. For region 1, $0.179 < \phi < 0.188$, neglect fringing and find: (*a*) $V(\phi)$; (*b*) $\mathbf{E}(\rho)$; (*c*) $\mathbf{D}(\rho)$; (*d*) ρ_S on the upper surface of the lower plane; (*e*) Q on the upper surface of the lower plane. (*f*) Repeat (*a*) to (*c*) for region 2 by letting the location of the upper plane be $\phi = 0.188 - 2\pi$, and then find ρ_S and Q on the lower surface of the lower plane. (*g*) Find the total charge on the lower plane and the capacitance between the planes.

7.16 (*a*) Solve Laplace's equation for the potential field in the homogeneous region between two concentric conducting spheres with radii a and b, $b > a$, if $V = 0$ at $r = b$, and $V = V_0$ at $r = a$. (*b*) Find the capacitance between them.

7.17 Concentric conducting spheres are located at $r = 5$ mm and $r = 20$ mm. The region between the spheres is filled with a perfect dielectric. If the inner sphere is at 100 V and the outer at 0 V: (*a*) find the location of the 20-V equipotential surface; (*b*) find $E_{r,\max}$; (*c*) find ϵ_R if the surface charge density on the inner sphere is $100\,\mu\text{C/m}^2$.

7.18 Concentric conducting spheres have radii of 1 and 5 cm. There is a perfect dielectric for which $\epsilon_R = 3$ between them. The potential of the inner sphere is 2 V and that of the outer is -2 V. Find: (*a*) $V(r)$; (*b*) $\mathbf{E}(r)$; (*c*) V at $r = 3$ cm; (*d*) the location of the 0-V equipotential surface; (*e*) the capacitance between the spheres.

7.19 Two coaxial conducting cones have their vertices at the origin and the z axis as their axis. Cone A has the point $A(1, 0, 2)$ on its surface, while cone B has the point $B(0, 3, 2)$ on its surface. Let $V_A = 100$ V and $V_B = 20$ V. Find: (*a*) α for each cone; (*b*) V at $P(1, 1, 1)$.

7.20 A potential field in free space is given as $V = 100\ln[\tan(\theta/2)] + 50$ V. (*a*) Find the maximum value of $|\mathbf{E}_\theta|$ on the surface $\theta = 40^\circ$ for $0.1 < r < 0.8$ m, $60^\circ < \phi < 90^\circ$. (*b*) Describe the surface $V = 80$ V.

7.21 In free space, let $\rho_v = 200\epsilon_0/r^{2.4}$. (*a*) Use Poisson's equation to find $V(r)$ if it is assumed that $r^2E_r \to 0$ when $r \to 0$, and also that $V \to 0$ as $r \to \infty$. (*b*) Now find $V(r)$ by using Gauss's law and a line integral.

7.22 Let the volume charge density in Fig. 7.3*a* be given by $\rho_v = \rho_{v0}(x/a)e^{-|x|a}$. (*a*) Determine $\rho_{v,\max}$ and $\rho_{v,\min}$ and their locations. (*b*) Find E_x and $V(x)$ if $V(0) = 0$ and $E_x \to 0$ as $x \to \infty$. (*c*) Use a development similar to that of Sec. 7.4 to show that $C = dQ/dV_0 = \epsilon_0 S/(4\sqrt{2}a)$.

7.23 A rectangular trough is formed by four conducting planes located at $x = 0$ and 8 cm and $y = 0$ and 5 cm in air. The surface at $y = 5$ cm is at a potential of 100 V, the other three are at zero potential, and the necessary gaps are placed at two corners. Find the potential at $x = 3$ cm, $y = 4$ cm.

7.24 The four sides of a square trough are held at potentials of 0, 20, −30, and 60 V; the highest and lowest potentials are on opposite sides. Find the potential at the center of the trough.

7.25 In Fig. 7.7 change the right side so that the potential varies linearly from 0 at the bottom of that side to 100 V at the top. Solve for the potential at the center of the trough.

7.26 If X is a function of x and $X'' + (x-1)X' - 2X = 0$, assume a solution in the form of an infinite power series and determine numerical values for a_2 to a_8 if $a_0 = 1$ and $a_1 = -1$.

7.27 It is known that $V = XY$ is a solution of Laplace's equation, where X is a function of x alone and Y is a function of y alone. Determine which of the following potential functions are also solutions of Laplace's equation: (*a*) $V = 100X$; (*b*) $V = 50XY$; (*c*) $V = 2XY + x - 3y$; (*d*) $V = xXY$; (*e*) $V = X^2Y$.

7.28 Assume a product solution of Laplace's equation in cylindrical coordinates, $V = PF$, where V is not a function of z, P is a function only of ρ, and F is a function only of ϕ. (*a*) Obtain the two separated equations if the separation constant is n^2. Select the sign of n^2 so that the solution of the ϕ equation leads to trigonometric functions. (*b*) Show that $P = A\rho^n + B\rho^{-n}$ satisfies the ρ equation. (*c*) Construct the solution $V(\rho, \phi)$. Functions of this form are called *circular harmonics*.