Inverse Trigonometry for θ in Terms of x

1 Problem Statement

Given the system of equations:

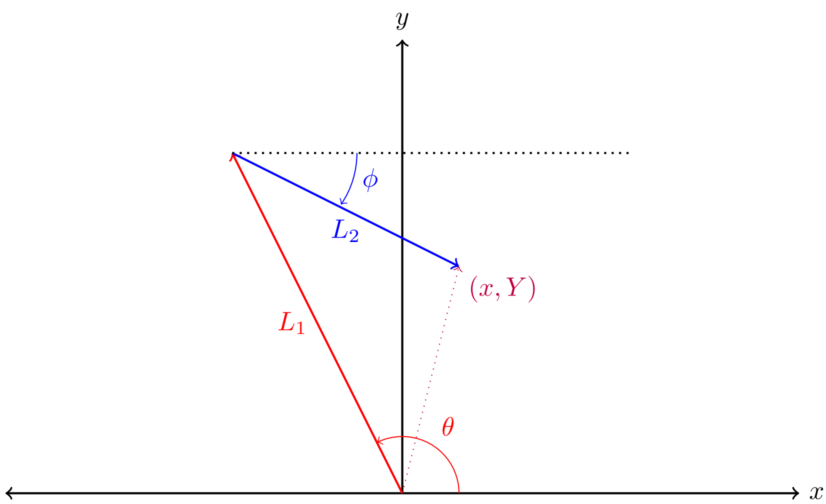
$$L_1 \sin\theta - L_2 \sin\phi = Y \text{ (constant)} \tag{1}$$
$$L_1 \cos\theta + L_2 \cos\phi = x \text{ (variable)} \tag{2}$$

we aim to solve for θ in terms of x, Y, L_1, and L_2.

1.1 Explanation of Variables

- L_1: Length of the first arm.
- L_2: Length of the second arm.
- x: Horizontal distance from origin of first arm (variable). Location of carriage.
- Y: Vertical distance from origin of first arm (constant).
- θ: Angle of the first arm with respect to its the horizontal axis. Controlled with a Servo.
- ϕ: Angle of the second arm with respect to its the horizontal axis.

1.2 Diagram of the Variables



2 Isolating $\sin\phi$ and $\cos\phi$

From equation (1):

$$L_2 \sin\phi = L_1 \sin\theta - Y \tag{3}$$
$$\sin\phi = \frac{L_1 \sin\theta - Y}{L_2} \tag{4}$$

From equation (2):

$$L_2 \cos\phi = x - L_1 \cos\theta \tag{5}$$
$$\cos\phi = \frac{x - L_1 \cos\theta}{L_2} \tag{6}$$

3 Using the Pythagorean Identity to Remove ϕ

Using the identity $\sin^2\phi + \cos^2\phi = 1$:

$$\sin^2\phi + \cos^2\phi = 1 \tag{7}$$

$$\left(\frac{L_1\sin\theta - Y}{L_2}\right)^2 + \left(\frac{x - L_1\cos\theta}{L_2}\right)^2 = 1 \tag{8}$$

Multiplying through by L_2^2:

$$(L_1\sin\theta - Y)^2 + (x - L_1\cos\theta)^2 = L_2^2 \tag{9}$$

Expanding and simplifying:

$$\underline{L_1^2\sin^2\theta} - 2L_1Y\sin\theta + Y^2 + x^2 - 2xL_1\cos\theta + \underline{L_1^2\cos^2\theta} = L_2^2 \tag{10}$$

$$\underline{L_1^2(\sin^2\theta + \cos^2\theta)} - 2L_1Y\sin\theta - 2xL_1\cos\theta + x^2 + Y^2 = L_2^2 \tag{11}$$

Using $\sin^2\theta + \cos^2\theta = 1$:

$$L_1^2 - 2L_1Y\sin\theta - 2xL_1\cos\theta + x^2 + Y^2 = L_2^2 \tag{12}$$

4 Isolating θ

Rearranging terms to isolate θ:

$$-2L_1Y\sin\theta - 2xL_1\cos\theta = L_2^2 - L_1^2 - x^2 - Y^2 \tag{13}$$

Dividing by $-2L_1$:

$$Y\sin\theta + x\cos\theta = \frac{L_1^2 + x^2 + Y^2 - L_2^2}{2L_1} \tag{14}$$

Let:

$$k = \frac{L_1^2 + x^2 + Y^2 - L_2^2}{2L_1} \tag{15}$$

Aside:

5 Trigonometric Identity Proof

Let:

$$< B, A > \tag{16}$$

$$R = \sqrt{A^2 + B^2} \tag{17}$$

$$\alpha = \arctan\left(\frac{A}{B}\right) \tag{18}$$

Solve for A and B in terms of α

$$A = R\sin\alpha \tag{19}$$

$$B = R\cos\alpha \tag{20}$$

Using $A\sin(\theta) + B\cos(\theta)$

$$A\sin(\theta) + B\cos(\theta) = [R\sin(\alpha)]\sin(\theta) + [R\cos(\alpha)]\cos(\theta) \tag{21}$$

$$= R[\sin(\alpha)\sin(\theta) + \cos(\alpha)\cos(\theta)] \tag{22}$$

$$= R[\sin(\theta)\sin(\alpha) + \cos(\theta)\cos(\alpha)] \tag{23}$$

$$A\sin(\theta) + B\cos(\theta) = R\cos(\theta - \alpha) \tag{24}$$

Substitute R and α

$$A\sin(\theta) + B\cos(\theta) = \sqrt{A^2 + B^2}\cos\left(\theta - \arctan\frac{A}{B}\right) \tag{25}$$

6 Using Trigonometric Identities

We recognize the equation $Y\sin\theta + x\cos\theta = k$ as a linear combination of $\sin\theta$ and $\cos\theta$. We convert to polar form:

$$A\sin\theta + B\cos\theta = \sqrt{A^2 + B^2}\cos\left(\theta - \arctan\left(\frac{A}{B}\right)\right) \tag{26}$$

where $A = Y$ and $B = x$.

7 Solving for θ

Let:

$$R = \sqrt{x^2 + Y^2} \tag{27}$$

$$\alpha = \arctan\left(\frac{Y}{x}\right) \tag{28}$$

Then:

$$Y\sin\theta + x\cos\theta = R\cos(\theta - \alpha) \tag{29}$$

So:

$$R\cos(\theta - \alpha) = k \tag{30}$$

Solving for θ:

$$\cos(\theta - \alpha) = \frac{k}{R} \tag{31}$$

Taking the inverse cosine:

$$\theta - \alpha = \arccos\left(\frac{k}{R}\right) \tag{32}$$

Thus:

$$\theta = \arccos\left(\frac{k}{R}\right) + \alpha \tag{33}$$

Substitute back k, R, and α:

$$\theta = \arccos\left(\frac{\frac{L_1^2 + x^2 + Y^2 - L_2^2}{2L_1}}{\sqrt{x^2 + Y^2}}\right) + \arctan\left(\frac{Y}{x}\right) \tag{34}$$

8 Conclusion

Plugging the equation into Desmos (click to see graph) and substituting in the constants L_1, L_2, and Y draws a curve showing us what angle we need for the carriage to be at a certain position.

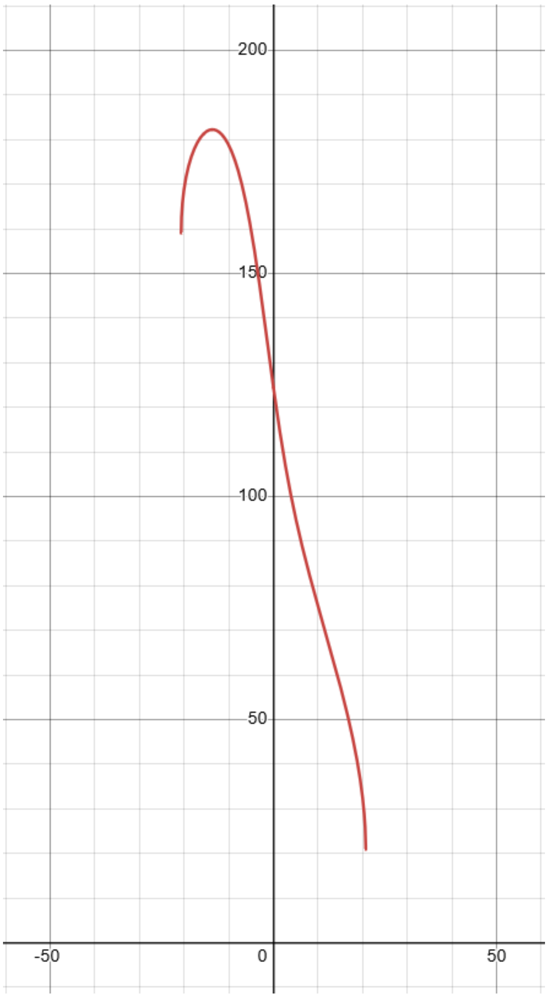


Figure 1: $\theta(x)$

Note that due to range of arctan $[\pi/2, -\pi/2]$ we must add π to the formula when x is negative to translate arctan's output to the 2nd quadrant in order for the curve to be continuous.

Desmos Equations

- $y = 57.2958 \left(\arccos \left(\frac{L_1^2 + x^2 + H^2 - L_2^2}{2L_1\sqrt{x^2+H^2}} \right) + \arctan\left(\frac{H}{x}\right) \right) \{x \geq 0\}$
  - Multiplying by 57.2958 to convert from radians to degrees
- $y = 57.2958 \left(\arccos \left(\frac{L_1^2 + x^2 + H^2 - L_2^2}{2L_1\sqrt{x^2+H^2}} \right) + \arctan\left(\frac{H}{x}\right) \right) + 180 \{x < 0\}$
  - Adding 180 (π) To ensure curve is continuous
- $L_1 = 13.685$ cm
- $L_2 = 8.447$ cm
- $H = 7.904$ cm
  - Changed Y to H