Computer-Controlled Systems

Third Edition

Solutions Manual

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Preface

This Solutions Manual contains solutions to most of the problems in the fourth edition of

> Åström, K. J. and B. Wittenmark (1997): Computer controlled Systems
> – Theory and Applications, Prentice Hall Inc., Englewood Cliffs, N. J.

Many of the problems are intentionally made such that the students have to use a simulation program to verify the analytical solutions. This is important since it gives a feeling for the relation between the pulse transfer function and the time domain. In the book and the solutions we have used Matlab/Simulink. Information about macros used to generate the illustrations can be obtained by writing to us.

It is also important that a course in digital control includes laboratory exercises. The contents in the laboratory experiments are of course dependent on the available equipment. Examples of experiments are

★ Illustration of aliasing

★ Comparison between continuous time and discrete time controllers

★ State feedback control. Redesign of continuous time controllers as well as controllers based on discrete time synthesis

★ Controllers based on input-output design

★ Control of systems subject to stochastic disturbances.

Finally we would like to thank collegues and students who have helped us to test the book and the solutions.

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Solutions to Chapter 2

Problem 2.1

The system is described by

$$\dot{x} = -ax + bu$$
$$y = cx$$

Sampling the system using (2.4) and (2.5) gives

$$x(kh + h) = e^{-ah}x(kh) + \left(1 - e^{-ah}\right)\left(\frac{b}{a}\right)u(kh)$$
$$y(kh) = cx(kh)$$

The pole of the sampled system is $\exp(-ah)$. The pole is real. For small values of h the pole is close to 1. If $a > 0$ then the pole moves towards the origin when h increases. If $a < 0$ then the pole moves along the positive real axis.

Problem 2.2

a. Using the Laplace transform method we find that

$$\Phi = e^{Ah} = \mathcal{L}^{-1}\left(\left(sI - A\right)^{-1}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2 + 1}\begin{pmatrix} s & 1 \\ -1 & s \end{pmatrix}\right) =$$

$$= \begin{pmatrix} \cos h & \sin h \\ -\sin h & \cos h \end{pmatrix}$$

$$\Gamma = \int_0^h e^{As}B\,ds = \int_0^h \begin{pmatrix} \sin s \\ \cos s \end{pmatrix} ds = \begin{pmatrix} 1 - \cos h \\ \sin h \end{pmatrix}$$

b. The system has the transfer function

$$G(s) = \frac{s + 3}{s^2 + 3s + 2} = \frac{s + 3}{(s + 1)(s + 2)} = \frac{2}{s + 1} - \frac{1}{s + 2}$$

Using the Table 2.1 gives

$$H(q) = 2\frac{1 - e^{-h}}{q - e^{-h}} - \frac{1 - e^{-2h}}{2\left(q - e^{-2h}\right)}$$

c. One state space realization of the system is

$$\dot{x} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}u$$

$$y = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}x$$

Now

$$A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad A^3 = 0$$

then

$$e^{Ah} = I + Ah + A^2 h^2/2 + \ldots = \begin{pmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ h^2/2 & h & 1 \end{pmatrix}$$

$$\Gamma = \int_0^h e^{As} B \, ds = \int_0^h \begin{pmatrix} 1 & s & s^2/2 \end{pmatrix}^T ds = \begin{pmatrix} h & h^2/2 & h^3/6 \end{pmatrix}^T$$

and

$$H(q) = C(qI - \Phi)^{-1}\Gamma = \frac{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}}{(q-1)^3}$$

$$\begin{pmatrix} x & x & x \\ x & x & x \\ h^2(q+1)/2 & h(q-1) & (q-1)^2 \end{pmatrix} \begin{pmatrix} h \\ h^2/2 \\ h^3/6 \end{pmatrix}$$

$$= \frac{h^3(q^2 + 4q + 1)}{6(q-1)^3}$$

Problem 2.3

$$\Phi = e^{Ah} \qquad \Rightarrow \qquad A = \frac{1}{h}\ln \Phi$$

a.

$$y(k) - 0.5y(k-1) = 6u(k-1)$$
$$y - 0.5q^{-1}y = 6q^{-1}u$$
$$qy - 0.5y = 6u$$

$$\begin{cases} x(kh+h) &= 0.5x(kh) + 6u(kh) \\ y(kh) &= x(kh) \end{cases} \qquad \begin{cases} \dot{x}(t) &= ax(t) + bu(t) \\ y(t) &= x(t) \end{cases}$$

(discrete-time system)   (continuous time system)

$$\begin{cases} \Phi &= e^{ah} = 0.5 \\ \Gamma &= \int_0^h e^{as}b \, ds = 6 \end{cases}$$

$$ah = \ln 0.5 \Rightarrow a = -\frac{\ln 2}{h}$$

$$\int_0^h e^{as} b \, ds = \frac{b}{a}e^{as}\Big|_0^h = \frac{b}{a}\left(e^{ah} - 1\right) = 6$$

$$\Rightarrow b = \frac{6a}{e^{ah} - 1} = \frac{12\ln 2}{h}$$

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b.

$$\begin{cases} x(kh + h) = \begin{pmatrix} -0.5 & 1 \\ 0 & -0.3 \end{pmatrix} x(kh) + \begin{pmatrix} 0.5 \\ 0.7 \end{pmatrix} u(kh) \\ y(kh) = \begin{pmatrix} 1 & 1 \end{pmatrix} x(kh) \end{cases}$$

Eigenvalue to Φ:

$$\det(sI - \Phi) = \begin{pmatrix} s + 0.5 & -1 \\ 0 & s + 0.3 \end{pmatrix} = 0 \Leftrightarrow (s + 0.5)(s + 0.3) = 0$$

$$\lambda_1 = -0.5 \quad \lambda_2 = -0.3$$

Both eigenvalues of Φ on the negative real axis $\Rightarrow$ No corresponding continuous system exists.

c.

$$y(k) + 0.5y(k - 1) = 6u(k - 1) \Leftrightarrow$$
$$y(k + 1) = -0.5y(k) + 6u(t)$$

$$H(q) = \frac{6}{q + 0.5} \qquad \text{one pole on the negative real axis.}$$

$\Rightarrow$ No equivalent continuous system exists.

Problem 2.4

Harmonic oscillator (cf. A.3 and 3.2a).

$$\begin{cases} x(k + 1) = \Phi x(k) + \Gamma u(k) \\ y(k) = Cx(k) \end{cases} \qquad \Phi(h) = \begin{pmatrix} \cos h & \sin h \\ -\sin h & \cos h \end{pmatrix}$$

$$\Gamma(h) = \begin{pmatrix} 1 - \cos h \\ \sin h \end{pmatrix}$$

a.

$$h = \frac{\pi}{2} \Rightarrow \Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \Gamma = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

Pulse transfer operator

$$H(q) = C(qI - \Phi)^{-1}\Gamma = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} q & -1 \\ 1 & q \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{q^2 + 1} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} q & 1 \\ -1 & q \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{q + 1}{q^2 + 1}$$

$$Y(z) = H(z)U(z) = \frac{z + 1}{z^2 + 1} \cdot \frac{z}{z - 1} = \frac{1}{z^2 + 1} + \frac{1}{z - 1}$$

$$y(k) = \sin \frac{\pi}{2}(k - 1)\theta(k - 1) + \theta(k - 1) = \left(1 - \cos \frac{\pi}{2}k\right)\theta(k - 1)$$

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where $\theta(k-1)$ is a step at $k=1$.

$$G(s) = [\text{see Probl. 2.2}] = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{s^2+1}$$

$$Y(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$y(t) = 1 - \cos t$$

$$y(kh) = 1 - \cos\frac{\pi}{2}k$$

b. The same way as a. $\Rightarrow$ $y(t) = 1 - \cos t$, and $y(kh) = 1 - \cos\frac{\pi}{4}k$ Notice that the step responses of the continuous time and the zero-order hold sampled systems are the same in the sampling points.

Problem 2.5

Do a partial fraction decomposition and sample each term using Table 2.1:

$$G(s) = \frac{1}{s^2(s+2)(s+3)} = \frac{1}{6}\frac{1}{s^2} - \frac{5}{36}\frac{1}{s} + \frac{1}{4}\frac{1}{s+2} - \frac{1}{9}\frac{1}{s+3}$$

$$H(q) = \frac{1}{12}\frac{q+1}{(q-1)^2} - \frac{5}{36}\frac{1}{q-1} + \frac{1}{8}\frac{1-e^{-2}}{q-e^{-2}} - \frac{1}{27}\frac{1-e^{-3}}{q-e^{-3}}$$

Problem 2.6

Integrating the system equation

$$\dot{x} = Ax + Bu$$

gives

$$x(kh+h) = e^{Ah}x(kh) + \int_{kh}^{kh+h} e^{As}B\delta(s-kh)u(kh)ds$$

$$= e^{Ah}x(kh) + Bu(kh)$$

Problem 2.7

The representation (2.7) is

$$x(kh+h) = \Phi x(kh) + \Gamma u(kh)$$

$$y(kh) = Cx(kh)$$

and the controllable form realization is

$$z(kh+h) = \tilde{\Phi}z(kh) + \tilde{\Gamma}u(kh)$$

$$y(kh) = \tilde{C}z(kh)$$

where

$$\Phi = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} h^2/2 \\ h \end{pmatrix} \qquad \tilde{\Phi} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \qquad \tilde{\Gamma} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad \tilde{C} = \begin{pmatrix} h^2/2 & h^2/2 \end{pmatrix}$$

From Section 2.5 we get

$$\tilde{\Phi} = T\Phi T^{-1} \qquad \text{or} \qquad \tilde{\Phi}T = T\Phi$$

$$\tilde{\Gamma} = T\Gamma$$

$$\tilde{C} = CT^{-1} \qquad \text{or} \qquad \tilde{C}T = C$$

This gives the following relations

$$
\begin{aligned}
t_{11} &= 2t_{11} - t_{21} \\
t_{21} &= t_{11} \qquad\qquad\qquad\qquad (1) \\
ht_{11} + t_{12} &= 2t_{12} - t_{22} \quad (2) \\
ht_{21} + t_{22} &= t_{12} \\
\frac{h^2}{2} t_{11} + ht_{12} &= 1 \\
\frac{h^2}{2} t_{21} + ht_{22} &= 0 \\
\frac{h^2}{2} t_{11} + \frac{h^2}{2} t_{21} &= 1 \qquad (3) \\
\frac{h^2}{2} t_{12} + \frac{h^2}{2} t_{22} &= 0 \qquad (4)
\end{aligned}
$$

Equations (1)–(4) now give

$$
\begin{aligned}
t_{11} &= t_{21} = 1/h^2 \\
t_{12} &= -t_{22} = 1/(2h)
\end{aligned}
\qquad \text{or} \qquad
T = \frac{1}{2h^2} \begin{pmatrix} 2 & h \\ 2 & -h \end{pmatrix}
$$

Problem 2.8

The pulse transfer function is given by

$$H(z) = C(zI - \Phi)^{-1}\Gamma = \frac{\begin{pmatrix} 1 & 0 \end{pmatrix}}{z(z - 0.5)} \begin{pmatrix} z & -0.2 \\ 0 & z - 0.5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$

$$= \frac{2z - 0.2}{z(z - 0.5)} = \frac{2(z - 0.1)}{z(z - 0.5)}$$

Problem 2.9

The system is described by

$$\dot{x} = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix} x + \begin{pmatrix} f \\ g \end{pmatrix} u = Ax + Bu$$

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The eigenvalues of A is obtained from

$$(\lambda + a)(\lambda + d) - bc = \lambda^2 + (a + d)\lambda + ad - bc = 0$$

which gives

$$\lambda = -\frac{a + d}{2} \pm \sqrt{\frac{(a - d)^2 + 4bc}{4}}$$

The condition that a, b, c, and d are nonnegative implies that the eigenvalues, λ_1 and λ_2, are real. There are multiple eigenvalues if both $a = d$ and $bc = 0$. Using the result in Appendix B we find that

$$e^{Ah} = \alpha_0 I + \alpha_1 A h$$

and

$$e^{\lambda_1 h} = \alpha_0 + \alpha_1 \lambda_1 h$$
$$e^{\lambda_2 h} = \alpha_0 + \alpha_1 \lambda_2 h$$

This gives

$$\alpha_0 = \frac{\lambda_1 e^{\lambda_2 h} - \lambda_2 e^{\lambda_1 h}}{\lambda_1 - \lambda_2}$$

$$\alpha_1 = \frac{e^{\lambda_1 h} - e^{\lambda_2 h}}{(\lambda_1 - \lambda_2)h}$$

$$\Phi = \begin{pmatrix} \alpha_0 - \alpha_1 a h & \alpha_1 b h \\ \alpha_1 c h & \alpha_0 - \alpha_1 d h \end{pmatrix}$$

To compute Γ we notice that α_0 and α_1 depend on h. Introduce

$$\beta_0 = \int_0^h \alpha_0(s)ds = \frac{1}{\lambda_1 - \lambda_2}\left(\frac{\lambda_1}{\lambda_2}\left(e^{\lambda_2 h} - 1\right) - \frac{\lambda_2}{\lambda_1}\left(e^{\lambda_1 h} - 1\right)\right)$$

$$\beta_1 = \int_0^h s\alpha_1(s)ds = \frac{1}{\lambda_1 - \lambda_2}\left(\frac{1}{\lambda_1}\left(e^{\lambda_1 h} - 1\right) - \frac{1}{\lambda_2}\left(e^{\lambda_2 h} - 1\right)\right)$$

then

$$\Gamma = \beta_0 B + \beta_1 A B$$

Problem 2.10

a. Using the result from Problem 2.9 gives

$$\lambda_1 = -0.0197 \qquad e^{\lambda_1 h} = 0.7895$$

$$\lambda_2 = -0.0129 \qquad e^{\lambda_2 h} = 0.8566$$

Further

$$\alpha_0 = 0.9839 \qquad \alpha_1 = 0.8223$$

and

$$\Phi = \alpha_0 I + 12\alpha_1 A = \begin{pmatrix} 0.790 & 0 \\ 0.176 & 0.857 \end{pmatrix}$$

$$\beta_0 = 11.9412$$

$$\beta_1 = 63.3824$$

$$\Gamma = (\beta_0 I + \beta_1 A)B = \begin{pmatrix} 0.281 \\ 0.0296 \end{pmatrix}$$

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b.   The pulse transfer operator is now given by

$$H(q) = C(qI - \Phi)^{-1}\Gamma = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} q - 0.790 & 0 \\ -0.176 & q - 0.857 \end{pmatrix}^{-1} \begin{pmatrix} 0.281 \\ 0.0297 \end{pmatrix}$$

$$= \frac{0.030q + 0.026}{q^2 - 1.65q + 0.68}$$

which agrees with the pulse transfer operator given in the problem formulation.

Problem 2.11

The motor has the transfer function $G(s) = 1/(s(s+1))$. A state space representation is given in Example A.2, where

$$A = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$\Phi = \exp(Ah) = \mathcal{L}^{-1}\left((sI - A)^{-1} \right) = \mathcal{L}^{-1}\left(\frac{1}{s(s+1)} \begin{pmatrix} s & 0 \\ 1 & s+1 \end{pmatrix} \right) =$$

$$= \begin{pmatrix} e^{-h} & 0 \\ 1 - e^{-h} & 1 \end{pmatrix}$$

$$\Gamma = \int_0^h e^{As} B\, ds = \int_0^h \begin{pmatrix} e^{-s} \\ 1 - e^{-s} \end{pmatrix} ds = \begin{pmatrix} 1 - e^{-h} \\ h + e^{-h} - 1 \end{pmatrix}$$

This gives the sampled representation given in Example A.2.

a.   The pulse transfer function is

$$H(z) = C(zI - \Phi)^{-1}\Gamma =$$

$$= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} z - e^{-h} & 0 \\ -1 + e^{-h} & z - 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 - e^{-h} \\ h + e^{-h} - 1 \end{pmatrix}$$

$$= \frac{\begin{pmatrix} 0 & 1 \end{pmatrix}}{(z - e^{-h})(z - 1)} \begin{pmatrix} z - 1 & 0 \\ 1 - e^{-h} & z - e^{-h} \end{pmatrix} \begin{pmatrix} 1 - e^{-h} \\ h + e^{-h} - 1 \end{pmatrix}$$

$$= \frac{(h + e^{-h} - 1)z + (1 - e^{-h} - he^{-h})}{(z - e^{-h})(z - 1)}$$

$$= \frac{(h + e^{-h} - 1)z + (1 - e^{-h} - he^{-h})}{z^2 - (1 + e^{-h})z + e^{-h}}$$

b.   The pulse response is

$$h(k) = \begin{cases} 0 & k = 0 \\ C\Phi^{k-1}\Gamma & k \geq 1 \end{cases}$$
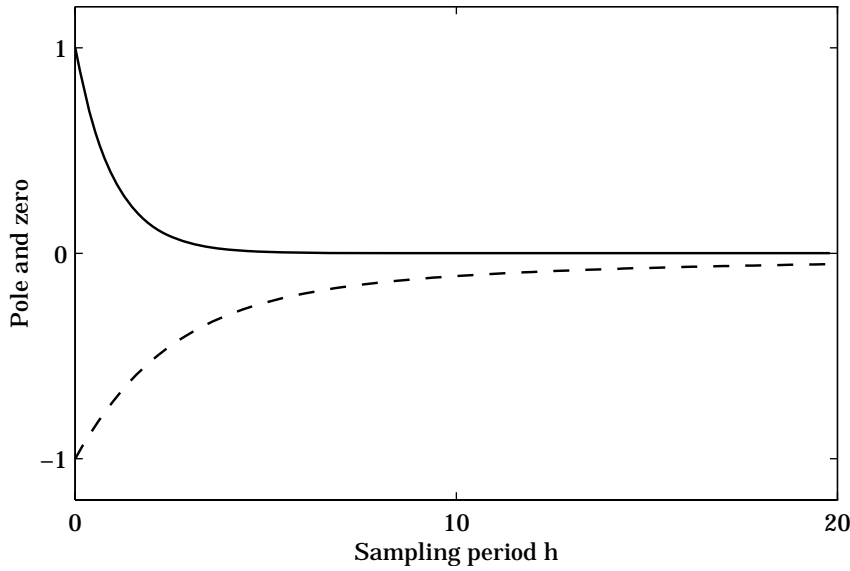
Now

$$\Phi^k = \left(e^{Ah} \right)^k = e^{Akh}$$

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Figure 2.1 The pole (solid) and zero (dashed) in Problem 2.11.d as function of the sampling period.

This gives

$$C\Phi^{k-1}\Gamma = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-(k-1)h} & 0 \\ 1 - e^{-(k-1)h} & 1 \end{pmatrix} \begin{pmatrix} 1 - e^{-h} \\ h + e^{-h} - 1 \end{pmatrix} =$$

$$= \left(1 - e^{-h}\right)\left(1 - e^{-(k-1)h}\right) + h + e^{-h} - 1 =$$

$$= h - e^{-(k-1)h} + e^{-kh}$$

An alternative way to find $h(k)$ is to take the inverse z-transform of $H(z)$.

c. A difference equation is obtained from $H(q)$

$$y(kh + 2h) - (1 + e^{-h})y(kh + h) + e^{-h}y(kh) =$$

$$= (h + e^{-h} - 1)u(kh + h) + (1 - e^{-h} - he^{-h})u(kh)$$

d. The poles are in $z = 1$ and $z = \exp(-h)$. The second pole will move from 1 to the origin when h goes from zero to infinity.
The zero is in

$$z = -\frac{1 - e^{-h} - he^{-h}}{h + e^{-h} - 1} = f(h)$$

The function $f(h)$ goes from -1 to 0 when h goes from zero to infinity. See Fig. 2.1.

Problem 2.12

Consider the following transfer operators of infinity order.

$$G(s) = \frac{1}{s} e^{-s\tau} \qquad h = 1, \quad \tau = 0.5$$

$$(\tau < h)$$

$$\dot{x} = u(t - \tau) = 0 \cdot x + 1 \cdot u(t - \tau) \qquad \text{(infinite order)}$$

a. Discrete time system is given by (cf. CCS p. 38–40).

$$x(k+1) \;=\; \Phi x(k) \;+\; \Gamma_0 u(k) \;+\; \Gamma_1 u(k-1) \text{ (Notice that } \tau < h)$$

$$\Phi = e^{Ah} = e^0 = 1$$

$$\Gamma_0 = \int_0^{h-\tau} e^{As}\, ds\, B = \int_0^{0.5} 1\, ds\, 1 = 0.5$$

$$\Gamma_1 = e^{A(h-\tau)} \int_0^{\tau} e^{As}\, ds\, B = \int_0^{0.5} ds = 0.5$$

$$\Rightarrow x(k+1) = x(k) + 0.5u(k) + 0.5u(k-1)$$

State space model:

$$\begin{pmatrix} x(k+1) \\ u(k) \end{pmatrix} = \begin{pmatrix} 1 & 0.5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ u(k-1) \end{pmatrix} + \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} u(k) = \bar{\Phi}\bar{x}(k) + \bar{\Gamma} u(k)$$

$$y(k) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ u(k-1) \end{pmatrix}$$

The system is of second order.

b. The pulse-transfer function is

$$H(z) = C(zI - \bar{\Phi})^{-1}\bar{\Gamma} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} z-1 & -0.5 \\ 0 & z \end{pmatrix}^{-1} \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{z(z-1)} \begin{pmatrix} z & 0.5 \\ 0 & z-1 \end{pmatrix} \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} = \frac{1}{z(z-1)} \begin{pmatrix} z & 0.5 \end{pmatrix} \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$$

$$= \frac{0.5(z+1)}{z(z-1)}$$

Invers transform of $H(z)$ using Table 2.3 gives

$$H(z) = \frac{0.5(z+1)}{z(z-1)} = 0.5 \left(\underbrace{z^{-1}\frac{z}{z-1}}_{=1;\ k\geq1} + \underbrace{z^{-2}\frac{z}{z-1}}_{=1;\ k\geq2} \right)$$

$$\Rightarrow h(kh) = \begin{cases} 0 & k = 0 \\ 0.5 & k = 1 \\ 1 & k > 1 \end{cases}$$

c. The poles are $z = 0, z = 1$ and the zeros: $z = -1$

Problem 2.13

a. This is the same system as in Example 2.6 but with $\tau = 1.5$. In this case we have $d = 2$ and $\tau' = 0.5$ and we get

$$x(k+1) = \Phi x(k) + \Gamma_0 u(k-1) + \Gamma_1 u(k-2)$$

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where
$$\Phi = e^{-1} = 0.37$$
$$\Gamma_0 = 1 - e^{-0.5} = 0.39$$
$$\Gamma_1 = e^{-0.5} - e^{-1} = 0.24$$

A state representation is obtained from (2.12)

$$\begin{pmatrix} x(k+1) \\ u(k-1) \\ u(k) \end{pmatrix} = \begin{pmatrix} \Phi & \Gamma_1 & \Gamma_0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ u(k-2) \\ u(k-1) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ u(k-2) \\ u(k-1) \end{pmatrix}$$

b.   The pulse transfer function is

$$H(z) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} z - \Phi & -\Gamma_1 & -\Gamma_0 \\ 0 & z & -1 \\ 0 & 0 & z \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \frac{\Gamma_0 z + \Gamma_1}{z^2(z - \Phi)} = \frac{0.39z + 0.24}{z^2(z - 0.37)}$$

Some calculations give

$$h(k) = \begin{cases} 0 & k \leq 1 \\ \Gamma_0 & k = 2 \\ \Gamma_0 e^{-(k-2)} + \Gamma_1 e^{-(k-3)} & k \geq 3 \end{cases}$$

Using Theorem 2.4 and the definition of the z-transform (2.27) we can also get the pulse response by expanding $H(z)$ in z^{-1}, i.e.,

$$H(z) = 0.39z^{-2} + 0.38z^{-3} + 0.14z^{-4} + 0.05z^{-5} + 0.02z^{-6} + \dots$$

The pulse response is now given by the coefficients in the series expansion.

c.   There are three poles $p_1 = p_2 = 0$ and $p_3 = 0.37$ and one zero $z_1 = -0.62$.

Problem 2.14

Sampling the given differential equation using the same procedure as in Problem 2.13 gives

$$a = e^{-\alpha h}$$

Using the definition of τ' and with $h = 1$ it follows that $d = 4$ and

$$\tau = 3 + \tau'$$

Further

$$b_3 = \int_0^{h-\tau'} e^{-\alpha s} b \, ds = \frac{b}{\alpha}\left(1 - e^{-\alpha(h-\tau')}\right)$$

$$b_4 = e^{-\alpha(h-\tau')} \int_0^{\tau'} e^{-\alpha s} b \, ds$$

$$= \frac{b}{\alpha}\left(e^{-\alpha(h-\tau')} - e^{-\alpha h}\right)$$

Straightforward calculations give

$$\frac{ab_3 + b_4}{b_3 + b_4} = e^{-\alpha(h-\tau')}$$

Thus

$$\tau' = \frac{1}{\alpha}\ln\left(\frac{ab_3 + b_4}{b_3 + b_4}\right) + h$$

and

$$\tau = 4 - \frac{1}{\ln a}\ln\left(\frac{ab_3 + b_4}{b_3 + b_4}\right)$$

where it has been used that

$$\alpha = -\ln a$$

Problem 2.15

$$y(k) - 0.5y(k-1) = u(k-9) + 0.2u(k-10) \Leftrightarrow$$
$$y(k+10) - 0.5y(k+9) = u(k+1) + 0.2u(k)$$
$$(q^{10} - 0.5q^9)y(k) = (q+0.2)u(k)$$
$$\begin{cases} A(q) = q^{10} - 0.5q^9 \\ \\ B(q) = q + 0.2 \end{cases} \qquad \text{System order= deg } A(q) = 10$$

Also

$$\left(1 - 0.5q^{-1}\right)y(k) = \left(1 + 0.2q^{-1}\right)u(k-9)$$

$$\Rightarrow \begin{cases} A^*\left(q^{-1}\right) = 1 - 0.5q^{-1} \\ \\ B^*\left(q^{-1}\right) = 1 + 0.2q^{-1} \end{cases} \qquad A^*\left(q^{-1}\right)y(k) = B^*\left(q^{-1}\right)u(k-d)$$

$$d = 9$$

Pole excess $= \deg A(q) - \deg B(q) = 9 = d$ (cf. CCS p. 49).

Remark

$$\frac{B(q)}{A(q)} = \frac{q+0.2}{q^{10} - 0.5q^9} = \frac{q^{-10}(q+0.2)}{q^{-10}\left(q^{10} - 0.5q^1\right)} = \frac{q^{-9} + 0.2q^{-10}}{1 - 0.5q^{-1}} =$$

$$= q^{-9}\frac{\left(1 + 0.2q^{-1}\right)}{\left(1 - 0.5q^{-1}\right)} = q^{-d}\frac{B^*\left(q^{-1}\right)}{A^*\left(q^{-1}\right)}$$

Problem 2.16

FIR filter:

$$H^*(q^{-1}) = b_0 + b_1 q^{-1} + \ldots + b_n q^{-n}$$

$$y(k) = \left(b_0 + b_1 q^{-1} + \ldots + b_n q^{-n} \right) u(k) =$$

$$= b_0 u(k) + b_1 u(k-1) + \ldots + b_n u(k-n)$$

$$\Rightarrow y(k+n) = b_0 u(k+n) + b_1 u(k+n-1) + \ldots + b_n u(k)$$

$$\Rightarrow q^n y(k) = \left(b_0 q^n + b_1 q^{n-1} + \ldots + b_n \right) u(k)$$

$$\Rightarrow H(q) = \frac{b_0 q^n + b_1 q^{n-1} + \ldots + b_n}{q^n} = b_0 + \frac{b_1 q^{n-1} + \ldots + b_n}{q^n}$$

n:th order system

$$H(q) = D + \frac{B(q)}{A(q)}$$

b. Observable canonical form:

$$\Phi = \begin{pmatrix} -a_1 & 1 & 0 & \ldots & 0 \\ -a_2 & 0 & 1 & \ldots & 0 \\ & & & & 1 \\ -a_n & 0 & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & & 0 \\ & & & & 1 \\ 0 & & & & 0 \end{pmatrix} \quad \Gamma = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} \qquad D = b_0$$

Problem 2.17

$$y(k+2) - 1.5y(k+1) + 0.5y(k) = u(k+1) \Leftrightarrow$$

$$q^2 y(k) - 1.5q y(k) + 0.5 y(k) = q u(k)$$

Use the z-transform to find the output sequence when

$$u(k) = \begin{cases} 0 & k < 0 \\ 1 & k \geq 0 \end{cases} \qquad \begin{cases} y(0) = 0.5 \\ y(-1) = 1 \end{cases}$$

Table 2.2 (page 57):

$$\mathcal{Z}\left(q^n f\right) = z^n \left(F - F_1 \right); \quad F_1(z) = \sum_{j=0}^{n-1} f(jh) z^{-j}$$

$$\Rightarrow z^2 \left(Y - y(0) - y(1)z^{-1} \right) - 1.5z \left(Y - y(0) \right) + 0.5Y = z\left(U - u(0) \right)$$

Compute

$$y(1) = u(0) - 0.5y(-1) + 1.5y(0) = 1 - 0.5 + 0.75 = 1.25$$

$$\Rightarrow z^2\left(Y - 0.5 - 1.25z^{-1}\right) - 1.5z(Y - 0.5) + 0.5Y = z(U - 1)$$

$$\left(z^2 - 1.5z + 0.5\right)Y - 0.5z^2 - 1.25z + 0.75z = zU - z$$

$$Y(z) = \frac{0.5z^2 - 0.5z}{z^2 - 1.5z + 0.5} + \frac{z}{z^2 - 1.5z + 0.5}\,U(z) =$$

$$= \frac{0.5z(z-1)}{(z-1)(z-0.5)} + \frac{z}{(z-1)(z-0.5)}\,U(z)$$

$$U = \frac{z}{z-1} \quad (\text{step}) \Rightarrow$$

$$Y(z) = \frac{0.5z}{z - 0.5} + \frac{z^2}{(z-1)^2(z-0.5)} = \frac{0.5z}{z - 0.5} + \frac{2}{(z-1)^2} + \frac{1}{z - 0.5}$$

Table 2.3 (page 59) gives via inverse transformation.

$$\mathcal{Z}^{-1}\left(\frac{z}{z - e^{-1/T}}\right) = e^{-k/T} \quad ; \quad e^{-1/T} = 0.5 \Rightarrow T = \frac{1}{\ln 2}$$

$$\mathcal{Z}^{-1}\left(z^{-1}\frac{z}{z - 0.5}\right) = e^{-(k-1)\cdot\ln 2}$$

$$\mathcal{Z}^{-1}\left(\frac{1}{(z-1)^2}\right) = \mathcal{Z}^{-1}\left(z^{-1}\frac{z}{(z-1)^2}\right) = k - 1 \qquad (h = 1)$$

$$\Rightarrow y(k) = 0.5e^{-k\cdot\ln 2} + 2(k-1) + e^{-(k-1)\ln 2}$$

$$\begin{cases} y(1) &= 1.25 \\ y(0) &= 0.5 \\ y(-1) &= 1 \end{cases}$$

Checking the result:

$$y(2) = 0.5e^{-2\ln 2} + 2 + e^{-\ln 2} = \frac{1}{8} + 2 + \frac{1}{2} = \frac{21}{8}$$

$$y(2) = u(1) + 1.5y(1) - 0.5y(0) = 1 + 1.5 \cdot 1.25 - 0.5^2$$

$$= 1 + \frac{3}{2} \cdot \frac{5}{4} - \frac{1}{4} = \frac{21}{8}$$

Problem 2.18:

Verify that

$$\mathcal{Z}\left(\frac{1}{2}(kh)^2\right) = \frac{h^2 z(z+1)}{2(z-1)^3} \qquad \text{(cf. Table 2.3)}$$

$$F(z) = \sum_{k=0}^{\infty} \frac{1}{2}(kh)^2 z^{-k} = \frac{h^2}{2}\sum_{k=0}^{\infty} k^2 z^{-k} = \frac{h^2}{2} f(z^{-1})$$

$$\sum_{k=0}^{\infty} z^{-k} = \frac{z}{z-1}; \qquad \text{differentiate twice}$$

$$\frac{d}{dz}\left(\Sigma z^{-k}\right) = \sum -k z^{-k-1} = \frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{(z-1)-z}{(z-1)^2} = \frac{-1}{(z-1)^2}$$

$$\Rightarrow (\text{multiply by } z) \quad \sum k z^{-k} = \frac{z}{(z-1)^2}$$

$$\begin{cases} \dfrac{d^2}{dz^2}\left(\Sigma z^{-k}\right) = \sum -k(-k-1)z^{-k-2} = \Sigma(k^2+k)z^{-k-2} \\[2mm] \dfrac{d^2}{dz^2}\left(\dfrac{z}{z-1}\right) = \dfrac{2(z-1)}{(z-1)^4} = \dfrac{2}{(z-1)^3} \end{cases}$$

$$\left(\text{multiplication by } z^2 \text{ gives} \Rightarrow \Sigma(k^2+k)z^{-k} = \frac{2z^2}{(z-1)^3}\right).$$

$$f(z^{-1}) = \sum_{0}^{\infty} k^2 z^{-k} = \frac{2z^2}{(z-1)^3} - \frac{z}{(z-1)^2} = \frac{z(z+1)}{(z-1)^3}$$

$$\Rightarrow F(z) = \frac{h^2}{2} f(z^{-1}) = \frac{h^2}{2}\frac{z(z+1)}{(z-1)^3}$$

Remark A necessary condition for convergence of $f(z^{-1})$ is that f is analytic for $|z| > 1$. In that case there exists a power series expansion and termwise differentiation is allowed.

Double integrator: The first step is to translate $G(s)$ to the corresponding pulse transfer operator $H(z)$. Use the method of page 58.
The sampled system.

$$u(k) = \text{step} \Rightarrow U(z) = \frac{z}{z-1} \Rightarrow U(s) = \frac{1}{s}$$

$$Y(s) = G(s) \cdot \frac{1}{s} = \frac{1}{s^3}$$

$$\Rightarrow Y(z) = \frac{h^2}{2}\frac{z(z+1)}{(z-1)^3} \qquad (Table\ 3.3)$$

$$Y(z) = H(z)U(z) \Rightarrow$$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{h^2}{2}\frac{z(z+1)}{(z-1)^3} \cdot \frac{z-1}{z} = \frac{h^2}{2}\frac{(z+1)}{(z-1)^2}$$

cf. example A1: $H(z) = C(zI - \Phi)^{-1}\Gamma$.

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Problem 2.19

a. The transfer function of the continuous time system is

$$G(s) = \frac{bc}{(s+a)}$$

Equation (2.30) gives

$$H(z) = \frac{1}{z - e^{-ah}} \operatorname{Res}_{s=-a} \left(\frac{e^{sh} - 1}{s} \frac{bc}{s+a} \right)$$

$$= \frac{1}{z - e^{-ah}} \frac{e^{-ah} - 1}{-a} bc =$$

$$= \frac{bc}{a} \frac{1 - e^{-ah}}{z - e^{-ah}}$$

This is the same result as obtained by using Table 2.1.

b. The normalized motor has the transfer function

$$G(s) = \frac{1}{s(s+1)}$$

and we get

$$H(z) = \sum_{s=s_i} \frac{1}{z - e^{sh}} \operatorname{Res} \left(\frac{e^{sh} - 1}{s} \frac{1}{s(s+1)} \right)$$

To compute the residue for $s = 0$ we can use the series expansion of $(e^{sh} - 1)/s$ and get

$$H(z) = \frac{1}{z - e^{-h}} \frac{e^{-h} - 1}{(-1)^2} + \frac{h}{z - 1} =$$

$$= \frac{(z-1)(e^{-h} - 1) + h(z - e^{-h})}{(z - e^{-h})(z - 1)} =$$

$$= \frac{(e^{-h} - 1 + h)z + (1 - e^{-h} - he^{-h})}{(z - e^{-h})(z - 1)}$$

Compare Problem 2.11.

Problem 2.21

$$\frac{s + \beta}{s + \alpha} \qquad \text{is lead if} \quad \beta < \alpha$$

Consider the discrete time system

$$\frac{z + b}{z + a}$$

$$\arg \left(\frac{e^{i\omega h} + b}{e^{i\omega h} + a} \right) = \arg \left(\frac{\cos \omega h + b + i \sin \omega h}{\cos \omega h + a + i \sin \omega h} \right) =$$

$$= \arctan \left(\frac{\sin \omega h}{b + \cos \omega h} \right) - \arctan \left(\frac{\sin \omega h}{a + \cos \omega h} \right)$$
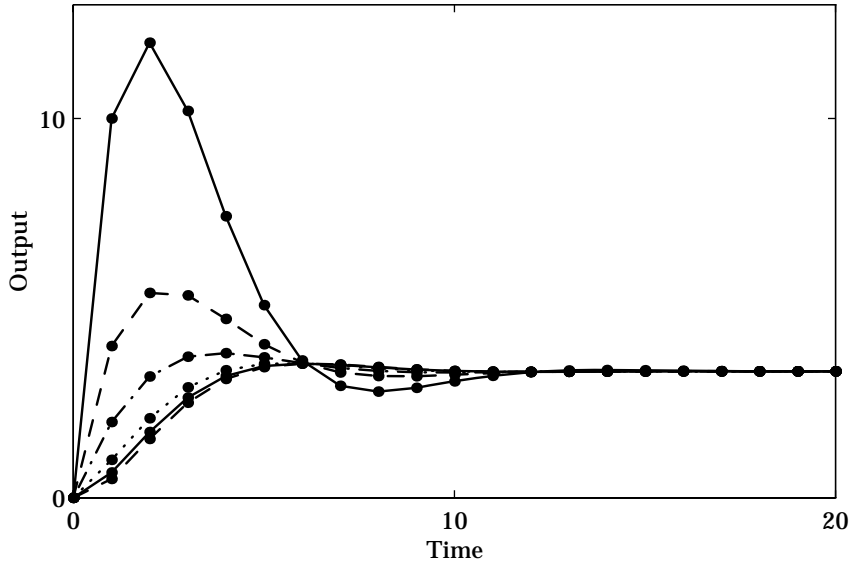
Figure 2.2 Simulation of the step response of the system in Problem 2.22 for $b = -0.9$ (upper solid), -0.75 (upper dashed), -0.50 (dash-dotted), 0 (dotted), 0.5 (lower solid), and 1 (lower dashed).

Phase lead if

$$\arctan\left(\frac{sin\omega h}{b + \cos \omega h}\right) > \arctan \frac{\sin \omega h}{a + \cos \omega h} \qquad 0 < \omega h < \pi$$

$$\frac{\sin \omega h}{b + \cos \omega h} > \frac{\sin \omega h}{a + \cos \omega h}$$

We thus get phase lead if $b < a$.

Problem 2.22

A state space representation of the system is

$$x(k + 1) = \begin{pmatrix} 1.1 & -0.4 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

$$y(k) = \frac{1}{b + 1} \begin{pmatrix} 1 & b \end{pmatrix} x(k)$$

Simulation of the system for $b = -0.9, -0.75, -0.5, 0, 0.5$ and 1 is shown in Fig. 2.2.

Problem 2.23

A state space representation of $G(s)$ is

$$\dot{x} = -ax + (b - a)u$$
$$y = x + u$$

The assumption that the system is stable implies that $a > 0$. Sampling this system gives

$$x(kh + h) = e^{-ah}x(kh) + (1 - e^{-ah})\frac{b - a}{a} u(kh)$$

$$y(kh) = x(kh) + u(kh)$$

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The pulse transfer operator is

$$H(q) = \frac{(b-a)(1 - e^{-ah})/a}{q - e^{-ah}} + 1 =$$

$$= \frac{q + \beta}{q - e^{-ah}}$$

where

$$\beta = \frac{b - a}{a}\left(1 - e^{-ah}\right) - e^{-ah} =$$

$$= \frac{b}{a}\left(1 - e^{-ah}\right) - 1$$

The inverse is stable if

$$\left|\frac{b}{a}\left(1 - e^{-ah}\right) - 1\right| < 1$$

or

$$0 < \frac{b}{a}(1 - e^{-ah}) < 2$$

Since $a > 0$ and $(1 - e^{-ah}) > 0$ then

$$0 < b < \frac{2a}{1 - e^{-ah}}$$

For $b > 0$ we have the cases

$$b \le 2a \qquad \text{The inverse is stable independent of } h.$$
$$b > 2a \qquad \text{Stable inverse if } ah < \ln(b/(b - 2a)).$$

Problem 2.28

$$y(k) = y(k - 1) + y(k - 2) \qquad y(0) = y(1) = 1$$

The equation has the characteristic equation

$$z^2 - z - 1 = 0$$

which has the solution

$$z_{12} = \frac{1 \pm \sqrt{5}}{2}$$

The solution has the form

$$y(k) = c_1\left(\frac{1 + \sqrt{5}}{2}\right)^k + c_2\left(\frac{1 - \sqrt{5}}{2}\right)^k$$

Using the initial values give

$$c_1 = \frac{1}{2\sqrt{5}}(\sqrt{5} + 1)$$

$$c_2 = \frac{1}{2\sqrt{5}}(\sqrt{5} - 1)$$

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Problem 2.29

The system is given by

$$\left(1 - 0.5q^{-1} + q^{-2}\right) y(k) = \left(2q^{-10} + q^{-11}\right) u(k)$$

Multiply by q^{11}. This gives

$$q^9 \left(q^2 - 0.5q + 1\right) y(k) = \left(2q + 1\right) u(k)$$

The system is of order 11 $\left(= \deg A(q)\right)$ and has the poles

$$p_{12} = \frac{1 \pm i\sqrt{15}}{4}$$

$$p_{3\ldots11} = 0 \qquad \text{(multiplicity 9)}$$

and the zero

$$z_1 = -0.5$$

Problem 2.30

The system H_1 has a pole on the positive real axis and can be obtained by sampling a system with the transfer function

$$G(s) = \frac{K}{s + a}$$

H_2 has a single pole on the negative real axis and has no continuous time equivalent.
The only way to get two poles on the negative real axis as in H_3 is when the continuous time system have complex conjugate poles given by

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

Further we must sample such that $h = \pi/\omega$ where

$$\omega = \omega_0\sqrt{1 - \zeta^2}$$

We have two possible cases

i) $$G_1(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

ii) $$G_2(s) = \frac{s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Sampling G_1 with $h = \pi/\omega$ gives, (see Table 2.1)

$$H(z) = \frac{(1 + \alpha)(q + \alpha)}{(q + \alpha)^2} = \frac{1 + \alpha}{q + \alpha}$$

where

$$\alpha = e^{-\zeta\omega_0 h}$$

i.e., we get a pole zero concellation. Sampling G_2 gives $H(z) = 0$. This implies that H_3 cannot be obtained by sampling a continuous time system. H_4 can be rewritten as

$$H_4(q) = 2 + \frac{0.9q - 0.8}{q(q - 0.8)}$$

The second part can be obtained by sampling a first order system with a time delay. Compare Example 2.8.

Problem 2.31

We can rewrite $G(s)$ as

$$G(s) = \frac{1}{s+1} + \frac{1}{s+3}$$

Using Table 2.1 gives

$$H(q) = \frac{1-e^{-h}}{q-e^{-h}} + \frac{1}{3}\cdot\frac{1-e^{-3h}}{q-e^{-3h}}$$

With $h = 0.02$ we get

$$H(q) = \frac{0.0392q - 0.0377}{(q-0.9802)(q-0.9418)}$$

Problem 2.32

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t-\tau) \qquad \tau = 0.2 \quad h = 0.3$$

$$\Phi = \mathcal{L}^{-1}\left((sI-A)^{-1}\right) = \mathcal{L}^{-1}\begin{pmatrix} s-1 & 0 \\ -1 & s-1 \end{pmatrix}^{-1} =$$

$$= \mathcal{L}^{-1}\begin{pmatrix} \dfrac{1}{s-1} & 0 \\ \dfrac{1}{(s-1)^2} & \dfrac{1}{s-1} \end{pmatrix} = \begin{pmatrix} e^h & 0 \\ he^h & e^h \end{pmatrix}$$

$$\Gamma_0 = \int_0^{h-\tau} \begin{pmatrix} e^s \\ se^s \end{pmatrix} ds = \begin{pmatrix} e^s \\ se^s \end{pmatrix}\Bigg|_0^{h-\tau} - \int_0^{h-\tau} \begin{pmatrix} 0 \\ e^s \end{pmatrix} ds =$$

$$= \begin{pmatrix} e^{h-\tau} - 1 \\ (h-\tau)e^{h-\tau} - e^{h-\tau} + 1 \end{pmatrix} = \begin{pmatrix} -1 + e^{h-\tau} \\ 1 + (h-\tau-1)e^{h-\tau} \end{pmatrix}$$

$$\Gamma_1 = e^{A(h-\tau)}\int_0^{\tau} \begin{pmatrix} e^{-s} \\ se^{-s} \end{pmatrix} ds = \begin{pmatrix} e^{h-\tau} & 0 \\ (h-\tau)e^{h-\tau} & e^{h-\tau} \end{pmatrix}\begin{pmatrix} -1 + e^{\tau} \\ 1 + (\tau-1)e^{\tau} \end{pmatrix} =$$

$$= \begin{pmatrix} e^{h-\tau}(-1 + e^{\tau}) \\ e^{h-\tau}(1 - h + \tau) + (h-1)e^h \end{pmatrix}$$

The pulse transfer operator is given by

$$H(q) = C(qI - \Phi)^{-1}(\Gamma_0 + \Gamma_1 q^{-1})$$

where

$$\Phi = \begin{pmatrix} 1.350 & 0 \\ 0.405 & 1.350 \end{pmatrix} \qquad \Gamma_0 = \begin{pmatrix} 0.105 \\ 0.005 \end{pmatrix} \qquad \Gamma_1 = \begin{pmatrix} 0.245 \\ 0.050 \end{pmatrix}$$

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Solutions to Chapter 3

Problem 3.1

Jury's criterion is used to test if the roots are inside the unit circle.

a.

$\underline{1}$	-1.5	0.9	
0.9	-1.5	1	$\alpha_2 = 0.9$

$\underline{0.19}$	-0.15	
-0.15	0.19	$\alpha_1 = -0.15/0.19 = -0.79$

$\underline{0.07}$

The roots are inside the unit circle since the underlined elements are positive. (The roots are $0.75 \pm 0.58i$.)

b.

$\underline{1}$	-3	2	-0.5	
-0.5	2	-3	1	$\alpha_3 = -0.5$

$\underline{0.75}$	-2	0.5	
0.5	-2	0.75	$\alpha_2 = \;\; 2/3$

$\underline{5/12}$	$-2/3$	
$-2/3$	$5/12$	$\alpha_1 = -8/5$

$-\underline{13/20}$

One of the underlined elements is negative and there is thus one root outside the unit circle. (The roots are 2.19 and $0.40 \pm 0.25i$.)

c.

$\underline{1}$	-2	2	-0.5	
-0.5	2	-2	1	$\alpha_3 = -0.5$

$\underline{0.75}$	-1	1	
1	-1	0.75	$\alpha_2 = \;\; 1.33$

$-\underline{0.58}$	0.33	
0.33	-0.58	$\alpha_1 = -0.57$

$-\underline{0.39}$

There are two roots outside the unit circle. (The roots are 0.35 and $0.82 \pm 0.86i$.)

d.

<u>1</u>	5	−0.25	−1.25	
−1.25	−0.25	5	1	$\alpha_3 = -1.25$

<u>−0.56</u>	4.69	6	
6	4.69	−0.56	$\alpha_2 = -10.71$

<u>63.70</u>	54.92	
54.92	63.70	$\alpha_1 = 0.86$

<u>16.47</u>

One root is outside the unit circle. (The roots are $-5, -0.5$, and 0.5.)

e.

<u>1</u>	−1.7	1.7	−0.7	
−0.7	1.7	−1.7	1	$\alpha_3 = -0.7$

<u>0.51</u>	−0.51	0.51	
0.51	−0.51	0.51	$\alpha_2 = 1$

<u>0</u>	0
0	0

The table breaks down since we can not compute α_1. There is one of the underlined elements that is zero which indicates that there is at least one root on the stability boundary. (The roots are 0.7 and $0.5 \pm 0.866i$. The complex conjugate roots are on the unit circle.)

Problem 3.2

The characteristic equation of the closed loop system is

$$z(z - 0.2)(z - 0.4) + K = 0 \qquad K > 0$$

The stability can be determined using root locus. The starting points are $z = 0, 0.2$ and 0.4. The asymptotes have the directions $\pm\pi/3$ and $-\pi$. The crossing point of the asymptotes is 0.2. To find where the roots will cross the unit circle let $z = a + ib$, where $a^2 + b^2 = 1$. Then

$$(a + ib)(a + ib - 0.2)(a + ib - 0.4) = -K$$

Multiply with $a - ib$ and use $a^2 + b^2 = 1$.

$$a^2 - 0.6a - b^2 + 0.08 + i(2ab - 0.6b) = -K(a - ib)$$

Equate real and imaginary parts.

$$a^2 - 0.6a - b^2 + 0.08 = -Ka$$
$$b(2a - 0.6) = Kb$$

If $b \neq 0$ then

$$a^2 - 0.6a - (1 - a^2) + 0.08 = -a(2a - 0.6)$$
$$4a^2 - 1.2a - 0.92 = 0$$

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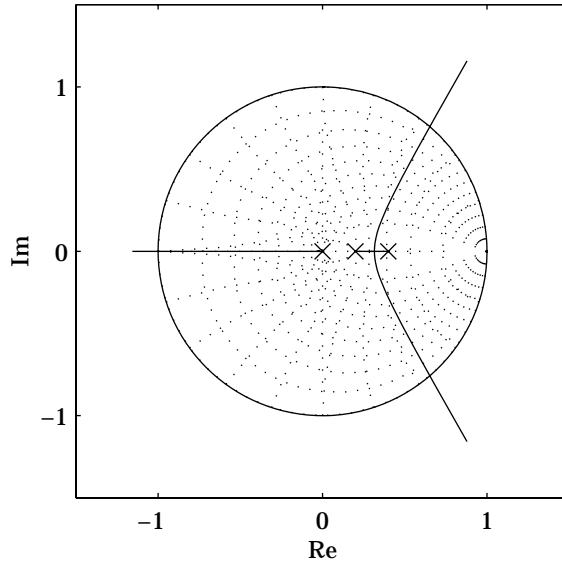
Figure 3.1 The root locus for the system in Problem 3.2.

The solution is

$$a_{12} = 0.15 \pm \sqrt{0.0225 + 0.23} = 0.15 \pm 0.502 = \left\{ \begin{array}{r} 0.652 \\ -0.352 \end{array} \right.$$

This gives $K = 2a - 0.6 = 0.70$ and -1.30. The root locus may also cross the unit circle for $b = 0$, i.e. $a = \pm 1$. A root at $z = -1$ is obtained when

$$-1(-1 - 0.2)(-1 - 0.4) + K = 0$$

or

$$K = 1.68$$

There is a root at $z = 1$ when

$$1(1 - 0.2)(1 - 0.4) + K = 0$$

or

$$K = -0.48$$

The closed loop system is thus stable for

$$K \leq 0.70$$

The root locus for $K > 0$ is shown in Fig. 3.1.

Problem 3.3

Sampling the system $G(s)$ gives the pulse transfer operator

$$H(q) = \frac{h}{q - 1}$$

The regulator is

$$u(kh) = K\left(u_c(kh - \tau) - y(kh - \tau)\right) = Ke(kh - \tau)$$

where $K > 0$ and $\tau = 0$ or h.

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a. When $\tau = 0$ then regulator is

$$u(kh) = Ke(kh)$$

and the characteristic equation of the closed loop system becomes

$$Kh + z - 1 = 0$$

The system is thus stable if

$$K < 2/h$$

When there is a delay of one sample $(\tau = h)$ then the regulator is

$$u(kh) = \frac{K}{q}e(kh)$$

and the characteristic equation is

$$Kh + z(z - 1) = z^2 - z + Kh = 0$$

The constant term is the product of the roots and it will be unity when the poles are on the unit circle. The system is thus stable if

$$K < 1/h$$

b. Consider the continuous system $G(s)$ in series with a time delay of τ seconds. The transfer function for the open loop system is

$$G_o(s) = \frac{K}{s}e^{-s\tau}$$

The phase function is

$$\arg G_o(i\omega) = -\frac{\pi}{2} - \omega\tau$$

and the gain is

$$|G_o(i\omega)| = \frac{K}{\omega}$$

The system is stable if the phase lag is less than π at the cross over frequency

$$|G_o(i\omega_c)| = 1$$
$$\Rightarrow \omega_c = K$$

The system is thus stable if

$$-\pi/2 - \omega_c\tau > -\pi$$

$$K < \frac{\pi/2}{\tau} = \begin{cases} \infty & \tau = 0 \\ \frac{\pi}{2h} & \tau = h \end{cases}$$

The continuous time system will be stable for all values of K if $\tau = 0$ and for $K < \pi/(2h)$ when $\tau = h$. This value is about 50% larger than the value obtained for the sampled system in b.
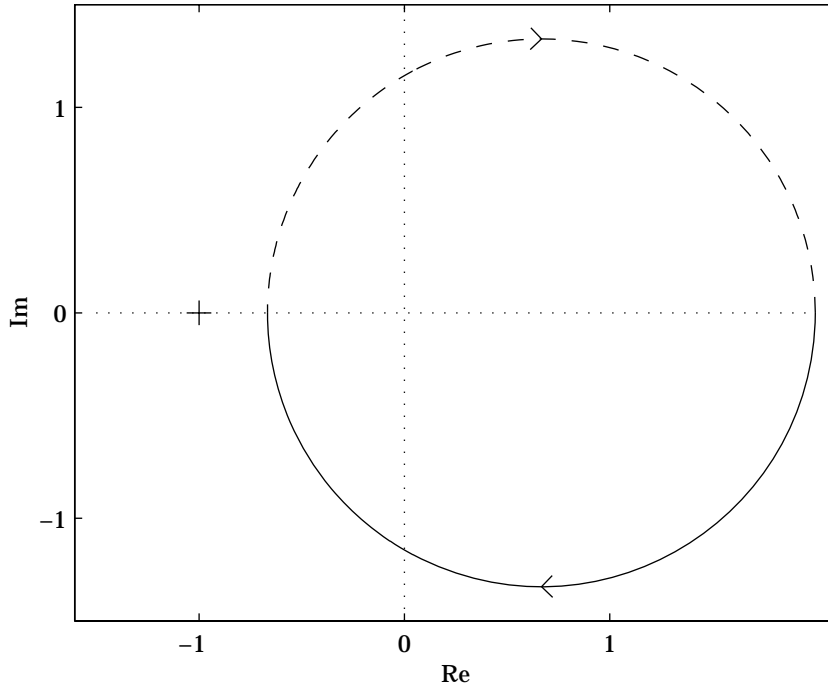
Figure 3.2 The Nyquist curve for the system in Problem 3.4.

Problem 3.4

The Nyquist curve is $H_o(e^{i\omega})$ for $\omega = [0, \pi]$. In this case

$$H_o(e^{i\omega}) = \frac{1}{e^{i\omega} - 0.5}$$

$$= \frac{1}{\cos\omega - 0.5 + i\sin\omega}$$

$$= \frac{\cos\omega - 0.5 - i\sin\omega}{\cos^2\omega + \sin^2\omega + 0.25 - \cos\omega}$$

$$= \frac{\cos\omega - 0.5 - i\sin\omega}{1.25 - \cos\omega}$$

For $\omega = 0$ then $H_o(1) = 2$ and for $\omega = \pi$ then $H_o(-1) = -2/3$. The argument is

$$\arg H_o = -\operatorname{arctg}\left(\frac{\sin\omega}{\cos\omega - 0.5}\right)$$

The argument is $-\pi/2$ for $\omega = \pi/3(\cos\omega - 0.5)$. The following table gives some values for the real and imaginary parts.

ω	$\mathrm{Re}\,H_o$	$\mathrm{Im}\,H_o$
0	2	0
$\pi/6$	0.97	-1.32
$\pi/3$	0	-1.16
$\pi/2$	-0.40	-0.80
$2\pi/3$	-0.57	-0.50
π	-0.67	0

The Nyquist curve is shown in Fig. 3.2.

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Problem 3.5

Consider the system

$$x(k+1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 0 & 1 \end{pmatrix} x(k)$$

We have

$$y(1) = x_2(1) = 0$$
$$y(2) = x_2(2) = x_1(1) + x_2(1) = 1 \qquad \Rightarrow \qquad x_1(1) = 1$$

Further

$$x(2) = \Phi x(1) + \Gamma u(1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot 1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x(3) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (-1) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The possibility to determine $x(1)$ from $y(1)$, $y(2)$ and $u(1)$ is called observability.

Problem 3.6

a.   The observability matrix is

$$W_o = \begin{pmatrix} C \\ C\Phi \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$$

The system is not observable since $\det W_o = 0$.

b.   The controllability matrix is

$$W_c = \begin{pmatrix} \Gamma & \Phi\Gamma \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & 1 \end{pmatrix}$$

$\det W_c = 2$ and the system is reachable.

Problem 3.7

The controllability matrix is

$$W_c = \begin{pmatrix} \Gamma & \Phi\Gamma \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0.5 & 0 \end{pmatrix}$$

For instance, the first two colums are linearly independent. This implies that W_c has rank 2 and the system is thus reachable. From the input u' we get the system

$$x(k+1) = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u'(k)$$

In this case

$$W_c = \begin{pmatrix} 0 & 0 \\ 1 & 0.5 \end{pmatrix}$$

$\operatorname{rank} W_c = 1$ and the system is not reachable from u'.

Problem 3.8

a.

$$x(k+1) = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u(k)$$

$$x(1) = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ u(0) \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 + u(0) \\ 0 \end{pmatrix}$$

$$x(2) = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 3 + u(0) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u(1) \\ 0 \end{pmatrix} = \begin{pmatrix} 3 + u(0) \\ u(1) \\ 0 \end{pmatrix}$$

$$\left. \begin{aligned} u(0) &= -3 \\ u(1) &= 0 \end{aligned} \right\} \quad \Rightarrow x(2) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$$

b.   Two steps, in general it would take 3 steps since it is a third order system.

c.

$$W_c = \begin{pmatrix} \Gamma & \Phi\Gamma & \Phi^2\Gamma \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

not full rank $\Rightarrow$ not reachable, but may be controlled.
$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ is not in the column space of W_c and can therefore not be reached from the origin. It is easily seen from the state space description that x_3 will be 0 for all $k > 0$.

Problem 3.11

The closed loop system is

$$y(k) = H_{cl}(q)u_c(k) = \frac{H_cH}{1 + H_cH} u_c(k)$$

a.   With $H_c = K$ where $K > 0$ we get

$$y(k) = \frac{K}{q^2 - 0.5q + K} u_c(k)$$

Using the conditions in Example 3.2 we find that the closed loop system is stable if $K < 1$. The steady state gain is

$$H_{cl}(1) = \frac{K}{K + 0.5}$$

b.   With an integral controller we get

$$H_{cl}(q) = \frac{Kq}{q(q-1)(q-0.5) + Kq} = \frac{Kq}{q(q^2 - 1.5q + 0.5 + K)}$$

The system is stable if

$$0.5 + K < 1 \qquad \text{and} \qquad 0.5 + K > -1 + 1.5$$

and we get the condition

$$0 < K < 0.5$$

The steady state gain is $H_{cl}(1) = 1$.

Problem 3.12

The z-transform for a ramp is given in Table 2.3, also compare Example 3.13, and we get

$$U_c(z) = \frac{z}{(z-1)^2}$$

Using the pulse transfer functions from Problem 3.11 and the final value theorem gives

$$\lim_{k \to \infty} e(k) = \lim_{k \to \infty} \left(u_c(k) - y(k) \right)$$

$$= \lim_{z \to 1} \left(\frac{z-1}{z} (1 - H_c(z)) U_c(z) \right)$$

if K is chosen such that the closed loop system is stable.

a.   To use the final value theorem in Table 2.3 it is required that $(1 - z^{-1})F(z)$ does not have any roots on or outside the unit circle. For the proportional controller we must then look at the derivative of the error, i.e.

$$\lim_{k \to \infty} e'(k) = \lim_{z \to 1} \frac{z-1}{z} \frac{z^2 - 0.5z + K - K}{z^2 - 0.5z + K} \frac{z}{(z-1)} = \frac{0.5}{K + 0.5}$$

The derivative is positive in steady-state and the reference signal and the output are thus diverging.

b.   For the integral controller we get

$$\lim_{k \to \infty} e(k) = \lim_{z \to 1} \frac{z-1}{z} \frac{z(z^2 - 1.5z + 0.5 + K - K)}{z(z^2 - 1.5z + 0.5 + K)} \frac{z}{(z-1)^2} = \frac{0.5}{K}$$

Problem 3.13

Consider the system

$$G(s) = \frac{s+1}{s^2 + 0.2s + 1}$$

The damping of the system is $\zeta = 0.1$ and the undamped natural frequency is $\omega_0 = 1$. The step response of the system will have an oscillation with the frequency

$$\omega = \omega_0 \sqrt{1 - \zeta^2} = \sqrt{0.99} \text{ rad/s}$$

The sampled system will not detect this oscillation if $h = k2\pi/\omega$. The frequency ω will then have the alias $\omega' = 0$.
Fig. 3.3 shows the continuous time and the sampled step responses when $h = 2\pi/\omega = 6.3148$. To formalize the analysis we can sample the system with $h = 2\pi/\omega$. The pulse transfer function is (Table 2.1)

$$H(q) = \frac{(1-\alpha)(q-\alpha)}{(q-\alpha)^2} = \frac{1-\alpha}{q-\alpha}$$

where $\alpha = \exp(-0.1h)$. There is a pole zero cancellation and only the first order exponential mode is seen at the sampling points.
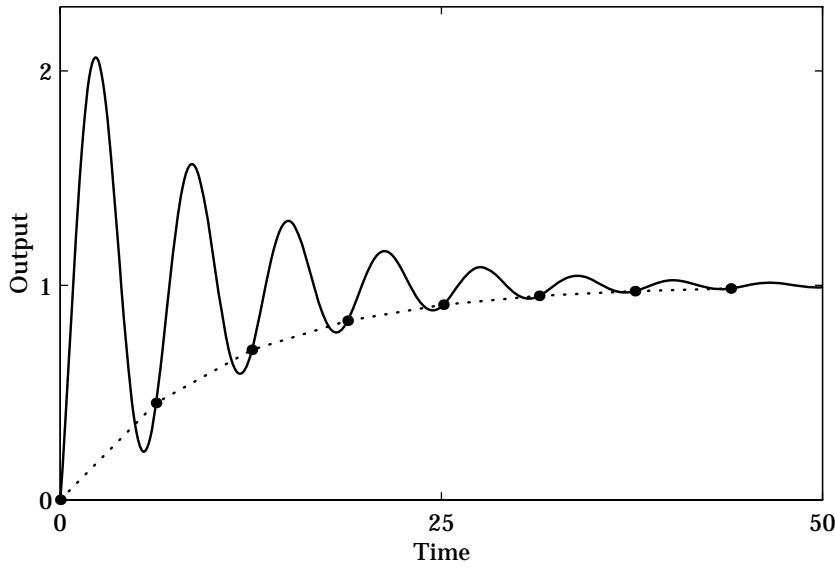
Figure 3.3 The continuous time (solid) and sampled step (dots) responses for the system in Problem 3.13.

Problem 3.14

a. When $H_r = K$ then the pulse transfer function for the closed loop system is

$$H_c(q) = \frac{K(0.030q + 0.026)}{q^2 - 1.65q + 0.68 + K(0.030q + 0.026)}$$

The characteristic equation is

$$z^2 + (0.030K - 1.65)z + 0.68 + 0.026K = 0$$

The closed loop system is stable if

$$0.68 + 0.026K < 1$$
$$0.68 + 0.026K > -1 - 1.65 + 0.030K$$
$$0.68 + 0.026K > -1 + 1.65 - 0.030K$$

This gives the stability region

$$-0.54 < K < 12.31$$

The steady state gain is

$$H_c(1) = \frac{0.056K}{0.03 + 0.056K}$$

The steady state error when the input is a unit step is

$$e(\infty) = 1 - H_c(1) = \frac{0.03}{0.03 + 0.056K}$$

The minimum error is about 4%.
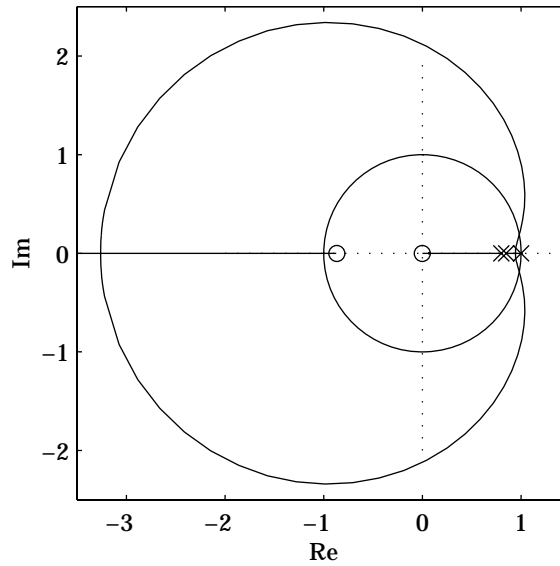
Figure 3.4 The root locus when the tank system in Problem 3.14 is controlled with an integral controller.
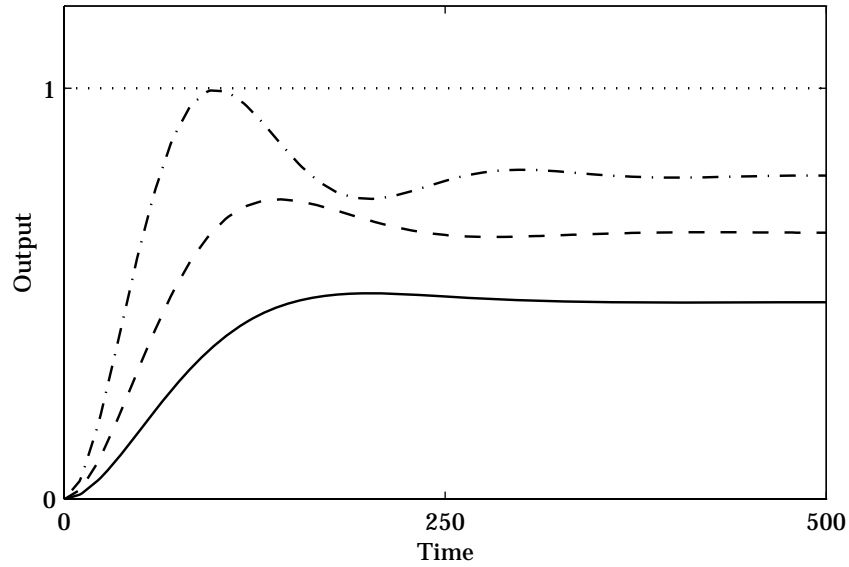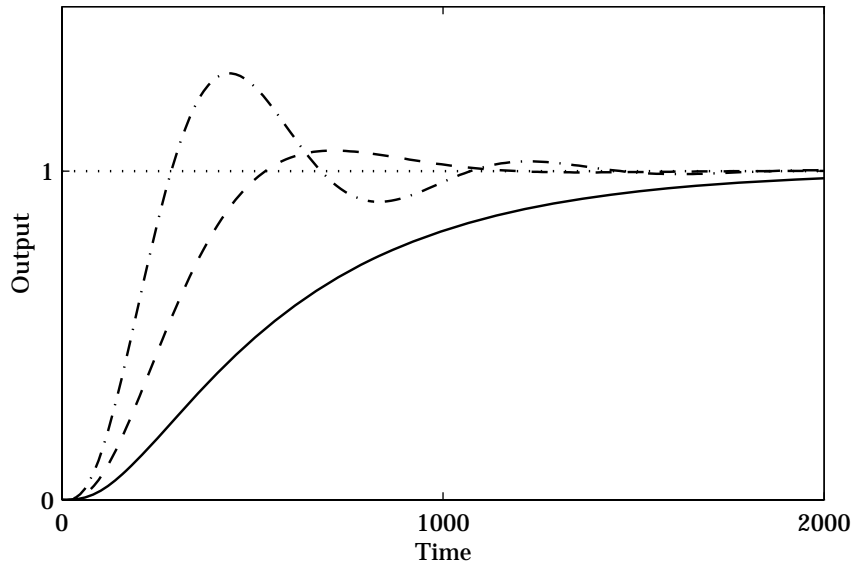

Figure 3.5 Step response of the system in Problem 3.14 when the proportional controller is used with $K = 0.5$ (solid), 1 (dashed), and 2 (dash-dotted).

b. If the closed loop system is stable then the steady state error is zero if the integral controller is used. The characteristic equation is then

$$(z^2 - 1.65z + 0.68)(z - 1) + Kz(0.03z + 0.026) = 0$$

A scetch of the root locus is shown in Fig. 3.4.

c. Fig. 3.5 and 3.6 show the step responses when the proportional controller with $K = 0.5$, 1, and 2 and the integral controller with $K = 0.01$, 0.025, and 0.05 are used on the tank system.

Figure 3.6 Step response of the system in Problem 3.14 when the integral controller is used with $K = 0.01$ (solid), 0.025 (dashed), and 0.05 (dash-dotted).

Problem 3.16

The pulse transfer function for the closed loop system is

$$H_c(z) = \frac{H_o(z)}{1 + H_o(z)}$$

and the characteristic equation is

$$z^2 + z + 0.16 + K(4z + 1) = z^2 + (1 + 4K)z + 0.16 + K = 0$$

Using the conditions in Example 3.2 give

$$0.16 + K < 1$$
$$0.16 + K > -1 + 1 + 4K$$
$$0.16 + K > -1 - 1 - 4K$$

This implies

$$K < 0.84$$
$$K < \frac{0.16}{3} = 0.053$$
$$K > -\frac{2.16}{5} = -0.432$$

Since it is assumed that $K > 0$ we get the condition

$$0 < K < 0.053$$

for stability.

Problem 3.17

Using (3.17) we find that the transformation is given by

$$T_c = \tilde{W}_c W_c^{-1}$$

where W_c is the controllability matrix of the original system and $\tilde{W}_c$ is the controllability of the transformed system.

$$W_c = \begin{pmatrix} \Gamma & \Phi\Gamma \end{pmatrix} = \begin{pmatrix} 3 & 11 \\ 4 & 11 \end{pmatrix}$$

The pulse transfer function of the system is

$$H(z) = C[zI - \Phi]^{-1}\Gamma = \begin{pmatrix} 5 & 6 \end{pmatrix} \begin{pmatrix} z-1 & -2 \\ -1 & z-2 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= \frac{39z + 4}{z^2 - 3z}$$

Thus the transformed system is given by

$$\tilde{\Phi} = \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix} \qquad \tilde{\Gamma} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \tilde{C} = \begin{pmatrix} 39 & 4 \end{pmatrix}$$

and

$$\tilde{W}_c = \begin{pmatrix} \tilde{\Gamma} & \tilde{\Phi}\tilde{\Gamma} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

The transformation to give the controllable form is thus

$$T_c = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 11 \\ 4 & 11 \end{pmatrix}^{-1} = \frac{1}{11} \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}$$

In the same way the tranformation to the observable form is given by

$$T_o = \tilde{W}_o^{-1} W_o = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 & 6 \\ 11 & 22 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ -4 & 4 \end{pmatrix}$$

Probem 3.18

a)   i   Poles are mapped as $z = e^{sh}$. This mapping maps the left half plane on the unit circle.

b)   i   see a)

c)   iii   When a system with relative degree > 1, "new" zeros appear as described on p. 63 CCS.

These zeros may very well be outside the unit circle.

d)   iii   See Example 2.18 p. 65 CCC

e)   ii   Sample the harmonic oscillator, Example A.3 p. 530 CCS.

For $h = 2\pi/\omega, \Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ not controllable

f)   ii   See e)

g)   iii   See p. 63, CCS

Problem 3.19

a. The controllability matrix is

$$W_c = \begin{pmatrix} \Gamma & \Phi\Gamma & \Phi^2\Gamma \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 0 \end{pmatrix}$$

Since one column is zero we find that the system is not reachable (det W_c= 0), see Theorem 3.7.

b. The system may be controllable if the matrix Φ is such that $\Phi^n x(0) = 0$. In this case

$$\Phi^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the origin can be reached from any initial condition $x(0)$ by the control sequence $u(0) = u(1) = u(2) = 0$.

Probem 3.20

$$H_o = \frac{1}{q^2 + 0.4q}$$

$$H_r = K$$

a.

$$H_{tot} = \frac{H_r H_o}{1 + H_r H_o} = \frac{K}{q^2 + 0.4q + K}$$

The system is stable if, see p. 82,

$$\left.\begin{array}{l} K < 1 \\ K > -1 + 0.4 \\ K > -1 - 0.4 \end{array}\right\} \qquad \Rightarrow -0.6 < K < 1$$

b.

$$e(k) = u_c - y$$

$$E(z) = \left(1 - H_{tot}(z)\right) U_c(z)$$

If K is chosen such that the closed loop system is stable (e.g. $K = 0.5$) the final-value theorem can be used.

$$\lim_{k\to\infty} e(k) = \lim_{z\to 1} \frac{z-1}{z} E(z) = \lim_{z\to 1} \frac{z-1}{z} \left(1 - \frac{K}{z^2 + 0.4z + K}\right) \frac{z}{z-1} =$$

$$= \lim_{z\to 1} \frac{z^2 + 0.4z}{z^2 + 0.4z + K} = \frac{1.4}{1.4 + K} = 0.74, \quad K = 0.5$$

Problem 3.21

$$y(k) = \frac{0.4q + 0.8}{q^2 - 1.2q + 0.5} u(k)$$
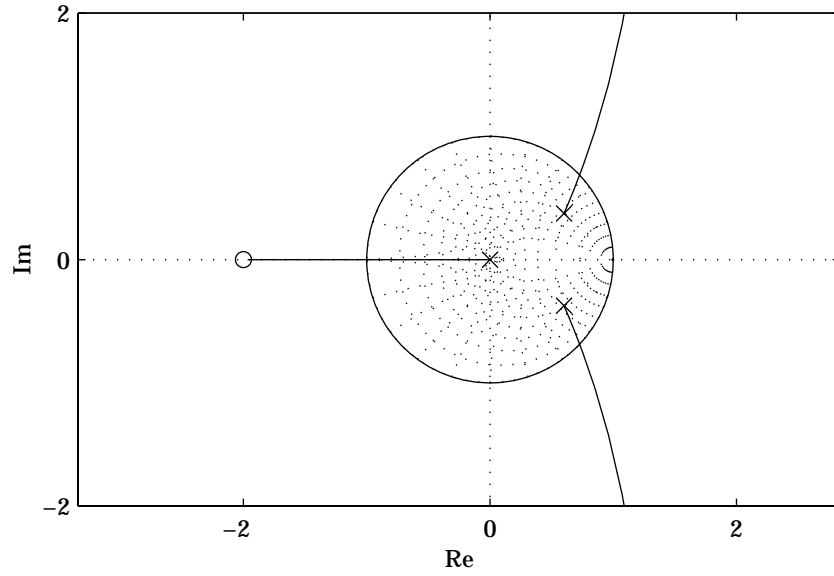
Figure 3.7 The root locus for Problem 3.21b.

a.

$$H_{tot} = \frac{H_r H_o}{1 + H_r H_o} = \frac{(0.4q + 0.8)K}{q^2 - 1.2q + 0.5 + K(0.4q + 0.8)} =$$

$$= \frac{(0.4q + 0.8)K}{q^2 + q(0.4K - 1.2) + 0.5 + 0.8K}$$

The system is stable if, see p. 82 CCS,

$$
\begin{aligned}
0.5 + 0.8K &< 1 & \Rightarrow && K &< \;\; 0.625 \\
0.5 + 0.8K &> -1 + 0.4K - 1.2 & \Rightarrow && K &> -6.75 \\
0.5 + 0.8K &> -1 - 0.4K + 1.2 & \Rightarrow && K &> -0.25
\end{aligned}
$$

$$-0.25 < K < 0.625$$

b.

$$H_r = \frac{K}{q}$$

$$H_{tot} = \frac{H_r H_o}{1 + H_r H_o} = \frac{(0.4q + 0.8)K}{q(q^2 - 1.2q + 0.5) + K(0.4q + 0.8)} =$$

$$= \frac{(0.4q + 0.8)K}{q^3 - 1.2q^2 + q(0.5 + 0.4K) + 0.8K}$$

Using root locus, Fig. 3.7, we can determine that the closed loop system is stable if

$$-0.25 < K < \frac{-17 + \sqrt{489}}{16}$$

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Solutions to Chapter 4

Problem 4.1

The characteristic equation of the closed loop system is

$$\det(zI - (\Phi - \Gamma L)) = z^2 - (a_{11} + a_{22} - b_2 \ell_2 - b_1 \ell_1)z + a_{11}a_{22} - a_{12}a_{21}$$
$$+ (a_{12}b_2 - a_{22}b_1)\ell_1 + (a_{21}b_1 - a_{11}b_2)\ell_2$$

Identifying with the desired characteristic equation gives

$$\begin{pmatrix} b_1 & b_2 \\ a_{12}b_2 - a_{22}b_1 & a_{21}b_1 - a_{11}b_2 \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = \begin{pmatrix} p_1 + \operatorname{tr} \Phi \\ p_2 - \det \Phi \end{pmatrix}$$

where $\operatorname{tr} \Phi = a_{11} + a_{22}$ and $\det \Phi = a_{11}a_{22} - a_{12}a_{21}$. The solution is

$$\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} a_{21}b_1 - a_{11}b_2 & -b_2 \\ -a_{12}b_2 + a_{22}b_1 & b_1 \end{pmatrix} \begin{pmatrix} p_1 + \operatorname{tr} \Phi \\ p_2 - \det \Phi \end{pmatrix}$$

where

$$\Delta = a_{21}b_1^2 - a_{12}b_2^2 + b_1 b_2(a_{22} - a_{11})$$

To check when $\Delta \neq 0$ we form the controllability matrix of the system

$$W_c = \begin{pmatrix} \Gamma & \Phi\Gamma \end{pmatrix} = \begin{pmatrix} b_1 & a_{11}b_1 + a_{12}b_2 \\ b_2 & a_{21}b_1 + a_{22}b_2 \end{pmatrix}$$

and we find that

$$\Delta = \det W_c$$

There exists a solution to the system of equations above if the system is controllable. For the double integrator with $h = 1$ and dead beat response we have $a_{11} = a_{12} = a_{22} = b_2 = 1$, $a_{21} = 0$, $b_1 = 0.5$, and $p_1 = p_2 = 0$.
This gives

$$\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = (-1) \begin{pmatrix} -1 & -1 \\ -0.5 & -0.5 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1.5 \end{pmatrix}$$

This is the same as given in Example 4.4.

Problem 4.2

In this case the desired characteristic equation is

$$(z - 0.1)(z - 0.25) = z^2 - 0.35z + 0.025.$$

Using the result from Problem 4.1 we find that

$$\Delta = 0.5$$

and L is obtained from

$$L^T = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} 0.5 & 0 \\ 0.1 & 1 \end{pmatrix} \begin{pmatrix} 0.75 \\ -0.025 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 0.1 \end{pmatrix}$$

To check the result we form

$$\Phi - \Gamma L = \begin{pmatrix} 1 & 0.1 \\ 0.5 & 0.1 \end{pmatrix} - \begin{pmatrix} 0.75 & 0.1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0.25 & 0 \\ 0.5 & 0.1 \end{pmatrix}$$

The matrix $\Phi - \Gamma L$ thus has the desired eigenvalues 0.25 and 0.1.

Problem 4.3

For the motor in Example A.2 we have

$$\Phi = \begin{pmatrix} \alpha & 0 \\ 1-\alpha & 1 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1-\alpha \\ h-1+\alpha \end{pmatrix}$$

where $\alpha = e^{-h}$. The dead beat controller is characterized by

$$p_1 = p_2 = 0.$$

From Problem 4.1 it follows that

$$
\begin{aligned}
L^T &= \frac{1}{\Delta} \begin{pmatrix} (1-\alpha)^2 - \alpha(h-1+\alpha) & 1-h-\alpha \\ 1-\alpha & 1-\alpha \end{pmatrix} \begin{pmatrix} 1+\alpha \\ -\alpha \end{pmatrix} \\
&= \frac{1}{\Delta} \begin{pmatrix} (1-\alpha)^2(1+\alpha) + \alpha^2(1-h-\alpha) \\ 1-\alpha \end{pmatrix} \\
&= \frac{1}{\Delta} \begin{pmatrix} 1-\alpha-h\alpha^2 \\ 1-\alpha \end{pmatrix}
\end{aligned}
$$

where

$$\Delta = (1-\alpha)^3 + (1-\alpha)^2(h-1+\alpha) = h(1-\alpha)^2$$

If $x(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$ then

$$u(0) = -\ell_1 - \ell_2 = -\frac{1-\alpha-h\alpha^2+1-\alpha}{h(1-\alpha)^2}$$

This function is shown in Fig. 4.1. We thus want to find h such that

$$\frac{1-\alpha-h\alpha^2+1-\alpha}{h(1-\alpha)^2} = 1$$

or

$$h = \frac{2(1-\alpha)}{2\alpha^2-2\alpha+1} = \frac{2(1-e^{-h})}{2e^{-2h}-2e^{-h}+1}$$

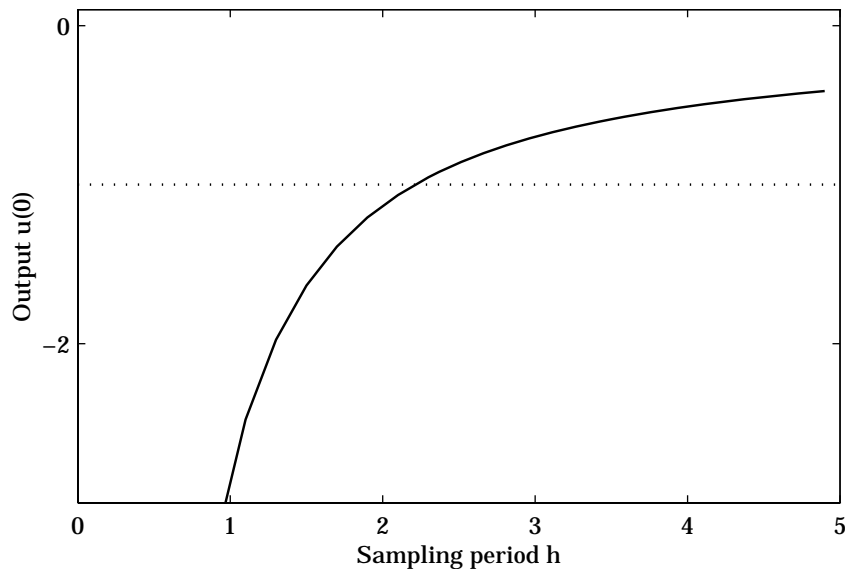This is a nonlinear equation of the form

$$h = f(h)$$

Figure 4.1 $u(0)$ s function of h in Problem 4.3.

One way to find h is to use the iterative scheme

$$h_{k+1} = f(h_k)$$

Starting with $h_0 = 2$ gives

k	h_{k+1}
0	2
1	2.26
2	2.20
3	2.21
4	2.21

with $h = 2.21$ we get $L = \begin{pmatrix} 0.49 & 0.51 \end{pmatrix}$. Fig. 4.2 shows the response of the motor controlled with a deadbeat controller when $h = 2.21$. We see that the system settles after two samples and that the constraint on the control signal is fulfilled.

Problem 4.4

a. Using the results in Problem 4.1 gives

$$L^T = \frac{1}{-0.0029} \begin{pmatrix} -0.0880 & -0.1600 \\ -0.0125 & 0.0100 \end{pmatrix} \begin{pmatrix} 0.5900 \\ -0.1595 \end{pmatrix} = \begin{pmatrix} 9.22 \\ 3.11 \end{pmatrix}$$

The problem is easily solved using Matlab by giving the commands

```
fi=[0.55 0.12; 0 0.67];
ga=[0.01; 0.16];
P=roots([1 -0.63 0.21])
L=place(fi,ga,P)
```
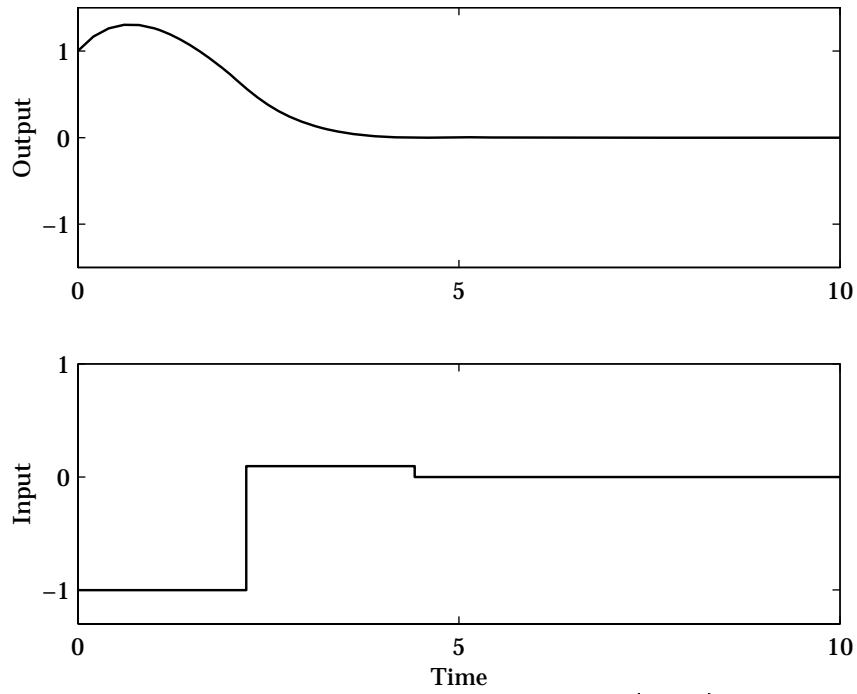
Figure 4.2 Deadbeat control of the motor when $x^T(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and $h = 2.21$.

b. From Example 4.4 we find that the continuous-time characteristic poly-
    nomial $s^2 + 2\zeta\omega s + \omega^2$ corresponds to $z^2 + p_1 z + p_2$ with

$$p_1 = -2e^{-\zeta\omega h}\cos(\omega h\sqrt{1-\zeta^2}) = -0.63$$
$$p_2 = e^{-2\zeta\omega h} = 0.21$$

This has the solution $\zeta \approx 0.7$ and $\omega \approx 5.6$, so the characteristic polynomial
becomes $s^2 + 7.8s + 31.7$. In Matlab you can do

```
rd=roots([1-0.63 0.21])
rc=Log(rd)/h
Ac=poly(rc)
```

The chosen sampling interval is higher than recommended by the rule of
thumb, since
$$\omega h \approx 1.1 > 0.6$$

c. The closed loop system when using $L = \begin{pmatrix} 9.22 & 3.11 \end{pmatrix}$ is shown in
    Fig. 4.3 when $x(0) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$.

Problem 4.5

a. In this case

$$W_o = \begin{pmatrix} C \\ C\Phi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0.22 & 1 \end{pmatrix}, \qquad W_o^{-1} = \begin{pmatrix} -4.55 & 4.55 \\ 1 & 0 \end{pmatrix}$$

$$\Omega = \begin{pmatrix} 0 \\ C\Gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0.03 \end{pmatrix},$$

$$\Psi = \Gamma - \Phi W_o^{-1}\Omega$$
$$= \begin{pmatrix} 0.22 \\ 0.03 \end{pmatrix} - \begin{pmatrix} 0.78 & 0 \\ 0.22 & 1 \end{pmatrix}\begin{pmatrix} -4.55 & 4.55 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0.03 \end{pmatrix} = \begin{pmatrix} 0.114 \\ 0 \end{pmatrix}$$
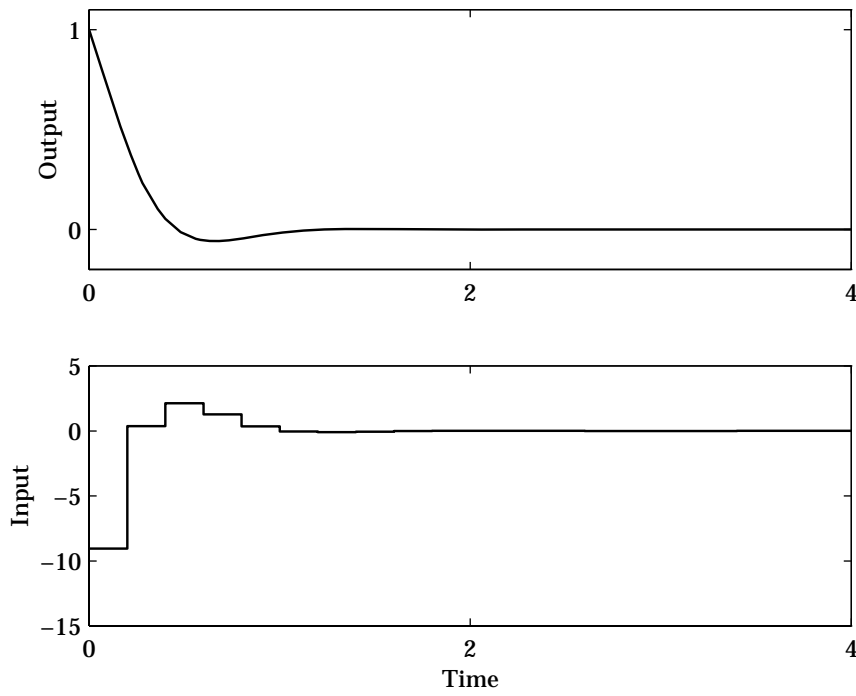
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Figure 4.3 The response and the control signal of the system in Problem 4.4 when $x(0) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and $L = \begin{pmatrix} 9.22 & 3.11 \end{pmatrix}$.

We get

$$\hat{x}(k) = \Phi W_o^{-1} \begin{pmatrix} y(k-1) \\ y(k) \end{pmatrix} + \Psi u(k-1)$$

$$= \begin{pmatrix} -3.55 & 3.55 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y(k-1) \\ y(k) \end{pmatrix} + \begin{pmatrix} 0.114 \\ 0 \end{pmatrix} u(k-1)$$

b. The dynamical observer (4.28) has the form

$$\hat{x}(k+1|k) = (\Phi - KC)\hat{x}(k|k-1) + \Gamma u(k) + Ky(k).$$

In this case we choose K such that the eigenvalues of $\Phi - KC$ are in the origin. Using the results from Problem 4.1 but with Φ^T and C^T instead of Φ and Γ gives

$$K = \begin{pmatrix} 2.77 \\ 1.78 \end{pmatrix}$$

c. The reduced order observer (4.32) has the form

$$\hat{x}(k|k) = (I - KC)(\Phi\hat{x}(k-1|k-1) + \Gamma u(k-1)) + Ky(k).$$

In this case we want to find K such that
i. $CK = 1$
ii. $(I - KC)\Phi$ has eigenvalues in the origin. The first condition implies that $k_2 = 1$. Further

$$(I - KC)\Phi = \begin{pmatrix} 1 & -k_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0.78 & 0 \\ 0.22 & 0 \end{pmatrix} = \begin{pmatrix} 0.78 - 0.22k_1 & -k_1 \\ 0 & 0 \end{pmatrix}$$

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The eigenvalues will be in the origin if

$$k_1 = 0.78/0.22 = 3.55.$$

The observer is then

$$\hat{x}(k|k) = \begin{pmatrix} 0 & -3.55 \\ 0 & 0 \end{pmatrix} \hat{x}(k-1|k-1) + \begin{pmatrix} 0.114 \\ 0 \end{pmatrix} u(k-1) + \begin{pmatrix} 3.55 \\ 1 \end{pmatrix} y(k)$$

Since $\hat{x}_2(k|k) = y(k)$ we get

$$\hat{x}(k|k) = \begin{pmatrix} -3.55 & 3.55 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y(k-1) \\ y(k) \end{pmatrix} + \begin{pmatrix} 0.114 \\ 0 \end{pmatrix} u(k-1)$$

which is the same as the observer obtained by direct calculation.

Problem 4.6

From Problem 2.10 we get for $h = 12$

$$x(kh + h) = \begin{pmatrix} 0.790 & 0 \\ 0.176 & 0.857 \end{pmatrix} x(kh) + \begin{pmatrix} 0.281 \\ 0.0296 \end{pmatrix} u(kh)$$

$$y(kh) = \begin{pmatrix} 0 & 1 \end{pmatrix} x(kh)$$

The continuous time poles of the system are -0.0197 and -0.0129. The observer should be twice as fast as the fastest mode of the open loop system. We thus choose the poles of the observer in

$$z = e^{-0.0394 \cdot 12} = 0.62$$

The desired characteristic equation of $\Phi - KC$ is thus

$$z^2 - 1.24z + 0.38 = 0$$

Using the results from Problem 4.1 gives

$$K = \begin{pmatrix} 0.139 \\ 0.407 \end{pmatrix}$$

Fig. 4.4 shows the states and the estimated states when $x(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$ and when $u(kh)$ is zero up to $t = 250$ and one thereafter.

Problem 4.7

The observer and the controller are described by

$$\hat{x}(k|k) = (I - KC)\Phi\hat{x}(k-1|k-1) + (I - KC)\Gamma u(k-1) + Ky(k)$$
$$u(k) = -L\hat{x}(k|k).$$

In the state equation both $\hat{x}$ and y have the time argument k. Introduce

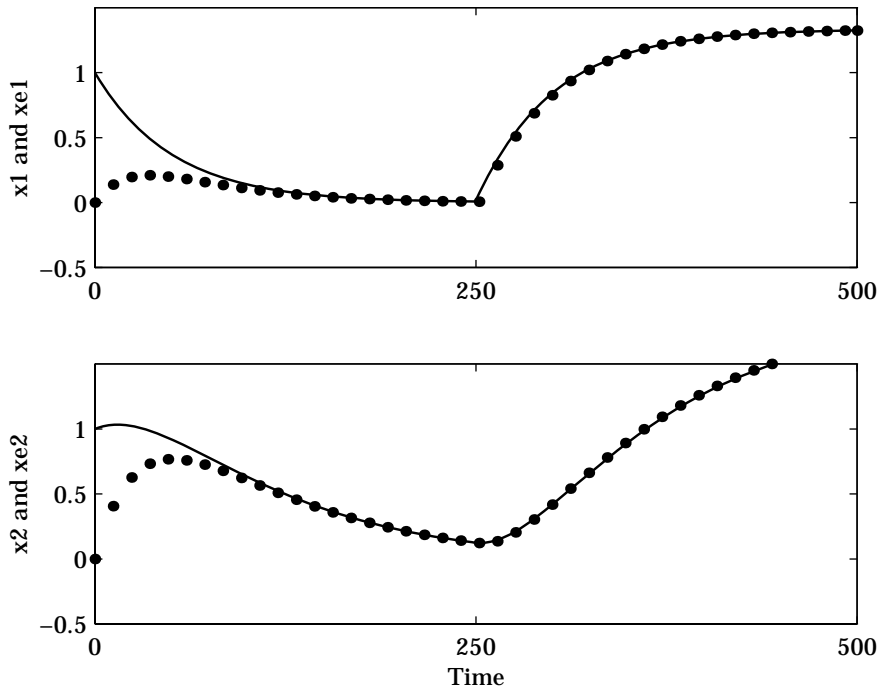$$\xi(k) = \hat{x}(k|k) - Ky(k)$$

Figure 4.4   The states (solid) and their estimates (dots) for the tank system in Problem 4.6

then

$$\begin{aligned}
\xi(k) &= (I - KC)\Phi[\xi(k-1) + Ky(k-1)] + (I - KC)\Gamma u(k-1) \\
&= (I - KC)\Phi\xi(k-1) + (I - KC)\Phi Ky(k-1) \\
&\quad - (I - KC)\Gamma L[\xi(k-1) + Ky(k-1)] \\
&= (I - KC)(\Phi - \Gamma L)\xi(k-1) + (I - KC)(\Phi - \Gamma L)Ky(k-1) \\
&= \Phi_o\xi(k-1) + \Gamma_o y(k-1)
\end{aligned}$$

The output of the regulator can be written

$$u(k) = -L\xi(k) - LKy(k) = C_o\xi(k) + D_o y(k).$$

The observer and the regulator can thus be written in the form given in the formulation of the problem.

Problem 4.8

The constant disturbance $v(k)$ can be described by the dynamical system

$$\begin{aligned}
w(k+1) &= w(k) \\
v(k) &= w(k)
\end{aligned}$$

The process can thus be described on the form given in (4.43) with

$$\Phi_w = 1, \qquad \Phi_{xw} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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a. If the state and v can be measured then we can use the controller

$$u(k) = -Lx(k) - L_w w(k).$$

This gives the closed loop system

$$x(k+1) = \Phi x(k) + \Phi_{xw} w(k) - \Gamma L x(k) - \Gamma L_w w(k)$$
$$= (\Phi - \Gamma L) x(k) + (\Phi_{xw} - \Gamma L_w) w(k)$$
$$y(k) = C x(k)$$

In general it is not possible to totally eliminate the influence of $w(k)$. This is only possible if $\Phi_{xw} - \Gamma L_w$ is the zero matrix. We will therefore only consider the situation at the output in steady state

$$y(\infty) = C[I - (\Phi - \Gamma L)]^{-1}(\Phi_{xw} - \Gamma L_w) w(\infty) = H_w(1) w(\infty)$$

The influence of w (or v) can be zero in steady state if

$$H_w(1) = 0$$

This will be the case if

$$L_w = \frac{1 - \varphi_{c22}}{\gamma_1(1 - \varphi_{c22}) + \gamma_2 \varphi_{c12}}$$

where φ_{cij} is the (i,j) element of $\Phi - \Gamma L$ and γ_i is the i:th element of Γ. Assume that L is determined to give a dead beat regulator then

$$L = \begin{pmatrix} 3.21 & 5.57 \end{pmatrix}$$

and

$$\Phi - \Gamma L = \begin{pmatrix} -0.142 & -0.114 \\ 0.179 & 0.142 \end{pmatrix}$$

and

$$L_w = 5.356$$

b. In this case is the state but not the disturbance measurable. The disturbance can now be calculated from the state equation

$$\Phi_{xw} w(k-1) = x(k) - \Phi x(k-1) - \Gamma u(k-1).$$

The first element in this vector equation gives

$$w(k-1) = [1 \ 0](x(k) - \Phi x(k-1) - \Gamma u(k-1))$$

Since $w(k)$ is constant and $x(k)$ is measurable it is possible to calculate $\hat{w}(k) = w(k-1)$. The following control law can now be used

$$u(k) = -Lx(k) - L_w \hat{w}(k)$$

where L_w is the same as in (a). Compared with the controller in (a) there is a delay in the detection of the disturbance.
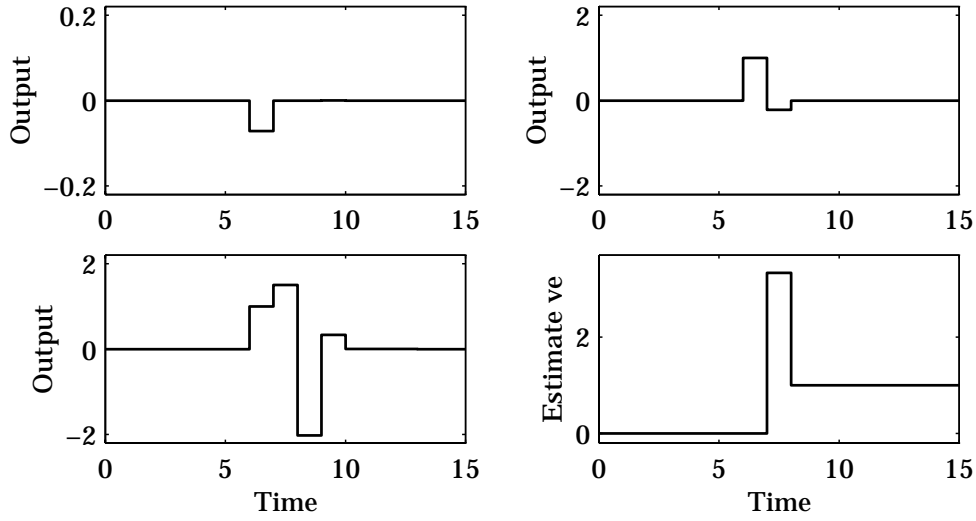
Figure 4.5 The output of the system in Problem 4.8 for the regulators in a) (upper left), b) (upper right) and c) (lower left and right). The estimate of v is also shown for case c). Notice the difference in scale in the upper left curve.

c.   If only the output is measurable then the state and the disturbance can be estimated using an observer of the form (4.41)

$$\begin{pmatrix} \hat{x}(k+1) \\ \hat{w}(k+1) \end{pmatrix} = \begin{pmatrix} \Phi & \Phi_{xw} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x}(k) \\ \hat{w}(k) \end{pmatrix} + \begin{pmatrix} \Gamma \\ 0 \end{pmatrix} u(k) + \begin{pmatrix} K \\ K_w \end{pmatrix} \varepsilon(k)$$

$$\varepsilon(k) = y(k) - C\hat{x}(k)$$

The gain vector can now be determined such that the error goes to zero provided the augmented system is observable. The error equation is

$$\begin{pmatrix} \tilde{x}(k+1) \\ \tilde{w}(k+1) \end{pmatrix} = \begin{pmatrix} \Phi - KC & \Phi_{xw} \\ -K_w C & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{w}(k) \end{pmatrix}$$

The characteristic equation of the system matrix for the error is

$$z^3 + (k_1 - 2.2)z^2 + (1.05 - 1.7k_1 + k_2 + k_w)z + 0.7k_1 + 0.15 - 0.7k_w - k_2 = 0.$$

The eigenvalues are in the origin if

$$k_1 = 2.2$$
$$k_2 = -0.6433$$
$$k_w = 3.3333.$$

The controller now has to be

$$u(k) = -L\hat{x}(k) - L_w\hat{w}(k)$$

where L and L_v are the same as in (a). The solutions above have the drawback that there may be an error in the output due to the disturbance if there are small errors in the model. Fig. 4.5 show that the output when the controllers in (a), (b) and (c) are used.

Problem 4.9

a. The state equation for the tank system when $h = 12$ was given in the solution to Problem 4.6. The desired characteristic equation is

$$z^2 - 1.55z + 0.64 = 0$$

Using the result in Problem 9.1 give

$$L = \begin{pmatrix} 0.251 & 0.8962 \end{pmatrix}$$

b. An integrator can be incorporated as shown in Section 4.5 by augmenting the system with

$$x_3(kh + h) = x_3(kh) + u_c(kh) - Cx(kh)$$

and using the control law

$$u(kh) = -Lx(kh) - \ell_3 x_3(kh) + \ell_c u_c(k)$$

The closed loop system is then

$$\begin{pmatrix} x(k+1) \\ x_3(k+1) \end{pmatrix} =$$

$$\begin{pmatrix} 0.790 - 0.281\ell_1 & -0.281\ell_2 & -0.281\ell_3 \\ 0.176 - 0.0296\ell_1 & 0.857 - 0.0296\ell_2 & -0.0296\ell_3 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x(k) \\ x_3(k) \end{pmatrix}$$

$$+ \begin{pmatrix} 0.281\ell_c \\ 0.0296\ell_c \\ 1 \end{pmatrix} u_c(k)$$

The characteristic equation is

$$z^3 + (-2.647 + 0.281\ell_1 + 0.0296\ell_2)z^2$$
$$+ (2.3240 - 0.5218\ell_1 - 0.0035\ell_2 - 0.0296\ell_3)z+$$
$$+ (-0.6770 + 0.2408\ell_1 - 0.0261\ell_2 - 0.0261\ell_3) = 0$$

Assume that two poles are placed in the desired location and that the third pole is in p. We get the following system of equations to determine the state feedback vector.

$$\begin{pmatrix} 0.281 & 0.0296 & 0 \\ -0.5218 & -0.0035 & -0.0296 \\ 0.2408 & -0.0261 & -0.0261 \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix} = \begin{pmatrix} 2.647 - 1.55 - p \\ -2.3240 + 1.55p + 0.64 \\ 0.6770 - 0.64p \end{pmatrix}$$

This gives

$$\begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix} = \begin{pmatrix} 3.279 - 3.028p \\ 5.934 - 5.038p \\ -1.617 + 1.617p \end{pmatrix}$$
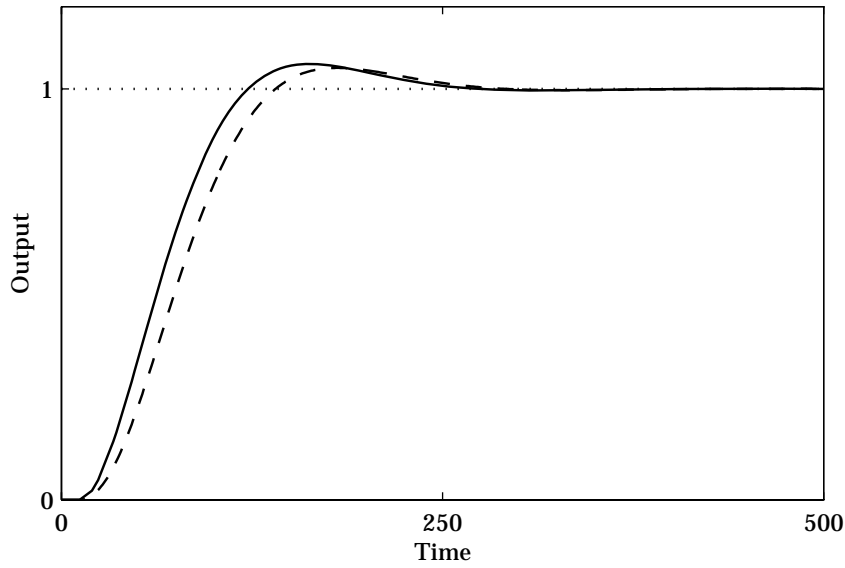
Figure 4.6 The stepresponse for the tank process when $p = 0$ (solid) and 0.5 (dashed), and when $\ell_c = 0$.
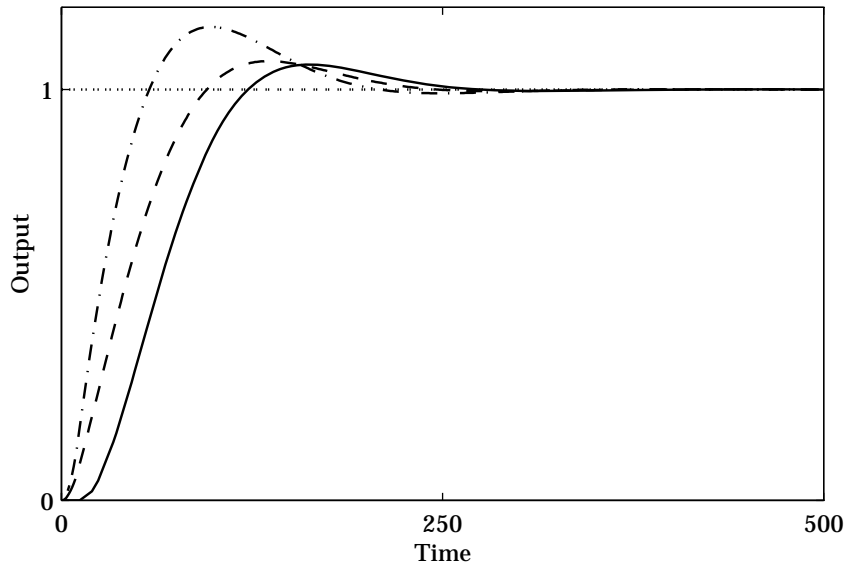

Figure 4.7 The step response for the tank process when $p = 0$ and when $\ell_c = 0$ (solid), 3 (dashed) and 6 (dash-dotted).

c.   The parameter ℓ_c will not influence the characteristic equation, but it is a feedforward term from the reference signal, see Fig. 4.11 CCS. Fig. 4.6 shows the step response for some values of p. Fig. 4.7 shows the influence of ℓ_c when $p = 0$.

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Problem 4.10

The process is

$$x(kh + h) = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} x(kh) + \begin{pmatrix} h^2/2 \\ h \end{pmatrix} u(k) + \begin{pmatrix} h^2/2 \\ h \end{pmatrix} v(k)$$

where $v(k)$ is a sinusoidal. I.e. it can be described by

$$w(kh + h) = \begin{pmatrix} \cos\omega_o h & -\sin\omega_o h \\ \sin\omega_o h & \cos\omega_o h \end{pmatrix} w(kh)$$

$$v(k) = \begin{pmatrix} 1 & 0 \end{pmatrix} w(kh)$$

The augmented system (9.33) is now

$$\begin{pmatrix} x(kh + h) \\ w(kh + h) \end{pmatrix} = \begin{pmatrix} 1 & h & h^2/2 & 0 \\ 0 & 1 & h & 0 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{pmatrix} \begin{pmatrix} x(kh) \\ w(kh) \end{pmatrix} + \begin{pmatrix} h^2/2 \\ h \\ 0 \\ 0 \end{pmatrix} u(kh)$$

where $\alpha = \cos\omega_o h$ and $\beta = \sin\omega_o h$. Assume first that the control law is

$$u(kh) = -Lx(kh) - L_w w(kh)$$

where

$$L = \begin{pmatrix} \frac{1}{h^2} & \frac{3}{2h} \end{pmatrix}$$

i.e., a dead beat controller for the states. Further if $L_w = \begin{pmatrix} 1 & 0 \end{pmatrix}$ we would totally eliminate v since v and u are influencing the system in the same way. Compare the discussion in the solution of Problem 4.8.

Since $x(kh)$ and $\xi(kh)$ cannot be measured we use the observer of the structure (9.35) where

$$\Phi_{xw} = \begin{pmatrix} h^2/2 & 0 \\ h & 0 \end{pmatrix} \quad \text{and} \quad \Phi_w = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

The error equation is then

$$\begin{pmatrix} \tilde{x}(k + 1) \\ \tilde{w}(k + 1) \end{pmatrix} = \begin{pmatrix} \Phi - KC & \Phi_{xw} \\ -K_w C & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{w}(k) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - k_1 & h & h^2/2 & 0 \\ -k_2 & 1 & h & 0 \\ -k_{w1} & 0 & \alpha & -\beta \\ -k_{w2} & 0 & \beta & \alpha \end{pmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{w}(k) \end{pmatrix}$$

Let the desired characteristic equation of the error be

$$(z - \gamma)^4 = 0$$

If $h = 1$ and $\omega_o = 0.1\pi$ then for $\gamma = 0.5$ we get the following system of equations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2.9022 & 1 & 0.5 & 0 \\ 2.9022 & -1.9022 & 0.0245 & -0.1545 \\ -1 & 1 & -0.4756 & -0.1545 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_{w1} \\ k_{w2} \end{pmatrix} = \begin{pmatrix} -4\gamma + 3.9022 \\ 6\gamma^2 - 5.8044 \\ -4\gamma^3 - 3.9022 \\ \gamma^4 - 1 \end{pmatrix}$$
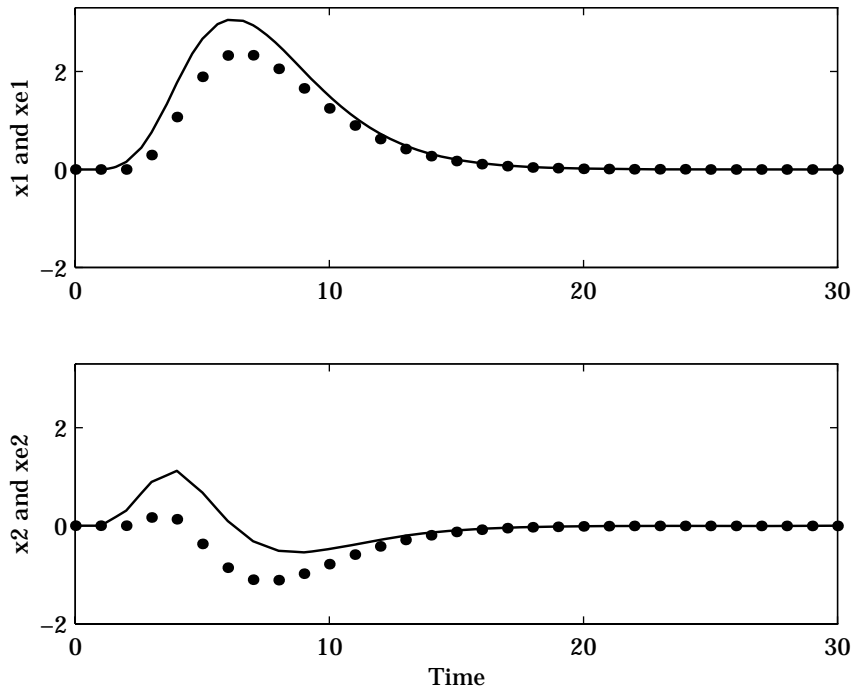
Figure 4.8 The states of the double integrator (solid) and their estimates (dots) when $v(t) = \sin(\omega_o t)$. The controller is defined by $L = \begin{pmatrix} 1 & 1.5 \end{pmatrix}$ and $L_w = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

which has the solution

$$\begin{cases} k_1 = 1.9022 \\ k_2 = 1.1018 \\ k_{w1} = 0.2288 \\ k_{w2} = 0.1877 \end{cases}$$

Fig. 9.8 shows the states of the double integrator and their estimates when $v(t) = \sin(\omega_o t)$. It is seen that the controller is able to eliminate the disturbance.

Problem 4.11

We have to determine the feedback vector L such that

a.

$$\Phi - \Gamma L = \begin{pmatrix} 0.9 - \ell_1 & -\ell_2 \\ 1 & 0.7 \end{pmatrix}$$

has all eigenvalues in the origin. This gives the condition

$$\det(\lambda I - \Phi + \Gamma L) = \lambda^2 + (-1.6 + \ell_1)\lambda + 0.63 - 0.7\ell_1 + \ell_2$$
$$= \lambda^2$$

I.e.,

$$-1.6 + \ell_1 = 0$$
$$0.63 - 0.7\ell_1 + \ell_2 = 0$$

or

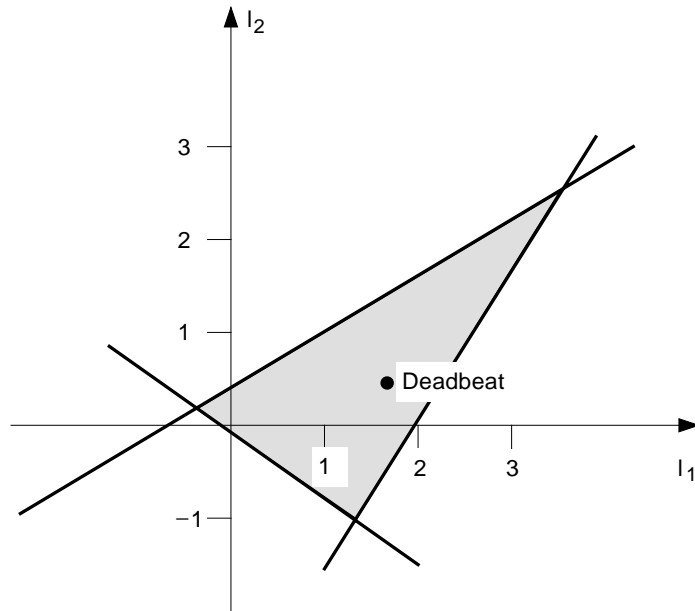$$L = \begin{pmatrix} 1.6 & 0.49 \end{pmatrix}$$

Figure 4.9 The stability area for L in Problem 4.11.

The stationary gain of the closed loop system is given by stationary gain of the

$$m \cdot C(I - \Phi + \Gamma L)^{-1}\Gamma = m$$

To get unit steady state gain we choose $m = 1$

b. The closed loop characteristic equation is stable if (See Example 3.2)

$$0.63 - 0.7\ell_1 + \ell_2 < 1$$

$$0.63 - 0.7\ell_1 + \ell_2 > -1 + (-1.6 + \ell_1)$$

$$0.63 - 0.67\ell_1 + \ell_2 > -1 - (-1.6 + \ell_1)$$

This gives

$$-0.7\ell_1 + \ell_2 < 0.37$$

$$-1.7\ell_1 + \ell_2 > -3.23$$

$$0.37\ell_1 + \ell_2 > -0.03$$

The stability area is shown in Fig. 4.9. Assume that the deadbeat control in a. is used. The closed loop system will be unstable if the feedback from x_1 is disconnected (i.e., if $\ell_1 = 0$), but the system will remain stable if x_2 is disconnected (if $\ell_2 = 0$).

Problem 4.12

a. The deadbeat requirement implies that the characteristic equation of the closed loop system is

$$\lambda^2 = \det(\lambda I - \Phi + \Gamma L) = \det \begin{pmatrix} \lambda - 0.25 + \ell_1 & \ell_2 - 0.5 \\ 4\ell_1 - 1 & \lambda - 2 + 4\ell_2 \end{pmatrix}$$

$$= \lambda^2 + \lambda(\ell_1 + 4\ell_2 - 2.25) \equiv \lambda^2$$

There are infinitely many solutions, one is

$$L = \begin{pmatrix} 2.25 & 0 \end{pmatrix}$$

b. The controllability matrix is

$$W_c = \begin{pmatrix} \Gamma & \Phi\Gamma \end{pmatrix} = \begin{pmatrix} 1 & 2.25 \\ 4 & 9 \end{pmatrix}$$

$\det W_c = 0$ implies that the system is not reachable and arbitrary states cannot be reached from the origin. However, $x(k) = \begin{pmatrix} 2 & 8 \end{pmatrix}^T$ is in the column space of W_c and the point can thus be reached.

$$x(2) = \Phi^2 x(0) + \Phi\Gamma u(0) + \Gamma u(1)$$

$$= \begin{pmatrix} 0.5625 & 1.125 \\ 2.25 & 4.5 \end{pmatrix} x(0) + \begin{pmatrix} 2.25 \\ 9 \end{pmatrix} u(0) + \begin{pmatrix} 1 \\ 4 \end{pmatrix} u(1)$$

With $x(2) = \begin{pmatrix} 2 & 8 \end{pmatrix}^T$ and $x(0) = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$ we get

$$2 = 2.25u(0) + u(1)$$
$$8 = 9u(0) + 4u(1)$$

One solution is

$$u(0) = 0$$
$$u(1) = 2$$

c. The observer should have the characteristic equation

$$(\lambda - 0.2)^2 = \lambda^2 - 0.4\lambda + 0.04 = 0$$

$$\det(\lambda I - \Phi + KC) = \det \begin{pmatrix} \lambda + k_1 - 0.25 & -0.5 \\ k_2 - 1 & \lambda - 2 \end{pmatrix}$$

$$= \lambda^2 + (k_1 - 2.25)\lambda + 0.5k_2 - 2k_1$$

Identifying coefficients give the system

$$k_1 - 2.25 = -0.4$$
$$0.5k_2 - 2k_1 = 0.04$$

which has the solution

$$K = \begin{pmatrix} 1.85 \\ 7.48 \end{pmatrix}$$

Solutions to Chapter 5

Problem 5.1

Euclid's algorithm defines a sequence of polynomials $A_0, \ldots, A_n$ where $A_0 = A$, $A_1 = B$ and

$$A_i = A_{i+1} Q_{i+1} + A_{i+2}$$

If the algorithm terminates with A_{n+1} then the greatest common factor is A_n. For the polynomials in the problem we get

$$A_0 = z^4 - 2.6z^3 + 2.25z^2 - 0.8z + 0.1$$
$$A_1 = z^3 - 2z^2 + 1.45z - 0.35.$$

This gives

$$Q_1 = z - 0.6$$
$$A_2 = -0.4z^2 + 0.42z - 0.11 = -0.4(z^2 - 1.05z + 0.275)$$

The next step of the algorithm gives

$$Q_2 = (z - 0.95)/(-0.4)$$
$$A_3 = 0.1775z - 0.08875 = 0.1775(z - 0.5)$$

Finally

$$Q_3 = (z - 0.55)/0.1775$$
$$A_4 = 0$$

The greatest common factor of A and B is thus $z - 0.5$.

Problem 5.2

a. To use Algorithm 5.1 we must know the pulse-transfer function $B(z)/A(z)$ and the desired closed-loop characteristic polynomial $A_{cl}(z)$. Since the desired pulse-transfer function from u_c to y is $H_m(z) = (1 + \alpha)/(z + \alpha)$, we know at least that $(z + \alpha)$ must be a factor in $A_{cl}(z)$.

Step 1. You easily see that, with A and A_{cl} being first order polynomials and B a scalar, you can solve the equation for the closed loop system using scalars $R(z) = r_0$ and $S(z) = s_0$:

$$A(z)R(z) + B(z)S(z) = A_{cl}(z)$$
$$(z + a) \cdot r_0 + 1 \cdot s_0 = z + \alpha$$

Identification of coefficients gives the following equation system:

$$\begin{cases} r_0 = 1 \\ ar_0 + s_0 = \alpha \end{cases}$$

with the solution $r_0 = 1$ and $s_0 = \alpha - a$.

Step 2. Factor $A_{cl}(z)$ as $A_c(z)A_o(z)$ where $\deg A_o \leq \deg R = 0$. Thus, $A_o = 1$ and $A_c = z + \alpha$. Choose

$$T(z) = t_0 A_o(z) = \frac{A_c(1)}{B(1)} A_o(z) = 1 + \alpha$$

With this choice of T, the static gain from u_c to y is set to 1 ($H_m(1) = 1$), and the observer dynamics are cancelled in the pulse-transfer function from u_c to y. In this case, there are no observer dynamics, though, since $\deg A_o = 0$.

The resulting control law becomes

$$R(q)u(k) = T(q)u_c(k) - S(q)y(k)$$
$$u(k) = (1 + \alpha)u_c(k) - (\alpha - a)y(k)$$

i.e., a (static) proportional controller.

Solution with higher order observer: The solution above is not the only one solving the original problem. We can, for example, decide to have another closed loop pole in $z = -\beta$, say.

Step 1. To solve the equation for the closed loop characteristic polynomial we must increase the order of R by one. This gives

$$(z + a)(r_0 z + r_1) + 1 \cdot s_0 = (z + \alpha)(z + \beta)$$

and the equation system becomes

$$\begin{cases} r_0 = 1 \\ ar_0 + r_1 = \alpha + \beta \\ ar_1 + s_0 = \alpha\beta \end{cases} \iff \begin{cases} r_0 = 1 \\ r_1 = \alpha + \beta - a \\ s_0 = \alpha\beta - a(\alpha + \beta - a) \end{cases}$$

Step 2. Splitting A_{cl} into factors $A_o = z + \beta$ and $A_c = z + \alpha$ gives

$$T(z) = t_0 A_o(z) = \frac{A_c(1)}{B(1)} A_o(z) = (1 + \alpha)(z + \beta)$$

The resulting control law becomes

$$R(q)u(k) = T(q)u_c(k) - S(q)y(k) \quad \Rightarrow$$

$$u(k) = -(\alpha + \beta - a)u(k - 1) + (1 + \alpha)\Big(u_c(k) + \beta u_c(k - 1)\Big)$$
$$- \big(\alpha\beta - a(\alpha + \beta - a)\big)y(k - 1)$$

The controller thus is a dynamical system. In this case there is a delay of one sample from the measurements y to the control signal u. This could have been avoided by choosing $\deg S = 1$.
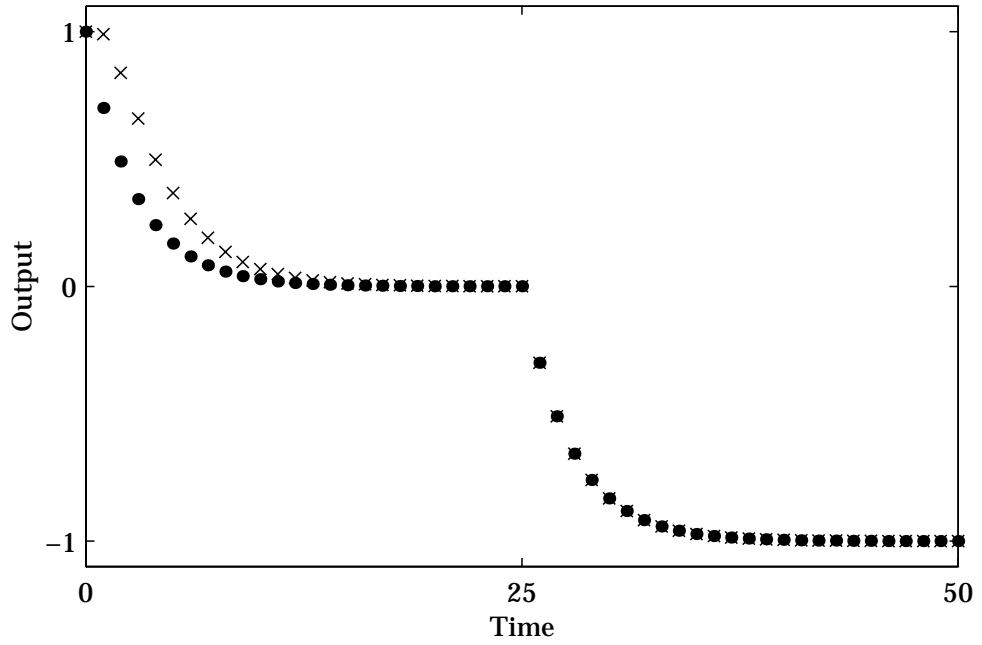
Figure 5.1 The output of the process with $y(0) = 1$, and $u_c(k)$ is 0 for $k < 25$ and -1 for $k >= 25$. The dots corresponds to the zero order controller, and the crosses to the first order controller.

b. The closed loop characteristic polynomial is given by $AR + BS$, i.e. $(z + \alpha)$ in the first solution, and $(z + \alpha)(z + \beta)$ in the second one. In both cases we get the same closed loop pulse-transfer function from u_c to y since the observer polynomial is cancelled by $T(z)$:

$$H_m(z) = \frac{B(z)T(z)}{A(z)R(z) + B(z)S(z)} = \frac{t_0 B(z)}{A_c(z)} = \frac{1 + \alpha}{z + \alpha}$$

Fig. 5.1 shows the response when the two different controllers are used. It is assumed in the simulations that $a = -0.99$, and that the design parameters are $\alpha = -0.7$ and $\beta = -0.5$. You can see the effect of the observer polynomial when regulating a nonzero initial state, but not in the response to a set point change.

Problem 5.3

a. The desired closed loop pulse-transfer function is

$$H_m(z) = \frac{B_m(z)}{z^2 - 1.5z + 0.7}$$

In this case, the process zero is cancelled by the controller, so $(z + 0.7)$ is not a factor of B_m. By choosing $B_m = 0.2z$ we make the steady state gain $H_m(1) = 1$, and the pole excess in H_m equals the pole excess in H. Cancellation of process poles and zeros is handled by Algorithm 5.3 or through the following discussion.

First, the process numerator is factored as $B = B^+ B^-$, where B^+ is the part of the numerator which should be cancelled by the controller, i.e., $B^+ = z + 0.7$ and $B^- = 1$. B^+ must be a part of the R polynomial as

well as A_{cl}. This gives the Diophantine equation

$$A(z)B^+(z)\bar{R}(z) + B^+(z)B^-(z)S(z) = B^+(z)\bar{A}_{cl}(z)$$
$$(z^2 - 1.8z + 0.81)\bar{R}(z) + S(z) = \bar{A}_{cl}(z)$$

If $\bar{R}(z)$ is a constant r_0, the left hand side is of second order, and so must $\bar{A}_{cl}$ be. With this choice of $\bar{R}$, the causality condition ($\deg S \leq \deg R = 1$) leads us to set $S(z) = s_0 z + s_1$. Now, we can solve the Diophantine equation above, since we have 3 indeterminates (r_0, s_0 and s_1) and 3 coefficients to set:

$$(z^2 - 1.8z + 0.81) \cdot r_0 + (s_0 z + s_1) = z^2 - 1.5z + 0.7 \Rightarrow$$

$$\begin{cases} r_0 = 1 \\ -1.8r_0 + s_0 = -1.5 \\ 0.81r_0 + s_1 = 0.7 \end{cases} \iff \begin{cases} r_0 = 1 \\ s_0 = 0.3 \\ s_1 = -0.11 \end{cases}$$

Thus, we have $R(z) = B^+(z)\bar{R}(z) = z + 0.7$ and $S(z) = 0.3z - 0.11$. To obtain the desired

$$H_m(z) = \frac{BT}{AR + BS} = \frac{B^+ B^- T}{B^+ (z^2 - 1.5z + 0.7)} = \frac{0.2z}{z^2 - 1.5z + 0.7}$$

we must select

$$T(z) = 0.2z$$

The controller is now

$$u(k) = -0.7u(k-1) + 0.2u_c(k) - 0.3y(k) + 0.11y(k-1).$$

b. In this case we do not want to cancel the process zero, so

$$H_m(z) = \frac{B_m(z)}{z^2 - 1.5z + 0.7} = \frac{\frac{0.2}{1.7}(z + 0.7)}{z^2 - 1.5z + 0.7}$$

in order to get $H_m(1) = 1$. The closed loop characteristic polynomial is now given by the identity

$$(z^2 - 1.8z + 0.81)R(z) + (z + 0.7)S(z) = A_{cl}(z)$$

The simplest choice, a zero order controller, will not suffice in this case since it would only give 2 parameters r_0 and s_0 to select the 3 parameters in the second order polynomial $z^2 - 1.5z + 0.7$. Thus, we must increase the order of the controller by one and, consequently, add an observer pole which is placed at the origin, i.e. $A_o = z$ and $A_{cl} = z^3 - 1.5z^2 + 0.7z$. Letting

$$R = r_0 z + r_1$$
$$S = s_0 z + s_1$$

the identity then gives the system of equations

$$\begin{cases} r_0 = 1 \\ -1.8r_0 + r_1 + s_0 = -1.5 \\ 0.81r_0 - 1.8r_1 + 0.7s_0 + s_1 = 0.7 \\ 0.81r_1 + 0.7s_1 = 0 \end{cases} \iff \begin{cases} r_0 = 1 \\ r_1 = 0.0875 \\ s_0 = 0.2125 \\ s_1 = -0.1012 \end{cases}$$
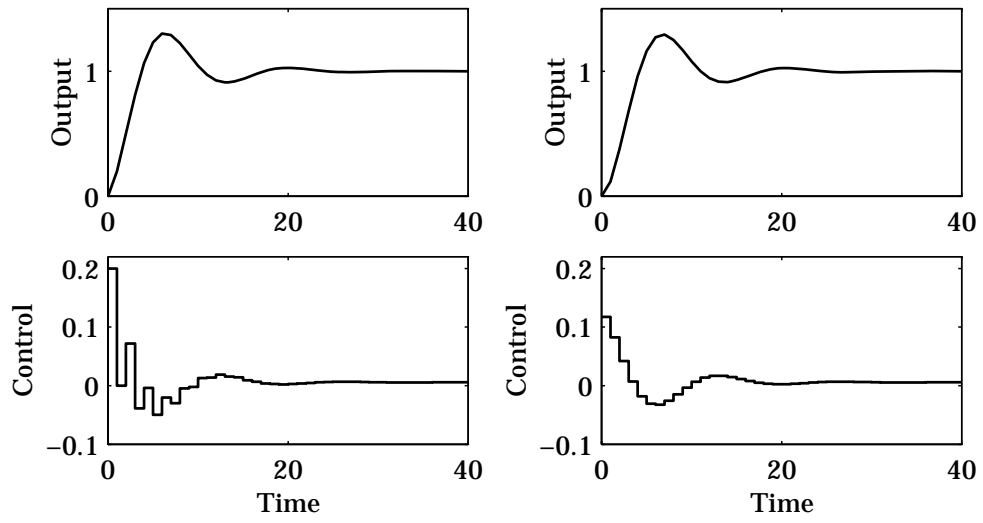
Figure 5.2 The output and the control signal for the controllers in Case a (left) and Case b (right) in Problem 5.3. The ringing in the control signal in Case a is due to the cancellation of the process zero on the negative real axis.

Further $T = t_0 A_o = \frac{0.2}{1.7}z$. The controller is thus

$$u(k) = -0.0875u(k-1) + 0.1176u_c(k) - 0.2125y(k) + 0.1012y(k-1)$$

c.   Fig. 5.2 shows the output and the control signal for the controllers in Case a and Case b. Case a should probably be avoided because of the ringing in the control signal.

Problem 5.4

a.   Using the controller

$$u(k) = \frac{S(q)}{R(q)}\bigl(u_c(k) - y(k)\bigr)$$

gives the closed loop system

$$\frac{BS}{AR + BS} = \frac{B_m}{A_m}$$

Section 5.10 gives one solution to the problem

$$R = B(A_m - B_m)$$
$$S = AB_m$$

With the given system and model we get

$$R = 1(z + \alpha - 1 - \alpha) = z - 1$$
$$S = (z + a)(1 + \alpha)$$

The controller contains an integrator. Further the pole of the process is cancelled.

b. The characteristic polynomial of the closed loop system is

$$AR + BS = (z + \alpha)(z + a)$$

The closed loop system will contain an unstable mode if $|a| > 1$. The controller can be written

$$\frac{S}{R} = \frac{A}{B} \frac{B_m}{A_m - B_m}$$

From this we can conclude that in order to get a stable closed loop we must fulfill the following constraints.

i. B_m must contain the zeros of B that are outside the unit circle.

ii. $A_m - B_m$ must contain the poles of the process that are outside the unit circle. The first constraint is the same as for the polynomial design discussed in Chapter 5.

Problem 5.5

a. Equation (5.33) gives the pulse transfer operator from u_c and v to y:

$$y(k) = \frac{B_m}{A_m} u_c(k) + \frac{BR}{AR + BS} v(k)$$

The design in Problem 5.2 gave

$$R = 1$$
$$S = \alpha - a$$

We thus get

$$\frac{BR}{AR + BS} = \frac{1}{z + \alpha}$$

If $v(k)$ is a step there will thus be a steady state error $1/(1 + \alpha)$ in the output.

b. By inspection of the transfer function from v to y we see that we must make $R(1) = 0$ in order to remove the steady state error after a load disturbance step. By forcing the factor $(z - 1)$ into $R(z)$ we thus have obtained integral action in the controller. The design problem is solved by using the general Algorithm 5.3 or through a discussion like the one below.

With $R(z) = (z - 1)\bar{R}(z)$ the closed loop characteristic equation becomes

$$A(z)(z - 1)\bar{R}(z) + B(z)S(z) = A_{cl}(z)$$
$$(z + a)(z - 1)\bar{R}(z) + 1 \cdot S(z) = A_{cl}(z)$$

If $\bar{R}(z)$ is a constant r_0, the left hand side is of second order, and so must A_{cl} be. With this choice of $\bar{R}$, the causality condition $(\deg S \leq \deg R = 1)$ leads us to set $S(z) = s_0 z + s_1$. Now, we can solve the Diophantine
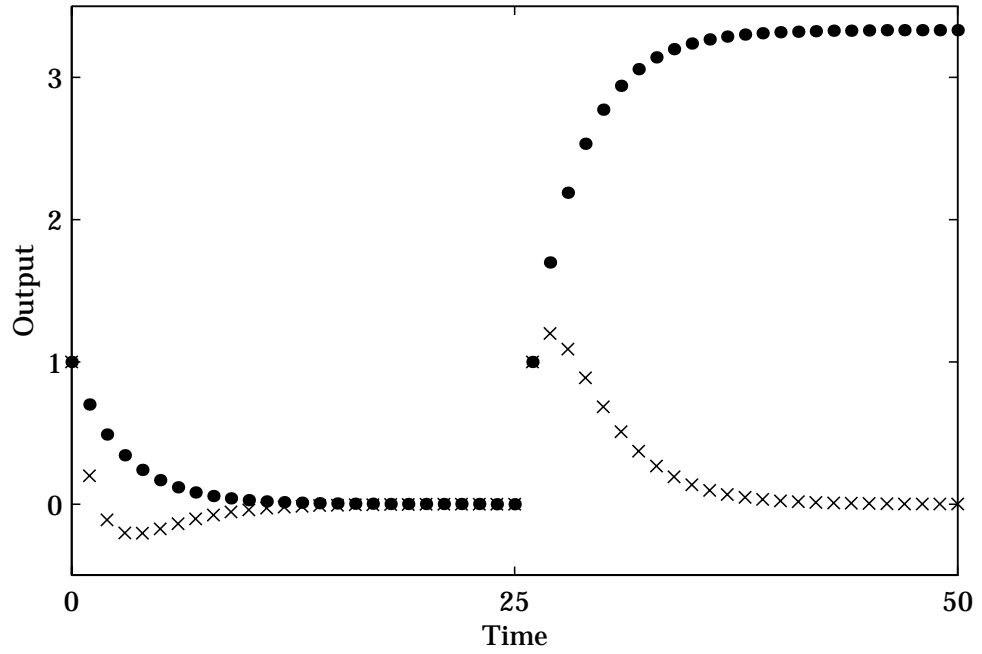
Figure 5.3 The output of the process in Problem 5.5 when the controller does not contain an integrator (dots) and when an integrator is introduced (crosses).

equation above, since we have 3 indeterminates $(r_0,\ s_0$ and $s_1)$ and 3 coefficients to set:

$$(z + a)(z - 1) \cdot r_0 + (s_0 z + s_1) = (z + \alpha)(z + \beta) \Rightarrow$$

$$\begin{cases} r_0 = 1 \\ (a - 1)r_0 + s_0 = \alpha + \beta \\ -ar_0 + s_1 = \alpha\beta \end{cases} \iff \begin{cases} r_0 = 1 \\ s_0 = \alpha + \beta - a + 1 \\ s_1 = \alpha\beta + a \end{cases}$$

To obtain the desired

$$H_m(z) = \frac{B(z)T(z)}{A(z)R(z) + B(z)S(z)} = \frac{T(z)}{(z + \alpha)(z + \beta)} = \frac{1 + \alpha}{z + \alpha}$$

we must select

$$T(z) = (1 + \alpha)(z + \beta)$$

The controller is now

$$u(k) = u(k - 1) - (\alpha + \beta - a + 1)y(k) - (a + \alpha\beta)y(k - 1)$$
$$+ (1 + \alpha)u_c(k) + \beta(1 + \alpha)u_c(k - 1)$$

Fig. 5.3 shows the controllers in Problem 5.2 and the controller with an integrator. The reference value is zero and there is an initial value of the state in the process. At $t = 25$ a constant load disturbance is introduced. It is assumed that $a = -0.99$, and the design parameters are chosen as $\alpha = -0.7$ and $\beta = -0.5$.

Problem 5.6

It is assumed that the design is based on the model $H = B/A$ while the true model is $H^0 = B^0/A^0$. The pulse transfer operator of the closed loop system is

$$H_{\text{cl}} = \frac{B^0 T}{A^0 R + B^0 S} = \frac{T/R}{A^0/B^0 + S/R}$$

The design gives

$$T = B'_m A_o$$

and

$$AR + BS = A_0 A_m B^+$$

or

$$\frac{S}{R} = \frac{B^+ A_m A_o}{BR} - \frac{A}{B}$$

This gives

$$H_{\text{cl}} = \frac{\dfrac{B'_m A_o}{R}}{\dfrac{A^0}{B^0} + \dfrac{B^+ A_o A_m}{BR} - \dfrac{A}{B}} = \frac{\dfrac{B'_m A_o}{R}}{\dfrac{A_o A_m}{B^- R} + \left(\dfrac{1}{H^0} - \dfrac{1}{H}\right)}$$

$$= \frac{B'_m B^-}{A_m} \cdot \frac{\dfrac{A_o}{R}}{\dfrac{A_o}{R} + \dfrac{B^-}{A_m}\left(\dfrac{1}{H^0} - \dfrac{1}{H}\right)} = H_m \frac{1}{1 + \dfrac{RB^-}{A_o A_m}\left(\dfrac{1}{H^0} - \dfrac{1}{H}\right)}$$

Problem 5.7

a.   The design in Problem 5.2 gives

$$R = 1$$
$$S = \alpha - a$$
$$T = 1 + \alpha$$

Assume that the true process is

$$\frac{1}{z + a^0}$$

Equation (5.41) gives

$$H_{\text{cl}} = \frac{1 + \alpha}{z + a^0 + \alpha - a}$$

The closed loop system is stable if

$$|a^0 + \alpha - a| < 1$$

With the numerical values in the problem formulation we get
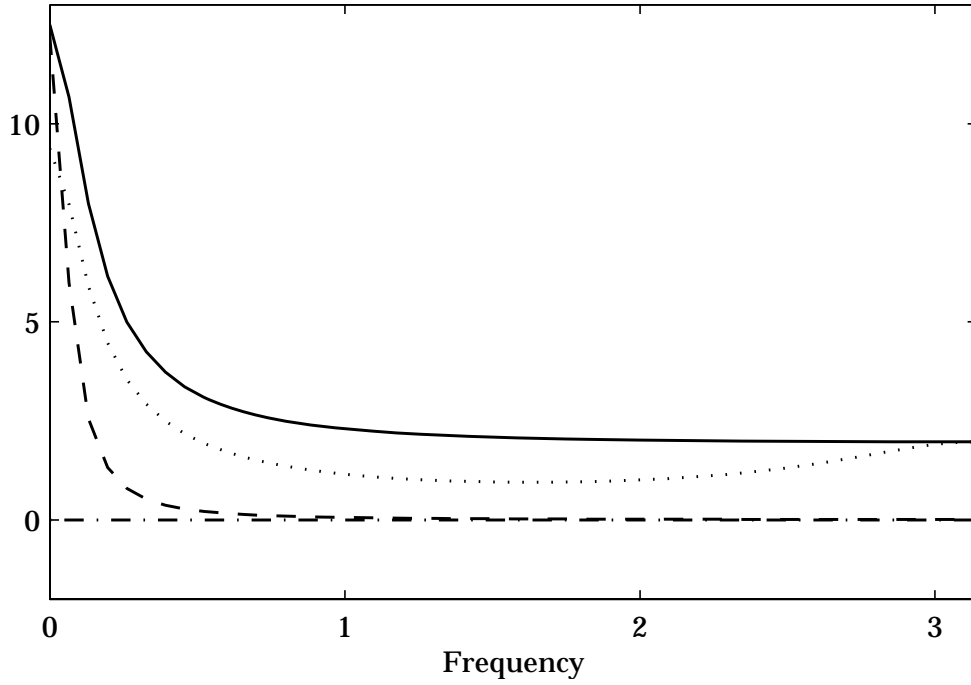
$$-1.4 < a^0 < 0.6$$

Figure 5.4 The left hand side of the inequality in Problem 5.7 when $z = e^{i\omega}$, $0 < \omega < \pi$ for $a^0 = -0.955$ (dashed), -0.9 (dash-dotted) and 0.6 (dotted). The right hand side of the inequality is also shown (solid).

b. Equation (5.40) gives the inequality

$$\left| H(z) - H^0(z) \right| < \left| \frac{H(z)}{H_m(z)} \right| \left| \frac{H_{ff}(z)}{H_{fb}(z)} \right| = \left| \frac{z + \alpha}{(1 + \alpha)(z + a)} \right| \left| \frac{1 + \alpha}{\alpha - a} \right|$$

$$= \left| \frac{z - 0.5}{z - 0.9} \right| \cdot 2.5$$

or

$$\left| \frac{1}{z - 0.9} - \frac{1}{z + a^0} \right| < \left| \frac{z - 0.5}{z - 0.9} \right| \cdot 2.5$$

Fig. 5.4 shows for $z = e^{i\omega}$ the left hand side of the inequality for different values of a^0. The right hand side is also shown.

Problem 5.8

Section 5.6 shows that the control signal is given by (5.52)

$$u(k) = \frac{H_m(q)}{H(q)} u_c(k) = \frac{(1 + \alpha)(q + a)}{q + \alpha} u_c(k)$$

We may assume that both the process and the model have a continuous time correspondence. This implies that a and α are less than zero. Further the desired model is stable, i.e. $|\alpha| < 1$. The control signal is now obtained by studying the step response of H_m/H, which is a stable first order system. The largest value is then either at the first step or the final value. The magnitude at the first step can be determined either through the initial value theorem or by using series expansion and the value is $1 + \alpha$. The final value is $1 + a$. If $|\alpha| < |a|$ then the closed loop system is faster than the open loop system and the control signal is largest at the first step. If the desired response is slower than the open loop system then the final value is the largest one.

Problem 5.14

a. The rule of thumb on p. 130 gives

$$\omega h = 0.1 - 0.6$$

Identifying with

$$s^2 + 2\zeta\omega s + \omega^2$$

gives $\omega = 0.1$. Thus an appropriate sampling interval is

$$h = 1 - 6$$

b. Using Example 2.16 we get sampled data characteristic equation

$$z^2 + a_1 z + a_2 = 0$$

where
$$a_1 = -2e^{-\zeta\omega h}\cos(\sqrt{1 - \zeta^2}\omega h) = -1.32$$
$$a_2 = e^{-2\zeta\omega h} = 0.5$$

The poles are in $0.66 \pm 0.25i$.

Problem 5.15

This solution demonstrates how to use Algorithm 5.3.

Data: The process is given by $A = q^2 - 1.6q + 0.65$ and $B = 0.4q + 0.3$. A_{cl} will at least contain $\bar{A}_c = q^2 - 0.7q + 0.25$, other factors may be added later on. $R_d = S_d = 1$ since no given factors are forced into the controller. The desired response to command signals is assumed to be $H_m = B_m/A_m = B_m/\bar{A}_c = 0.55/(q^2 - 0.7q + 0.25)$ (cancelled process zero, $H_m(1) = 1$).

Pole excess condition:

$$\deg A_m - \deg B_m \geq \deg A - \deg B$$
$$2 - 0 \geq 2 - 1$$

Remark: The fact that we cancel one zero and do not introduce any other zero in B_m causes the delay from the command signal to be one time unit more than the delay of the process.

Model following condition:

$$B_m = B^- \bar{B}_m \Rightarrow \bar{B}_m = 0.55/0.4 = 1.375$$

Degree condition:

$$\deg A_{cl} = 2 \deg A + \deg A_m + \deg R_d + \deg S_d - 1 =$$
$$= 2 \cdot 2 + 2 + 0 + 0 - 1 = 5$$

with $A_{cl} = A^+ B^+ A_m \bar{A}_{cl}$ and $\bar{A}_{cl} = \bar{A}_c \bar{A}_o$.

Step 1. $A^+ = 1$, $A^- = A = q^2 - 1.6q + 0.65$, $B^+ = q + 0.75$ and $B^- = 0.4$ achives cancellation of the process zero, but no cancellation of process poles.

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Step 2. Using the degree condition above we may conclude that

$$\deg \bar{A}_o = \deg A_{cl} - \deg A^+ - \deg B^+ - \deg A_m - \deg \bar{A}_c =$$
$$= 5 - 0 - 1 - 2 - 2 = 0$$

The Diophantine equation to solve thus becomes

$$A^- R_d \bar{R} + B^- S_d \bar{S} = \bar{A}_{cl}$$
$$(q^2 - 1.6q + 0.65)\bar{R} + 0.4\bar{S} = q^2 - 0.7q + 0.25$$

Since this is of second order, $\bar{R}$ must be a constant, r_0, say. In order to solve the identity we must have two more parameters, so we let $\bar{S} = s_0 q + s_1$:

$$(q^2 - 1.6q + 0.65)r_0 + (s_0 q + s_1) \cdot 0.4 = q^2 - 0.7q + 0.25$$

This gives the system of equations

$$
\begin{aligned}
r_0 &= 1 \\
-1.6r_0 + 0.4s_0 &= -0.7 \\
0.65r_0 + 0.4s_1 &= 0.25
\end{aligned}
\qquad \Longleftrightarrow \qquad
\begin{aligned}
r_0 &= 1 \\
s_0 &= 2.25 \\
s_1 &= -1
\end{aligned}
$$

Step 3. The controller polynomials are now given by (5.45):

$$R = A_m B^+ R_d \bar{R} = A_m(q + 0.75)$$
$$S = A_m A^+ S_d \bar{S} = A_m(2.25q - 1)$$
$$T = \bar{B}_m A^+ \bar{A}_{cl} = 1.375 \cdot \bar{A}_c$$

Since, in this case, $A_m = \bar{A}_c$, this factor can (and should) of course be cancelled in all controller polynomials, giving

$$R = q + 0.75$$
$$S = 2.25q - 1$$
$$T = 1.375$$

The corresponding degree of the closed-loop polynomial $AR + BS$ will thus be 3 instead of 5.

Problem 5.16

In this case we want to have an integrator in the controller, i.e., $R_d = (q - 1)$. This will increase the degree of the closed loop by one compared to Problem 5.15 (see (5.42)), which is done by having $\bar{A}_o = (q + a_o)$, say. This gives the Diophantine equation

$$A^- R_d \bar{R} + B^- S_d \bar{S} = \bar{A}_{cl}$$
$$(q^2 - 1.6q + 0.65)(q - 1)\bar{R} + 0.4\bar{S} = (q^2 - 0.7q + 0.25)(q + a_o)$$

$\bar{R}$ must still be a constant (which as usual will be 1) and $\bar{S}$ must be of second order:

$$(q^2 - 1.6q + 0.65)(q - 1) + (s_0 q^2 + s_1 q + s_2) \cdot 0.4 = (q^2 - 0.7q + 0.25)(q + a_0)$$

This gives
$$-2.6 + 0.4s_0 = a_0 - 0.7$$
$$2.25 + 0.4s_1 = -0.7a_0 + 0.25$$
$$-0.65 + 0.4s_2 = 0.25a_0$$

and
$$\bar{S}(q) = 2.5\left((a_0 + 1.9)q^2 - (0.7a_0 + 2)q + 0.25a_0 + 0.65\right)$$

Using $a_0 = -0.25$, (5.45) gives (after cancelling the common factor A_m):

$$R = B^+(q - 1) = q^2 - 0.25q - 0.75$$
$$S = 4.125q^2 - 4.5625q + 1.46875$$
$$T = \bar{B}_m \bar{A}_o = 1.375q - 0.34375$$

Problem 5.17

The minimum degree solution has deg $A_0 = 1$ and gives a unique solution to the Diophantine equation. Let us instead use deg $A_0 = 2$ and deg $S = $ deg $R - 1$. This gives the equation

$$(z + 1)(z + 2)(z^2 + r_1 z + r_2) + z \cdot (s_1 z + s_0) = z^2 \cdot z^2$$

with the solution
$$R_0 = z^2 - 3z$$
$$S_0 = 7z + 6$$

The controller is causal. Using Theorem 5.1 we also have the solutions

$$R = R_0 + Qz$$
$$S = S_0 - Q(z - 1)(z - 2)$$

where Q is an arbitrary polynomial. Choose for instance $Q = -1$. This gives

$$R = z^2 - 3z - z = z^2 - 4z$$
$$S = 7z + 6 + (z^2 - 3z + 2) = z^2 + 4z + 8$$

This is also a causal controller. The closed loop systems when using R_0, S_0, $T_0 = S_0$ and $R, S, T = S$ respectively are

$$\frac{BS_0}{AR_0 + BS_0} = \frac{z(7z + 6)}{z^4}$$
$$\frac{BS}{AR + BS} = \frac{z(z^2 + 4z + 8)}{z^4}$$

The number of zeros are different.

Solutions to Chapter 6

Problem 6.1

In the first case it is assumed that we have a control structure as in Fig. 6.1. There are three subsystems each with the transfer function

$$G_i(s) = \frac{K_i}{s + K_i}$$

and the total transfer function from u_c to y is

$$G = \frac{G_1 G_2 G_3}{s + K_4 G_1 G_2 G_3} = \frac{K_1 K_2 K_3}{s(s + K_1)(s + K_2)(s + K_3) + K_1 K_2 K_3 K_4}$$

If either of the gains K_i is increased sufficiently much the closed system will become unstable. Fig. 6.2 shows the response when u_c is an impulse and when $K_1 = K_2 = K_3 = 1$ and $K_4 = 0.1, 0.25$, and 0.75.
A disturbance in the process will propagate in the direction of the flow. In the case of control in the direction opposite to the flow each of the subprocesses
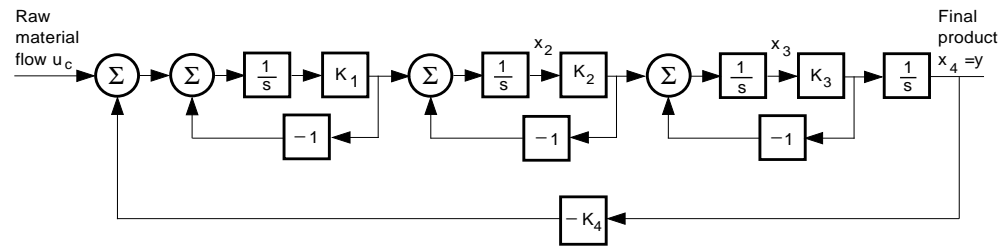
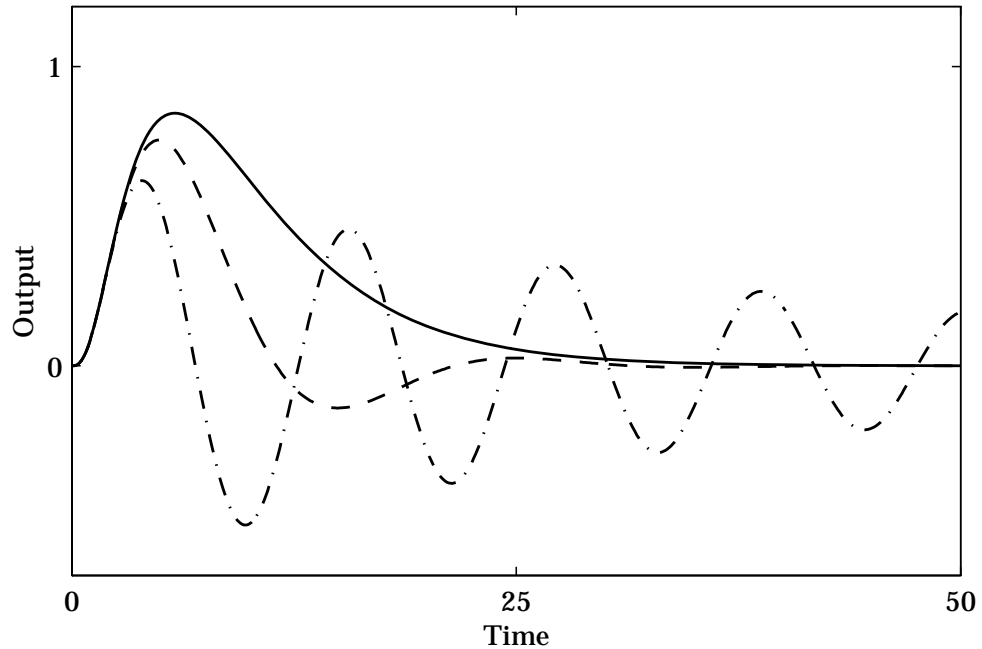**Figure 6.1** Block diagram for the control in the direction of the flow in Problem 6.1.

Figure 6.2 Impulse response for the control in the direction of the flow when $K_4 = 0.1$ (solid), 0.25 (dashed), and 0.75 (dash-dotted).
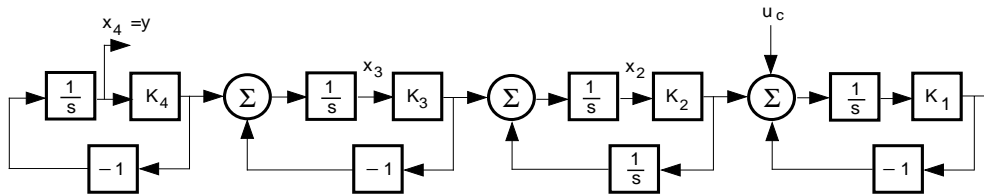
Figure 6.3 Block diagram for the control in the direction opposite of the flow in Problem 6.1.

has a transfer function of the type G_i. The system is then represented with the block diagram in Fig. 6.3. Notice the order of the states. The system will remain stable for all positive values of K_i. A disturbance will now propagate in the direction opposite the flow. A disturbance in u_c will now only influence the first subprocess and will not propagate along with the flow. The reader is strongly recommended to compare with the case where the disturbance appears at the final product storage instead.

Problem 6.2

Fig. 6.3 in CCS contains several examples of couplings of simple control loops.

a. Cascade control loops are found for the cooling media flow and for the output product flow.

b. Feedforward is used for the level control loop where the input flow is used as a measurable disturbance. The input flow is also used as feedforward for the cooling of the jacket.

c. Nonlinear elements are used in the flow control loops of the product output and the coolant flow. The flow is probably measured using differential pressure which is proportional to the square of the flow. The square root device is thus used to remove the nonlinearity of the measurement device. An intentional nonlinearity is introduced in the selector. Either the temperature or the pressure is used to control the coolant flow depending on the status of the process.

Solutions to Chapter 7

Problem 7.1

Which frequencies will the signal

$$f(t) = a_1 \sin 2\pi t + a_2 \sin 20 t$$

give rise to when sampled with $h = 0.2$?

Since sampling is a linear operation we consider each component of $f(t)$ separately. The sampled signal has the Fourier transform, see (7.3)

$$F_s = \frac{1}{h} \sum_{k=-\infty}^{\infty} F(\omega + k\omega_s) \qquad \omega_s = \frac{2\pi}{h}$$

where $F(\omega)$ is the Fourier transform of the time continuous signal. The Fourier transform of $\sin \omega_0 t$ has its support (i.e., the set where it is $\neq 0$) in the two points $\pm \omega_0$. More precisely, it equals $\pi i \left(\delta \left(\omega + \omega_0 \right) - \delta \left(\omega - \omega_0 \right) \right)$. Thus, if the signal $\sin \omega_0 t$ is sampled with the sample interval h its Fourier transform will be $\neq 0$ in the points

$$\pm \omega + k \omega_s \quad ; \quad k = 0, \pm 1, \pm 2, \ldots$$

For $\omega_0 = 2\pi$ and $\omega_s = 2\pi/0.2 = 10\pi$ we get the angular frequencies

$$\pm 2\pi \pm k \cdot 10\pi = \pi \left(\pm 2, \pm 8, \pm 12, \pm 18, \pm 22, \ldots \right)$$

$\omega = 20$ gives rise to

$$\pm 20 \pm k \cdot 10\pi \approx \pi \left(\pm 3.63, \pm 6.37, \pm 13.63, \pm 16.37, \ldots \right)$$

The output of the sampler is composed of the frequencies

$$\pi \left(2, 3.63, 6.37, 8, 12, 13.63, 16.37, \ldots \right)$$

Problem 7.2

We have the following specifications on the choice of sampling period and presampling filter:

1. All frequencies in the interval $(-f_1, f_1)$ should be possible to reproduce from the samples of the continuous time signal.

2. We want to eliminate the disturbance with the known and fixed frequency $f_2 \approx 5 f_1$.
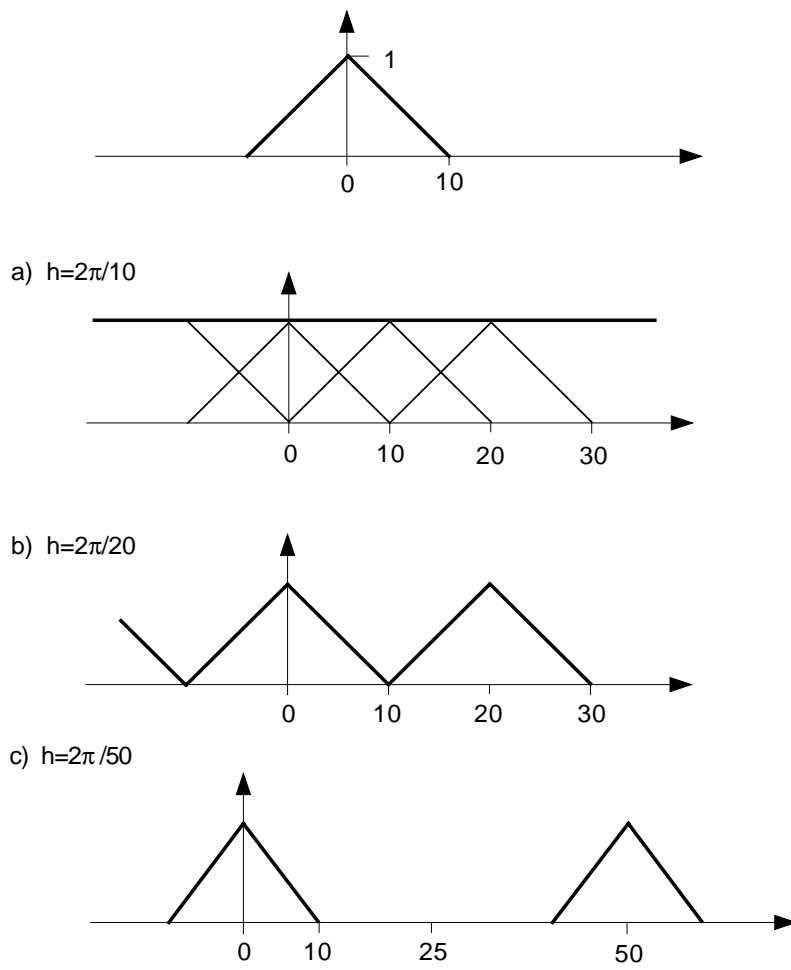
a) h=2π/10

b) h=2π/20

c) h=2π/50

Figure 7.1 Folding for different frequencies in Problem 7.5.

The sampling theorem states that the first specification will be satisfied if and only if the sample frequency f_s is chosen such that

$$f_s > 2f_1$$

Moreover, for the disturbance f_2 not to fold on the data signal

$$(f_s/2 - f_1) > f_2 - f_s/2 \qquad \Rightarrow \qquad f_s > f_2 + f_1 = 6f_1$$

Two cases:

1. Filter out the disturbance using an antialiasing filter. Then sample with $f_s > 2f_1$. Suppose the disturbance should be attenuated 20 dB without effecting the datasignal. A n:th order filter gives maximally $n \cdot 20$ dB/decade. So to achieve 20 dB in $\log \frac{f_2}{f_1} = 0.699$ decades takes $n \approx 2$.

2. If $f_s > 6f_1$, the disturbance does not mix with the data signal. It can instead be removed using digital filters.

Problems 7.5 and 7.6

The magnitude of the spectrum of the sampled signal can be obtained by folding the spectrum of the time continuous signal around the angular frequency $\omega_N = \omega_s/2 = \pi/h$. See Fig. 7.1 and Fig. 7.2.
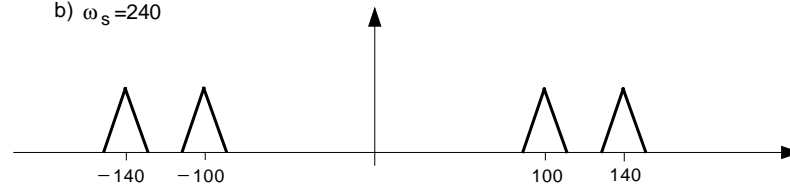
a) ω_s=120
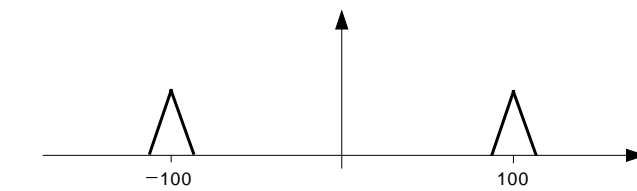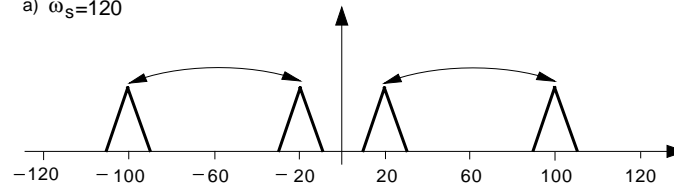
b) ω_s =240

Figure 7.2 Folding for different frequencies in Problem 7.6.

$-\omega_s$ $-\omega_r$ 0 ω_r ω_s
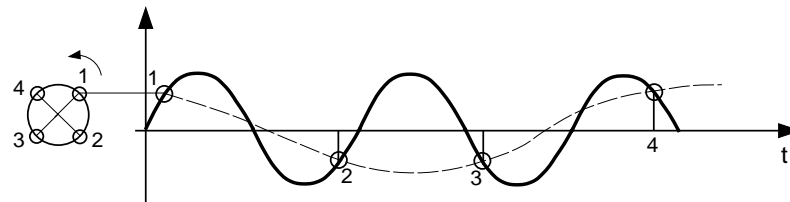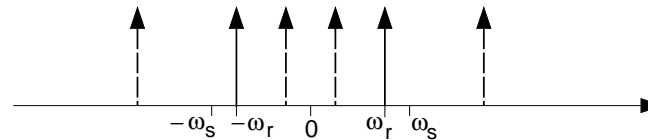

Figure 7.3 Folding in Problem 7.7 when $\omega_s < 2\omega_r$.

Problem 7.7

The rotation frequency of the wheel $\omega_r = 2\pi r$.

The frequency of the camera shutter $\omega_s = 2\pi/h$.

The picture will not move if $\omega_r = n \cdot \omega_s$; for integer values n.

A correct picture will be seen, if $\omega_s > 2\omega_r$ according to the sampling theorem. (The eye acts like a low pass filter).

The wheel will appear to rotate with a frequency lower than r if $\omega_s < 2\omega_r$. See Fig. 7.3. For instance let $\omega_s = 4/3\,\omega_r$. Aliasing will give a frequency $\omega = 1/3\,\omega_r$. The wheel then appears to rotate three times slower and in the wrong direction. If $\omega_s = \omega_r$ the wheel will appear to stand still. Compare the stroboscope.

Problem 7.9

The signal is

$$u(t) = \sin(4\omega_0 t)\cos(2\omega_0 t) = \frac{1}{2}\left[\sin(6\omega_0 t) + \sin(2\omega_0 t)\right]$$

Sampling the signal with

$$\omega_s = \frac{2\pi}{h} = 6\omega_0$$

gives the Nyquist frequency $\omega_N = 3\omega_0$. Sampling the signal $u(t)$ gives the alias of $\sin(6\omega_0 t)$ in $\omega = 0$. We thus get the frequencies

$$f_1 = 0$$
$$f_2 = \frac{\omega_0}{\pi}$$

in the sampled signal.

Solutions to Chapter 8

Problem 8.1

The three transformations Euler's method (forward difference (8.4)), backward difference (8.5) and Tustin's approximation (8.6) have different stability properties. This can be seen by finding how the left half s-plane is transformed into the z-plane. For Euler's method we have

$$z = sh + 1$$

This implies that the stability boundary in the sh-plane (the imaginary axis) is translated one unit to the right, see Fig. 8.1a. When the backward difference is used then

$$z = \frac{1}{1 - sh}$$

For $s = i\omega$ we get

$$\frac{1}{1 - \omega h}$$

This represents a circle with radius 0.5 and going through the points 0 and 1, see Fig. 8.1b.
Finally for Tustin's approximation with $s = i\omega$
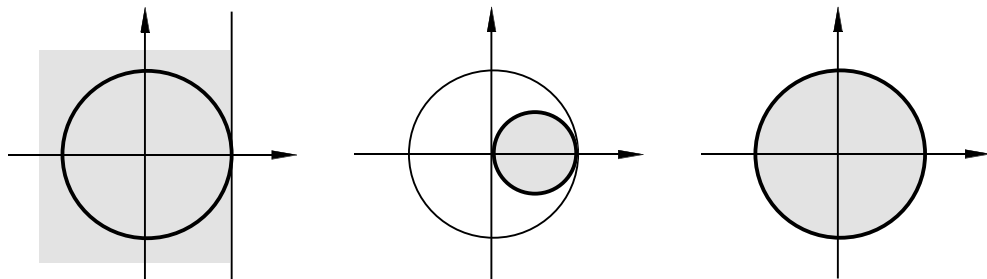
$$z = \frac{1 + i\omega h/2}{1 - i\omega h/2}$$

Now

$$|z| = 1$$
$$\arg z = 2\arctan \omega h$$

The imaginary axis is thus transformed into the unit circle in the z-plane. If a transfer function is stable in the s-plane it will be translated into a stable discrete time system if using the backward difference or Tustin's approximation.

Problem 8.2

$$G(s) = \frac{a}{s + a}, \qquad a > 0$$

Forward differences   Backward differences   Tustin

Figure 8.1 Transformation of the left half s-plane when using a. Eulers method, b. Backward difference and c. Tustin's approximation.

a. Using Euler's method we get

$$H(z) = \frac{a}{(z-1)/h + a} = \frac{ah}{z - 1 + ah}$$

This corresponds to the difference equation

$$y(kh + h) + (ah - 1)y(kh) = ahu(kh).$$

The difference equation is stable if

$$|ah - 1| < 1$$

or

$$0 < h < 2/a.$$

The approximation may, however, be poor even if the difference equation is stable.

b. For Tustin's approximation we get

$$H(z) = \frac{a}{\dfrac{2}{h}\dfrac{z-1}{z+1} + a} = \frac{(z+1)ah/2}{(1 + ah/2)z + (ah/2 - 1)}$$

$$= \frac{ah/2}{1 + ah/2} \frac{z + 1}{z + \dfrac{ah/2 - 1}{ah/2 + 1}}$$

The pole of the discrete time system will vary from 1 to -1 when h vary from 0 to infinity. The discrete time approximation is always stable when $a > 0$.

c. Using Tustin's approximation with prewarping gives

$$H(z) = \frac{a}{\alpha\dfrac{z-1}{z+1} + a} = \frac{a/\alpha}{1 + a/\alpha} \frac{z + 1}{z + \dfrac{a/\alpha - 1}{a/\alpha + 1}}$$

where

$$\alpha = \frac{a}{\tan(ah/2)}$$

Thus

$$H(z) = \frac{\tan(ah/2)}{1 + \tan(ah/2)} \cdot \frac{z + 1}{z + \dfrac{\tan(ah/2) - 1}{\tan(ah/2) + 1}}$$

Problem 8.3

The lead network

$$G_k(s) = 4\frac{s + 1}{s + 2}$$

should be approximated using different methods. At $\omega = 1.6$ rad/s it has the argument $19°$ and the gain 2.95.

a. Euler's method gives

$$H_E(z) = 4\frac{(z-1)/h + 1}{(z-1)/h + 2} = 4\frac{z - 1 + h}{z - 1 + 2h} = 4\frac{z - 0.75}{z - 0.5}$$

b. Backward differences

$$H_B(z) = 4\frac{(z-1)/(zh) + 1}{(z-1)/(zh) + 2} = 4\frac{z(1 + h) - 1}{z(1 + 2h) - 1} = 3.333\frac{z - 0.80}{z - 0.667}$$

c. Tustin's approximation

$$H_T(z) = 4\frac{\dfrac{2}{h}\dfrac{z-1}{z+1} + 1}{\dfrac{2}{h}\dfrac{z-1}{z+1} + 2} = 4\frac{z(1 + h/2) - (1 - h/2)}{z(1 + h) - (1 - h)} = 3.6\frac{z - 0.778}{z - 0.6}$$

d. Tustin's approximation with prewarping

$$H_{TW}(z) = 4\frac{\alpha\dfrac{z-1}{z+1} + 1}{\alpha\dfrac{z-1}{z+1} + 2} = 4\frac{z(1 + 1/\alpha) - (1 - 1/\alpha)}{z(1 + 2/\alpha) - (1 - 2/\alpha)} = 3.596\frac{z - 0.775}{z - 0.596}$$

where

$$\alpha = \frac{\omega_1}{\tan(\omega_1 h/2)} \approx 7.893$$

Within two decimals this is the same as in (c).

e. Zero order hold sampling gives

$$H_{ZOH}(z) = 4 - 4\cdot\frac{1}{2}\frac{1 - e^{-2h}}{z - e^{-2h}} = 4\frac{z - e^{-2h} - (1 - e^{-2h})/2}{z - e^{-2h}} = 4\frac{z - 0.803}{z - 0.607}$$

All five approximations have all the form

$$H(z) = K\frac{z + a}{z + b}$$

The gain and the phase at $\omega = 1.6$ are obtained from

$$H(e^{i\omega h}) = K\frac{e^{i\omega h} + a}{e^{i\omega h} + b} = K\frac{(e^{i\omega h} + a)(e^{-i\omega h} + b)}{(e^{i\omega h} + b)(e^{-i\omega h} + b)}$$
$$= K\frac{1 + ab + (a + b)\cos(\omega h) + i(b - a)\sin(\omega h)}{1 + b^2 + 2b\cos(\omega h)}$$
$$\arg H(e^{i\omega h}) = \arctan\frac{(b - a)\sin(\omega h)}{1 + ab + (a + b)\cos(\omega h)}$$
$$|H(e^{i\omega h})| = K\sqrt{\frac{1 + a^2 + 2a\cos(\omega h)}{1 + b^2 + 2b\cos(\omega h)}}$$

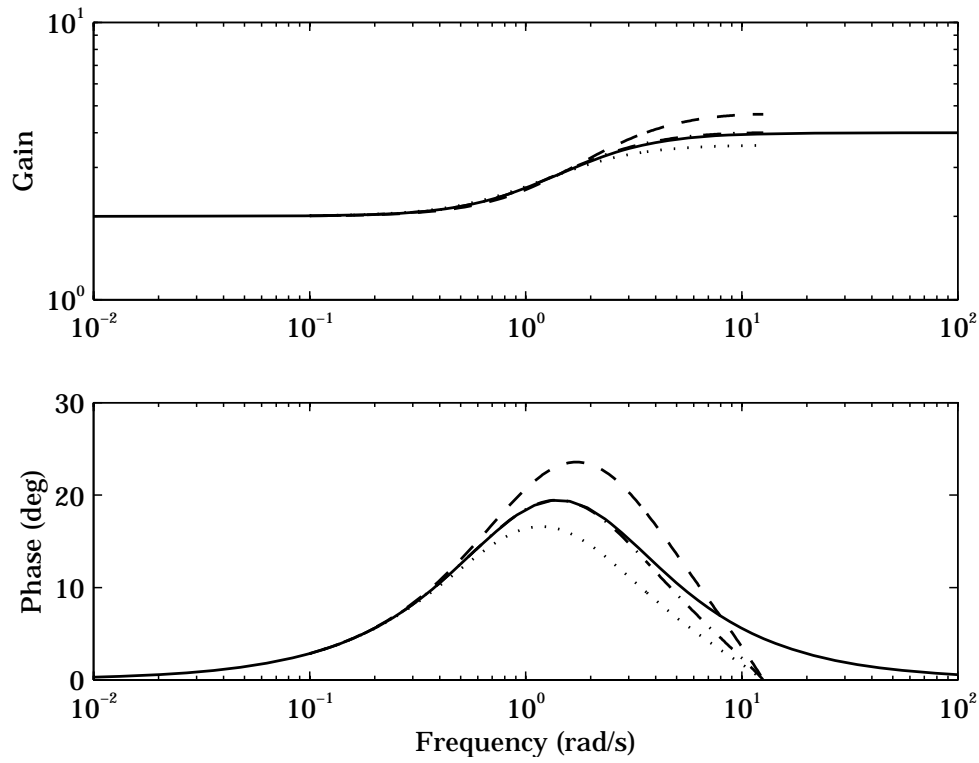The different approximations give at $\omega = 1.6$ rad/s.

Figure 8.2 The Bode diagrams for the filter in Example 8.3 when $h = 0.25$ continuous time filter (full); Euler's method (dashed); backward difference (dotted); Tustin (dash-dotted).

	$|H(e^{i\omega})|$	$\arg H(e^{i\omega})$
Euler	2.97	24°
Backward	2.92	16°
Tustin	2.96	19°
Tustin with prewarping	2.96	19°
Zero order hold	3.25	22°

Fig. 8.2 shows the Bode diagrams for the continuous time system and for the Euler, backward and Tustin approximations. Fig. 8.3 is the same as Fig. 8.2 but with $h = 0.05$. The shorter sampling period gives a better approximation.

Problem 8.4

It is assumed that the sample and hold circuit can be approximated by a delay of $h/2$ seconds. Further we will allow a decrease of the phase margin of $5° - 15°$. This approximately corresponds to a decrease of the damping by $0.05 - 0.15$. A time delay of $h/2$ seconds gives at the crossover frequency a decrease of

$$\Delta\varphi = \omega_c h/2\,[\text{rad}] = \frac{180°\omega_c h}{2\pi} = \frac{\omega_c h}{0.035} = 5° - 15°$$

This gives

$$\omega_c h = 0.17 - 0.52$$

or approximately
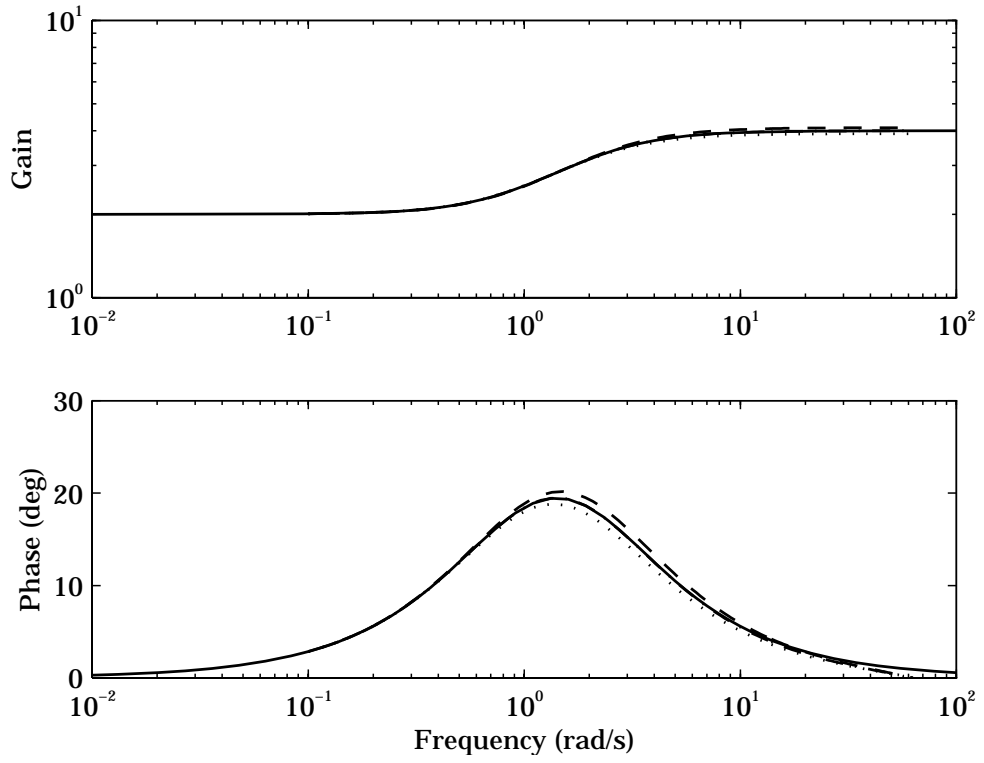
$$\omega_c h \approx 0.15 - 0.5.$$

Figure 8.3 The same as Fig. 8.2 but when $h = 0.05$.

Problem 8.5

The transfer function of the integral part of the PID-controller (8.22) is

$$G_I(s) = \frac{K}{T_i s}$$

Using Euler's approximation (8.4) gives

$$H_I(z) = \frac{Kh}{T_i(z-1)}$$

which is the same integral part in (8.23). The derivative part of (8.22) has the transfer function

$$G_D(s) = \frac{K T_d s}{1 + T_d s / N}$$

Using backward difference gives

$$H_D(z) = \frac{\dfrac{K T_d (z-1)}{zh}}{1 + \dfrac{T_d (z-1)}{zhN}} = \frac{K T_d (z-1)}{z(h + T_d/N) - T_d/N}$$

$$= \frac{K T_d}{h + T_d/N} \frac{z-1}{z - \dfrac{T_d}{Nh + T_d}}$$

which is the same as the derivative part on page 308.

a. Approximation of the integral part with backward difference gives

$$H_I(z) = \frac{Khz}{T_i(z-1)}$$

An error will then directly influence the computation of the integral part. Euler's approximation gives a delay of one sampling interval before an error will influence the integral part. The sampling interval is, however, usually short for digital PID-algorithms.

b. Euler's approximation for the derivative part gives

$$H_d(z) = \frac{KN(z-1)}{z-1+hN/T_d}$$

A small value of T_d can make H_d unstable. Since the D-part of a PID-controller sometimes is not used it is necessary that the regulator remains stable when $T_d = 0$.

Problem 8.6

Using the bilinear transformation gives

$$H_T(z) = K\left(1 + \frac{1}{T_i\dfrac{2}{h}\dfrac{z-1}{z+1}}\right) = K\left(1 + \frac{h}{2T_i} + \frac{h}{T_i}\frac{1}{z-1}\right)$$

$$= K\left(1 + \frac{h}{2T_i}\right)\left(1 + \frac{2h}{(2T_i+h)(z-1)}\right)$$

This is of the same form as (8.24) with

$$K_d = K\left(1 + \frac{h}{2T_i}\right)$$

$$T_{id} = T_i + h/2$$

Problem 8.7

The tank process in Problem 2.10 has the transfer function

$$G(s) = \frac{0.000468}{(s+0.0197)(s+0.0129)}$$

a. At the desired cross over frequency we have

$$|G(i\omega_c)| = 0.525$$
$$\arg G(i\omega_c) = -115°$$

We will use a PI controller of the form

$$G_r(s) = \frac{K(Ts+1)}{Ts}$$

and we want the gain $1/0.523$ and the phase -15 degrees at ω_c. This gives
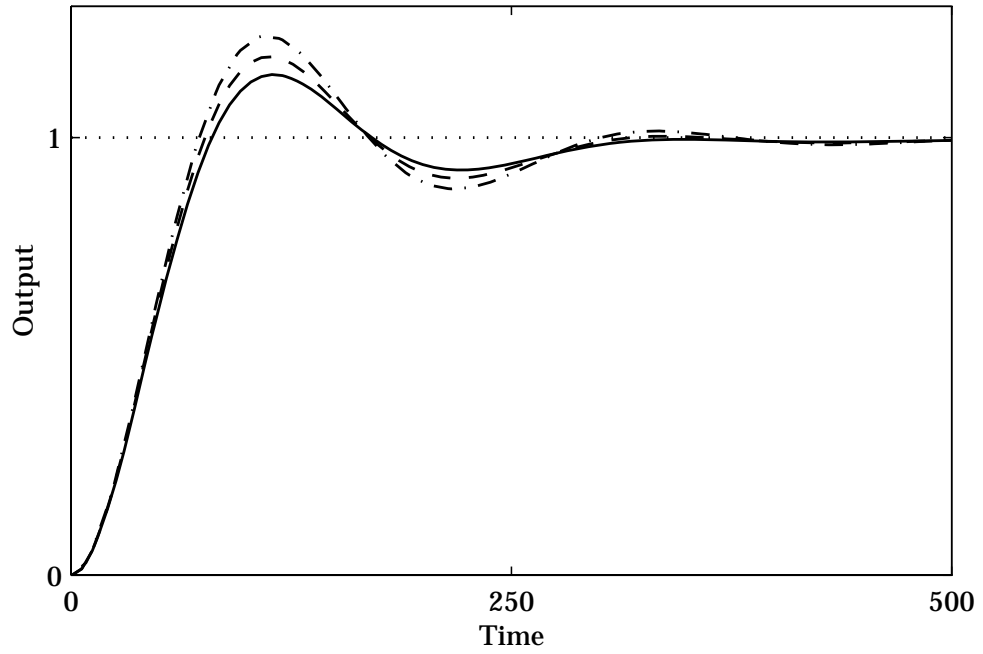
$$K = 1.85$$
$$T = 149$$

Figure 8.4 Step response of the tank process when controlled with a continuous time (solid) and a discrete time PI controller. The sampling interval is 6 (dash-dotted) and 12 seconds (dashed).

b. The characteristic equation of the closed loop system is

$$s^3 + 0.0326s^2 + 0.00112s + 0.00000581 = 0$$

The roots are $s_{1,2} = -0.0135 \pm 0.0281i$ and $s_3 = -0.006$. The complex poles have a damping $\zeta = 0.43$. The zero of the closed loop system is -0.0062.

c. Tustin's approximation with warping gives with $\alpha = \omega_c / \tan(\omega_c h / 2)$

$$H_r(z) = \frac{1.85 \left(\alpha \dfrac{z-1}{z+1} + 0.0067 \right)}{\alpha \dfrac{z-1}{z+1}}$$

$$= \frac{1.85(\alpha + 0.0067)}{\alpha} \left(1 + \frac{0.0134}{(\alpha + 0.0067)(z - 1)} \right)$$

Using the rule of thumb from Section 8.2 for choosing the sampling period gives

$$h \approx 6 - 20 \text{ seconds}$$

The choice $h = 12$ seems to be reasonable. This gives $\alpha = 0.165$ and

$$H_r(z) = 1.925 \left(1 + \frac{0.0778}{z - 1} \right)$$

d. Fig. 8.4 shows simulations of the step response of the system controlled with the continuous time and the approximate discrete time PI-controller when $h = 6$ and 12 seconds.
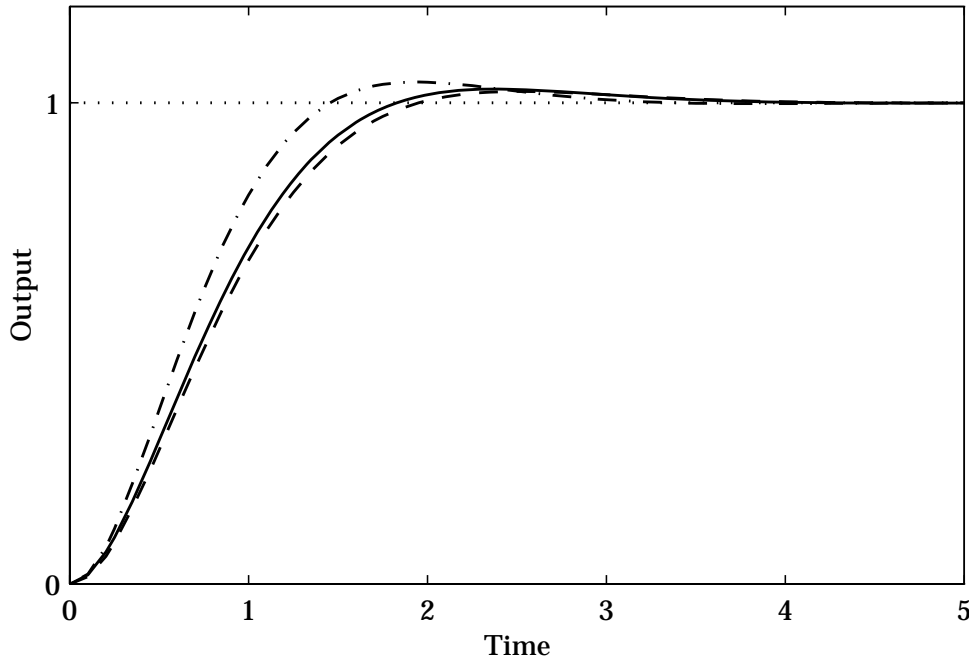
Figure 8.5 Stepresponses for the motor in Problem 8.9 when a continuous time
(solid), a discretized (dash-dotted) and a modified discretized state (dashed) feed-
back controller is used when $h = 0.25$.

Problem 8.9

a. The continuous time controller is

$$u(t) = M u_c(t) - L x(t).$$

A discretization is obtained by sampling u_c and x and letting u be constant
between the sampling period points i.e. we get

$$u(kh) = M u_c(kh) - L x(kh)$$

b. Using (8.24) and (8.25) give

$$\tilde{L} = L(I + (A - BL)h/2) = \begin{pmatrix} 2 & 4 \end{pmatrix} \begin{pmatrix} 1 - 3h/2 & -2h \\ h/2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 - h & 4 - 4h \end{pmatrix}$$

$$\tilde{M} = (I - LBh/2)M = 4(1 - h)$$

c. Fig. 8.5 shows the stepresponse of the system when using the continuous
controller and the controllers in a) and b) when $h = 0.25$. It is possible to
calculate backwards to find out the corresponding damping and natural
frequency for the controllers in a) and b). A discrete time state space
representation of the motor is given in (A.6). Using $L = \begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix}$ gives

$$\Phi - \Gamma L = \begin{pmatrix} e^{-h} - \ell_1(1 - e^{-h}) & -\ell_2(1 - e^{-h}) \\ 1 - e^{-h} - \ell_1(h - 1 + e^{-h}) & 1 - \ell_2(h - 1 + e^{-h}) \end{pmatrix}$$

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For $h = 0.25$ and $L = \begin{pmatrix} 2 & 4 \end{pmatrix}$ we get

$$\Phi - \Gamma L = \begin{pmatrix} 0.336 & -0.885 \\ 0.164 & 0.885 \end{pmatrix}$$

and for $h = 0.25$ and $L = \begin{pmatrix} 1.75 & 3 \end{pmatrix}$ we get

$$\Phi - \Gamma L = \begin{pmatrix} 0.392 & -0.664 \\ 0.171 & 0.914 \end{pmatrix}$$

These two matrices have the characteristic equations

$$z^2 - 1.221z + 0.442 = 0$$

and

$$z^2 - 1.305z + 0.471 = 0.$$

From the equations given in Example 2.16 we can calculate the corresponding continuous time systems. For the discretized controller ($L = \begin{pmatrix} 2 & 4 \end{pmatrix}$) we get

$$\zeta = 0.71$$
$$\omega_0 = 2.31$$

and for the modified controller ($L = \begin{pmatrix} 2 - h & 4 - 4h \end{pmatrix}$) we get

$$\zeta = 0.77$$
$$\omega_0 = 1.96$$

The change in the damping is smaller when the modified controller is used and the change in the undamped natural frequency is also smaller.

Problem 8.10

a. We first want to compute a state feedback such that $A - BL$ has the characteristic equation
$$s^2 + 8s + 32 = 0.$$
Assume $L = \begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix}$ then

$$A - BL = \begin{pmatrix} -3 & 1 \\ -\ell_1 & -2 - \ell_2 \end{pmatrix}$$

The characteristic equation of $A - BL$ is

$$(s + 3)(s + 2 + \ell_2) + \ell_1 = s^2 + (5 + \ell_2)s + 6 + 3\ell_2 + \ell_1 \equiv s + 8s + 32.$$

This gives

$$L = \begin{pmatrix} 17 & 3 \end{pmatrix}$$

b. Modifying L using (8.16) gives

$$L = L(I + (A - BL)h/2)$$

$$= \begin{pmatrix} 17 & 3 \end{pmatrix} \begin{pmatrix} 1 - 3h/2 & h/2 \\ -17h/2 & 1 - 5h/2 \end{pmatrix}$$

$$= \begin{pmatrix} 17(1 - 3h) & 3 + h \end{pmatrix}$$

Fig. 8.6 shows the output when using the discrete time controller in a) for different values of h. The response when using the modified discrete time controller from b) is shown in Fig. 8.7.

Problem 8.12

a. Using (8.16) and (8.17) give

$$\tilde{L} = L\left(I + (A - BL)\frac{h}{2}\right) = L \begin{pmatrix} 1 & h/2 \\ -h/2 & 1 - h \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & h/2 \\ -h/2 & 1 - h \end{pmatrix} = \begin{pmatrix} 1 - h & 2 - \frac{3}{2}h \end{pmatrix} = \begin{pmatrix} 0.8 & 1.7 \end{pmatrix}$$

$$\tilde{M} = \begin{pmatrix} I - LBh/2 \end{pmatrix} \quad M = 2 - 2h = 1.6$$

b. Using the backward difference approximation gives

$$\frac{1 - q^{-1}}{h} I \hat{x}(k) = (A - KC)\hat{x}(k) + Bu(k) + Ky(k)$$

$$(I - Ah + KCh)\hat{x}(k) = q^{-1}\hat{x}(k) + Bhu(k) + Khy(k)$$

Introduce

$$\Phi_0 = (I - Ah + KCh)^{-1} = \frac{1}{1 + h + h^2} \begin{pmatrix} 1 & h \\ -h & 1 + h \end{pmatrix}$$

This gives

$$\hat{x}(k) = \Phi_0 \hat{x}(k - 1) + \Phi_0 Bhu(k) + \Phi_0 Khy(k)$$

$$= \begin{pmatrix} 0.81 & 0.16 \\ -0.16 & 0.97 \end{pmatrix} \hat{x}(k - 1) + \begin{pmatrix} 0.03 \\ 0.19 \end{pmatrix} u(k) + \begin{pmatrix} 0.19 \\ 0.16 \end{pmatrix} y(k)$$
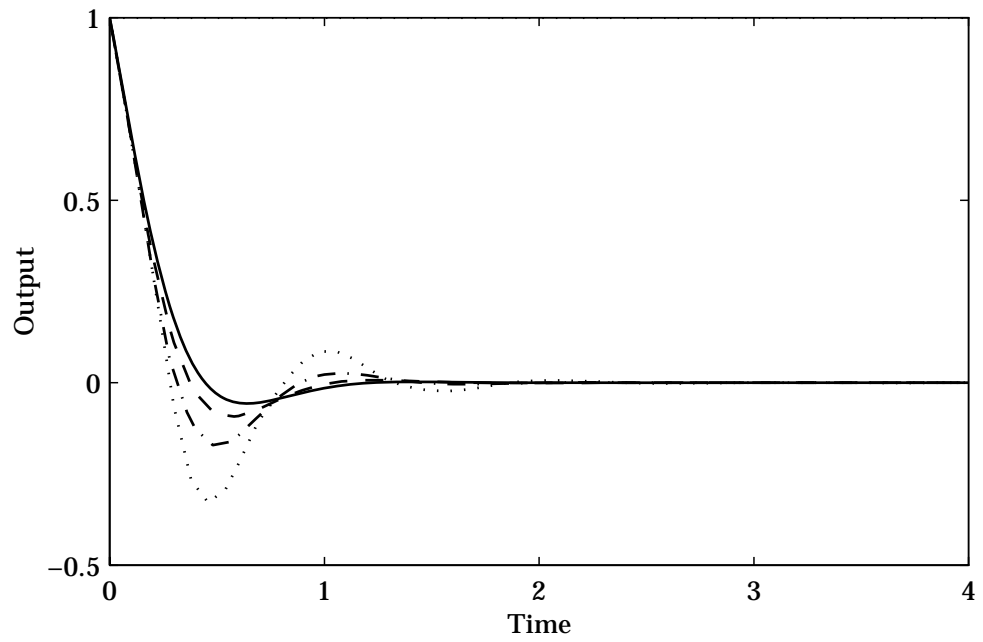
Figure 8.6 The response of the system in Problem 8.10 when the state feedback controller in a) is used with $h = 0.1$ (dashed) , 0.2 (dash-dotted) and 0.3 (dotted). The response for the continuous-time controller is also shown (solid).

Figure 8.7 The response of the system in Problem 8.10 when the modified state feedback controller in b) is used with $h = 0.1$ (dashed), 0.2 (dash-dotted) and 0.3 (dotted). The response for the continuous-time controller is also shown (solid).
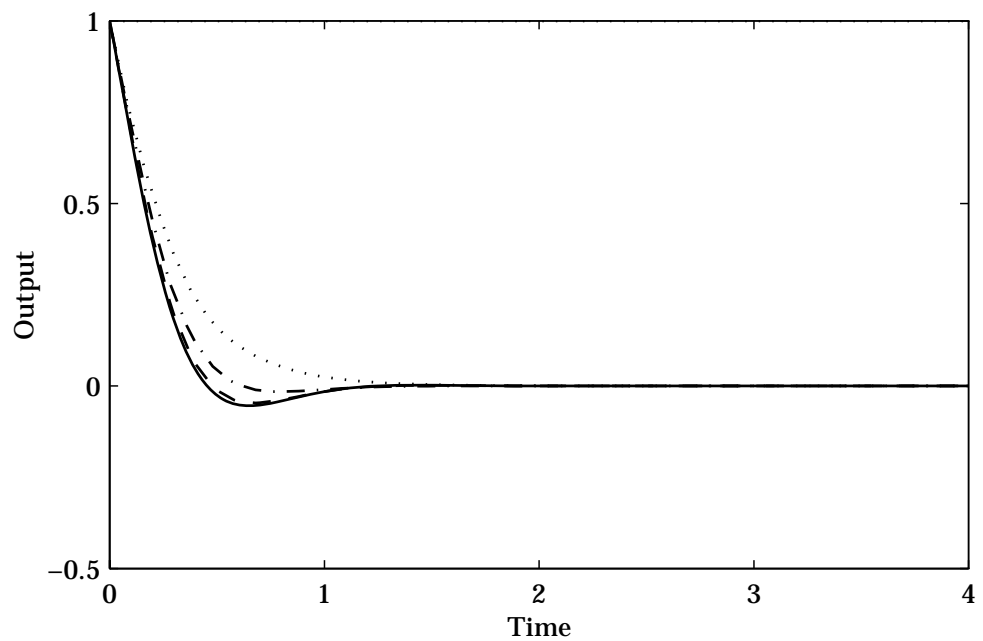
Solutions to Chapter 10

Problem 10.1

a.   Better sensors, for instance a tachometer with less noise.

b.   Flow control with local feedback.

c.   Temperature control in houses.

Problem 10.2

a.   The time function $y(t) = \sin(\omega t)$ has the z-transform

$$Y(z) = \frac{z \sin \omega h}{z^2 - 2z \cos \omega h + 1}$$

See Table 2.1. Consider a system with

$$H_d(z) = Y(z)$$

The impulse response of H_d will thus be $\sin(kh\omega)$. That this is the correct answer is easily seen by making long division.

$$H_d(z) = \sin(h\omega)z^{-1} + \sin(2h\omega)z^{-2} + \sin(3h\omega)z^{-3} + \cdots$$

b.   The time function $t \cdot e^{-t}$ has the z-transform

$$Y(z) = \frac{he^{-h}z}{(z - e^{-h})^2}$$

This can be found by looking in a table of z-transforms. The desired system thus has the z-transform

$$H_d(z) = \frac{he^{-h}z}{(z - e^{-h})^2}$$

Long division gives

$$H_d(z) = he^{-h}z^{-1} + 2he^{-2h}z^{-2} + \cdots$$

Problem 10.3

Using the model of the disturbance gives

$$y(k + m) = \frac{C(q)}{A(q)}w(k + m).$$

Introduce the identity

$$q^{m-1}C(q) = A(q)F(q) + G(q)$$

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where $\deg F = m - 1$ and $\deg G = n - 1$. Then

$$y(k + m) = F(q)w(k + 1) + \frac{qG(q)}{A(q)}\, w(k) = F(q)w(k + 1) + \frac{qG(q)}{C(q)}\, y(k)$$

If $w(k + 1), \ldots, w(k + m)$ are assumed to be zero then the best prediction of $y(k + m)$ is

$$\hat{y}(k + m) = \frac{qG(q)}{C(q)} y(k).$$

The operator $qG(q)/C(q)$ is casual since $\deg C = \deg G + 1$.
Let $A(q) = q - 0.5$, $C(q) = q$ and $m = 3$ then

$$q^2 \cdot q = (q - 0.5)(q^2 + f_1 q + f_2) + g_0$$

This gives the system of equations

$$0 = -0.5 + f_1$$
$$0 = -0.5 f_1 + f_2$$
$$0 = -0.5 f_2 + g_0$$

with solution

$$f_1 = 0.5 \quad f_2 = 0.25 \quad g_0 = 0.125$$

The predictor at $k + 3$ given data up to and including k is thus

$$\hat{y}(k + 3|k) = \frac{0.125q}{q} y(k) = 0.125 y(k)$$

Let $w(k)$ be zero except at $k = 0$ and 5 when it is assumed to be one then

k	$y(k) = 0.5y(k - 1) + w(k)$	$\hat{y}(k|k - 3)$
-1	0	0
0	1	0
1	0.5	0
2	0.25	0
3	0.125	0.125
4	0.063	0.063
5	1.031	0.031
6	0.516	0.016
7	0.258	0.008
8	0.129	0.129
9	0.064	0.064
10	0.032	0.032

Problem 10.4

Using (10.11) we find that the stationary variance of x fulfils (Φ stable)

$$P = \Phi P \Phi^T + R_1$$

The stationary covariance exists since the system is stable. Since R_1 is symmetric P is also symmetric. Introduce

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

then

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = \begin{pmatrix} 0.4 & 0 \\ -0.6 & 0.2 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 0.4 & -0.6 \\ 0 & 0.2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

This gives

$$p_{11} = 0.16p_{11} + 1$$
$$p_{12} = -0.24p_{11} + 0.08p_{12}$$
$$p_{22} = 0.36p_{11} - 0.24p_{12} + 0.04p_{22} + 2$$

The solution is

$$P = \begin{pmatrix} 1.19 & -0.31 \\ -0.31 & 2.61 \end{pmatrix}$$

The stationary covariance function is

$$E x(k+\tau)x(k)^T = \Phi^\tau P$$

It remains to compute Φ^τ. The eigenvalues of Φ are $\lambda_1 = 0.4$ and $\lambda_2 = 0.2$. Using the results on matrix functions in Appendix B we get

$$\Phi^\tau = \alpha_0 I + \alpha_1 \Phi$$

where α_0 and α_1 are obtained from

$$\lambda_1^\tau = \alpha_0 + \alpha_1 \lambda_1$$
$$\lambda_2^\tau = \alpha_0 + \alpha_1 \lambda_2$$

The solution is

$$\alpha_0 = \frac{-\lambda_1 \lambda_2 (\lambda_1^{\tau-1} - \lambda_2^{\tau-1})}{\lambda_1 - \lambda_2}$$
$$\alpha_1 = \frac{\lambda_1^\tau - \lambda_2^\tau}{\lambda_1 - \lambda_2}$$

and

$$\Phi^\tau = \begin{pmatrix} 0.4^\tau & 0 \\ -3(0.4^\tau - 0.2^\tau) & 0.2^\tau \end{pmatrix}$$

Finally

$$r_x(\tau) = \begin{pmatrix} 1.19 \cdot 0.4^\tau & -0.31 \cdot 0.4^\tau \\ -3.57 \cdot 0.4^\tau + 3.27 \cdot 0.2^\tau & 0.93 \cdot 0.4^\tau + 1.68 \cdot 0.2^\tau \end{pmatrix} \quad \tau \geq 0$$

Problem 10.5

From the state space description we get the input-output description of the process

$$y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) = c_1 v(k-1) + \cdots + c_n v(k-n)$$

where $a_i,\quad i = 1, \ldots, n$ are the coefficients of the characteristic polynomial of the matrix Φ. Multiply the equation above by $y(k - \tau)$ and take the mathematical expectation. $y(k - \tau)$ is independent of all the terms on the right hand if $\tau > n + 1$. Thus

$$r_y(\tau) + a_1 r_y(\tau - 1) + \cdots + a_n r_y(\tau - n) = 0.$$

This is called the Yule-Walker equation.

Problem 10.6

There are two noise sources v_1 and v_2 that is used to generate $y(k)$. Using Theorem 10.2 we find that the spectral density of y is

$$\phi_y = H(z)\phi_v H^T(z^{-1})$$

where $z = e^{i\omega}$

$$H(z) = C(zI - \Phi)^{-1}\Gamma = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{bmatrix} z+a & 0 \\ 0 & z+b \end{bmatrix}^{-1} 1 = \begin{pmatrix} \frac{1}{z+a} & \frac{1}{z+b} \end{pmatrix}$$

and

$$\phi_v = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

Thus

$$\phi_y = \begin{pmatrix} \frac{1}{z+a} & \frac{1}{z+b} \end{pmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{pmatrix} \frac{1}{z^{-1}+a} \\ \frac{1}{z^{-1}+b} \end{pmatrix}$$

$$= \frac{\sigma_1^2}{(z+a)(z^{-1}+a)} + \frac{\sigma_2^2}{(z+b)(z^{-1}+b)}$$

$$= \frac{\sigma_1^2(z+b)(z^{-1}+b) + \sigma_2^2(z+a)(z^{-1}+a)}{(z+a)(z+b)(z^{-1}+a)(z^{-1}+b)}$$

Using the spectral factorization theorem (Theorem 10.3) we find that we can generate the same spectral density by sending white noise through

$$H_1(z) = \lambda \frac{z+c}{(z+a)(z+b)}$$

this gives the spectral density

$$\phi_1 = \lambda^2 \frac{(z+c)(z^{-1}+c)}{(z+a)(z^{-1}+a)(z+b)(z^{-1}+b)}$$

Identification with ϕ_y gives the relationship

$$\lambda^2(1+c^2) = \sigma_1^2(1+b^2) + \sigma_2^2(1+a^2)$$
$$\lambda^2 c = \sigma_1^2 b + \sigma_2 a$$

Problem 10.7

The process is
$$x(k+1) = ax(k) + v(k)$$
$$y(k) = x(k) + e(k)$$

This is the same as in Example 10.3 with the exception that $Ev(k)e(s) = r_{12}\delta(k-s)$. The covariance function for x will thus be the same but the covariance of y will contain an additional term due to the correlation between v and e. From Example 10.3 we know that

$$r_x(\tau) = a^{|\tau|}\frac{r_1}{1-a^2}$$

The covariance of y is

$$r_y(\tau) = E\{y(k+\tau)y(k)\} = E\{[x(k+\tau)+e(k+\tau)][x(k)+e(k)]\} =$$
$$= r_x(\tau) + r_{xe}(\tau) + r_{ex}(\tau) + r_e(\tau) = r_x(\tau) + r_{xe}(\tau) + r_{xe}(-\tau) + r_e(\tau)$$

where it has been used that $r_{ex}(\tau) = r_{xe}(-\tau)$ in stationarity.

$$r_{xe}(\tau+1) = E\{x(k+\tau+1)e(k)\} = E\{[ax(k+\tau)+v(k+\tau)]e(k)\} =$$
$$= ar_{xe}(\tau) + r_{ve}(\tau)$$

The last term is zero except for $\tau = 0$, where it is r_{12}. The first term is zero for $\tau \le 0$. It is natural that white noise in the future is uncorrelated with the present value of x. This gives

$$r_{xe}(\tau) = \begin{cases} 0 & \tau \le 0 \\ r_{12} & \tau = 1 \\ a^{\tau-1}r_{12} & \tau > 1 \end{cases} = \begin{cases} 0 & \tau \le 0 \\ a^{\tau-1}r_{12} & \tau \ge 1 \end{cases}$$

and

$$r_y(\tau) = \begin{cases} a^{-\tau}\frac{r_1}{1-a^2} + 0 + a^{-\tau-1}r_{12} + 0 & \tau < 0 \\ \frac{r_1}{1-a^2} + 0 + 0 + r_2 & \tau = 0 \\ a^{\tau}\frac{r_1}{1-a^2} + a^{\tau-1}r_{12} + 0 + 0 & \tau > 0 \end{cases} =$$

$$= \begin{cases} a^{|\tau|}\frac{r_1}{1-a^2} + a^{|\tau|-1}r_{12} & \tau \ne 0 \\ \frac{r_1}{1-a^2} + r_2 & \tau = 0 \end{cases}$$

The definition of spectral density gives

$$\phi_y(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} r_y(\tau)e^{-i\omega\tau} =$$

$$= \frac{1}{2\pi}\left\{\left[\frac{r_1}{1-a^2} - \frac{r_{12}}{a}\right]\frac{1-a^2}{(e^{i\omega}-a)(e^{-i\omega}-a)} - \frac{r_1}{1-a^2} - \frac{r_{12}}{a} + \frac{r_1}{1-a^2} + r_2\right\} =$$

$$= \frac{1}{2\pi}\frac{r_1 a + r_{12}(1-a^2) - r_{12}(e^{i\omega}-a)(e^{-i\omega}-a) + r_2 a(e^{i\omega}-a)(e^{-i\omega}-a)}{a(e^{i\omega}-a)(e^{-i\omega}-a)} \tag{1}$$

where it has been used that

$$\sum_{\tau=-\infty}^{\infty} a^{|\tau|}e^{-i\omega\tau} = \frac{1-a^2}{(e^{i\omega}-a)(e^{-i\omega}-a)}$$

The spectral density for

$$y(k) = \lambda\frac{q-c}{q-a}\varepsilon(k)$$

is (see Theorem 10.2)

$$\phi_y(\omega) = \frac{\lambda^2}{2\pi}\frac{(e^{i\omega}-c)(e^{-i\omega}-c)}{(e^{i\omega}-a)(e^{-i\omega}-a)} \tag{2}$$

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Identification of (1) with (2) gives the relation

$$
\begin{cases}
r_1 + r_{12}\dfrac{1-a^2}{a} - r_{12}\dfrac{1+a^2}{a} + r_2 a\dfrac{1+a^2}{a} = \lambda^2(1+c^2) \\[2mm]
r_{12} - r_2 a = -\lambda^2 c
\end{cases}
$$

$$\Leftrightarrow$$

$$
\begin{cases}
r_2(1+a^2) + r_1 - 2ar_{12} = \lambda^2(1+c^2) \\[2mm]
r_2 a - r_{12} = \lambda^2 c
\end{cases}
$$

A more elegant solution   The ouput can be written as

$$
y(k) = \begin{pmatrix} H(q) & 1 \end{pmatrix} \begin{pmatrix} v(k) \\ e(k) \end{pmatrix}
$$

where $H(z) = \frac{1}{z-a}$. The spectral density of y is given by

$$
\begin{aligned}
\phi_y &= \frac{1}{2\pi} \begin{pmatrix} H(z) & 1 \end{pmatrix} \begin{pmatrix} r_1 & r_{12} \\ r_{12} & r_2 \end{pmatrix} \begin{pmatrix} H(z^{-1}) \\ 1 \end{pmatrix} \\[2mm]
&= \frac{1}{2\pi} r_1 H(z)H(z^{-1}) + r_{12}H(z) + r_{12}H(z^{-1}) + r_2 \\[2mm]
&= \frac{1}{2\pi}\frac{r_1}{(z-a)(z^{-1}-a)} + r_{12}\left(\frac{1}{z-a} + \frac{1}{z^{-1}-a}\right) + r_2 \\[2mm]
&= \frac{1}{2\pi}\frac{z(r_{12}-r_2 a) + r_1 - 2ar_{12} + r_2(1+a^2) + z^{-1}(r_{12} - r_2 a)}{(z-a)(z^{-1}-a)}
\end{aligned}
$$

which gives the same equations as in the previous method

$$
\begin{cases}
r_2(1+a^2) + r_1 - 2ar_{12} = \lambda^2(1+c^2) \\[2mm]
ar_2 - r_{12} = \lambda^2 c
\end{cases}
$$

Problem 10.8

The process can be written as

$$
y(k) = \frac{q-c}{q-a} e(k) = \frac{a-c}{q-a} e(k) + e(k) = x(k) + e(k)
$$

where

$$
x(k+1) = ax(k) + (a-c)e(k)
$$

Using Problem 10.7 we find that

$$
\begin{aligned}
r_1 &= (a-c)^2 \\
r_{12} &= a - c \\
r_2 &= 1
\end{aligned}
$$

Thus

$$
r_y(\tau) = \frac{a^{|\tau|}r_1}{1-a^2} + a^{|\tau|-1}r_{12}
$$

with $a = 0.7$ and $c = 0.5$ we get for $\tau \neq 0$

$$r_y(\tau) = (0.7)^{|\tau|}\frac{0.04}{1 - 0.49} + \frac{0.2}{0.7}(0.7)^{|\tau|} = 0.36(0.7)^{|\tau|}$$

For $\tau = 0$

$$r_y(0) = \frac{r_1}{1 - a^2} + r_2 = \frac{(a - c)^2}{1 - a^2} + 1 = 1.08$$

The variance can also be obtained from Theorem 10.4.

$$r_y(0) = I_1 = \frac{1 + c^2 - 2ac}{1 - a^2} = 1.08$$

Further (10.17) gives

$$\phi_y = \frac{1}{2\pi}\frac{(z - c)(z^{-1} - c)}{(z - a)(z^{-1} - a)}) = \frac{1}{2\pi}\frac{1.25 - \cos\omega}{1.49 - 1.4\cos\omega}$$

where $z = e^{i\omega}$.

Problem 10.9

The variance of the process

$$y(k) = \frac{q^2 + 0.2q}{q^2 - 1.5q + 0.7}e(k) = \frac{b_0q^2 + b_1q + b_2}{a_0q^2 + a_1q + a_2}e(k)$$

can be determined using Theorem 10.4. The formula for I_2 gives

$$B_0 = 1 + (0.2)^2 + 0 = 1.04$$
$$B_1 = 2(0.2 + 0) = 0.4$$
$$B_2 = 0$$
$$e_1 = 1.7$$

and

$$r_y(0) = I_2 = \frac{1.04 \cdot 1.7 + 0.4 \cdot 1.5}{(1 - 0.49)1.7 - 1.5 \cdot 1.5(1 - 0.7)} = 12.33$$

The recursive scheme in Section 10.4 can also be used to get

1	-1.5	0.7		1	0.2	0	
0.7	-1.5	1	$\alpha_2 = 0.7$	0.7	-1.5	1	$\beta_2 = 0$
0.51	-0.45			1	0.2		
-0.45	0.51		$\alpha_1 = -0.8824$	-0.45	0.51		$\beta_1 = 0.3922$
0.1129				1.1765			
				0.1129			$\beta_0 = 10.4167$

This gives

$$I_2 = 10.4167 \cdot 1.1765 + 0.3922 \cdot 0.2 \approx 12.33$$

These calculations have been performed in high precision and thereafter rounded.

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Problem 10.10

The process is

$$y(k) = e(k) - 2e(k-1) + 3e(k-2) - 4e(k-3)$$
$$r_y(\tau) = Ey(k+\tau)y(k)$$

This gives

$$r_y(0) = E\left\{(e(k) - 2e(k-1) + 3e(k-2) - 4e(k-3))^2\right\}$$
$$= E\left\{e(k)^2 + 4e(k-1)^2 + 9e(k-2)^2 + 16e(6-3)^2 + \text{crossterms}\right\}$$
$$= 1 + 4 + 9 + 16 = 30$$

The mean value of the crossterms are zero since

$$E\left\{e(k+\tau)e(k)\right\} = 0 \qquad \tau \neq 0$$

In the same way

$$r_y(1) = 1 \cdot (-2) + (-2) \cdot 3 + 3 \cdot (-4) = -20$$
$$r_y(2) = 1 \cdot 3 + (-2) \cdot (-4) = 11$$
$$r_y(3) = 1 \cdot (-4) = -4$$
$$r_y(4) = 0 \qquad k \geq 4$$

Problem 10.11

$$\Phi_y = \frac{1}{1.36 + 1.2\cos\omega}$$

a.

$$H(z) = \frac{b}{z - a}$$

Using Theorem 10.2 we get

$$\phi_y(\omega) = H(e^{i\omega})\phi_u(\omega)H(e^{-i\omega}) = H(e^{i\omega})\frac{1}{2\pi}H(e^{-i\omega})$$

$$= \frac{\frac{1}{2\pi}b^2}{\left(e^{i\omega} - a\right)\left(e^{-i\omega} - a\right)} = \frac{b^2/2\pi}{1 + a^2 - a \cdot 2\left(\dfrac{e^{i\omega} + e^{-i\omega}}{2}\right)}$$

$$= \frac{b^2/2\pi}{1 + a^2 - 2a\cos\omega} = \frac{1}{2\pi\dfrac{1+a^2}{b^2} - 2\pi\dfrac{2a}{-b^2}\cos\omega}$$

Identifying with the desired spectral density gives

$$2\pi(1 + a^2) = b^2 \cdot 1.36$$
$$2\pi(2a) = -1.2b^2$$

Divide the two equations

$$\frac{1+a^2}{2a} = -\frac{1.36}{1.2} \quad \Rightarrow \quad a^2 + \frac{2.72}{1.2}a + 1 = 0$$

$$a = -\frac{1.36}{1.2} \pm \sqrt{\left(\frac{1.36}{1.2}\right)^2 - 1} = -0.6$$

$$b = \sqrt{\frac{2a}{-1.2}} \cdot 2\pi = \sqrt{2\pi}$$

The desired filter is

$$H(z) = \frac{\sqrt{2\pi}}{z + 0.6}$$

b.   Theorem 10.4 CCS $\text{Var } y = I_1 = \dfrac{2\pi}{1-0.6^2} = \dfrac{2\pi}{0.64}$

Problem 10.12

$$x(k+1) = \begin{pmatrix} 0.3 & 0.2 \\ 0 & 0.5 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) + v(k)$$

The stationary covariance is given by

$$P = \Phi P \Phi^T + R_1$$

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.2 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 0.3 & 0 \\ 0.2 & 0.5 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}$$

$$p_{11} = 0.09p_{11} + 0.12p_{12} + 0.04p_{22} + 1$$

$$p_{12} = 0.15p_{12} + 0.1p_{22}$$

$$p_{22} = 0.25p_{22} + 0.5$$

$$P = \begin{pmatrix} 1.1385 & 0.0784 \\ 0.0784 & 0.667 \end{pmatrix}$$

Problem 10.13

Example 10.4 shows that the filter

$$H(z) = \frac{b}{z - a}$$

gives the spectral density

$$\Phi(\omega) = \frac{r_1}{2\pi} \cdot \frac{b^2}{1 + a^2 - 2a \cos \omega}$$

In this case $r_1 = 1$. Identify with the desired spectral density gives

$$\frac{1}{2\pi} \cdot \frac{3}{5.43 - 5.40 \cos \omega} = \frac{1}{2\pi} \quad \frac{1}{1.81 - 1.8 \cos \omega}$$

This gives $a = 0.9$ and $b = 1$.

Solutions to Chapter 11

Problem 11.1

For the system we have

$$\frac{d}{dt}\Phi(t, kh) = -a\Phi(t, kh), \qquad \Phi(kh, kh) = 1$$

This differential equation has the solution

$$\Phi(t, kh) = e^{-a(t-kh)}$$

and

$$\Gamma(t, kh) = \int_{kh}^{t} e^{-a(t-s)} b\, ds = \frac{b}{a}(1 - e^{-a(t-kh)})$$

The discrete time loss function is obtained from CCS (11.6)-(11.8), which gives

$$Q_1 = \int_{kh}^{kh+h} e^{-2a(s-kh)}\, ds = \frac{1}{2a}(1 - e^{-2ah})$$

$$Q_{12} = \int_{kh}^{kh+h} e^{-a(s-kh)} \frac{b}{a}(1 - e^{-a(s-kh)})\, ds = \frac{b}{2a^2}(1 - e^{-ah})^2$$

$$Q_2 = \int_{kh}^{kh+h} \left(\frac{b^2}{a^2}(1 - e^{-a(s-kh)})^2 + \varrho \right)\, ds$$

$$= \left(\frac{b^2}{a^2} + \varrho \right) h - \frac{b^2}{2a^3} \left(3 - 4e^{-ah} + e^{-2ah} \right)$$

Notice that there will be a Q_{12} term even if $Q_{12c} = 0$.

Problem 11.2

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

Sample the system

$$\Phi = e^{Ah} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$$

$$\Gamma = \int_{0}^{h} e^{A\tau} B\, d\tau = \begin{pmatrix} h^2/2 \\ h \end{pmatrix}$$

Sample the loss function $Q_{1c} = I$ $Q_{2c} = 1$ $Q_{12c} = 0$. Using (11.6)-(11.8) we get

$$Q_1 = \int_0^h \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix} \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} d\tau = \int_0^h \begin{pmatrix} 1 & \tau \\ \tau & \tau^2 + 1 \end{pmatrix} d\tau = \begin{pmatrix} h & h^2/2 \\ h^2/2 & h^3/3 + h \end{pmatrix}$$

$$Q_{12} = \int_0^h \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix} \begin{pmatrix} \tau^2/2 \\ \tau \end{pmatrix} d\tau = \int_0^h \begin{pmatrix} \tau^2/2 \\ \tau^3/2 + \tau \end{pmatrix} d\tau = \begin{pmatrix} h^3/6 \\ h^4/8 + h^2/2 \end{pmatrix}$$

$$Q_2 = \int_0^h \left(\begin{pmatrix} \tau^2/2 & \tau \end{pmatrix} \begin{pmatrix} \tau^2/2 \\ \tau \end{pmatrix} + 1 \right) d\tau = \int_0^h \left(1 + \tau^2 + \frac{\tau^4}{4} \right) d\tau = h + \frac{h^3}{3} + \frac{h^5}{20}$$

Problem 11.3

The Riccati equation(11.17) gives

$$s(k) = a^2 s(k+1) + 1 - a^2 b^2 \frac{s(k+1)^2}{b^2 s(k+1)} = 1 \quad k = N-1, .., 1$$

and (11.19) gives

$$L(k) = b^{-2} ba = \frac{a}{b}$$

which gives the controller

$$u(k) = -L(k)x(k) = -\frac{ab}{b^2} x(k) = -\frac{a}{b} x(k)$$

The minimum loss is given by

$$\min J = x(0)^2 s(0) = x(0)^2$$

and the closed loop system is

$$x(k+1) = 0$$

The state is zero after one step and the resulting controller is thus a dead-beat controller.

Problem 11.5

a.   The loss function is $\Sigma y^2 + \rho u$, i.e.

$$Q_1 = C^T C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The steady state value of the Riccati equation is obtained from

$$S = \Phi^T S \Phi + Q_1 - \Phi^T S \Gamma (Q_2 + \Gamma^T S \Gamma)^{-1} \Gamma^T S \Phi$$

Let

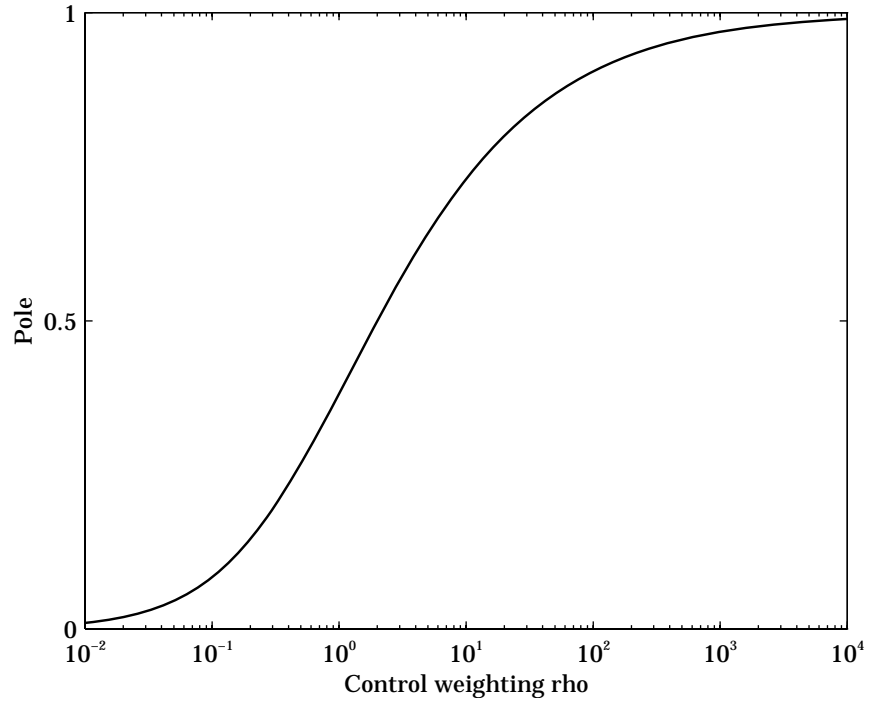$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}$$

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Figure 11.1 The pole in Problem 11.5 as a function of the control weighting ϱ.

For the inventory model we get

$$s_{11} = s_{11} + 1 - \frac{s_{12}^2}{\varrho + s_{22}}$$

$$s_{12} = s_{11} - \frac{s_{12}^2}{\varrho + s_{22}}$$

$$s_{22} = s_{11} - \frac{s_{12}^2}{\varrho + s_{22}}$$

The solution is

$$s_{12} = s_{22} = \frac{1 + \sqrt{1 + 4\varrho}}{2}$$

$$s_{11} = 1 + s_{12}$$

The feedback vector is

$$L = K(\varrho) \begin{pmatrix} 1 & 1 \end{pmatrix}$$

where

$$K(\varrho) = \frac{s_{12}}{\varrho + s_{22}} = \frac{1 + \sqrt{1 + 4\varrho}}{2\varrho + 1 + \sqrt{1 + 4\varrho}}$$

b.   The dynamics of the closed loop system is

$$\Phi - \Gamma L = \begin{pmatrix} 1 & 1 \\ -K(\varrho) & -K(\varrho) \end{pmatrix}$$

The poles are obtained from

$$(\lambda - 1)(\lambda + K(\varrho)) + K(\varrho) = \lambda(\lambda - 1 + K(\varrho)) = 0$$

There is one pole in the origin and one in $1 - K(\varrho)$. For $\varrho = 0$ then $1 - K(\varrho) = 0$ and as $\varrho \to \infty$ then $1 - K(\varrho) \to 1$. Fig 11.1 shows how the pole varies as a function of ϱ.
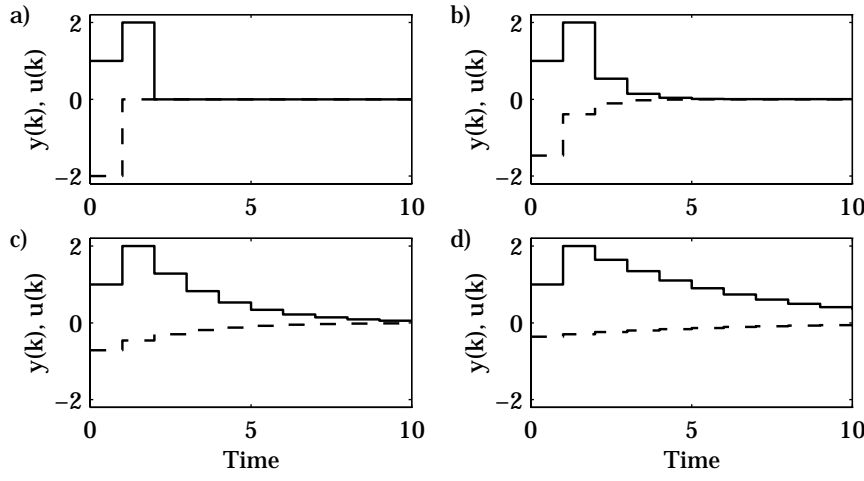
Figure 11.2 The output (solid) and the control signal (dashed) for the system in Problem 11.5 when a) $\varrho = 0$, b) $\varrho = 0.5$, c) $\varrho = 5$ and d) $\varrho = 25$.

The poles of the closed loop system can also be determined from (11.40). For the inventory system

$$A(z) = q(q - 1)$$
$$B(z) = 1$$

and we get

$$\varrho(z^{-2} - z^{-1})(z^2 - z) + 1 = r(z^2 + p_1 z + p_2)(z^{-2} + p_1 z^{-1} + p_2)$$

or

$$0 = rp_2$$
$$-\varrho = r(p_1 + p_1 p_2)$$
$$2\varrho + 1 = r(1 + p_1^2 + p_2^2)$$

Since $r = \Gamma^T S \Gamma + Q_2 \neq 0$ then the first equation implies that $p_2 = 0$ and we get

$$\begin{cases} -\varrho = rp_1 \\ 2\varrho + 1 = r(1 + p_1^2) \end{cases}$$

which has the solution

$$r = \frac{2\varrho^2}{2\varrho + 1 - \sqrt{1 + 4\varrho}} = \frac{2\varrho + 1 + \sqrt{1 + 4\varrho}}{2}$$
$$p_1 = -\frac{2\varrho + 1 - \sqrt{1 + 4\varrho}}{2\varrho}$$

The poles of the closed loop system are thus one at the origin and one at

$$-p_1 = \frac{2\varrho + 1 - \sqrt{1 + 4\varrho}}{2\varrho}$$

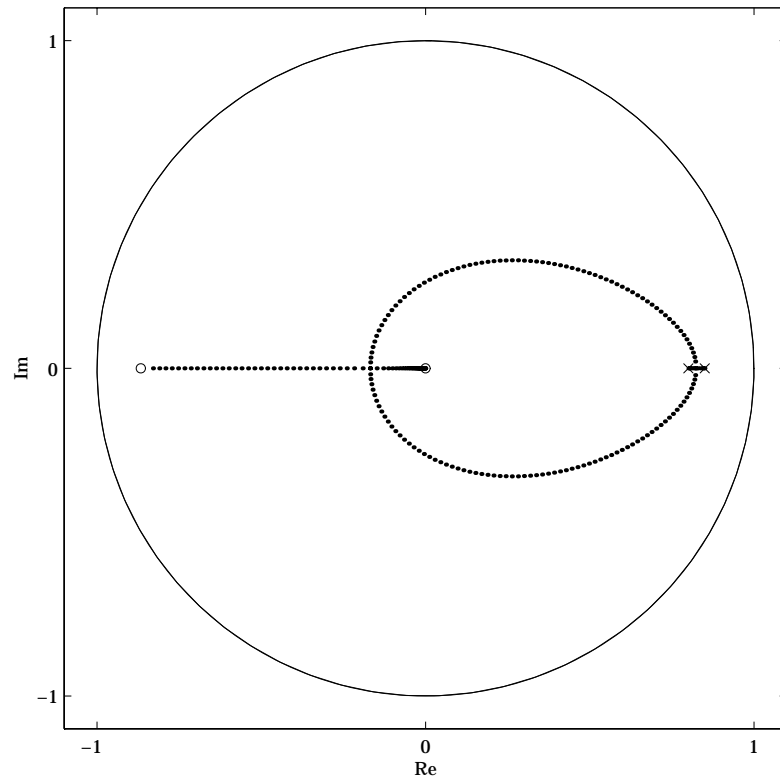It is easily seen that this is the pole in $1 - K(\rho)$.

Figure 11.3   The closed loop poles in Problem 11.6 when ϱ is varying.

Problem 11.6

The system has the transfer function

$$H(z) = \frac{0.030z + 0.026}{z^2 - 1.65z + 0.68}$$

Only the output and the control signals are penalized in the loss function. The closed loop system has poles that are determined by the stable roots of

$$\varrho + H(z)H(z^{-1}) = 0$$

This gives

$$\varrho + \frac{0.030z + 0.026}{z^2 - 1.65z + 0.68} \cdot \frac{0.030z^{-1} + 0.026}{z^{-2} - 1.65z^{-1} + 0.68}$$

$$= \varrho + \frac{0.030z + 0.026}{z^2 - 1.65z + 0.68} \cdot \frac{0.030z + 0.026z^2}{1 - 1.65z + 0.68z^2} = 0$$

or

$$0.00078z^3 + 0.001576z^2 + 0.00078z +$$

$$\varrho(0.68z^4 - 2.7720z^3 + 4.1849z^2 - 2.7720z + 0.68) = 0$$

Fig. 11.3 shows the closed loop poles when ϱ is varying.

Problem 11.9

Solving LQ-problem for system of higher order than one by hand causes much work. With numerical packages, like Control toolbox for Matlab, the design is significantly simplified for the control engineer. The following Matlab-macro illustrates the solution of Problem 11.9, i.e. sampling of the system, sampling of the loss function and solving the Riccati equation.

```
%Macro for Problem 11.9 CCS


alpha=0.001;
k=0.0005;
rho=0.08;
h=5;


A=[0 1; 0 -alpha];
B=[0; k];


Q1c=[1 0; 0 0];
Q12c=[0; 0];
Q2c=rho;


%Transform continuous-time LQG problem to the corresponding
%discrete-time LQG problem
[Phi,Gam,Q1,Q2,Q12,R1,Je] = lqgsamp(A,B,h,Q1c,Q2c,Q12c,zeros(2,2));


%Linear quadratic regulator design for discrete-time system
[L,Lv,S] = lqrd(Phi,Gam,Q1,Q2,Q12);


L
```

The design gives
$$L = \begin{bmatrix} 3.055 & 108.7 \end{bmatrix}$$

Problem 11.10

In Problem 11.5 we have determined r, then

$$\frac{\varrho}{r} = \frac{2\varrho}{2\varrho + 1 + \sqrt{1 + 4\varrho}}$$

Equation (11.37) may be used to get the exact values of the gain margin. With

$$A(z) = z^2 - z$$
$$P(z) = z^2 + p_1 z$$

we get

$$z^2 - z + \beta(z^2 + p_1 z - z^2 + z) = z(z + (\beta p_1 + \beta - 1)) = 0$$

I.e., the system is stable if

$$-1 < \beta p_1 + \beta - 1 < 1 \quad \Rightarrow$$
$$\beta_{\min} = 0 \leq \beta \leq \frac{4\rho}{-1 + \sqrt{1 + 4\varrho}} = \beta_{\max}$$

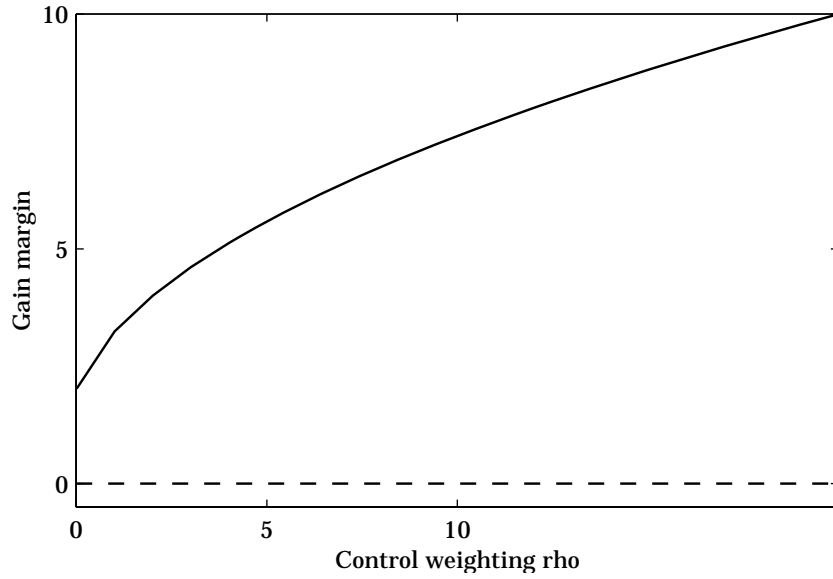$\beta_{\min}$ and $\beta_{\max}$ are also shown in Fig. 11.4.

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Figure 11.4 The gain margin $\beta_{\min}$ (dashed) and $\beta_{\max}$ (solid) from (11.37).

Problem 11.11

The system is

$$x(k+1) = 0.5x(k) + v(k)$$
$$y(k) = x(k) + e(k)$$

Theorem 11.5 gives that the Kalman filter is defined by

$$
\begin{cases}
\hat{x}(k+1|k) = 0.5\hat{x}(k|k-1) + K(k)(y(k) - \hat{x}(k|k-1)), \quad \hat{x}(0|-1) = 0 \\[2mm]
K(k) = \dfrac{0.5P(k)}{r_2 + P(k)} \\[3mm]
P(k+1) = 0.25P(k) + r_1 - \dfrac{0.25P(k)^2}{r_2 + P(k)}, \quad P(0) = r_0
\end{cases}
$$

The dynamics of the filter is determined by

$$\Phi - KC = 0.5 - K(k) = \frac{0.5r_2}{r_2 + P(k)}$$

The steady state variance is given from

$$P^2 + (0.75r_2 - r_1)P = r_1 r_2$$

Consider three cases $r_1 \gg r_2$, $r_1 = r_2$ and $r_1 \ll r_2$. In the first case $P \approx r_1$ and $\Phi - KC \approx 0$. In the second case $P = 1.13r_1$ and $\Phi - KC = 0.23$. Finally if $r_1 \ll r_2$ then $P \approx 1.33r_1$ and $\Phi - KC \approx 0.5$. Fig. 11.5 shows $\Phi - KC$ for different values of r_1 when $r_2 = 1$.

Additional problem

Suppose that the system in Problem 11.11 has a control signal $u(k)$, i.e. the system is

$$x(k+1) = 0.5x(k) + v(k) + u(k)$$
$$y(k) = x(k) + e(k)$$

Determine a steady-state LQG-controller when $Q_1 = 1$, $Q_{12} = 0$ and $Q_2 = \rho$.
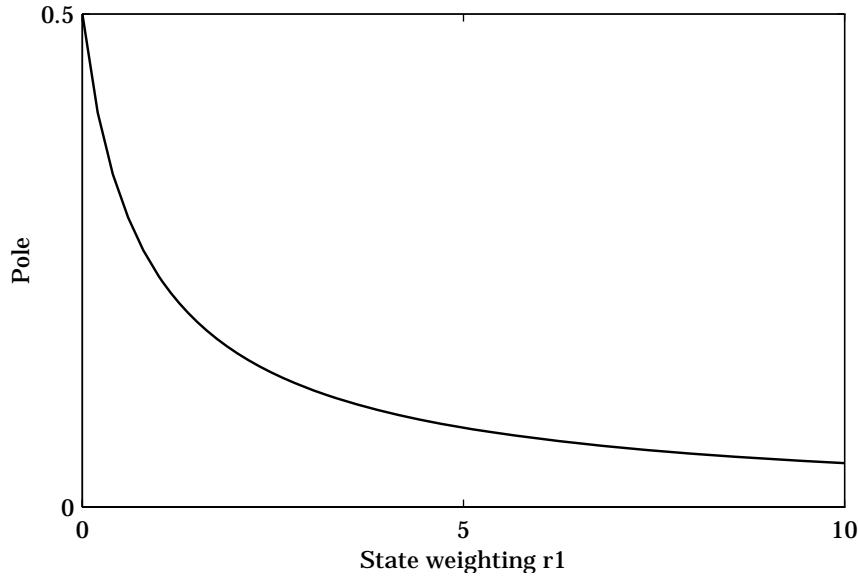
Figure 11.5 The pole of the Kalman filter in Problem 11.11 for different values of r_1 when $r_2 = 1$.

Solution to the additional problem

Equation (11.17) and (11.19) gives

$$S = 0.25S + 1 - \frac{0.25\,S^2}{\rho + S}$$

$$L = \frac{0.5S}{\rho + S}$$

which has the solution

$$L = \frac{-0.75\rho + 1 + \sqrt{(0.75\rho - 1)^2 + 4\rho}}{2.5\rho + 2 + 2\sqrt{(0.75\rho - 1)^2 + 4\rho}}$$

Using the Kalman filter from Problem 11.11 and the LQ-controller gives

$$
\begin{aligned}
\hat{x}(k+1|k) &= \Phi\hat{x}(k|k-1) + \Gamma u(k) + K\left(y(k) - C\hat{x}(k|k-1)\right) \\
&= \Phi\hat{x}(k|k-1) + \Gamma\left(-L\hat{x}(k|k-1)\right) + K\left(y(k) - C\hat{x}(k|k-1)\right) \\
&= (\Phi - \Gamma L - KC)\hat{x}(k|k-1) + Ky(k)
\end{aligned}
$$

and thus

$$U(q) = -L(qI - \Phi + \Gamma L + KC)^{-1}K\,Y(q)$$

or

$$H_{reg}(q) = \frac{-LK}{q - 0.5 + K + L}$$

Problem 11.12

a. Equation (11.47) gives

$$P(k+1) = \Phi P(k)\Phi^T + R_1 - \Phi P(k)C^T(R_2 + CP(k)C^T)^{-1}CP(k)\Phi^T$$
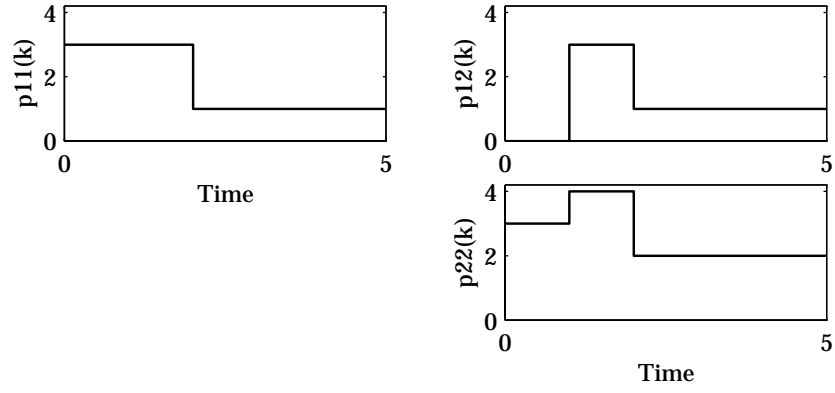
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Figure 11.6 The elements of the variance matrix in Problem 11.12.

with

$$\Phi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$R_1 = \Gamma_v \Gamma_v^T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

we get

$$p_{11}(k+1) = p_{11}(k) + 2p_{12}(k) + p_{22}(k) - \frac{(p_{11}(k) + p_{12}(k))^2}{p_{11}(k)}$$

$$= \frac{p_{11}(k)p_{22}(k) - p_{12}(k)^2}{p_{11}(k)}$$

$$p_{12}(k+1) = p_{12}(k) + p_{22}(k) - \frac{p_{12}(k)(p_{11}(k) + p_{12}(k))}{p_{11}(k)} = p_{11}(k+1)$$

$$p_{22}(k+1) = p_{22}(k) + 1 - \frac{p_{12}^2}{p_{11}(k)} = 1 + p_{11}(k+1)$$

Further

$$K(k) = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} p_{11}(k) + p_{12}(k) \\ p_{12}(k) \end{pmatrix} \frac{1}{p_{11}(k)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad k > 0$$

For $k = 0$ $K(0) = [1\ 0]^T$ i.e. K is timevarying. The steady state value of P is

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

The poles of the filter are found from $\det(\lambda I - (\Phi - KC)) = \lambda^2 = 0$ The filter has a dead beat response.

b. The initial values of the filter are

$$\hat{x}(0|-1) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$$

and assume that $P(0) = 3I$. Fig. 11.5 shows the elements of the covariance matrix as a function of time.

Problem 11.14

Introduce the notation

$$\Phi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \Gamma_2 = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$$

The state at time $k+3$ can now be determined

$$\begin{aligned}
x(k+3) &= \Phi x(k+2) + \Gamma_1 v(k+2) + \Gamma_2 \\
&= \Phi^2 x(k+1) + \Phi \Gamma_1 v(k+1) + \Phi \Gamma_2 + \Gamma_1 v(k+2) + \Gamma_2 \\
&= \Phi^3 x(k) + \Phi^2 \Gamma_1 v(k) + \Phi \Gamma_1 v(k+1) + \Gamma_1 v(k+2) \\
&\quad + \Phi^2 \Gamma_2 + \Phi \Gamma_2 + \Gamma_2
\end{aligned}$$

The best estimate of $x(k)$ given $y(k)$ is determined from (11.50). Since $v(k)$, $v(k+1)$ and $v(k+2)$ are independent of $y(k)$ then the best estimate of $x(k+3)$ is given by

$$\hat{x}(k+3|k) = \Phi^3 \hat{x}(k|k) + (\Phi^2 + \Phi + I)\Gamma_2 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \hat{x}(k|k) + \begin{pmatrix} 4.5 \\ 3 \end{pmatrix}$$

The variance of the estimation error is

$$\begin{aligned}
P(k+3|k) &= \Phi^3 P(k|k)(\Phi^3)^T + 0.01(\Phi^2 \Gamma_1 \Gamma_1^T (\Phi^2)^T + \Phi \Gamma_1 \Gamma_1^T \Phi^T + \Gamma_1 \Gamma_1^T) \\
&= \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} P(k|k) \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} + 0.01 \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}
\end{aligned}$$

If $x(0)$ is known then $P(0|0) = 0$ and

$$\hat{y}(3) = \begin{pmatrix} 1 & 3 \end{pmatrix} x(0) + 4.5$$

and the variance of the prediction error is 0.05.

Problem 11.15

$$\begin{aligned}
x(k+1) &= ax(k) + v(k) & \text{cov } v &= 1 \\
y(k) &= x(k) + e(k) & \text{cov } e &= \sigma
\end{aligned}$$

We use the exponential smoothing estimator

$$\hat{x}(k|k) = \alpha \hat{x}(k-1|k-1) + (1-\alpha)y(k)$$

$$\begin{aligned}
[\hat{x}(k|k) - x(k)] &= \frac{(1-\alpha)q}{q-\alpha} y(k) - x(k) \\
&= \frac{(1-\alpha)q}{q-\alpha} \left(\frac{1}{q-a} v(k) + e(k) \right) - \frac{1}{q-a} v(k) \\
&= -\frac{\alpha(q-1)}{(q-a)(q-\alpha)} v(k) + \frac{(1-\alpha)q}{q-\alpha} e(k)
\end{aligned}$$

Using Theorem 10.4 we get

$$\text{var } \tilde{x}(k|k) = \frac{2\alpha^2}{(1+a)(1+\alpha)(1-a\alpha)} + \frac{1-\alpha}{\alpha+1}\sigma$$

Minimize with respect to α, use Maple.

$$\alpha_{min} = \frac{\sigma a(1+a) + 1 - \sqrt{\sigma(1+a)^2 + 1}}{\sigma a^2(1+a) + a - 1}$$

Kalman with direct term

$$\hat{x}(k|k) = \Phi\hat{x}(k-1|k-1) + K\big(y(k) - C\Phi\hat{x}(k-1|k-1)\big)$$
$$= (I - KC)\Phi\hat{x}(k-1|k-1) + Ky(k)$$

This will have the same form as the exponential smoothing estimator only when $a = 1$.

Kalman variance

$$P = a^2 P + 1 - \frac{a^2 P^2}{P + \sigma}$$

$$\big((1 - a^2)P - 1\big)(P + \sigma) = -a^2 P^2$$

This gives

$$P = \frac{1 - \sigma(1 - a^2)}{2} + \frac{1}{2}\sqrt{1 + 2\sigma(1 + a^2) + \sigma^2(1 - a^2)^2}$$

The gain in the Kalman filter is

$$K = \frac{aP}{P + \sigma}$$

$$\operatorname{var}\tilde{x}(k|k) = P - \frac{P^2}{P + \sigma} = \frac{P\sigma}{P + \sigma} = \frac{\sigma}{a}K$$

Numerical values: $\sigma = 1, a = 0.5$.

Exp. smoothing: $\alpha = 0.4222,$ $\operatorname{var}\tilde{x}(k|k) = 0.6181$

Kalman: $K = 0.2650,$ $\operatorname{var}\tilde{x}(k|k) = 0.5311$

Problem 11.16

The scalar state equations are

$$\begin{cases} x(k+1) = ax(k) + u(k) + v(k) \\ y(k) = x(k) + e(k) \end{cases} \quad \begin{aligned} v(k) &= v_1(k) + m_v; \quad Ev_1 = 0 \\ e(k) &= e_1(k) + m_e; \quad Ee_1 = 0 \end{aligned}$$

Let

$$X = \begin{pmatrix} x & m_v & m_e \end{pmatrix}^T$$

and

$$\begin{cases} X(k+1) = \begin{pmatrix} a & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} X(k) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u(k) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} v_1(k) \\ y(k) = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} X(k) + e_1 \end{cases}$$

The observability matrix is then

$$W_o = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ a & 1 & 1 \\ a^2 & a+1 & 1 \end{pmatrix}$$

with

$$\text{rank } W_o = 2$$

This means that m_e and m_v are not both observable and no Kalman-filter can be designed. It is, however, possible to design a second order observer with reconstruction of a linear combination of m_e and m_v. Redefining the state vector X as

$$X = \begin{pmatrix} x_1 \\ m \end{pmatrix}$$

where

$$\begin{cases} x_1 = x + m_e \\ m = (a-1)m_e + m_v \end{cases}$$

gives

$$\begin{cases} X(k+1) = \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix} X(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_1(k) \\ y(k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} X(k) + e_1(k) \end{cases}$$

Reconstruction of x_1 and m is possible if these states are observable. The observability matrix is

$$W_o = \begin{pmatrix} C \\ C\Phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

from which follows that rank $W_o = 2$.

Problem 11.17

The constants a_1 and a_2 are determined by the following two conditions
1. The correct mean value should be obtained.
2. The variance should be minimized.
Condition 1 gives that

$$a_1 + a_2 = 1$$

The variance of the estimation error is

$$V = E(x(k) - \hat{x}(k))^2 = E(x(k) - a_1 x(k) - a_1 e_1(k) - a_2 x(k) - a_2 e_2(k))^2$$
$$= a_1^2 \cdot 1 + a_2^2 \cdot 9 = a_1^2 + (1-a_1)^2 \cdot 9 = 10a_1^2 + 9 - 18a_1$$

Taking the derivative with respect to a_1 gives the condition

$$20a_1 - 18 = 0 \quad \Longleftrightarrow \quad a_1 = \frac{9}{10}$$

The estimator is thus

$$\hat{x}(k) = \frac{9}{10} y_1(k) + \frac{1}{10} y_2(k)$$

The minimum value of V is
$$V_{\min} = \frac{9}{10}$$
Using only y_1 gives the variance 1 and only y_2 the variance 9. Thus, a combination of the measurements gives a better result than using only the best measurement. Assume that the a priori estimate of x is zero and the variance of x is p, i.e.
$$\hat{x}(0|0) \qquad \text{and} \qquad P(0|0) = p$$
From $(11.50) - (11.54)$ we get
$$P(1|0) = p$$
and
$$K(1) = P(1|0)C^T \left(R_2 + CP(1|0)C^T \right)^{-1} = \frac{p}{10p + 9} \begin{pmatrix} 9 & 1 \end{pmatrix}$$
This gives
$$\hat{x}(k|k) = \frac{9p}{10p + 9} y_1(k) + \frac{p}{10p + 9} y_2(k) + \frac{9}{10p + 9} \hat{x}(k|k - 1)$$
If p is large then the weights for y_1 and y_2 will be those calculated above. In the example $R_1 = 0$ the steady state gain will then be zero since the estimate will approach the true value of x.

Problem 11.20
$$x(k + 1) = \begin{pmatrix} 1.45 & -0.45 \\ 1 & 0 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k)$$
$$y(k) = \begin{pmatrix} 0.5 & 0.38 \end{pmatrix} x(k)$$
$$Q_1 = C^T C = \begin{pmatrix} 0.25 & 0.19 \\ 0.19 & 0.1444 \end{pmatrix}$$
$$Q_{12} = 0$$
$$Q_2 = 0$$
The steady state solution is obtained from (11.17). Using Matlab we get the solution
$$\Rightarrow \qquad S = \begin{pmatrix} 0.25 & 0.19 \\ 0.19 & 0.1444 \end{pmatrix}$$
$$L = \begin{pmatrix} 2.21 & -0.45 \end{pmatrix}$$
An alternative solution is to use (11.40)
$$\rho A(z)A(z^{-1}) + B(z)B(z^{-1}) = rP(z)P(z^{-1})$$
$$\rho = 0 \quad \Rightarrow \quad P(z) = \frac{1}{b_1} z B(z) = 2z(0.5z + 0.38) = z(z + 0.76)$$
Now we have to find L such that
$$(\Phi - \Gamma L) = \begin{pmatrix} -0.76 & 0 \\ 1 & 0 \end{pmatrix}$$
where controllable canonical form have been used. This gives
$$L = \begin{pmatrix} 2.21 & -0.45 \end{pmatrix}$$

Problem 11.21

a. First assume that $\eta = 0$ and use linear quadratic design with

$$Q_1 = Q_{12} = 0$$
$$Q_2 = \varrho$$
$$Q_0 = 1$$
$$N = 2$$

Theorem 11.1 gives the Riccati equation

$$S(k) = 0.5\left(0.5 - L(k)\right)S(k+1)$$
$$L(k) = \frac{0.5S(k)}{\varrho + S(k+1)}$$

This gives

$$S(N) = S(2) = Q_0 = 1 \qquad \Rightarrow \qquad L(1) = \frac{0.5}{\varrho + 1}$$

$$S(1) = \frac{0.5^2\varrho}{\varrho + 1} \qquad \Rightarrow \qquad L(0) = \frac{0.5^3}{\varrho + 1 + 0.5^2}$$

The control law is $u(k) = -L(k)x(k) = -Ly(k)$. For different values of ϱ we get

ϱ	1.0	0.1	0
$L(0)$	0.056	0.093	0.1
$L(1)$	0.250	0.455	0.5

b. In this case $\eta = 1$ and $x(k)$ is reconstructed using a Kalman filter

$$\hat{x}(k+1) = 0.5\hat{x}(k) + u(k) + K(k)\left(y(k) - \hat{x}(k)\right)$$

with

$$R_1 = R_{12} = 0$$
$$R_2 = \eta = 1$$
$$R_0 = E\left(x^2(0)\right) = 1$$

Theorem 11.5 gives

$$K(h) = \frac{0.5P(k)}{1 + P(k)}$$
$$P(k+1) = 0.25P(k) - 0.5P(k)K(k)$$

with $P(0) = R_0 = 1$. This gives

k	$P(k)$	$K(k)$
0	1	0.25
1	0.125	0.056

The control law is $u(k) = -L(u)\hat{x}(k)$.

Problem 11.22

$$x(k+1) = x(k) + v(k)$$

$$y(k) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x(k) + e(k) \qquad R_1 = 0.01, R_2 = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

a.

$$\hat{x}(k+1|k) = \left(1 - K \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \hat{x}(k|k-1) + K y(k)$$

b. $(11.47) \Rightarrow$

$$p = p + 0.01 - p^2 \begin{pmatrix} 1 & 1 \end{pmatrix} \left(\begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} p \begin{pmatrix} 1 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$p^2 \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1^2 + p & p \\ p & \sigma_2^2 + p \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0.01$$

$$p^2 \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 \sigma_2^2 + (\sigma_1^2 + \sigma_2^2)p} = 0.01$$

$$p^2 - 0.01p - 0.01\frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} = 0$$

$$p = 0.005 \pm \sqrt{0.005^2 + 0.01\frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}}$$

$$= 0.005 + \sqrt{0.005^2 + 0.01\frac{4}{5}} = 0.09458$$

c.

$$K = p \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1^2 + p & p \\ p & \sigma_2^2 + p \end{pmatrix}^{-1} = \frac{\begin{pmatrix} \sigma_2^2 & \sigma_1^2 \end{pmatrix}}{\sigma_1^2 \sigma_2^2 + p(\sigma_1^2 + \sigma_2^2)} p$$

$$= \begin{pmatrix} 4 & 1 \end{pmatrix} \frac{0.09458}{4 + 5 \cdot 0.09458} = \begin{pmatrix} 0.0846 & 0.0211 \end{pmatrix}$$

Solutions to Chapter 12

Problem 12.1

The m-step ahead predictor is given by

$$\hat{y}(k+m|k) = \frac{qG(q)}{C(q)} y(k)$$

where G is obtained from the identity (12.17) and the variance of the prediction error is given by (12.19). For $m = 1$ we get

$$q^2 - 1.4q + 0.5 = q^2 - 1.2q + 0.4 + g_0 q + g_1$$

which gives

$$G(q) = g_0 q + g_1 = -0.2q + 0.1$$

The predictor is then given by

$$\hat{y}(k+1|k) = \frac{-0.2q^2 + 0.1q}{q^2 - 1.4q + 0.5} y(k)$$

and the variance of the prediction error

$$E\tilde{y}^2(k+1|k) = \sigma^2 = 4$$

For $m = 2$

$$q(q^2 - 1.4q + 0.5) = (q^2 - 1.2q + 0.4)(q + f_1) + g_0 q + g_1$$

This gives

$$F(q) = q - 0.2$$
$$G(q) = -0.14q + 0.08$$

and

$$E\tilde{y}^2(k+2|k) = \sigma^2(1 + f_1^2) = 4.16$$

For $m = 3$ we get

$$q^2(q^2 - 1.4q + 0.5) = (q^2 - 1.2q + 0.4)(q^2 + f_1 q + f_2) + g_0 q + g_1$$

which gives

$$F(q) = q^2 - 0.2q - 0.14$$
$$G(q) = -0.088q + 0.056$$
$$E\tilde{y}^2(k+3|k) = \sigma^2(1 + f_1^2 + f_2^2) = 4.24$$

Using Theorem 10.4 we can compute the variance of y to

$$\text{var}(y) =$$

$$2^2 \frac{(1+1.4^2+0.5^2)(1+0.4)+2(-1.4-1.4\cdot 0.5)\cdot 1.2+2\cdot 0.5(1.2^2-0.4(1+0.4))}{(1-0.4^2)(1+0.4)+(-1.2+1.2\cdot 0.4)\cdot 1.2} =$$

$$= 4.28$$

This implies that the prediction variance is almost the same as the variance of y when $m \geq 3$.

Problem 12.2

The identity (12.17) is

$$q^{m-1}(q+c) = (q+a)(q^{m-1} + f_1 q^{m-2} + \ldots + f_{m-1}) + g_0$$

This gives

$$c = a + f_1$$
$$0 = af_1 + f_2$$
$$\vdots$$
$$0 = af_{m-2} + f_{m-1}$$
$$0 = af_{m-1} + g_0$$

The solution is

$$f_1 = c - a$$
$$f_2 = (-a)(c-a)$$
$$f_3 = (-a)^2(c-a)$$
$$\vdots$$
$$f_{m-1} = (-a)^{m-2}(c-a)$$
$$g_0 = (-a)^{m-1}(c-a)$$

The m-step ahead predictor is

$$\hat{y}(k+m|k) = \frac{(-a)^{m-1}(c-a)q}{q+c} y(k)$$

and the variance of the prediction error is

$$E\tilde{y}(k+m|k) = \sigma^2(1 + (c-a)^2 + a^2(c-a)^2 + \cdots + a^{2(m-2)}(c-a)^2)$$
$$= \sigma^2 \left(1 + (c-a)^2 \frac{1 - a^{2(m-1)}}{1-a^2} \right)$$

Problem 12.3

a. The C-polynomial has a zero outside the unit circle. Example 12.1 shows how the C-polynomial should be changed. It is replaced by

$$C^*(z) = 5z + 1 = 5(z + 0.2)$$

The equivalent process is thus

$$y(k) - 0.9y(k-1) = 5(e(k) + 0.2e(k-1))$$

b. The two-step-ahead predictor is obtained from

$$q(q + 0.2) = (q - 0.9)(q + f_1) + g_0$$

This gives

$$F(q) = q + 1.1$$
$$G(g) = 0.99$$

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This predictor is

$$\hat{y}(k+2|k) = \frac{0.99q}{q+0.2}\,y(k)$$

and

$$E\tilde{y}^2(k+2|k) = 25(1+1.1^2) = 55.25$$

Problem 12.4

Using the data given up to time $k = 7$ it is possible to calculate $y(k)$ and $z_d(k) = z(k) - y(k)$. z_d is the deterministic part of z.

k	z(k)	y(k)	$z_d(k)$
1	320	10	310
2	320	0	320
3	325	−5	330
4	330	−10	340
5	350	0	350
6	370	10	360
7	375	5	370

The prediction of the demand for August through November is

$$\hat{z}(8|7) = z_d(8) + \hat{y}(8|7)$$

$$\vdots$$

$$\hat{z}(11|7) = z_d(11) + \hat{y}(11|7)$$

We thus need the 1, 2, 3 and 4 step ahead predictors of y. Those are given by solving the identity (12.17) and give

m	F(q)	G(g)
1	1	$0.7q + 0.1$
2	$q + 0.7$	$0.59q + 0.07$
3	$q^2 + 0.7q + 0.59$	$0.48q + 0.06$
4	$q^3 + 0.7q^2 + 0.59q + 0.48$	$0.40q + 0.05$

The prediction is

$$\hat{y}(k+m|k) = \frac{qG(q)}{C(q)}\,y(k) = g_0 y(k) + g_1 y(k-1)$$

which gives the predicted values and their standard deviation σ.

m	$\hat{y}(7+m|7)$	$z_d(7+m)$	$\hat{z}(7+m|7)$	σ
1	4.5	380	384.5	5
2	3.7	390	393.7	6.1
3	3.0	400	403.0	6.8
4	2.5	410	402.6	7.2

Problem 12.5

The polynomials are
$$A = q^3 - q^2 + 0.5q$$
$$B = q + 0.5$$
$$C = q^3 + 0.8q^2 + 0.25q$$

It is easily seen that C is a stable polynomial, e.g. by the stability triangle. This is a necessary condition for the minimum variance design method. The pole excess is
$$d = \deg A - \deg B = 2$$

The Diophantine equation is
$$q^{d-1} C(q) = A(q) F(q) + G(q)$$
$$q(q^3 + 0.8q^2 + 0.25q) = (q^3 - q^2 + 0.5q)(q + f_1) + (g_0 q^2 + g_1 q + g_2)$$

Identifying coefficients of powers of q gives
$$0.8 = -1 + f_1$$
$$0.25 = 0.5 - f_1 + g_0$$
$$0 = 0.5 f_1 + g_1$$
$$0 = g_2$$

Solving these equations
$$f_1 = 1.8$$
$$g_0 = -0.25 + f_1 = 1.55$$
$$g_1 = -0.9$$
$$g_2 = 0$$

The minimum variance regulator is obtained from (12.27)
$$u(k) = -\frac{G(q)}{B(q)F(q)} y(k) = \frac{-1.55q^2 + 0.9q}{(q + 0.5)(q + 1.8)} y(k)$$

The loss function is
$$Ey^2 = 0.5^2(1 + 1.8^2) = 1.06$$

Problem 12.6

The noise sequence has a non zero mean value. Introduce
$$e(k) = 2 + \epsilon(k)$$
$$u(k) = \overline{u} + \tilde{u}(k)$$

where $\epsilon(k)$ is zero mean white noise. The process is now
$$y(k) - 0.5y(k-1) = \overline{u} + \tilde{u}(k-2) + 2 + \epsilon(k) - 0.7(2 + \epsilon(k-1))$$
$$= \tilde{u}(k-2) + \epsilon(k) - 0.7\epsilon(k-1) + \overline{u} + 0.6$$

Choose $\overline{u} = -0.6$ and the problem is reduced to the standard problem. The identity gives
$$\tilde{u} = \frac{0.1q}{q - 0.2} y(k)$$

and
$$u(k) = \frac{0.1q}{q - 0.2} y(k) - 0.6$$

Problem 12.7

a.   The identity gives
$$F(q) = q + c - a$$
$$G(q) = a(a - c)q$$
and the minimum variance controller is
$$u(k) = -\frac{a(a - c)q}{q + (c - a)} y(k)$$

b.   The expression above gives the optimal controller $u(k) = 0$ if $a = 0$. The process is then a moving average process of first order. This implies that $u(k - 2)$ cannot be used to decrease the variance of $y(k)$.

Problem 12.8

a.   The identity gives for $d = 1$
$$F(q) = 1$$
$$G(q) = 3.2q + 0.2$$
and for $d = 2$
$$F(q) = q + 3.2$$
$$G(q) = 5.64q^2 - 2.24q$$
The minimum variance controller is
$$u(k) = -\frac{G(q)}{B(q)F(q)} y(k)$$
and the minimum variance in the two cases are
$$d = 1 : \qquad Ey^2 = 1$$
$$d = 2 : \qquad Ey^2 = 1 + 3.2^2 = 11.24$$

b.   Fig. 12.1 shows the output of the open loop system for one realization of the noise sequence. The output is drifting since the A-polynomial contains an integrator. Fig. 12.2 and Fig. 12.3 shows the output and the control signal for the same noise realization when the minimum variance controller is used with $d = 1$ and $d = 2$.

Problem 12.9

a.   Assume that
$$H(z) = \lambda \frac{1}{z + a}$$
Sending white noise $e(k)$ through this filter gives the spectral density (see Theorem 10.2)
$$\phi(\omega) = \frac{\lambda^2}{2\pi} \frac{1}{1 + a^2 + 2a \cos \omega}$$
This implies that $\lambda = 1$ and $a = 0.6$ gives the desired spectral density. The process is now described by
$$y(k) = \frac{1}{1 - 0.5q^{-1}} \left(\frac{1}{q + 0.6} e(k) + \frac{1}{q} u(k) \right)$$
or
$$(q^2 + 0.1q - 0.3)\, y(k) = (q + 0.6)\, u(k) + q\, e(k)$$
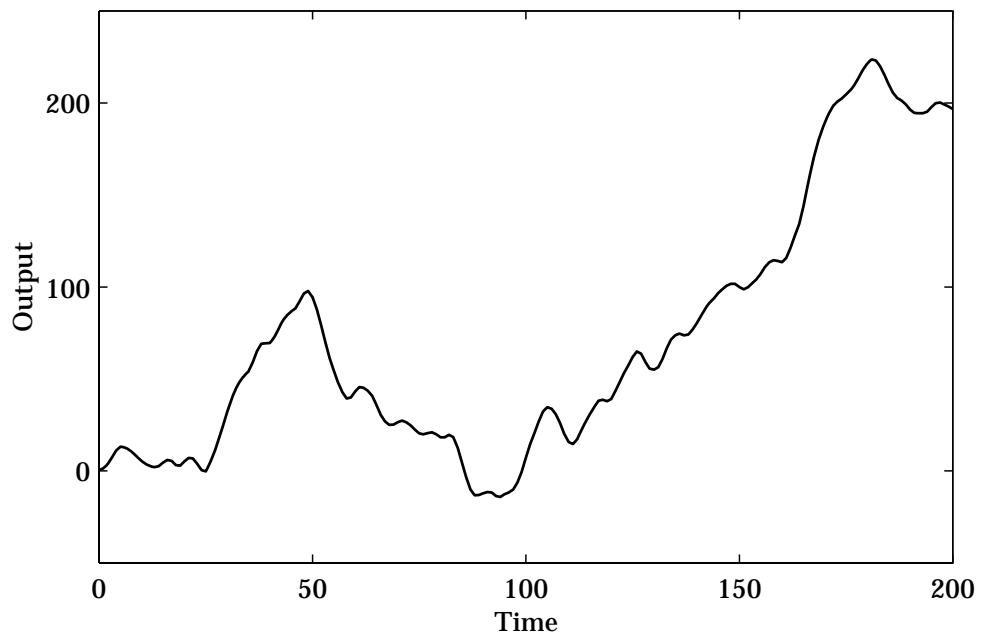
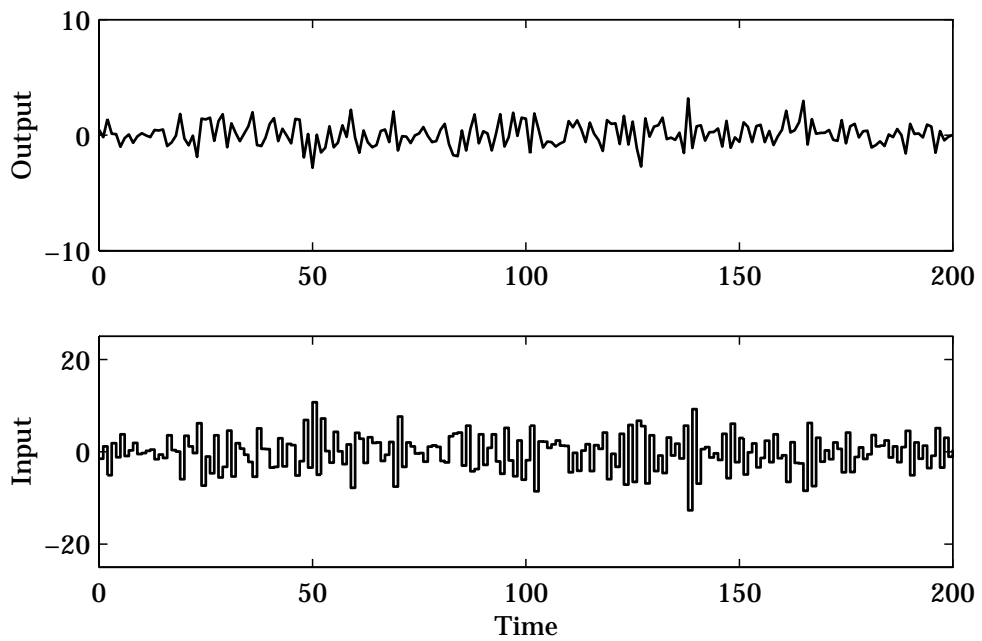**Figure 12.1** The output of the open loop system in Problem 12.8.

Figure 12.2 The output and the control signal when $d = 1$ and the minimum variance controller is used for the process in Problem 12.8.

b. Use the controller

$$u(k) = -K y(k)$$

This gives

$$(q^2 + 0.1q - 0.3) y(k) = -(q + 0.6) K y(k) + q e(k)$$

$$y(k) = \frac{q}{q^2 + (0.1 + K)q + (0.6K - 0.3)} e(k)$$
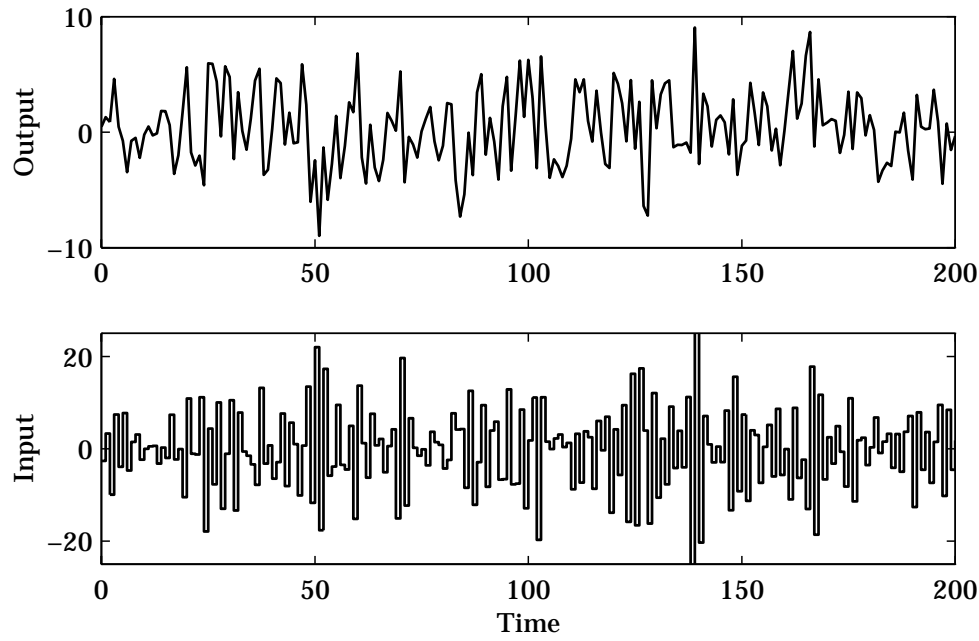
The system is stable if

$$-0.5 < K < 1.5$$

Figure 12.3 Same as Fig. 12.2 but when $d = 2$.

Theorem 10.4 gives an expression for the variance of a second order process.

$$I_2(K) = \frac{0.7 + 0.6K}{(1.3 - 0.6K)((0.7 + 0.6K)^2 - (0.1 + K)^2)}$$

For $K = 1$ we get $I_2 = 3.87$.

c.   The minimum value of I_2 is obtained from

$$\frac{dI_2}{dK} = 0$$

This gives the third order equation

$$72K^3 + 12K^2 - 266K + 1 = 0$$

which has the roots $K \approx -2.009$, 1.839 and 0.004. Only the root $K = 0.004$ gives a stable closed loop system. The value of the variance is

$$I_2(0.004) = 1.12$$

d.   The minimum variance is $Ey^2 = 1$ since $d = 1$.

Problem 12.10

a.   With the proportional controller

$$u(k) = -Ky(k)$$

we get the closed loop system

$$y(k) = \frac{q^2 + 0.5q}{q^2 + (K - 0.25)q + 0.5} e(k)$$

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Using the results in Theorem 10.4 gives

$$B_0 = 1 + 0.5^2 = 1.25$$
$$B_1 = 2(1 \cdot 0.5 + 0.5 \cdot 0) = 1$$
$$B_2 = 0$$
$$e_1 = 1.5$$

and

$$
\begin{aligned}
I_2(K) &= \frac{1.25 \cdot 1.5 - (K - 0.25)}{(1 - 0.25) \cdot 1.5 - (K - 0.25)^2(1 - 0.5)} \\
&= \frac{2.125 - K}{0.5(1.75 - K)(1.25 + K)}
\end{aligned}
$$

Taking the derivative of I_2 and putting the derivative equal to zero leads to the equation

$$K^2 - 4.25K + 3.25 = 0$$

with the solutions $K = 1$ and $K = 3.25$. $K = 1$ gives the variance

$$I_2(1) = \frac{4}{3}$$

This is minimal variance for the present control law. With minimum variance control we would get $E\, y^2 = 1$.

b.   From Example 3.2 we find that the closed loop system is stable if

$$0.5 > -1 + K - 0.25$$
$$0.5 > -1 - K + 0.25$$

or

$$-1.25 < K < 1.75$$

Both $K = 3.25$ (the second root in a.) and $K = 2.125$ give an unstable closed loop system and the calculation of I_2 is no longer valid.

Problem 12.11

$$y(k) - 1.5y(k - 1) + 0.7y(k - 2) = u(k - 2) - 0.5u(k - 3) + v(k)$$

gives

$$A(q) = q^3 - 1.5q^2 + 0.7q$$
$$B(q) = q - 0.5$$

Note that the process zero (0.5) is stable and well damped. This means that the process zero can be cancelled without problem. The degree conditions gives

$$\deg A_m - \deg B_m \geq \deg A - \deg B = 2$$
$$\deg A_o \geq 2 \deg A - \deg A_m - \deg B^+ - 1$$

a. $v(k) = 0$; *Deadbeat Control*

$$A_m(q) = q^2$$
$$B^+(q) = q - 0.5$$
$$B^-(q) = 1$$
$$B'_m(q) = 1$$
$$A_o(q) = q^2$$

The polynomials R_1 and S are obtained from the Diophantine equation

$$A(z)R_1(z) + B^-(z)S(z) = A_m(z)A_o(z)$$

Recalling the condition $\deg S = \deg A - 1$ the equation becomes

$$(z^3 - 1.5z^2 + 0.7z)(z + r_1) + s_0z^2 + s_1z + s_2 = z^4$$

with solution
$$r_1 = 1.5$$
$$s_0 = 1.5r_1 - 0.7 = 1.55$$
$$s_1 = -0.7r_1 = -1.05$$
$$s_2 = 0$$

This gives the regulator

$$u(k) = \frac{T(q)}{R(q)} u_c(k) - \frac{S(q)}{R(q)} y(k)$$

where
$$R(q) = R_1(q)B^+(q) = (q + 1.5)(q - 0.5)$$
$$T(q) = B_m A_o = q^2$$

Assuming that $u_c(k) = 0$ gives

$$u(k) = -\frac{S(q)}{R(q)} y(k) = -\frac{1.55q^2 - 1.05q}{(q + 1.5)(q - 0.5)}$$

b. $v(k) = e(k) - 0.2e(k - 1)$; *Minimum variance* The polynomial C is given by
$$C(q) = q^3 - 0.2q^2$$

The minimum variance control law can be written as

$$u(k) = -\frac{G(q)}{B(q)F(q)} y(k)$$

where the polynomials F and G are obtained from

$$q^{d-1}C(q) = A(q)F(q) + G(q), \qquad \deg F = d - 1 = 1, \quad \deg G = 2$$

which in this case is

$$q(q^3 - 0.2q^2) = (q^3 - 1.5q^2 + 0.7q)(q + f_1) + g_0q^2 + g_1q + g_2$$

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which yields the equations

$$f_1 - 1.5 = -0.2$$
$$0.7 - 1.5 f_1 + g_0 = 0$$
$$0.7 f_1 + g_1 = 0$$
$$g_2 = 0$$

with solution

$$f_1 = 1.3$$
$$g_0 = 1.25$$
$$g_1 = -0.91$$
$$g_2 = 0$$

The minimum variance controller is thus given by

$$u(k) = -\frac{1.25 q^2 - 0.91 q}{(q - 0.5)(q + 1.3)} y(k)$$

c.   The output is in the deadbeat case given by

$$
\begin{aligned}
y(k) &= \frac{CR}{AR + BS} e(k) = \frac{CR_1}{A_m A_o} e(k) \\
&= q^{-4} C(q) R_1(q) e(k) = C^*(q^{-1}) R_1^*(q^{-1}) e(k) \\
&= (1 - 0.2 q^{-1})(1 + 1.5 q^{-1}) e(k) = (1 + 1.3 q^{-1} - 0.3 q^{-2}) e(k)
\end{aligned}
$$

which is a moving average (MA) process. The variance of the output is then simply calculated as

$$E(y^2) = (1 + 1.3^2 + 0.3^2)\sigma^2 = 2.78\sigma^2$$

In the minimum variance case the output is

$$y(k) = q^{-(d-1)} F(q) e(k) = F^*(q^{-1}) e(k) = (1 + 1.3 q^{-1}) e(k)$$

which also is an MA process. The output variance is

$$E(y^2) = (1 + 1.3^2)\sigma^2 = 2.69\sigma^2$$

d.   Fig. 12.5 shows the output and the sum of the square of the output for the regulators in a and b.

Problem 12.12

Introduce the polynomials

$$A_1(q) = A(q)D(q)$$
$$B_1(q) = B(q)D(q)$$
$$C_1(q) = A(q)C(q)$$

and the noise $e_1 = \lambda e$. The system can then be written as

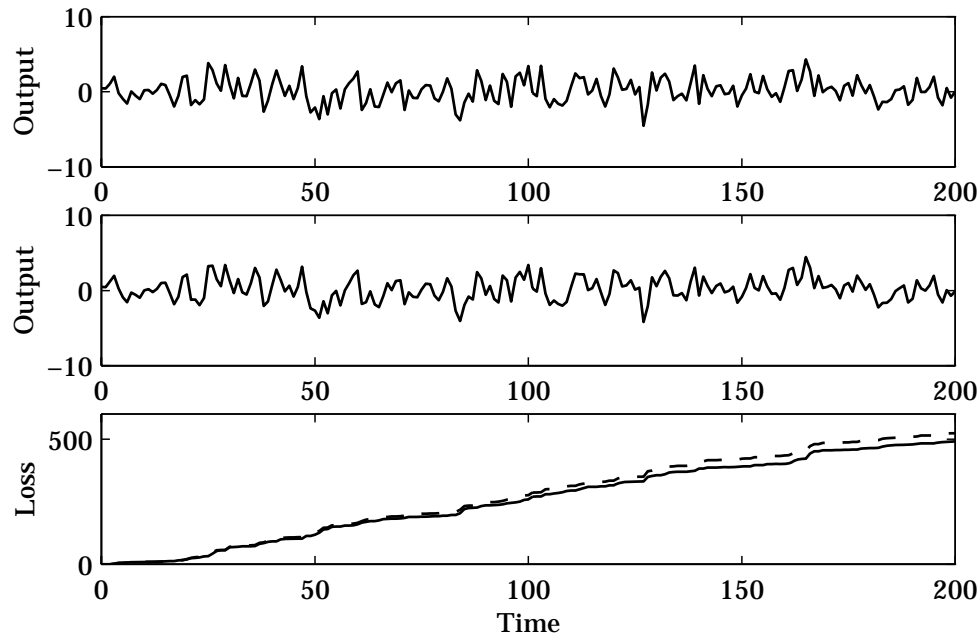$$y(k) = \frac{B_1}{A_1} u(k) + \lambda \frac{C_1}{A_1} e_1(k) \tag{12.1}$$

Figure 12.4 The output and the sum of the square of the output for the two regulators in Problem 12.11. The deadbeat controller is used in the upper diagram and the minimum variance controller in the middle diagram. The accumulated loss functions are shown in the lower diagram, deeadbeat (dashed) and minimum variance (solid).

Since A, C and D are monic, $A_1 = AD$ and $C_1 = AC$ will also bo monic. Let $d_1 = \deg A_1 - \deg B_1 = \deg A - \deg B = d$. The minimum-variance controller for the rewritten system (12.1) is then calculated as

$$u(k) = -\frac{G_1(q)}{B_1(q)F_1(q)}\, y(k) \tag{12.2}$$

where

$$q^{d_1 - 1} C_1(q) = A_1(q)F_1(q) + G_1(q) \tag{12.3}$$

Equation (12.3) is equivalent to

$$q^{d-1} A(q)C(q) = A(q)D(q)F_1(q) + G_1(q)$$

which implies that A must divide G_1, i.e. $G_1(q) = A(q)G(q)$. Putting $F = F_1$ gives the minimum-variance control law

$$u(k) = -\frac{G(q)A(q)}{B(q)D(q)F(q)}\, y(k) = -\frac{G^*(q^{-1})A^*(q^{-1})}{B^*(q^{-1})D^*(q^{-1})F^*(q^{-1})}\, y(k)$$

where F and G satisfy the equation

$$q^{d-1} C(q) = D(q)F(q) + G(q)$$

with $\deg F = d - 1$ and $\deg G = \deg D - 1$. We see that if $A = D$ the controller reduces to the normal minimum variance controller.

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Problem 12.13

In this case

$$A(q) = q + a$$
$$B(q) = b$$
$$C(q) = q + c$$
$$D(q) = q$$
$$d = 1$$

The identity in Problem 12.12 gives

$$q + c = q \cdot 1 + g_0$$

$$\Rightarrow \quad g_0 = c$$

The minimum variance controller is thus

$$u(k) = -\frac{c(q + a)}{bq} y(k) = -\frac{c}{b}y(k) - \frac{ac}{b}y(k - 1)$$

Problem 12.15

We have

$$y_1(k) = \frac{1}{1 - 0.7q^{-1}} u(k - 1) + \frac{1 - 0.5q^{-1}}{1 - 0.7q^{-1}} e(k)$$

$$y_2(k) = \frac{q^{-1}}{1 + \alpha q^{-1}} y_1(k)$$

$$= \frac{1}{A_1^*}\left(\frac{B^*}{A^*} u(k - 2) + \frac{C^*}{A^*} e(k - 1)\right)$$

$$= \frac{1}{(1 + \alpha q^{-1})(1 - 0.7q^{-1})} u(k - 2) + \frac{1 - 0.5q^{-1}}{(1 + \alpha q^{-1})(1 - 0.7q^{-1})} e(k - 1)$$

To normalize the notations it is convenient to introduce a new noise $\varepsilon(k) = e(k - 1)$.

a. Assume that y_1 can be measured. The minimum variance controller for y_1 is then

$$u(k) = -0.2y_1(k)$$

b. The variances of y_1 and y_2 are

$$E y_1^2 = 1$$

$$E y_2^2 = \frac{1}{1 - \alpha^2} \approx 2.78$$

c. The minimum variance controller for y_2 when y_2 is measurable is obtained by using the identity

$$1 - 0.5q^{-1} = (1 + \alpha q^{-1})(1 - 0.7q^{-1})(1 + f_1 q^{-1}) + q^{-2}(g_0 + g_1 q^{-1})$$

This gives

$$F^*(q^{-1}) = 1 + q^{-1}$$
$$G^*(q^{-1}) = 0.94 - 0.56q^{-1}$$

and the controller

$$u(k) = -\frac{0.94 - 0.56q^{-1}}{1 + q^{-1}} y_2(k)$$

This gives

$$y_2(k) = (1 + q^{-1})\varepsilon(k) = \varepsilon(k-1) + e(k-2)$$

and

$$y_1(k) = (1 + q^{-1})(1 + \alpha q^{-1})e(k+1)$$

The variances of the two signals are

$$Ey_1^2 = 1 + (\alpha + 1)^2 + \alpha^2 = 1.68$$
$$Ey_2^2 = 1 + 1 = 2$$

d. In this case both y_1 and y_2 are measurable and $y_1(k)$ will contain more recent information about the noise process than $y_2(k)$. It is now only necessary to predict one step ahead. Introduce the identity

$$C^* = A^* A_1^* + q^{-1}G_1^*$$

This gives

$$y_2(k+2) = \frac{C^*}{A_1^* A^*} e(k+2) + \frac{B^*}{A_1^* A^*} u(k)$$

$$= e(k+2) + \frac{G_1^*}{A_1^* A^*} e(k+1) + \frac{B^*}{A_1^* A^*} u(k)$$

But

$$\varepsilon(k+1) = \frac{A^*}{C^*} y_1(k) - \frac{B^*}{C^*} u(k-1)$$

This gives

$$y_2(k+2) = \varepsilon(k+2) + \frac{G_1^*}{A_1^* A^*} \left(\frac{A^*}{C^*} y_1(k) - \frac{B^*}{C^*} u(k-1) \right) + \frac{B^*}{A_1^* A^*} u(k)$$

$$= \varepsilon(k+2) + \frac{G_1^*}{A_1^* A^*} y_1(k) + \frac{B^*}{A_1^* A^* C^*} \left(C^* - G_1^* q^{-1} \right) u(k)$$

$$= \varepsilon(k+2) + \frac{1}{C^*} \left(\frac{G_1^*}{A_1^*} y_1(k) + B^* u(k) \right)$$

The variance is thus minimized when

$$u(k) = -\frac{G_1^*}{A_1^* B^*} y_1(k)$$

which is an admissible control law. For the specific system we get

$$u(k) = -\frac{1 - 0.56q^{-1}}{1 - 0.8q^{-1}} y_1(k)$$

With this control law we get

$$y_2(k) = \varepsilon(k) = e(k-1)$$

and

$$y_1(k) = (1 + \alpha q^{-1})y_2(k+1) = (1 + \alpha q^{-1})\varepsilon(k+1) = (1 + \alpha q^{-1})e(k)$$

The variances of the two output signals are

$$Ey_1^2 = 1 + \alpha^2 = 1.64$$

and

$$Ey_2^2 = 1$$

We can thus conclude that the extra measurement will greatly improve the control of the output of the system.

Problem 12.16

The same arguments can be used in this case as when deriving the normal minimum variance controller. Assume that the system has a stable inverse. Also assume that $\deg A = \deg B = \deg D$ i.e. that the process can be written as

$$A^*(q^{-1})y(k) = B^*(q^{-1})u(k-d) + C^*(q^{-1})e(k) + D^*(q^{-1})v(k-d)$$

The identity (12.17) can be used here also and gives

$$\begin{aligned}
y(k+d) &= \frac{C^*}{A^*}e(k+d) + \frac{B^*}{A^*}u(k) + \frac{D^*}{A^*}v(k) \\
&= F^*e(k+d) + \frac{G^*}{A^*}e(k) + \frac{B^*}{A^*}u(k) + \frac{D^*}{A^*}v(k) \\
&= F^*e(k+d) + \frac{G^*}{A^*}\left(\frac{A^*}{C^*}y(k) - \frac{B^*}{C^*}u(k-d) - \frac{D^*}{C^*}v(k-d)\right) \\
&\quad + \frac{B^*}{A^*}u(k) + \frac{D^*}{A^*}v(k) \\
&= F^*e(k+d) + \frac{G^*}{C^*}y(k) + \frac{B^*}{A^*C^*}\left(C^* - q^{-d}G^*\right)u(k) \\
&\quad + \frac{D^*}{A^*C^*}\left(C^* - q^{-d}G^*\right)v(k) \\
&= F^*e(k+d) + \frac{1}{C^*}\left(G^*y(k) + B^*F^*u(k) + D^*F^*v(k)\right)
\end{aligned}$$

The minimum variance controller is thus

$$u(k) = -\frac{G^*}{B^*F^*}y(k) - \frac{D^*}{B^*}v(k)$$

Problem 12.17

$$A(q)y(k) = B(q)u(k) + C(q)e(k)$$
$$A(q) = q - 0.9, \quad B(q) = q, \quad C(q) = q - 0.5$$

LQG-Control: Minimize $E(y^2 + \rho u^2)$. Let $P(z)$ be the closed loop system characteristic equation (12.45)

$$rP(z)P(z^{-1}) = \rho A(z)A(z^{-1}) + B(z)B(z^{-1})$$

$P(z)$ contains stable zeros of the right hand expression (Lemma 12.1)

$$r(z + p_1)(z^{-1} + p_1) = \rho(z - 0.9)(z^{-1} - 0.9) + 1 \cdot 1$$
$$r(1 + p_1^2) + rp_1 z + rp_1 z^{-1} = 1.81\rho + 1 - 0.9\rho z^{-1} - 0.9\rho z$$

This gives the system of equations

$$rp_1 = -0.9\rho$$
$$r(1 + p_1^2) = 1 + 1.81\rho$$

which has two solutions, one of which gives a stable $P(z)$

$$p_1 = -\frac{1 + 1.81\rho}{1.8\rho} + \sqrt{(\frac{1 + 1.81\rho}{1.8\rho})^2 - 1}$$

Determine the LQG-regulator by means of pole placement:

$$A_m = P(z) = z + p_1$$
$$A_o = C(z) = z - 0.5$$

Control law:

$$u(k) = -\frac{S(q)}{R(q)} y(k)$$

where $S(0) = 0$. See computational procedure in Section 12.5.
The Diophantine equation to be solved is

$$PC = AR + BS$$

or

$$(z + p_1)(z - 0.5) = (z - 0.9)(z + r_1) + zs_0$$

This gives

$$p_1 - 0.5 = -0.9 + r_1 + s_0$$
$$-0.5p_1 = -0.9r_1$$

The solution is given by

$$r_1 = \frac{5}{9}p_1$$
$$s_0 = 0.4 + \frac{4}{9}p_1$$

which results in the controller

$$u(k) = -\frac{s_0 q}{q + r_1} y(k)$$

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For the closed loop system we get

$$y(k) = \frac{B}{A} u(k) + \frac{C}{A} e(k) = \frac{B}{A} \left(-\frac{S}{R} y(k) \right) + \frac{C}{A} e(k)$$

or

$$y(k) = \frac{CR}{AR + BS} e(k) = \frac{CR}{PC} e(k) = \frac{R}{P} e(k) = \frac{q + r_1}{q + p_1} e(k)$$

Theorem 10.4 gives

$$Var\, y = \frac{1 + r_1^2 - 2r_1 p_1}{1 - p_1^2}$$

The input u is

$$u(k) = -\frac{S}{R} y(k) = -\frac{S}{R} \cdot \frac{R}{P} e(k) = -\frac{S}{P} e(k) = -\frac{s_0 q}{q + p_1} e(k)$$

Theorem 10.4 gives

$$Var\, u = \frac{s_0^2}{1 - p_1^2}$$

In the following table the calculations are summarized for $\rho = 0.1, 1$ and 10.

ρ	p_1	$Var\, y$	$Var\, u$
0.1	-0.077	1.0012	0.135
1	-0.36	1.030	0.0658
10	-0.70	1.197	0.0148

Problem 12.24

From Example 12.16 we get a polynomial description of the process

$$A(q)y(k) = B(q)u(k) + C(q)e(k)$$

where

$$A(q) = q - 1$$
$$B(q) = h$$
$$C(q) = q + c$$

The *minimum variance* regulator is given by the Diophantine

$$PC = AR + BS$$

where

$$P(q) = q^{d-1} B(q) = h$$

The solution is

$$r_0 = h$$
$$s_0 = c + 1$$

and the minimum variance regulator is

$$u(k) = -\frac{c + 1}{h} y(k)$$

In $LQ-design$ we use a state-space description of the process (see Example 12.16)

$$x(kh+h) = x(kh) + hu(kh) + v(kh+h) - v(kh)$$
$$y(kh) = x(kh) + \epsilon(kh)$$

To obtain the LQ-controller we have to sample the loss function.
From (11.6) - (11.8) we get

$$Q_1 = \int_0^h 1 \cdot ds = h$$

$$Q_{12} = \int_0^h s \cdot ds = \frac{h^2}{2}$$

$$Q_2 = \int_0^h s^2 \cdot ds = \frac{h^3}{3}$$

The Riccati equation is

$$\begin{cases} S = \Phi^T S \Phi + Q_1 - L^T (Q_2 + \Gamma^T S \Gamma) L \\ L = (Q_2 + \Gamma^T S \Gamma)^{-1} (\Gamma^T S \Phi + Q_{12}) \end{cases}$$

and the solution is

$$\begin{cases} S = \frac{h}{\sqrt{12}} \\ L = \frac{1}{h} \frac{3+\sqrt{3}}{2+\sqrt{3}} \end{cases}$$

The closed loop system has a pole in $-p_1$

$$-p_1 = \Phi - \Gamma L = 1 - h \frac{1}{h} \frac{3+\sqrt{3}}{2+\sqrt{3}} = \sqrt{3} - 2$$

and

$$P(z) = z + p_1$$

To get the regulator we solve the Diophantine (see p. 481)

$$PC = AR + BS \qquad S(0) = 0$$
$$(q+p_1)(q+c) = (q-1)(q+r_1) + hs_0 q$$

which has the solution

$$r_0 = -p_1 c$$
$$s_0 = \frac{1}{h}(p_1 + c + 1 + p_1 c)$$

and thus

$$u(k) = -\frac{\frac{1}{h}(p_1 + c + 1 + p_1 c)q}{q - p_1 c} y(k)$$

Problem 12.29

The process is described by

$$A(z) = z^2 - 1.4z + 0.65$$
$$B(z) = z - 0.2$$
$$C(z) = z^2 + 0.4z$$

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a. To get the minimum variance controller we use the identity

$$A(z)\,F(z) + G(z) = z^{d-1}\,C(z)$$
$$z^2 - 1.4z + 0.65 + g_0 z + g_1 = z^2 + 0.4z$$

This gives
$$G(z) = 1.8z - 0.65$$

The control law is

$$u = -\frac{G}{BF}\,y = -\frac{1.8q - 0.65}{q - 0.2}\,y$$

b. The dead-beat controller is obtained from the identity

$$A(z)F(z) + G(z) = z^2$$

which gives
$$G(z) = 1.4z - 0.65$$
$$u = -\frac{1.4q - 0.65}{q - 0.2}\,y$$

c. Minimum variance control

$$\mathrm{var}(y) = 1 \cdot \sigma^2 = 4$$

Dead-beat control

$$\mathrm{var}(y) = (1 + 0.4^2)\sigma^2 = 4.64$$

Problem 12.30

a. Assume that $C = 0$ and use the identity

$$C = AF + G$$

This gives
$$F = 1$$
$$G = -a$$

and the controller
$$u = -\frac{G}{BF}\,y = ay$$

The closed loop system becomes

$$y(k) = e(k) + ce(k - 1)$$

and the variance
$$\mathrm{var}(y) = 1 + c^2$$

b. Assume that $C(z) = z + \hat{c}$. The minimum variance controller for this model is given by

$$F = 1$$

$$G = \hat{c} - a$$

The closed loop system is now

$$y(k) = \frac{q + c}{q + \hat{c}} e(k)$$

which has the variance, see Theorem 6.4

$$\mathrm{var}(y) = \frac{(1 + c^2) - 2c\hat{c}}{1 - \hat{c}^2}$$

It is better to use a guessed value $\hat{c}$ if

$$0 < \hat{c} < \frac{2c}{1 + c^2}$$

Problem 12.31

The C polynomial has a pole outside the unit circle. Replace C by a new polynomial obtained from spectral factorization

$$C(z)C(z^{-1}) = (z - 1.25)(z^{-1} - 1.25) = 1.25^2(z - 0.8)(z^{-1} - 0.8)$$

The new process is

$$y(k) = \frac{q^2 - 0.8q}{q^2 - 1.1q + 0.3} \varepsilon(k)$$

where ε has the standard deviation 1.25.

a. The 2-step ahead predictor is given by

$$z^{m-1}C(z) = A(z)F(z) + G(z)$$

$$z(z^2 - 0.8z) = (z^2 - 1.1z + 0.3)(z + f_1) + g_0 z + g_1$$

which gives

$$F(z) = z + 0.3$$

$$G(z) = 0.03z - 0.09$$

and

$$\hat{y}(k + 2|k) = \frac{qG(q)}{C(q)} y(k) = \frac{0.03(q - 3)}{q - 0.8} y(k)$$

b. The prediction error variance is

$$E\tilde{y} = 1.25^2(1 + 0.3^2) = 1.70$$

Problem 12.32

a. $d = 2$. This gives the identity

$$zC(z) = A(z)F(z) + G(z)$$

where $\deg F = 1$ and $\deg G = 2$, i.e.,

$$z^3(z - 0.1) = (z^3 - 1.7z^2 + 0.8z - 0.1)(z + f_1) + g_0 z^2 + g_1 z + g_2$$

The solution is
$$F(z) = z + 1.6$$
$$G(z) = 1.92z^2 - 1.18z + 0.16$$

and the controller is

$$u(k) = -\frac{G}{BF}y(k) = -\frac{1.92q^2 - 1.18q + 0.16}{2(q - 0.9)(q + 1.6)}y(k)$$

b. The output variance when using the minimum variance controller in a. is given by
$$\text{var}(y) = 1 + f_1^2 = 1 + 1.6^2 = 3.56$$

c. Since $B(z)$ has a root outside the unit circle we use the procedure in Theorem 12.3. Factor $B(z)$ as

$$B(z) = B^+(z)\,B^-(z)$$

with $B^{-*}(q)$ monic, so

$$B^+(z) = -2$$
$$B^-(z) = -0.9q + 1$$

The Diophantine to solve is

$$q^{d-1}C(q)B^{-*}(q) = A(q)F(q) + B^-(z)G(q)$$
$$q^3(q - 0.1)(q - 0.9) = (q^3 - 1.7q^2 + 0.8q - 0.1)(f_0 q^2 + f_1 q + f_2)$$
$$+ (-0.9q + 1)(g_0 q^2 + g_1 q + g_2)$$

This gives the system of equations

$$1 = f_0$$
$$-1 = -1.7f_0 + f_1$$
$$0.09 = 0.8f_0 - 1.7f_1 + f_2 + 1.8g_0$$
$$0 = -0.1f_0 + 0.8f_1 - 1.7f_2 - 2g_0 + 1.8g_1$$
$$0 = -0.1f_1 + 0.8f_2 - 2g_1 + 1.8g_2$$
$$0 = -0.1f_2 - 2g_2$$

which has the solution
$$f_0 = 1.000$$
$$f_1 = 0.700$$
$$f_2 = 2.721$$
$$g_0 = 2.490$$
$$g_1 = -1.862$$
$$g_2 = 0.272$$

Equation (12.31) gives

$$u(k) = -\frac{G(q)}{B^+(q)F(q)}y(k) = -\frac{-1.245q^2 + 0.931q - 0.136}{q^2 + 0.7q + 2.721}y(k)$$

The closed loop system is

$$Ay(k) = Bu(k) + Ce(k) = B\left(-\frac{G}{B^+F}y(k)\right) + Ce(k)$$

$$= -\frac{GB^-}{F}y(k) + Ce(k)$$

or

$$y(k) = \frac{CF}{AF + GB^-}e(k) = \frac{CF}{q^{d-1}CB^{-*}}e(k) = \frac{F}{q^{d-1}B^{-*}}e(k)$$

$$= \frac{q^2 + 0.7q + 2.721}{q(q - 0.9)}e(k)$$

Theorem 10.4 gives the output variance

$$Var\ y = 94.7$$

Problem 12.33

$$y(k) = \frac{0.9q + 1}{(q - 1)(q - 0.7)}u(k) + \frac{q(q - 0.7)}{(q - 1)(q - 0.7)}e(k)$$

Determine the controller from

$$AR + BS = q^{d-1}C$$

$$(q - 1)(q - 0.7)(q + r_1) + (0.9q + 1)(s_0q + s_1) = q^2(q - 0.7)$$

This gives the system of equations

$$-1.7 + r_1 + 0.9s_0 = -0.7$$

$$0.7 - 1.7r_1 + 0.9s_1 + s_0 = 0$$

$$0.7r_1 + s_1 = 0$$

with the solution

$$r_1 = 0.526$$

$$s_0 = 0.526$$

$$s_1 = -0.368$$

$$y(k) = \frac{CR}{AR + BS}e(k) = \frac{CR}{qC} = (1 + 0.526q^{-1})e(k)$$

$$var\ y(k) = (1 + 0.526^2)\sigma^2 = 1.27\sigma^2$$

Compare to Example 12.9 p. 468

$$var\ y(k) = \frac{20}{19}\sigma^2 = 1.053\sigma^2$$