Discrete Applied Mathematics 323 (2022) 184–200

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Revising Johnson's table for the 21st century

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A R T I C L E   I N F O

Article history:
Received 29 May 2020
Received in revised form 21 December 2020
Accepted 17 May 2021
Available online 7 June 2021

Keywords:
Computational complexity
Parameterized complexity
NP-complete
Steiner tree
Dominating set

A B S T R A C T

What does it mean today to study a problem from a computational point of view? We focus on parameterized complexity and on Column 16 "Graph Restrictions and Their Effect" of D.S. Johnson's Ongoing guide, where several puzzles were proposed in a summary table with 30 graph classes as rows and 11 problems as columns. Several of the 330 entries remain unclassified into Polynomial or NP-complete after 35 years. We provide a full dichotomy for the Steiner Tree column by proving that the problem is NP-complete when restricted to Undirected Path graphs. We revise Johnson's summary table according to the granularity provided by the parameterized complexity for NP-complete problems.

1. Graph restrictions and their effect 35 years later

The 1979 book *Computers and Intractability, A Guide to the Theory of* NP*-completeness* by Michael R. Garey and David S. Johnson [53] is considered the single most important book by the computational complexity community and it is known as The Guide, which we cite by [GJ]. The book was followed by *The* NP*-completeness Column: An Ongoing Guide* where, from 1981 until 2007, D.S. Johnson continuously updated The Guide in 26 columns published first in the *Journal of Algorithms* and then in the *ACM Transactions on Algorithms*. The Guide has an appendix where 300 NP-complete problems are organized into 13 categories according to subject matter. The first, "A1 Graph Theory", contains 65 problems and the second, "A2 Network Design", contains 51 problems. Category "A13 Open Problems" lists 12 problems in NP, at the time not classified into polynomial or NP-complete, and it is surprising that since then 5 have been classified into polynomial and 5 into NP-complete. Garey and Johnson were amazingly able to foresee a list of challenging problems which would evenly be split into tractable and intractable problems. The goal of the present paper is to propose an answer to the question: *What does it mean today to study a problem from a computational complexity point of view?* In search of an answer, we focus on parameterized complexity and on Column 16 "Graph Restrictions and Their Effect" [66], which we cite by [OG], where several puzzles were proposed by D.S. Johnson and many remain unsolved after 35 years. Consider in Table 1 the summary table from [OG] with 30 graph classes as rows and 11 columns, the first of which is Membership, followed by 10 well-known NP-complete problems, listed in The Guide's appendix as GT20, GT19, GT15, GT13, Open5, GT37, GT2, ND16, ND12, and Open1. The entries follow the notation of [OG], where the complexity of the problem restricted to the graph class is N = NP-complete, P = polynomial, or O = open. Following our convention, reference [GJ] stands for "The Guide" and [OG] stands for "Column 16", and we highlight in bold the reference updates with the corresponding new

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https://doi.org/10.1016/j.dam.2021.05.021
0166-218X/

Table 1
The updated NP-Completeness Column: An Ongoing Guide table 35 years later. Depicted in bold are the references that correspond to unresolved entries in [OG] and [GJ]. The references not in bold confirm resolved entries from [OG] or [GJ], that we updated either because they cited private communications, because the cited reference is not easily accessible, or could not be confirmed. There is one entry highlighted in italic that corrects the entry for HamCirc restricted to Circle graphs. We keep the abbreviations used by [OG], namely for entries: P = Polynomial-time solvable; N = NP-complete; I = Open, but equivalent in complexity to general Graph Isomorphism; O? = Apparently open, but possibly easy to resolve; and O = Open, and may well be hard; and for references [T] = Restriction trivializes the problem; [GJ] = the Guide [53]; and [OG] = the Ongoing guide [66], please refer to this reference for the entry.

Graph Class	Member	IndSet	Clique	CliPar	ChrNum	ChrInd	HamCir	DomSet	MaxCut	StTree	GraphIso
Trees/Forests	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [GJ]	P [T]	P [GJ]
Almost Trees (k)	P [OG]	P [OG]	P [T]	**P** [16]	**P** [5]	**P** [17]	**P** [5]	P [5]	**P** [20]	**P** [76]	**P** [17]
Partial k-trees	P [OG]	P [5]	P [T]	**P** [16]	P [5]	**P** [17]	P [5]	P [5]	**P** [20]	**P** [76]	**P** [17]
Bandwidth-k	P [OG]	P [OG]	P [T]	**P** [16]	P [5]	**P** [17]	**P** [5]	P [5]	P [OG]	**P** [76]	P [OG]
Degree-k	P [T]	N [GJ]	P [T]	N [29]	N [GJ]	N [OG]	N [GJ]	N [GJ]	N [GJ]	N [GJ]	P [OG]
Planar	P [GJ]	N [GJ]	P [T]	N [78]	N [GJ]	O	N [GJ]	N [GJ]	P [GJ]	N [OG]	P [GJ]
Series Parallel	P [OG]	P [OG]	P [T]	**P** [16]	P [5]	P [17]	P [5]	P [OG]	P [GJ]	P [OG]	P [GJ]
Outerplanar	P [OG]	P [OG]	P [T]	P [OG]	P [OG]	P [OG]	P [T]	P [OG]	P [GJ]	P [OG]	P [GJ]
Halin	P [OG]	P [OG]	P [T]	P [OG]	P [5]	P [17]	P [T]	P [OG]	P [GJ]	**P** [118]	P [GJ]
k-Outerplanar	P [OG]	P [OG]	P [T]	P [OG]	P [5]	**P** [17]	P [OG]	P [OG]	P [GJ]	**P** [76]	P [GJ]
Grid	P [OG]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [OG]	N [32]	P [T]	N [OG]	P [GJ]
$K_{3,3}$-Free*	P [OG]	N [GJ]	P [T]	N [78]	N [GJ]	O?	N [GJ]	N [GJ]	P [OG]	N [GJ]	**P** [40]
Thickness-k	N [OG]	N [GJ]	P [T]	N [78]	N [GJ]	N [OG]	N [GJ]	N [GJ]	N [119]	N [GJ]	**I** Proposition 3
Genus-k	P [OG]	N [GJ]	P [T]	N [78]	N [GJ]	O?	N [GJ]	N [GJ]	O?	N [GJ]	P [OG]
Perfect	**P** [34]	P [OG]	P [OG]	P [OG]	P [OG]	**N** [28]	N [OG]	N [OG]	**N** [20]	N [GJ]	I [84]
Chordal	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	N [OG]	**N** [20]	N [OG]	I [84]
Split	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	N [93]	N [OG]	**N** [20]	N [OG]	I [108]
Strongly Chordal	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	**N** [93]	P [OG]	**N** [109]	P [OG]	**I** [111]
Comparability	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	**N** [28]	N [OG]	N [94]	**N** [102]	N [GJ]	I [22]
Bipartite	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	N [OG]	N [94]	P [T]	N [GJ]	I [22]
Permutation	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	**P** [44]	P [OG]	O?	P [OG]	P [OG]
Cographs	P [T]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	**P** [20]	P [OG]	P [OG]
Undirected Path	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	**N** [13]	N [OG]	**N** [20]	**N** Theorem 4	I [22]
Directed Path	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	**N** [99]	P [OG]	**N** [1]	P [OG]	**P** [7]
Interval	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	**N** [1]	P [OG]	P [OG]
Circular Arc	P [OG]	P [OG]	P [OG]	P [OG]	N [OG]	O?	**P** [106]	P [OG]	**N** [1]	P [11]	**P** [80]
Circle	P [OG]	P [GJ]	P [OG]	**N** [73]	N [OG]	O?	*N [39]*	**N** [71]	**N** [26]	P [OG]	**P** [68]
Proper Circ. Arc	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	O?	P [11]	**P** [82]
Edge (or Line)	P [OG]	P [GJ]	P [T]	N [95]	N [OG]	**N** [28]	N [OG]	N [GJ]	**P** [59]	N [19]	I [OG]
Claw-Free	P [T]	P [OG]	**N** [103]	N [85]	N [OG]	**N** [28]	N [OG]	N [GJ]	**N** [20]	N [19]	I [OG]

recent references. Every reference associated with each entry of Table 1 has been checked, and the updated entries are precisely those that needed to be updated. It is surprising that several O? entries remain stubbornly open. At the time, D.S. Johnson proposed only one "O! = famous open problem", Membership for Perfect graphs, which we know today to be in P, and two entries "O = may well be hard", Hamiltonian Circuit restricted to Permutation graphs, known today to be in P, and Chromatic Index restricted to Planar graphs, which is still open. We remark that in the original summary table [OG], there was only one entry co-authored by a Brazilian researcher among 330 entries, namely Hamiltonian Circuit restricted to Grids [64], and today we have two additional such entries: Maximum Cut restricted to Strongly Chordal [109] and Graph Isomorphism restricted to Proper Circular Arc [82].

We depict in Figs. 1 and 2 the relations between the graph classes, and use the convention from [OG] that an arrow from Class A to Class B means that Class A contains Class B. Since the only O! entry was Membership for Perfect graphs, the chosen 30 classes were classified into the following four categories: Trees and Near-Trees, Planarity and its Relations, A Catalog of Perfect Graphs, and Intersection Graphs. Although very similar to the figures presented in [OG], our Figs. 1 and 2 present some additional relations, that were either unknown or unobserved, and about which we comment next. Unless explicitly mentioned, we follow the definitions from [OG]. According to it, Undirected Path graphs can be modeled by a set of paths in a tree, and Directed Path graphs are undirected path graphs whose representation is such that for some root vertex in the tree, all paths are subpaths of paths from the root to a leaf. We refer to [91] for the variations called UV, DV and RDV of intersection graphs of a family of undirected or directed vertex paths in an undirected or in a directed tree.

Fig. 1 highlights the key property that 7 graph classes are subclasses of Partial k-Trees [18], also known as Bounded Treewidth graphs. Also, although Table 1 follows the same organization as the one used in [OG], our proposed new Table 2 is organized in a way as to highlight this relationship, with the first 8 rows being exactly Partial k-Trees and its subclasses. In the summary table, D.S. Johnson used entry "P? = appears to be polynomial-time solvable by standard techniques, but I haven't checked the details". D.S. Johnson is correct, since all P? entries are known today to be P entries. All former P? entries used to appear in these 8 rows, Partial$k$-Trees and the 7 subclasses [18].

Another difference is that we use $K_{3,3}$-Free* to denote the graph class referred to in [OG] by $K_{3,3}$-Free. This is to avoid confusion, since nowadays it is standard to use H-Free to refer to the class of graphs that do not contain H as

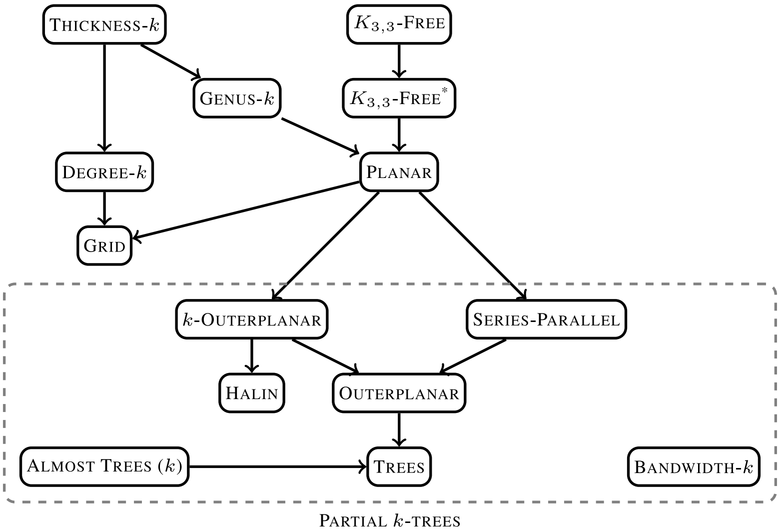

Fig. 1. Containment relations for classes from [OG], where, in particular, the subclasses of Partial k-Trees are highlighted. A graph class Class A has an arrow to a graph class Class B if Class A contains Class B.
Source: Adapted from [OG].

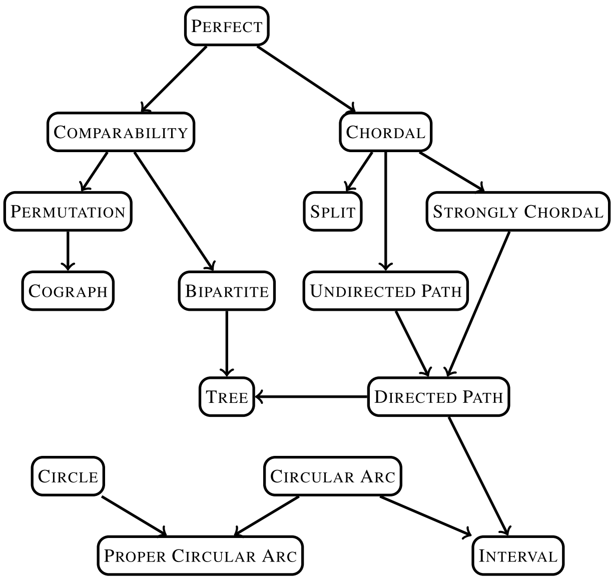

Fig. 2. Containment relations for classes from [OG], our target class is Undirected Path.
Source: Adapted from [OG].

an *induced subgraph*. However, the class investigated in [OG] is instead the class of graphs that contain no subgraphs *homeomorphic* to $K_{3,3}$; in other words, the class of graphs that do not contain $K_{3,3}$ as a topological minor. Observe that, using our notation, we have that $K_{3,3}$-Free* is a proper subclass of $K_{3,3}$-Free. Also, we mention that this confusion does not occur for Claw-Free graphs, since we, as well as [OG], use it to denote the class of graphs that do not contain $K_{1,3}$ (also known as the claw) as an induced subgraph.

Finally, we mention two relations that do not appear in [OG], both involving the class Thickness-k. A graph G is said to have *thickness at most k* if $E(G)$ can be partitioned into at most k subsets, each of which forms a planar subgraph of G. In the same way as all the other graph classes that have a parameter in their names such as the class of Partial k-Trees that is also known as Bounded Treewidth graphs, the class Thickness-k means Bounded Thickness graphs. First, note that if G has degree at most $\Delta(G)$, then by Vizing's Theorem, we get that $E(G)$ can be colored with at most $\Delta(G)+1$ colors. In other words, this means that the edge set of G can be partitioned into $\Delta(G)+1$ matchings, which are planar graphs, and hence G has thickness at most $\Delta(G)+1$. Therefore, we get that Degree-k is a subclass of Thickness-k. Another non-trivial

Table 2
The parameterized NP-Completeness Column: An Ongoing Guide table revised for the 21st century. The parameterized puzzle is to classify every O entry, every O? entry and every N entry into FPT = Fixed parameter tractable, W_1= W [1]-hard, W_2= W [2]-hard, and pN = paraNP-complete, where the considered parameterization is with respect to the natural parameter of each corresponding problem. We highlight as O* the N entry of Table 1 that constitutes the parameterized puzzle, for which so far we were not able to provide a parameterized complexity classification.

Graph Class	Member	IndSet	Clique	CliPar	ChrNum	ChrInd	HamCir	DomSet	MaxCut	κ-StTree	GraphIso
Partial k-trees	P [OG]	P [5]	P [T]	P [16]	P [5]	P [17]	P [5]	P [5]	P [20]	P [76]	P [17]
Trees/Forests	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	P [GJ]	P [T]	P [GJ]
Almost Trees (k)	P [OG]	P [OG]	P [T]	P [16]	P [5]	P [17]	P [5]	P [5]	P [20]	P [76]	P [17]
Bandwidth-k	P [OG]	P [OG]	P [T]	P [16]	P [5]	P [17]	P [5]	P [5]	P [OG]	P [76]	P [OG]
Series Parallel	P [OG]	P [OG]	P [T]	P [16]	P [5]	P [17]	P [5]	P [OG]	P [GJ]	P [OG]	P [GJ]
Outerplanar	P [OG]	P [OG]	P [T]	P [OG]	P [OG]	P [OG]	P [T]	P [OG]	P [GJ]	P [OG]	P [GJ]
Halin	P [OG]	P [OG]	P [T]	P [OG]	P [5]	P [17]	P [T]	P [OG]	P [GJ]	P [118]	P [GJ]
k-Outerplanar	P [OG]	P [OG]	P [T]	P [OG]	P [5]	P [17]	P [OG]	P [OG]	P [GJ]	P [76]	P [GJ]
Planar	P [GJ]	FPT [96]	P [T]	FPT [T]	pN [55]	O	FPT [89]	FPT [47]	P [GJ]	FPT [101]	P [GJ]
Grid	P [OG]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	FPT [89]	FPT [47]	P [T]	FPT [101]	P [GJ]
$K_{3,3}$-Free*	P [OG]	FPT [42]	P [T]	FPT [T]	pN [55]	O?	FPT [89]	FPT [100]	P [OG]	XP [T]	P [40]
Thickness-k	pN [OG]	FPT [74]	P [T]	FPT [T]	pN [55]	pN [63]	FPT [89]	XP [T]	FPT [86]	XP [T]	FPT [6]
Genus-k	P [OG]	FPT [30]	P [T]	FPT [T]	pN [55]	O?	FPT [89]	FPT [51]	FPT [86]	FPT [101]	P [OG]
Degree-k	P [T]	FPT [48]	P [T]	FPT [T]	pN [55]	pN [63]	FPT [89]	FPT [3]	FPT [86]	FPT [67]	P [OG]
Perfect	P [34]	P [OG]	P [OG]	P [OG]	P [OG]	pN [28]	FPT [89]	W_2 [104]	FPT [86]	W_2 [104]	FPT [6]
Chordal	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	FPT [89]	W_2 [104]	FPT [86]	W_2 [104]	FPT [6]
Split	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	FPT [89]	W_2 [104]	FPT [86]	W_2 [104]	FPT [6]
Strongly Chordal	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	FPT [89]	P [OG]	FPT [86]	P [OG]	FPT [6]
Comparability	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	pN [28]	FPT [89]	W_2 [104]	FPT [86]	W_2 Proposition 1	FPT [6]
Bipartite	P [T]	P [GJ]	P [T]	P [GJ]	P [T]	P [GJ]	FPT [89]	W_2 [104]	P [T]	W_2 Proposition 1	FPT [6]
Permutation	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [44]	P [OG]	FPT [86]	P [OG]	P [OG]
Cographs	P [T]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	P [20]	P [OG]	P [OG]
Undirected Path	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	FPT [89]	XP [T]	FPT [86]	XP [T]	FPT [6]
Directed Path	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	FPT [89]	P [OG]	FPT [86]	P [OG]	P [7]
Interval	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	FPT [86]	P [OG]	P [OG]
Circular Arc	P [OG]	P [OG]	P [OG]	P [OG]	FPT [54]	O?	P [106]	P [OG]	FPT [86]	P [11]	P [80]
Circle	P [OG]	P [GJ]	P [OG]	XP [73]	pN [112]	O?	FPT [89]	W_1 [24]	FPT [86]	P [OG]	P [68]
Proper Circ. Arc	P [OG]	P [OG]	P [OG]	P [OG]	P [OG]	O?	P [OG]	P [OG]	FPT [86]	P [11]	P [82]
Edge (or Line)	P [OG]	P [GJ]	P [T]	O* [95]	pN [63]	pN [28]	FPT [89]	FPT [38]	P [59]	XP [T]	FPT [6]
Claw-Free	P [T]	P [OG]	FPT [38]	pN [85]	pN [63]	pN [28]	FPT [89]	FPT [38]	FPT [86]	XP [T]	FPT [6]

relation involving this class is with Genus-k graphs. A *k-book embedding* of a graph G is a linear ordering of its vertices along the spine of a book and an assignment of its edges to k pages so that edges assigned to the same page can be drawn on that page without crossings. The *pagenumber* of a graph G is the minimum k for which G admits a k-book embedding. Clearly, the pagenumber of G is an upper bound for the thickness of G. Additionally, in [87] the authors prove that the pagenumber of G is bounded above by a function of the genus of G. This means that if G has bounded genus, then G also has bounded thickness. Therefore, Genus-k is a subclass of Thickness-k. On the other hand, to the best of our knowledge, it is not known whether graphs with bounded thickness have bounded genus.

Our contribution. In the summary table, the Steiner Tree column had 6 unresolved entries: 5 P? entries, all of which are now known to be subclasses of Partial k-Trees and henceforth are in P, and one O? entry for Undirected Path graphs. Upon close investigation of the references given in [GJ] and [OG], we found that many consist of "private communication" or could not be confirmed. In the particular case of the Steiner Tree column, we found that this happens for the lines Circular Arc, Circle, Proper Circular Arc, Edge (or Line), and Claw-Free. We were able to find a recent reference for Edge (or Line) and Claw-Free. Additionally, based on the facts that Circular Arc and, consequently, Proper Circular Arc graphs have bounded mim-width and that Steiner tree is polynomial-time solvable for graphs of bounded mim-width, we were able to resolve such entries as well. However, we could not find any reference for Circle graphs, and therefore we underline the corresponding [OG] reference in Table 1. Moreover, the entry Undirected Path is said to be NP-complete in [107], but again with a "private communication" reference (we comment more on this in Section 4). Believing in the need to have explicit proofs for these important problems, we here give a proof of NP-completeness for Undirected Path graphs, which would provide a full dichotomy Polynomial versus NP-complete for the Steiner Tree column. Actually, we provide a second dichotomy for the Steiner Tree problem restricted to Undirected Path graphs, according to the diameter of the input graph. For the Graph Isomorphism column we also provide a full dichotomy Polynomial versus NP-complete by giving an explicit proof of GI-completeness for Thickness-k graphs (please refer to Section 3).

Besides providing a full dichotomy Polynomial versus NP-complete for the Steiner Tree column, in Table 1 we have thoroughly revised the summary table that 35 years later has 54 new resolved entries depicted in bold. Additionally, there are 36 citations for references not in bold that confirm resolved entries from [OG] or [GJ], that we updated because they cited private communications, or because the cited reference is not easily accessible, or could not be confirmed. There is one entry highlighted in italic that corrects the entry for Hamiltonian Circuit restricted to Circle graphs originally P but that actually is N [39].

In addition, we consider the parameterized complexity of hard problems to revise Table 1 into a new Table 2, a proposed summary table of what it means today to study a problem from a computational complexity point of view. This is of course just a sample of what it means, since we could even consider other classifications (e.g., the approximability complexity theory and the space complexity theory). We have kept the same 30 classes but have drawn the horizontal lines so that the Partial k-Trees subclasses appear together, and we may focus on the remaining rows, where the NP-complete entries appear. In Section 2, we discuss in detail Table 2, also presenting the basic definitions of parameterized complexity, in order to draw the reader's attention to the granularity provided by the parameterized complexity for the NP-complete problems into XP, FPT, W_1, W_2, and pN. We depict in Table 2 as O* the only N entry of Table 1 that constitutes the parameterized puzzle for which so far we were not able to provide a parameterized complexity classification. This is to show how rich the original problems posed by Garey and Johnson are, and how their initial classification continues to develop into ever evolving complexity classes, with the NP-complete class being now just the beginning of a very interesting story.

2. The parameterized complexity of hard problems

In this section, we discuss in detail new Table 2. We also further discuss some of the differences that arise between our updated Table 1 and the table presented in [OG] thirty-five years ago. We start by giving some basic definitions of parameterized complexity and its complexity hierarchy classes. After that, we discuss each of the 11 columns separately.

Parameterized complexity. We refer the reader to [37,48,49,52,96] for all basic formal definitions, as well as many techniques employed in the parameterized complexity theory. Formally, given a fixed finite alphabet Σ, a language $L \subseteq \Sigma^* \times \mathbb{N}$ is called a *parameterized problem*; given an instance $(x, \kappa) \in L$, we call κ the *parameter*. Also, we denote the *size* of an instance (x, κ) by $|(x, \kappa)|$. Observe that each possible parameter defines a different parameterized problem; for example, when considering the Clique problem on graphs, it can be parameterized by the size of the desired clique, or by the maximum degree of the input graph. In both cases, the input consists of a graph G, an integer c, and the corresponding parameter, and the problem consists of deciding whether G has a clique of size at least c, except that in the former the parameter is also c, while in the latter the parameter is the maximum degree $\Delta(G)$. When the parameterized problem is a decision problem having as parameter the size of the solution, we say that the problem is parameterized by the *natural parameter*. Table 2 is filled taking into account the natural parameter, whenever possible. We give more details about this when analyzing each of the 11 columns.

We say that a parameterized problem L is *fixed parameter tractable* (from now on denoted by FPT) if there exists an algorithm $\mathcal{A}$ that solves L on input (x, κ) in time $f(\kappa) \cdot |(x, \kappa)|^{\mathcal{O}(1)}$, where f is a computable function. In this case, the algorithm $\mathcal{A}$ is said to be an FPT algorithm for L, and we also use FPT to denote the set of FPT problems. Observe that P $\subseteq$ FPT.

Intuitively, one could describe the FPT class as the parallel, in the parameterized complexity theory, of the P class in the traditional complexity theory. Another "approachable" class is that of the slice-wise polynomial. We say that a parameterized problem L is *slice-wise polynomial* (denoted by XP) if there exists an algorithm that solves L in time $f(\kappa) \cdot |(x, \kappa)|^{g(\kappa)}$, where f and g are two computable functions. Observe that, for each fixed value of κ, this is a polynomial algorithm.

Concerning a parallel of the NP-complete class, there are two main hard classes in the parameterized complexity, the paraNP-complete and the W-hard. A parameterized problem L is *paraNP-complete* if it is NP-complete for some fixed value of the parameter κ. For instance, the Vertex Coloring problem is paraNP-complete when parameterized by the number of colors. Note that, unless P = NP, a paraNP-complete problem cannot be XP, hence it cannot be FPT either.

Now, before defining our last parameterized complexity class, we need another definition. Given an instance (x, κ) of a parameterized problem L, a *parameterized reduction* from L to another parameterized problem L' is an algorithm that computes, in time $f(\kappa) \cdot |x|^{\mathcal{O}(1)}$ for some computable function f, an equivalent instance (x', κ') of L' such that $\kappa' \leq g(\kappa)$ for some computable function g. The class of W-hard problems can be formally defined based on a hierarchy of nested classes called W[i], for each $i \in \mathbb{N} \setminus \{0\}$. However, for our purposes it suffices to define the W[1]-hard and W[2]-hard classes in terms of their "base" problems; think of it as defining the NP-hard class in terms of SAT. A parameterized problem L is *W[1]-hard* if there is a parameterized reduction from Clique, parameterized by the size of the clique, to L; and it is *W[2]-hard* if there is a parameterized reduction from Dominating Set, parameterized by the size of the dominating set, to L. Observe that a parameterized version of an NP-hard problem can be classified in any of these classes, unless of course P = NP, in which case all the classes collapse to P.

Tree decompositions are an important tool in the parameterized complexity theory, as well as in the traditional computational complexity, since good algorithms can often be obtained for graphs with bounded treewidth, and also because even graphs with unbounded treewidth can sometimes be approached by applying the bidimensionality technique (see e.g. [37]). Since the publication of [OG], in addition to treewidth, many other width parameters have been introduced (see [113] for a nice Hasse diagram containing 32 graph parameters; we also refer to the survey [58]). In particular, the clique-width [69] and the mim-width [113] parameters are of special interest to us, since they are bounded for some of our proposed classes, and because some of the proposed problems can be solved in polynomial time when these parameters are bounded. More specifically, Partial kP-trees and Cographs have bounded clique-width [36], while the following have bounded mim-width: graphs with bounded clique-width [113], Permutation [10], Circular Arc [10], Directed Path [25] (this is because they are a subclass of leaf powers [65]). Regarding the proposed problems, the following can be solved in polynomial time on graphs with bounded mim-width, provided a branch decomposition of bounded mim-width is given: Independent Set, Dominating Set [27], and Steiner Tree [11]; while the following can always be solved in polynomial time on graphs with bounded clique-width, since a construction sequence of bounded clique-width can be found in polynomial (and even FPT) time [62,97,98]: Clique [35], Chromatic Number [75], Hamiltonian Circuit, Maximum Cut [116], and Clique Partition [105]. Although the problem of finding in polynomial-time a branching decomposition of bounded mim-width is still open for general graphs with bounded mim-width, it has been proven to be polynomial-time solvable for some graph classes, including Permutation and Circular Arc graphs [10]. These observations help to solve an issue about the complexity of Steiner Tree restricted to Circular Arc and Proper Circular Arc, which we discuss later on. Also, because Maximum Cut is NP-complete on Interval graphs [1], and this is a subclass of Circular Arc, it follows that the problem is NP-complete on graphs with bounded mim-width, which is in contrast with the complexity of the problem restricted to bounded clique-width [116].

We now discuss Tables 1 and 2, dividing the discussion by the problems.

Membership. The entries P are inherited from Table 1; hence the only class that can be further refined in the parameterized complexity is the class Thickness-k. It is known that deciding whether a given graph G has thickness at most k is NP-complete even if $k = 2$ (for $k = 1$ it coincides with deciding planarity, which is polynomial) [OG]. We therefore get that, considering k as the parameter, the related parameterized problem is paraNP-complete.

We mention that the reference cited by [OG] for Partial k-trees was a technical report that now has a published version [4].

Independent set. The natural parameter considered is the size of the desired independent set. The problem is trivially in XP in general: by enumerating all vertex subsets of size κ, one can readily check in time $\mathcal{O}(n^{\kappa})$ whether a graph on n vertices admits an independent set of size at least κ. Nevertheless, Independent Set is unlikely to be FPT, since it is known to be W[1]-hard [48]. In fact, by considering the complement graph, we obtain that Independent Set and Clique, for general graphs, are equivalent to each other from the parameterized complexity perspective. On the positive side, as can be seen in Table 2, the problem is FPT for Planar [96], Genus-k [30] and Degree-k [48] graphs. More generally, the problem is also FPT for Thickness-k graphs [74]: This follows from the fact that Independent Set is FPT when restricted to graphs with bounded clique number, as first observed in [74] as a special case of a more general framework *cf.* [52,104]. Additionally, Independent Set is also FPT for $K_{3,3}$-Free*. Indeed, it is well known that, if a graph H of maximum degree at most 3 is a minor of a graph G, then H is a topological minor of G as well *cf.* [46,77]. Thus, $K_{3,3}$-Free* coincides with the class $K_{3,3}$-Minor-Free* of graphs that do not contain $K_{3,3}$ as a minor. Therefore, since Independent Set was proven to be FPT for $K_{3,3}$-Minor-Free* graphs [42,43], we obtain that the problem is also FPT for $K_{3,3}$-Free*. We remark that, although Independent Set is FPT for $K_{3,3}$-Free* graphs, it remains W[1]-hard for the larger class $K_{3,3}$-Free. This latter result follows from the fact that Independent Set was proven to be W[1]-hard for another subclass of $K_{3,3}$-Free, the C_4-free graphs [21], which consists of graphs that do not contain C_4 as an induced subgraph.

We observe that, for the entry Partial k -Trees, we cite Ref. [5], which is different from Ref. [2] cited in [OG]. This is because, upon checking [2], we were not able to find any mention to the Independent Set problem restricted to Partial k -Trees.

An interesting fact is that D.S. Johnson mentions, in the caption of his summary table, that Vertex Cover was not included as a column because its complexity will always be the same as the complexity of Independent Set. While this is true for traditional complexity theory, one could say that Vertex Cover is a canonical problem in FPT since it is FPT in general [31] and many of the known techniques can be successfully applied to it, whereas Independent Set can be regarded as a canonical W[1]-hard problem.

Clique. Analogously, the natural parameter considered is the size of the desired clique. All entries here are in P, except for Claw-Free graphs, which is FPT [38].

Partition into cliques. The problem has as input a graph G and a positive integer κ, and it consists of deciding whether the vertex set of G can be partitioned into κ disjoint cliques. The natural parameter considered is the integer κ. One can straightforwardly verify that Partition into Cliques is polynomially equivalent to Chromatic Number by considering the complement graph $\overline{G}$ of the input graph G, i.e. the problem of deciding whether $\overline{G}$ can be proper colored with at most κ colors. Nevertheless, it is worth mentioning that the respective particular cases of Partition into Cliques and Chromatic Number restricted to specific graph classes are not necessarily polynomially equivalent to each other, as usually these graph classes are not closed under taking the complement. As an example, observe that in Table 1, Partition into Cliques is in P for Circular Arc graphs, while Chromatic Number remains NP-complete for Circular Arc graphs.

The problem is trivially FPT for graphs that only contain cliques whose size can be upper-bounded by a computable function f of κ. Indeed, if the size of the maximum clique of G is at most $f(\kappa)$, then either G contains at most $f(\kappa)\cdot\kappa$ vertices, or the vertex set of the input graph cannot be partitioned into at most κ disjoint cliques, and thus we are dealing with a no-instance. Based on that, we immediately obtain that Partition into Cliques is FPT for Planar and $K_{3,3}$-Free* graphs (observe that the latter graphs cannot have cliques of size bigger than 5). Additionally, one can verify that Thickness-k, Genus-k and Degree-k graphs also have cliques whose size depends only on k, and since k is constant by definition, we get that these graphs have maximum clique bounded by a constant value. As a result, we obtain that Partition into Cliques is also FPT for Thickness-k, Genus-k and Degree-k graphs.

On the other hand, Circle graphs may have cliques of unbounded size. J. Keil and L. Stewart proved that Partition into Cliques is XP for Circle graphs [73], however it remains unknown whether the problem is FPT in this class. The paraNP-completeness of Partition into Cliques for Claw-Free graphs follows from the fact that Chromatic Number is NP-complete, even when $\kappa = 3$, for Triangle-Free graphs (*i.e.,* graphs that do not contain K_3 as an induced subgraph) [85] — since the complement graphs of Triangle-Free graphs do not have independent sets of size larger than 2, such complement graphs are Claw-free.

In what follows, we discuss some discrepancies between our Table 1 and the table presented in [OG], with respect to the Partition into Cliques entries. First, [OG] cites [GJ] for the Degree-k entry. However it is not immediate how the results presented in [GJ] (or in the references cited by [GJ]) lead to the NP-completeness of Partition into Cliques for Degree-k. In fact, [GJ] proves that Partition into Cliques remains NP-complete for graphs that contain no cliques of size larger than 4; nevertheless the graph constructed in their reduction does not have bounded maximum degree. For this reason, we cite [29] instead, which gives an explicit NP-completeness proof for Partition into Cliques restricted to Cubic graphs. As for Planar graphs, [OG] cites [12], which actually proves that the following problem is NP-hard: given a planar graph G, and a fixed connected outerplanar graph H with at least three vertices, maximizes the number of vertices of G that can be covered with copies of H. This does not immediately imply that Partition into Cliques is NP-complete for Planar graphs since they limit the number of vertices in each clique to 3 (by considering $H = K_3$) when a planar graph could have a partition into cliques with cliques also of size 4. We then cite the more explicit construction given in [78]. Note that this also impacts the entries $K_{3,3}$-Free*, Thickness-k, and Genus-k, as these contain Planar graphs. Finally, we remark that the references cited by [OG] for Line graphs and Claw-Free graphs are actually private communications. Therefore, we cite [95] for Line graphs, and [85] for Claw-Free graphs, instead of [OG]. We remark that although the NP-completeness of Partition into Cliques for such graph classes directly follows from [95], we have decided to cite the oldest known reference for each result.

A problem closely related to Partition into Cliques is the so-called Clique Edge-Partition, which is defined as follows: given a graph G and a positive integer k, decide whether the edge set of G can be partitioned into at most k subsets such that each subset induces a complete subgraph of G. However, it is worth mentioning that, since the line graph of a complete graph is not necessarily a complete graph, this problem is not the same as deciding whether the vertex set of the line graph $L(G)$ of a graph G can be partitioned into at most k disjoint cliques. As a matter of fact, while Partition into Cliques is paraNP-complete [85], Clique Edge-Partition is FPT in general [92].

Chromatic number. The problem has as input a graph G and a positive integer κ, with κ being the considered parameter, and it consists of deciding whether the chromatic number of G is at most κ. Also one of Karp's 21 NP-complete problems [70], Chromatic Number is NP-complete for fixed values of κ for very restricted graph classes. Since it is NP-complete to decide whether a planar graph with maximum degree 4 is 3-colorable [55], we get that Chromatic Number is paraNP-complete for Planar, $K_{3,3}$-Free*, Genus-k, Thickness-k and Degree-k graphs. Additionally, since it is NP-complete to decide whether a circle graph is 4-colorable [112], we obtain that Chromatic Number is paraNP-complete for Circle graphs. Finally, it is also NP-complete to decide whether the line graph of a 3-regular graph is 3-colorable [63], implying the paraNP-completeness for Line and Claw-Free graphs. On the positive side, we mention that, even though the parameterized complexity theory had not yet been defined by the time of publication of [54], the algorithm presented in [54] for Circular Arc is actually an FPT algorithm. Therefore, Chromatic Number parameterized by κ, the number of colors, is FPT for Circular Arc graphs.

As for the differences between our tables and [OG]'s Table, they cite [2] for the entry Partial k-Trees. However this reference does not mention the Chromatic Number problem, and this is why we cite [5]. As for Bandwidth-k graphs, they cite [90], which treats only the case $k = 3$. The same happens for the entry k-Outerplanar [8]. Also, we could not find the reference cited by [OG] for Series–Parallel graphs [110], which is the same as the one cited for Halin graphs. Nevertheless, the polynomial results for all these classes follow from the fact they all have bounded treewidth; therefore we cite [5].

Chromatic index. This is by far the hardest problem of the table, with the largest number of open entries, remaining open even for classes considered "easy" as for instance Cographs, a graph class for which all problems besides Chromatic Index have been classified as P [75]. The problem has as input a graph G, and a positive integer κ, with κ being the considered parameter, and one wants to decide whether the chromatic index of G, denoted by $\chi'(G)$, is at most κ. Observe that, since Vizing's Theorem tells us that $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$, we can then consider κ as being equal to the maximum degree of G as otherwise the answer is trivial. As already mentioned, deciding whether $\chi'(G) = 3$ is NP-complete even for Cubic graphs [63], which implies that Chromatic Index is paraNP-complete for Degree-k graphs, and that it is also paraNP-complete for Thickness-k graphs. In addition, L. Cai and J. A. Ellis proved that deciding whether $\chi'(G) = 3$ is NP-complete even for Comparability graphs and for Line graphs [28]. Therefore, Chromatic Index remains paraNP-complete when restricted to Comparability, Perfect, Line and Claw-Free graphs.

The reference cited by [OG] for Series–Parallel graphs and Halin graphs is the same as the one cited for these graphs on column Chromatic Number [110]. Again, even though we could not find the reference, the results follow from the fact that these graphs have bounded treewidth [17].

Hamiltonian circuit. Here, the size of any solution is the size of the input graph; therefore, we consider the problem of deciding whether a given graph G has an Hamiltonian cycle parameterized by $n = |V(G)|$. It is known that Longest Cycle parameterized by the size κ of the cycle is FPT [89]. If follows that Hamiltonian Circuit is FPT when parameterized by n.

The reference cited by [OG] for Series–Parallel graphs [110] is the same as the one cited for these graphs on Chromatic Number (see comments above); thus, we cite [5] instead. Also, we were not able to access the reference for Split and Chordal graphs [33], therefore we cite Ref. [93]. For Circle graphs, we find a more serious discrepancy between our Table 1 and the table presented in [OG]. They cite [13] as providing a polynomial algorithm for Circle graphs. However, the paper only provides a polynomial algorithm for Interval graphs, a subclass of Circle graphs. And actually, the problem has been shown to be NP-complete for Circle graphs in [39].

Dominating set. Given a graph G and a positive integer κ, the problem consists of deciding whether G has a dominating set of size at most κ (a set $D \subseteq V(G)$ is *dominating* if, for every vertex $v \in V(G)$, $v \in D$ or v has a neighbor that belongs to D). The natural parameter is of course κ, and as previously mentioned, this is the canonical problem in the class W[2]-hard. Because of this, Dominating Set is among the most investigated problems in the parameterized complexity theory, deserving a survey of its own. Here, we make a short compilation of the results that concern the classes of interest. The problem is FPT for: Planar graphs [47], and therefore also for Grid graphs; for $K_{3,3}$-Free* graphs [100]; for Genus-k graphs [51]; for k-Degenerate graphs [3], and therefore also for Degree-k graphs; and for Claw-Free graphs [38], and therefore also for Line graphs. In addition, it is W[2]-hard for Split and Bipartite graphs [104], and therefore also for Chordal, Perfect, and Comparability graphs. Finally, it is W[1]-hard for Circle graphs [24]. Observe that this problem is trivially in XP, since it suffices to test all the $\mathcal{O}(n^{\kappa})$ subsets of size κ. However, this is not completely refined, and the entries for Thickness-k and Undirected Path graphs can be regarded as the only open cases in this column.

Regarding the differences between our Table 1 and the table in [OG], they cite [57] for Almost Trees (k), but the paper does not seem to attack this class. Something similar happens with entry Bandwidth-k, where they cite [90] but the paper attack domination-related problems, but not Dominating Set itself. Nevertheless, we now know that the problem is indeed polynomial for these classes since they have bounded treewidth [5]. Also, the entry for Grids is cited as a private communication in [OG]; this is why we provide Ref. [32]. As for the Bipartite and Comparability entries, we were not able to find Ref. [45], cited by [OG], this is why we also provide [94].

Maximum cut. This is another problem that is FPT in general. Given a graph G, and an integer κ, we consider the problem of deciding whether there exists $S \subseteq E(G)$ that separates G and has size at least κ, parameterized by κ. The best known algorithm so far runs in time $\mathcal{O}(m + n + \kappa \cdot 4^{\kappa})$ [86], where m denotes the number of edges and n denotes the number of vertices of the input graph G.

We were not able to find the reference cited by [OG] for Thickness-k graphs [9], and therefore we provide the same reference given by [GJ] for Degree-k graphs [119].

Steiner tree. Given a graph G, a subset $X \subseteq V(G)$, called *terminal* set, and a positive integer t, the problem consists of deciding whether there exists a subset $S \subseteq V(G) \setminus X$ such that $|S \cup X| \leq t$ and $G[S \cup X]$ is connected — and hence $G[S \cup X]$ contains a tree subgraph $\mathcal{T}$ with $X \subseteq V(\mathcal{T})$, called a *Steiner tree* of G for X. The vertices in S are commonly called *Steiner vertices*. Steiner Tree parameterized by the number of terminal vertices $|X|$ is well-known to be FPT [50], with the current best algorithm running in time $\mathcal{O}(2^{|X|} \cdot n^{\mathcal{O}(1)})$ [15], where n denotes the number of vertices of the input graph G. This clearly implies that Steiner Tree parameterized by the natural parameter, *i.e.* by the size of the sought solution $|S \cup X| \leq t$, is also FPT. Interestingly enough, the problem is W[2]-hard when parameterized by the number of Steiner vertices $|S|$ as shown in [88]. We denote by κ the maximum number of Steiner vertices allowed in a given instance of the problem, and then we write κ-Steiner Tree to denote Steiner Tree parameterized by κ. Since the other parameterized versions of the problem are already known to be FPT for general graphs, this latter is the version considered in Table 2.

The κ-Steiner Tree problem is FPT for Genus-k graphs [101], and therefore for Planar and Grid graphs; and for k-Degenerate graphs [67], and therefore for Degree-k graphs. Also, the proof given in [104] for W[2]-hardness of Dominating Set for Split graphs actually holds for Connected Dominating Set. Moreover, in [117], the authors give a parameterized reduction from Connected Dominating Set to κ-Steiner Tree that works for any subclass of Chordal graphs, without changing the input graph. Therefore, based on the results presented in [104], we obtain that κ-Steiner Tree is W[2]-hard for Split, Chordal and Perfect graphs. In the next section, we give in Proposition 1 a simple reduction to prove that the problem is W[2]-hardness for Bipartite graphs (and therefore also for Comparability graphs). Finally, observe that a simple XP algorithm can be obtained by simply testing all $\mathcal{O}(n^{\kappa})$ possible vertex subsets $S \subseteq V(G) \setminus X$ of size at most κ.

Regarding the differences between our tables and [OG]'s Table, Ref. [114] cited in [OG] for Outerplanar graphs could not be found, but we mention that the reference cited in [OG] for Series–Parallel [115] is indeed correct, and that it can be used for Outerplanar as well. Also, [OG] cites a private communication with Schäffer for the entries Circle, Line, and Claw-Free graphs, and cites [117] for Circular Arc graphs and Proper Circular Arc. We were not able to find any mention to Circular Arc graphs in [117]. Also, in his book [107], Spinrad writes:

> "The status of Steiner Tree is slightly unclear; Schäffer sketched a proof that this is polynomially solvable (for both Circle and Circular Arc graphs), and thus it appeared as polynomial in the table of '[OG]', though no algorithm solving the problem appears in the general literature".

Nevertheless, because Circular Arc graphs have bounded mim-width and a branch decomposition with bounded mim-width of these graphs can be computed in polynomial time [10], it follows from [11] that Steiner Tree can be solved in polynomial time for Circular Arc, and consequently also for Proper Circular Arc. As for entries Line and Claw-Free, they have been recently filled [19], while the situation remains the same for Circle graphs. We add that in [71], J. Keil proves that Connected Dominating Set is NP-complete for Circle graphs. Thus if a proof of polynomiality of Steiner tree indeed exists for Circle graphs, this will be a nice example of class that separates Connected Dominating Set from Steiner tree.

Graph isomorphism. This is problem Open1 from [GJ], and perhaps the most controversial problem in Graph Theory, being regarded as the only naturally defined problem with a high chance to be an NP-intermediate problem, thus having deserved a classification of its own. Given two graphs G and H, it consists of deciding whether G and H are isomorphic, i.e., whether there exists a bijection of the vertex sets that preserves adjacencies. A problem is GI-complete if it is equivalent in complexity to general Graph Isomorphism. As it happened with Hamiltonian Circuit, the natural parameter here is the size of the input graph. This problem is FPT in general, with the best known algorithm running in time $O^*(2^{\sqrt{n}\log n})$ [6].

There are again some discrepancies between our Table 1 and the table presented in [OG]. In [OG], they cite [GJ] as a reference for the entries Perfect and Chordal graphs to be GI-complete; however, Chordal graphs are cited as an open case in [GJ]. Nevertheless, these classes are indeed GI-complete as proven in [84] for Chordal graphs; this construction was noticed to work also for Split graphs [108]. Something similar happens in entries Bipartite and Undirected Path graphs, with them being cited as open cases in [GJ], instead of GI-complete, as cited in [OG]. Nevertheless, these are indeed GI-complete as proven in [22]. This also impacts the Comparability graphs entry. Finally, Graph Isomorphism is also GI-complete for Thickness-k *cf.* [41]. Since [41] cites an unpublished paper due to De Biasi, we provide a proof in Proposition 3, for the sake of completeness.

3. Some simple reductions

For the sake of completeness, here we present two simple proofs. First, we prove that κ-Steiner Tree is W[2]-hard when restricted to Bipartite graphs. Indeed, this result follows from a standard parameterized reduction from Dominating Set, described in Proposition 1. We remark that Raman and Saurabh present a similar reduction to prove that Dominating Set is W[2]-hard for Bipartite graphs [104].

Proposition 1. *κ-Steiner Tree remains W[2]-hard for Bipartite graphs.*

Proof. Let $I = (G, \kappa)$ be an instance of Dominating Set. We let $I' = (G', X, \kappa)$ be the instance of κ-Steiner Tree such that G' is defined as follows:

- $V(G') = \{r\} \cup \{v' : v \in V(G)\} \cup V(G)$, where r denotes a new vertex, and we add a new vertex v' for each $v \in V(G)$;
- $E(G') = \{rv : v \in V(G)\} \cup \{v'u : u \in N_G[v],\ u, v \in V(G)\}$; and

the terminal set is defined as $X = \{r\} \cup \{v' : v \in V(G)\}$. Note that X and $V(G)$ are independent sets of G'. Thus, G' is a bipartite graph.

Suppose that G admits a dominating set $D \subseteq V(G)$ of size at most κ. It is not hard to check that $D \cup X$ induces a connected subgraph of G'. Therefore, I' is a yes-instance of κ-Steiner Tree.

Conversely, suppose that G' admits a Steiner tree T for X such that $|V(T) \setminus X| \leq \kappa$. Note that neighbors of X in G' belong to $V(G)$. Thus, $V(T) \cap V(G)$ is a dominating set of G, otherwise either T would not be connected, or there would exist some terminal vertex belonging to $X \setminus \{r\}$ that is not in T. Moreover, since $|V(T) \setminus X| \leq \kappa$, we obtain that $|V(T) \cap V(G)| \leq \kappa$. Therefore, $V(T) \cap V(G)$ is a dominating set of G of size at most κ, and I is a yes-instance of Dominating Set. □

Now, we present a proof that Graph Isomorphism is GI-complete when restricted to Thickness-k graphs. This result actually follows from a simple adaptation of an argument described in [14] by De Biasi, which we present in Proposition 3. The *subdivision* of a graph G is defined as the graph $\mathfrak{s}(G)$ obtained from G by replacing each edge $e = uv \in E(G)$ with the path $\langle u, w_e, v\rangle$, where w_e denotes a new vertex. More formally, $\mathfrak{s}(G)$ is the graph with vertex set $V(\mathfrak{s}(G)) = V(G) \cup \{w_e : e \in E(G)\}$ and edge set $E(\mathfrak{s}(G)) = \{uw_e, w_e v : e = uv \in E(G)\}$.

Lemma 2. *For each graph G, the subdivision $\mathfrak{s}(G)$ of G has thickness at most 2.*

Proof. Assume without loss of generality that $V(G) = \{v_1, \ldots, v_n\}$, for some positive integer n. Let H_1 and H_2 be the spanning subgraphs of $\mathfrak{s}(G)$ defined as follows: for each edge $e = v_i v_j \in E(G)$ with $i < j$, add the edge $v_i w_e$ to H_1 and add the edge $w_e v_j$ to H_2. Note that, for each $e \in E(G)$, the degree of w_e in H_1, and in H_2, is exactly 1. Moreover, $V(G)$ is an independent set in H_1, and in H_2. Thus, H_1 and H_2 are forests whose components are stars, which implies that H_1 and H_2 are planar graphs. Therefore, since $\mathfrak{s}(G) = H_1 \cup H_2$, we obtain that $\mathfrak{s}(G)$ has thickness at most 2. □

Proposition 3. Graph Isomorphism *is* GI*-complete for* Thickness-k.

Proof. Let G_1 and G_2 be two arbitrary graphs. It follows from Lemma 2 that $\mathfrak{s}(G_1)$ and $\mathfrak{s}(G_2)$ have thickness at most 2. Moreover, one can easily verify that G_1 and G_2 are isomorphic if and only if $\mathfrak{s}(G_1)$ and $\mathfrak{s}(G_2)$ are isomorphic. □

4. Steiner tree for undirected path graphs

In this section, we prove that the Steiner Tree problem is NP-complete for Undirected Path graphs, which provides a full dichotomy Polynomial versus NP-complete for the Steiner Tree column. Our proof holds even if the input graph has diameter 3, and we show that Steiner Tree is in P when restricted to Undirected Path graphs of diameter 2, thus getting another dichotomy for the problem in terms of the diameter.

We mention that Spinrad writes in his book [107] that he was unable to find any work on the Steiner Tree problem restricted to Undirected Path graphs, but that Dieter Kratsch told him this should be NP-complete as a simple extension of a proof that Connected Dominating Set is NP-complete for Undirected Path graphs. Haynes et al. cite a paper [72], submitted in 1997, in their book [60], where the NP-completeness proof of Connected Dominating Set supposedly appears. However, we were not able to find any version of [72]. Thus, in order to fill this gap, we provide a non-trivial adaptation of the NP-completeness proof presented in [23] for the Dominating Set problem restricted to Undirected Path graphs, which finally explicitly shows that the Connected Dominating Set problem restricted to Undirected Path graphs is indeed NP-complete. Then, we use a transformation by White et al. [117] between the Steiner Tree and the Connected Dominating Set problems to obtain the desired result as a corollary.

A closely related variant of Connected Dominating Set that should be mentioned is the so-called Total Dominating Set problem, which, rather than a dominating set inducing a connected subgraph, simply requires a dominating set having no isolated vertices. Through a different non-trivial adaptation of the proof presented in [23], Total Dominating Set restricted to Undirected Path graphs was proven to be NP-complete [81]. However, it is worth noticing that the construction described in [81] cannot be used so as to further obtain the NP-completeness of Connected Dominating Set for Undirected Path graphs. Therefore, we emphasize the merit of our contribution.

We start by giving some formal definitions. A *chordal* graph can also be described as the intersection graph of subtrees of a tree: given a tree T, each vertex u of G corresponds to a subtree T_u of T, and $uv \in E(G)$ if and only if $V(T_u) \cap V(T_v) \neq \emptyset$. We call $(T, \{T_u\}_{u \in V(G)})$ a *tree model of* G. One can verify that a tree decomposition of G of width $\omega(G)$ can be obtained from this tree model. The subclasses of Undirected Path, Directed Path and Interval graphs can be derived from this definition as follows. An *undirected path graph* is a chordal graph that has a tree model $(T, \{T_u\}_{u \in V(G)})$ where each $u \in V(G)$ corresponds to a subpath of T. A *directed path graph* is an undirected path graph that has a tree model $(T, \{T_u\}_{u \in V(G)})$ such that T is rooted at a vertex r, and every subpath T_u is from a node $t \in V(T)$ to a node $t' \in V(T)$ where t belongs to the (r, t')-path of T. Finally, an *interval graph* is a chordal graph that has a tree model $(T, \{T_u\}_{u \in V(G)})$ where T is a path. From Fig. 2, we know that these classes are nested.

We recall that, given a graph G, a subset $D \subseteq V(G)$ is a *dominating set* of G if, for every vertex $v \in V(G) \setminus D$, $N_G(v) \cap D \neq \emptyset$. Additionally, D is said to be *connected* if $G[D]$ is a connected subgraph of G. Given also a subset $X \subseteq G$ of *terminals*, a *Steiner tree* of G for X is a tree subgraph $\mathcal{T}$ of G such that $X \subseteq V(\mathcal{T})$. Next, we formally state the Steiner Tree and Connected Dominating Set problems. Although the usual question for Steiner Tree asks for the minimum tree, it is more convenient for our reduction to ask for the minimum set of non terminal vertices. Notice that this gives a polynomially equivalent problem.

Steiner Tree
Input: A connected graph G, a non-empty subset $X \subseteq V(G)$, and a positive integer k.
Question: Does there exist a subset $S \subseteq V(G) \setminus X$ with $|S| \leq k$, such that $G[S \cup X]$ is connected?

Connected Dominating Set
Input: A graph G and a positive integer k.
Question: Does there exist a subset $D \subseteq V(G)$ with $|D| \leq k$, such that $N_G[D] = V(G)$ and $G[D]$ is connected?

As we said before, we first prove that Connected Dominating Set is NP-complete for Undirected Path graphs. We do this with a reduction from the following problem, which is one of Karp's 21 NP-complete problems [70].

3D-Matching
Input: Disjoint sets P, Q and R each of cardinality n, for some positive integer n, and a subset $\mathcal{S} \subseteq P \times Q \times R$.
Question: Does there exist a subset $D \subseteq \mathcal{S}$ such that $|D| = n$ and $\mathbf{s} \cap \mathbf{s}' = \emptyset$ for every two triples $\mathbf{s}, \mathbf{s}' \in D$?

Theorem 4. Connected Dominating Set *remains* NP-*complete when restricted to undirected path graphs of diameter at most 3.*

Proof. Let $P = \{p_1, \ldots, p_n\}$, $Q = \{q_1, \ldots, q_n\}$ and $R = \{r_1, \ldots, r_n\}$ be disjoint sets each of cardinality n, for some positive integer n, and let $\mathcal{S} = \{\mathbf{s}_1, \ldots, \mathbf{s}_m\}$ be a subset of $P \times Q \times R$ of cardinality m, for some positive integer m. We let $I = (P, Q, R, \mathcal{S})$ be the instance of 3D-Matching constituted by P, Q, R and $\mathcal{S}$. Then, we let G be the graph obtained from I as follows (Fig. 3 shows a tree model of the constructed graph):

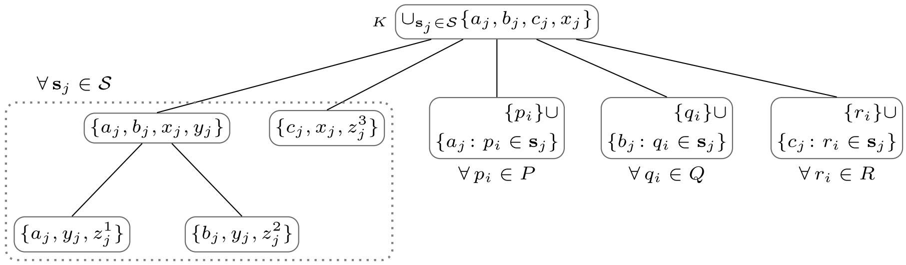

Fig. 3. A tree model $(T, \{T_u\}_{u \in V(G)})$ associated with the graph G, constructed from a given instance $I = (P, Q, R, \mathcal{S})$ of the 3D-Matching problem.

- For each $\mathbf{s}_j \in \mathcal{S}$, we let $V_j = \{a_j, b_j, c_j, x_j, y_j, z_j^1, z_j^2, z_j^3\}$. We remark that, for each two sets $\mathbf{s}_j, \mathbf{s}_\ell \in \mathcal{S}$, $V_j \cap V_\ell = \emptyset$ if and only if $j \neq \ell$;
- $V(G) = \cup_{j=1}^{m} V_j \cup P \cup Q \cup R$;
- $K = \bigcup_{\mathbf{s}_j \in \mathcal{S}} \{a_j, b_j, c_j, x_j\}$ is a clique of G;
- for each $\mathbf{s}_j \in \mathcal{S}$, $\{a_j, b_j, x_j, y_j\}$, $\{a_j, y_j, z_j^1\}$, $\{b_j, y_j, z_j^2\}$ and $\{c_j, x_j, z_j^3\}$ are cliques of G;
- for each $p_i \in P$, $\{p_i\} \cup \{a_j : p_i \in \mathbf{s}_j, \mathbf{s}_j \in \mathcal{S}\}$ is a clique of G;
- for each $q_i \in Q$, $\{q_i\} \cup \{b_j : q_i \in \mathbf{s}_j, \mathbf{s}_j \in \mathcal{S}\}$ is a clique of G;
- for each $r_i \in R$, $\{r_i\} \cup \{c_j : r_i \in \mathbf{s}_j, \mathbf{s}_j \in \mathcal{S}\}$ is a clique of G.

Fig. 3 illustrates a tree model $(T, \{T_u\}_{u \in V(G)})$ associated with the graph G. We depict inside a node $t \in V(T)$, the set of vertices of G that contains node t in its corresponding subtree; more formally, denoting by X_t the subset of $V(G)$ drawn inside node t, and given $v \in V(G)$, we define T_v as the subtree of T induced by $\{t \in V(T) : v \in X_t\}$. Based on $(T, \{T_u\}_{u \in V(G)})$, one can verify that G is an undirected path graph. Indeed, for each vertex $u \in V(G)$, we get that T_u is a path of T. Moreover, one can readily verify that K is a dominating clique of G. Therefore, G has diameter at most 3.

We now prove that I is a yes-instance of 3D-Matching if and only if G admits a connected dominating set D of size at most $2m + n$.

First, suppose that I is a yes-instance of 3D-Matching, and let M be a 3d-matching of I. Then, we define $D = \{a_j, b_j, c_j : \mathbf{s}_j \in M\} \cup \{x_j, y_j : \mathbf{s}_j \notin M\}$. Note that $|D| = 3n + 2(m - n) = 2m + n$. We claim that D is a connected dominating set of G. Indeed, since M is a 3d-matching of I, we have that the following holds:

- for each $p_i \in P$, there exists (exactly) one triple $\mathbf{s}_j \in M$ such that $p_i \in \mathbf{s}_j$, which implies that $a_j \in D$ and that p_i is dominated in G by a_j;
- for each $q_i \in Q$, there exists (exactly) one triple $\mathbf{s}_j \in M$ such that $q_i \in \mathbf{s}_j$, which implies that $b_j \in D$ and that q_i is dominated in G by b_j;
- for each $r_i \in R$, there exists (exactly) one triple $\mathbf{s}_j \in M$ such that $r_i \in \mathbf{s}_j$, which implies that $c_j \in D$ and that r_i is dominated in G by c_j.

Additionally, it directly follows from the construction of G that $\{a_j, b_j, c_j\}$ dominates all vertices belonging to V_j for each $\mathbf{s}_j \in M$, and that $\{x_\ell, y_\ell\}$ dominates all vertices belonging to V_ℓ for each $\mathbf{s}_\ell \in \mathcal{S} \setminus M$. Consequently, D is a dominating set of G. To verify that D induces a connected subgraph of G, first note that $K' = \bigcup_{\mathbf{s}_j \in M} \{a_j, b_j, c_j\}$ induces a connected subgraph of G since K is a clique of G and $K' \subseteq K$. In addition, it follows from the facts that $\bigcup_{\mathbf{s}_j \in \mathcal{S}} \{x_j\} \subseteq K$, and $x_j y_j \in E(G)$ for each $\mathbf{s}_j \in \mathcal{S}$, that the set $K'' = \bigcup_{\mathbf{s}_j \in \mathcal{S} \setminus M} \{x_j, y_j\}$ induces a connected subgraph of G. Therefore, the result follows since $D = K' \cup K''$ is a connected subgraph of G.

Conversely, suppose now that G admits a connected dominating set D of size at most $2m + n$. By the construction of G, for each $\mathbf{s}_j \in \mathcal{S}$, the following holds (see Fig. 4):

- $D \cap \{a_j, y_j, z_j^1\} \neq \emptyset$, otherwise z_j^1 would not be dominated in G by D;
- $D \cap \{b_j, y_j, z_j^2\} \neq \emptyset$, otherwise z_j^2 would not be dominated in G by D;
- $D \cap \{c_j, x_j, z_j^3\} \neq \emptyset$, otherwise z_j^3 would not be dominated in G by D.

We recall that $V_j = \{a_j, b_j, c_j, x_j, y_j, z_j^1, z_j^2, z_j^3\}$ for each $\mathbf{s}_j \in \mathcal{S}$.

Then, based on the above, one can verify that $|D \cap V_j| \geq 2$. Moreover, we prove that if $|D \cap V_j| = 2$ for some $\mathbf{s}_j \in \mathcal{S}$, then $D \cap V_j = \{x_j, y_j\}$. This follows from the fact that the only other possibilities for $D \cap V_j$ with $|D \cap V_j| = 2$ would be $D \cap V_j = \{c_j, y_j\}$ and $D \cap V_j = \{z_j^3, y_j\}$. However, note that, $\{a_j, b_j, x_j\}$ is a separator of c_j and y_j in G for each $\mathbf{s}_j \in \mathcal{S}$. Thus, if $D \cap V_j = \{c_j, y_j\}$, then D would not induce a connected subgraph of G. Analogously, $\{c_j, x_j\}$ is a separator of z_j^3 and y_j in G for each $\mathbf{s}_j \in \mathcal{S}$. Thus, if $D \cap V_j = \{z_j^3, y_j\}$, then D would not induce a connected subgraph of G. As a result, we have that $D \cap V_j = \{x_j, y_j\}$ whenever $|D \cap V_j| = 2$.

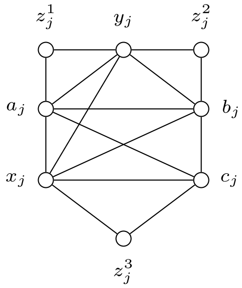

Fig. 4. Subgraph of G induced by V_j, for $\mathbf{s}_j \in \mathcal{S}$.

We prove now that we can assume that $D \cap V_j = \{a_j, b_j, c_j\}$ for each $\mathbf{s}_j \in \mathcal{S}$ with $|D \cap V_j| \geq 3$. Indeed, for each $\mathbf{s}_j \in \mathcal{S}$, $\{a_j, b_j, c_j\}$ dominates at least the same vertices of G as any other subset of V_j, which implies $N_G[D \cap V_j] \subseteq N_G[\{a_j, b_j, c_j\}]$. Moreover, since by hypothesis D induces a connected subgraph of G, we obtain that $(D \setminus V_j) \cup \{a_j, b_j, c_j\}$ also induces a connected subgraph of G for each $\mathbf{s}_j \in \mathcal{S}$. Thus, assume without loss of generality that $D \cap V_j = \{a_j, b_j, c_j\}$ whenever $|D \cap V_j| \geq 3$.

Now, let $M = \{\mathbf{s}_j \in \mathcal{S} : |D \cap V_j| = 3\}$. Based on the definition of M, we obtain that $|D| \geq 3|M| + 2(m - |M|) = 2m + |M|$. On the other hand, we have by hypothesis that $|D| \leq 2m + n$. Consequently, $|M| \leq n$. Towards a contradiction, suppose that $|M| < n$. Let $D' = \bigcup_{\mathbf{s}_j \in \mathcal{S}} (D \cap V_j)$. Note that, for each $U \in \{P, Q, R\}$, there are at most $|M|$ vertices from U that are dominated in G by some vertex in D'. As a result, there exist at least $3(n - |M|)$ distinct vertices from $P \cup Q \cup R$ that are not dominated in G by any vertex belonging to D', *i.e.* $|(P \cup Q \cup R) \setminus N_G[D']| \geq 3(n - |M|)$. This implies that D must further contain each of such vertices from $P \cup Q \cup R$ that are not dominated in G by D'. Then, we obtain that actually

$$|D| \geq 3|M| + 2(m - |M|) + 3(n - |M|) = 2m + n + 2(n - |M|) > 2m + n,$$

which contradicts the hypothesis of D being a connected dominating set of G of size at most $2m + n$. Consequently, $|M| = n$ and $|D| = 2m + n$. This implies that, if u is a vertex in D, then $u \in V_j$ for some $\mathbf{s}_j \in \mathcal{S}$. Hence, we obtain that M is a 3d-matching of I, otherwise there would exist a vertex $v \in P \cup Q \cup R$ such that, for every triple $\mathbf{s}_j \in \mathcal{S}$ with $v \in \mathbf{s}_j$, $|D \cap V_j| = 2$, which would imply that v is not dominated in G by any vertex belonging to D. Therefore, I is a `yes`-instance of 3D-Matching. □

As previously mentioned, in [117], the authors give a reduction from Connected Dominating Set to Steiner Tree that works in any subclass of Chordal graphs without changing the input graph. We then get the following corollary.

Corollary 5. Steiner Tree *is* NP*-complete when restricted to undirected path graphs of diameter at most 3.*

In [79], the author proves that deciding whether an undirected path graph has a dominating clique of size at most k can be done in polynomial time. We apply his result to get a dichotomy for Steiner Tree restricted to Undirected Path graphs in terms of the diameter of the input graph. For this, we first need some definitions and tool lemmas, which are presented below.

Let G be a connected graph and $X \subseteq V(G)$ be a non-empty set. We denote by $ST(G, X)$ the minimum size of a subset $S \subseteq V(G) \setminus X$ such that $S \cup X$ induces a connected subgraph of G. Throughout the remainder of this section, we assume without loss of generality that $|X| \geq 3$ and that X does not induce a connected subgraph of G, as otherwise $ST(G, X)$ would be easily determined: if $|X| = 1$ or $G[X]$ is connected, then trivially $ST(G, X) = 0$; and, if $X = \{u, v\}$, then $ST(G, X)$ is equal to the number of vertices in any minimum path between u and v in G.

We say that two distinct vertices $u, v \in V(G)$ are *twins* in G if they have the same neighborhood in G, *i.e.* either $N_G(u) = N_G(v)$ or $N_G[u] = N_G[v]$. We prove in the next lemma that we can suppose without loss of generality that G has no twins. Observe that the hypothesis involving u and v can be assumed without loss of generality. It is included in order to write the equation in a more concise way.

Lemma 6. *Let G be a connected graph, containing twin vertices u and v, and $X \subseteq V(G)$ with $|X| \geq 3$. Also, suppose that $u \in X$ implies $v \in X$. Then,*

$$ST(G, X) = ST(G - u, X - u).$$

Proof. First, let $S \subseteq V(G - u) \setminus X$ be a minimum set such that $S \cup (X - u)$ induces a connected subgraph of $G - u$. We want to prove that $S \cup X$ induces a connected subgraph of G, in which case we get $ST(G, X) \leq ST(G - u, X - u)$. Suppose first that $v \in S$. Since $X \setminus \{u\} \neq \emptyset$, v must have some neighbor $w \in S \cup X$ in $G - u$. Then, it follows from the hypothesis that u and v are twins in G that w is also a neighbor of u in G. This implies that $S \cup X$ induces a connected subgraph of G. A similar argument can be applied when $v \in X$. Indeed, since $|X| \geq 3$, $X \setminus \{u, v\} \neq \emptyset$. Thus, v must have some neighbor

$w \in S \cup (X - u)$, which is also a neighbor of u in G, and consequently $S \cup X$ induces a connected subgraph of G. Finally, if $v \notin S \cup X$, then by hypothesis we also know that $u \notin X$, in which case trivially $S \cup X$ induces a connected subgraph of G.

Now, let $S \subseteq V(G) \setminus X$ be a minimum set such that $S \cup X$ induces a connected subgraph of G. Since S is minimum, $|S \cap \{u, v\}| \leq 1$. If $u \in S$, then, by the minimality of S, $v \notin S$ and $(S \setminus \{u\}) \cup \{v\}$ witnesses $ST(G - u, X - u) \leq ST(G, X)$. On the other hand, if $u \notin S$, then we trivially get that $S \cup (X - u)$ induces a connected subgraph of $G - u$, and therefore $ST(G - u, X - u) \leq ST(G, X)$. □

Based on Lemma 6, we assume from now on that the input graph G has no twin vertices. Also, in the remainder of the text, $(T, \{T_u\}_{u\in V(G)})$ is a tree model of G. Moreover, given a node $t \in V(T)$, we denote by V_t the set $\{u \in V(G): t \in V(T_u)\}$. We say that $u \in V(G)$ is a *leafy vertex* if $V(T_u) = \{\ell_u\}$ and ℓ_u is a leaf in T; denote by $\mathcal{L}$ the set of leafy vertices, and for every $u \in \mathcal{L}$, denote by ℓ_u the unique node in T_u. We also say that $(T, \{T_u\}_{u\in V(G)})$ is *minimal* if there are no two adjacent nodes $t, t' \in V(T)$ such that $V_t \subseteq V_{t'}$. It is known that such a tree model can be computed in polynomial time [56]. We prove in the following lemma that, for any minimal tree model $(T, \{T_u\}_{u\in V(G)})$, there is a one-to-one correspondence between the leaves of T and the leafy vertices associated with T.

Lemma 7. *Let G be a connected undirected path graph without twin vertices, and $(T, \{T_u\}_{u\in V(G)})$ be a minimal tree model for G. Then, for every leaf ℓ of T, there exists a unique $u \in \mathcal{L}$ such that $\mathcal{L} \cap V_\ell = \{u\}$.*

Proof. Since G has no twin vertices, for every leaf t of T, there exists at most one leafy vertex u of G associated with T such that $\ell_u = t$. On the other hand, suppose that there exists a leaf t of T such that there is no leafy vertex of G associated with T corresponding to t, i.e., for every leafy vertex u of G associated with T, we have that $\ell_u \neq t$. Then, let t' be the parent of t in T. One can readily verify that $V_t \subseteq V_{t'}$, contradicting the fact that we are on a minimal tree model. □

A vertex u is called *simplicial* if $N_G(u)$ is a clique in G. Note that every leafy vertex is simplicial. Moreover, note that a simplicial vertex that is not in X certainly is not contained in any minimum Steiner tree for X; therefore we can suppose that X contains every simplicial vertex of G and, in particular, $\mathcal{L} \subseteq X$. In the next lemma, we prove that we can suppose that every $x \in X$ is either leafy, or is such that T_x contains no leaf of T.

Lemma 8. *Let G be a connected undirected path graph without twin vertices. Also, let $(T, \{T_u\}_{u\in V(G)})$ be a minimal tree model of G, $\mathcal{L}$ be the set of leafy vertices associated with T, and let $X \subseteq V(G)$ be a set of terminals such that $\mathcal{L} \subseteq X \subseteq V(G)$. Suppose that $x \in X \setminus \mathcal{L}$ is such that $\ell \in V(T_x)$ for some leaf ℓ of T. Then, $ST(G, X) = ST(G - u, X - u)$, where $\mathcal{L} \cap V_\ell = \{u\}$.*

Proof. By Lemma 7, $G - u$ is the graph related to the tree model $T - \ell$. Suppose that there exists a set $S \subseteq V(G - u) \setminus X$ such that $S \cup (X - u)$ induces a connected subgraph of $G - u$. Since $\ell \in T_u \cap T_x$, $ux \in E(G)$. Thus, $S \cup X$ induces a connected subgraph of G, and consequently $ST(G, X) \leq ST(G - u, X - u)$. Conversely, suppose that there exists a set $S \subseteq V(G) \setminus X$ such that $S \cup X$ induces a connected subgraph of G. Since u is a simplicial vertex of G, we obtain that $S \cup (X - u)$ induces a connected subgraph of $G - u$, and therefore $ST(G, X) \geq ST(G - u, X - u)$. □

In the proof, we modify a subset S that gives a solution in order to ensure that the new set is a clique. The following lemma will help us do that.

Lemma 9. *Let G be a connected undirected path graph, $(T, \{T_u\}_{u\in V(G)})$ be a tree model of G, $X \subseteq V(G)$ be a set of terminals, $S \subseteq V(G)\setminus X$ be a set such that $G[S\cup X]$ is connected, and let $u, v \in S$. If $y, z \in V(G)$ are such that $N_G(u)\cup N_G(v) \subseteq N_G(y)\cup N_G(z)$, then $S' \cup X$ is connected, where $S' = ((S \setminus \{u, v\}) \cup \{y, z\}) \setminus X$.*

Proof. Suppose otherwise, and let H, H' be distinct components of $G[S' \cup X]$. This means that there exist $w \in V(H)$ and $w' \in V(H')$ such that every path P between w and w' goes through u and/or v; but since $N_G(u) \cup N_G(v) \subseteq N_G(y) \cup N_G(z)$, it means that u and/or v can be replaced by y and/or z. □

We are now ready to prove our theorem. In [79], the author proves that deciding whether an undirected path graph has a dominating clique of size at most k can be done in polynomial time. We make a polynomial reduction from Connected Dominating Set to Dominating Clique, thus getting the desired polynomial algorithm by the equivalence given in [117]. This and Theorem 4 give a dichotomy of both Connected Dominating Set and Steiner Tree in terms of the diameter of the input graph G. We observe that, in order to prove that Dominating Clique is polynomial-time solvable for Undirected Path graphs, it is used in [79] an alternative notion of tree model called *characteristic tree*, where the nodes of the model are the maximal cliques of the graph G (see [56,91]).

Theorem 10. Steiner Tree *and* Connected Dominating Set *can be solved in polynomial time when restricted to undirected path graphs of diameter at most 2.*

Proof. In view of the reduction from Connected Dominating Set to Steiner Tree presented in [117], it suffices to prove that Steiner Tree can be solved in polynomial time. Thus, let G be a connected undirected path graph with diameter at most 2, $X \subseteq V(G)$ be a terminal set such that $|X| \geq 3$, and let κ be a positive integer.

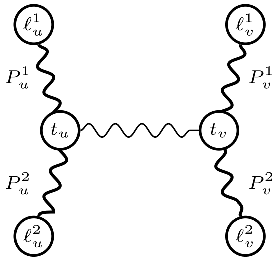

Fig. 5. The bold lines represent the paths T_u and T_v.

As usual, we consider a minimal tree model $(T, \{T_u\}_{u\in V(G)})$ of G. There is no loss of generality making these assumptions, since, as previously mentioned, it is known that such a tree model can be computed in polynomial time [56]. Let $\mathcal{L}$ denote the set of leafy vertices associated with T, and assume that $\mathcal{L} \subseteq X$.

In what follows, we prove that: $ST(G, X) \leq \kappa$ if and only if there exists a clique $S \subseteq V(G) \setminus X$ in G of size at most κ that dominates $\mathcal{L}$. Our theorem follows since this is exactly what is computed in the algorithm presented in [79].

For the sufficiency part of our claim, we just note that if S is a clique of G that dominates $\mathcal{L}$, then $\bigcup_{u\in S} V(T_u) = V(T)$, which means that in fact S is a dominating clique of G and therefore $S \cup X$ induces a connected subgraph of G. Because $|S| \leq \kappa$, it follows that $ST(G, X) \leq \kappa$.

Now, to prove necessity, suppose first that $ST(G, X) \leq \kappa$, and let $S \subseteq V(G)\setminus X$ be a set such that $S\cup X$ induces a connected subgraph of G. Suppose that S is minimum and that, among all such subsets of minimum cardinality, S maximizes the number of edges in $E(G[S])$. In addition, by Lemma 8, we can suppose that there are no edges in G between the vertices belonging to $X \setminus \mathcal{L}$ and the vertices belonging to $\mathcal{L}$. Moreover, note that $\mathcal{L}$ is an independent set. Thus, since $G[S \cup X]$ is connected, we get that every vertex in $\mathcal{L}$ must be adjacent to some vertex in S; in other words, S dominates $\mathcal{L}$. Thus, it remains to prove that S is a clique of G. Suppose for the sake of contradiction that there exist two distinct vertices $u, v \in S$ such that $uv \notin E(G)$. We prove that one can find a pair x, y of adjacent vertices in G such that $N_G(u)\cup N_G(v) \subseteq N_G(x)\cup N_G(z)$. Then, based on Lemma 9, by letting $S' = ((S\setminus\{u, v\})\cup\{y, z\})\setminus X$, we obtain a contradiction, since in this case either $|S'| < |S|$, or $|E(G[S'])| > |E(G[S])|$.

Let $t_u \in V(T_u)$ and $t_v \in V(T_v)$ be nodes whose distance from each other in T is the smallest possible (observe Fig. 5). Note that $t_u \neq t_v$ since $V(T_u) \cap V(T_v) = \emptyset$. Also, let P_u^1, P_u^2 be the two subpaths defined by t_u in T_u, and define P_v^1, P_v^2 similarly. For each $i \in \{1, 2\}$, let ℓ_u^i be the end vertex of P_u^i different from t_u, if it exists; otherwise, let ℓ_u^i be equal to t_u. Define ℓ_v^1, ℓ_v^2 similarly. Note that, if $\ell_u^1 \neq t_u$, then we can suppose that there exist $w_u^1 \in V(G)$ such that $\ell_u^1 \in T_{w_u^1}$, and $t \notin T_{w_u^1}$, where t is the neighbor of ℓ_u^1 in P_u^1 (otherwise, we could contract the edge $\ell_u^1 t$ and still have a tree model of G). Define w_u^2, w_v^1, w_v^2 similarly. There are two possible cases to be considered.

Case 1. Suppose that all the vertices $w_u^1, w_u^2, w_v^1, w_v^2$ exist and are well-defined. Since G has diameter at most 2, there must exist vertices $y \in N_G(w_u^1) \cap N_G(w_v^1)$ and $z \in N_G(w_u^2) \cap N_G(w_v^2)$. Clearly, the path between t_u and t_v in T is contained in the paths T_y and T_z (which means that y and z are adjacent in G), and $V(P_u^1 \cup P_v^1) \subseteq V(T_y)$, and $V(P_u^2 \cup P_v^2) \subseteq V(T_z)$. Thus, $N_G(u) \cup N_G(v) \subseteq N_G(y) \cup N_G(z)$, as desired.

Case 2. Now, suppose that some of the vertices $w_u^1, w_u^2, w_v^1, w_v^2$ do not exist or are not well-defined. Note that, since $S\cup X$ induces a connected subgraph of G, there must exist a path in $G[S \cup X]$ between u and v, which means that there must exist $w \in (S \cup X) \setminus \{u\}$ such that $t_u \in V(T_w)$. This implies that at least one of the vertices w_u^1, w_u^2 is well-defined, as otherwise $V(T_u) = \{t_u\} \subseteq V(T_w)$ and we could just remove u from S. The same argument can be applied with respect to w_v^1, w_v^2. Thus, suppose without loss of generality that w_u^1, w_v^1 are well-defined, and that w_u^2 is not well-defined (which means that $V(P_u^2) = \{t_u\}$). Pick y as before, and note that $V(T_u) \subseteq V(T_y)$, and that $yv \in E(G)$ since $\ell_v^1 \in V(T_y)\cap V(T_v)$. We can then apply Lemma 9 to $\{u, v\}$ and $\{y, v\}$ since $N_G(u)\cup N_G(v) \subseteq N_G(y)\cup N_G(v)$. □

5. Stubborn puzzles 35 years later

After 40 years, two open problems from [GJ] are still unsolved, namely: Open1 Graph Isomorphism, and Open8 Precedence constrained 3-processor schedule. In STOC 2016, László Babai announced that Graph Isomorphism could be solved in Quasipolynomial Time. Only one O entry from [OG] remains stubbornly open for 35 years: the complexity of Chromatic Index for Planar graphs. It is a puzzle to understand why still today the Chromatic Index column has the majority of thirteen O? entries, for instance Chromatic Index for Cographs is a long-standing open problem, as mentioned in [75]. We invite the reader to find a reference or a proof for the underlined [OG] entry in Table 1 corresponding to a "private communication" which would classify as polynomial Steiner Tree restricted to Circle graphs. Our proposed Table 2 leaves as open the parameterized complexity classification of Partition into Cliques for Line graphs. We invite the reader to further study the eight XP entries, observing that five of them belong to our target Steiner Tree column. In particular, we highlight that, even though the closely related problem Connected Dominating Set is known to be

FPT for Claw-Free graphs [61], it is open whether κ-Steiner Tree is also FPT for Line and Claw-Free graphs. Regarding the obtained second dichotomy for the Steiner Tree problem restricted to Undirected Path graphs, according to the diameter of the input graph, we should mention that Connected Dominating Set was proven to be NP-complete and W[2]-hard even when restricted to Split graphs of diameter 2 [83]. A straightforward modification of their proof leads to the NP-completeness of Steiner Tree (and to the W[2]-hardness of κ-Steiner Tree) when restricted to Split graphs of diameter 2.

Acknowledgments

We would like to thank anonymous reviewers for their thorough reading and numerous insightful suggestions and comments; in particular, we would like to thank one of the reviewers for the references related to mim-width. We are grateful to Vinicius F. Santos for sharing Ref. [1], for the NP-completeness proof of MaxCut for Interval graphs. This work was partially supported by the Brazilian agencies CNPq, Brazil (Grant Nos.: 407635/2018-1, 140399/2017-8, 407430/2016-4, 437841/2018-9, 401519/2016-3, and 304576/2017-4), CAPES, Brazil (Finance Code 001), FAPERJ, Brazil (Grant Nos.: CNE E-26/202.793/2017), and CNPq/FUNCAP, Brazil (PNE-0112-00061.01.00/16).

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