1. Consider the HMM of Homework #2 again, with the parameter matrices $p(s'|s)$, $q(s'|s)$, $q(\mathbf{0}|s \to s')$ and $q(\mathbf{1}|s \to s')$ as previously specified.

   Perform the following calculations *by hand*, again retaining adequate numerical precision of the intermediate answers.

   (a) Draw a 4-stage trellis for this HMM, showing *only* the paths which could have resulted in the output **0110**. (You may copy your solution from Homework #2.)

$$A = \begin{bmatrix} p(1|1) & p(2|1) & p(3|1) \\ p(1|2) & p(2|2) & p(3|2) \\ p(1|3) & p(2|3) & p(3|3) \end{bmatrix} \quad p(s'|s) = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{1}{3} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} q(1|1) & q(2|1) & q(3|1) \\ q(1|2) & q(2|2) & q(3|2) \\ q(1|3) & q(2|3) & q(3|3) \end{bmatrix} \quad q(s'|s) = \begin{bmatrix} 0 & 0 & \frac{1}{6} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$B(0) = \begin{bmatrix} q(Y=0|1,1) & q(Y=0|2,1) & q(Y=0|3,1) \\ q(Y=0|1,2) & q(Y=0|2,2) & q(Y=0|3,2) \\ q(Y=0|1,3) & q(Y=0|2,3) & q(Y=0|3,3) \end{bmatrix} \quad q(\mathbf{0}|s \to s') = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$B(1) = \begin{bmatrix} q(Y=1|1,1) & q(Y=1|2,1) & q(Y=1|3,1) \\ q(Y=1|1,2) & q(Y=1|2,2) & q(Y=1|3,2) \\ q(Y=1|1,3) & q(Y=1|2,3) & q(Y=1|3,3) \end{bmatrix} \quad q(\mathbf{1}|s \to s') = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 1 & \frac{2}{3} \\ 1 & 1 & 1 \end{bmatrix}$$

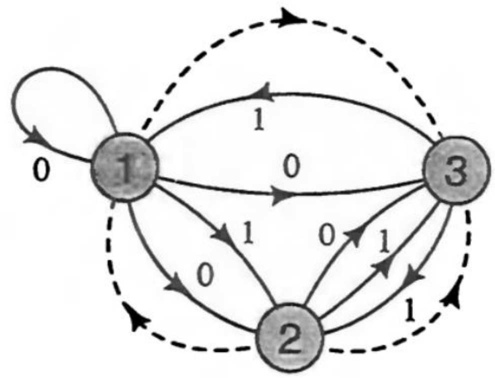


Figure 2.8
Addition of null transitions to the HMM of figure 2.3

(b) Calculate the *a posteriori* probabilities $P(t^i = t \,|\, y_1y_2y_3y_4 = \mathbf{0110}, s_0 = 1)$ for each arc in the trellis. Show your answers on the trellis.

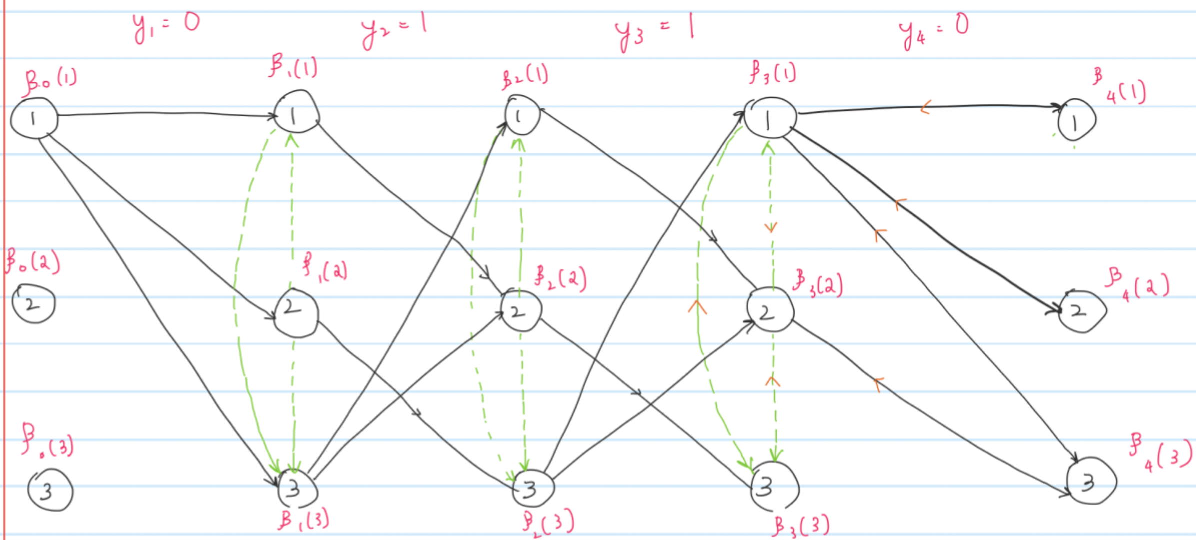
from previous HW,

```
ALPHA 0
 [[1 0 0]]
ALPHA 1
 [[0.52777778]
 [0.08333333]
 [0.28240741]]
ALPHA 2
 [[0.25      ]
 [0.11458333]
 [0.09837963]]
ALPHA 3
 [[0.08892747]
 [0.04542824]
 [0.05542695]]
ALPHA 4
 [[0.04446373]
 [0.00741062]
 [0.01986883]]
```

Marginal probability

```
[0.07174318]
```

Now I have to find β, which is Backward Algorithm



$$\beta_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Now calculate β_3

$$\beta_3 = \begin{pmatrix} \beta_4(1)\, p(1|1)\, q(0|1\to 1) + \beta_4(2)\, p(2|1)\, q(0|1\to 2) + \beta_4(3)\, p(3|1)\, q(0|1\to 3) + \beta_1(3)\, \gamma(3|1) \\ \beta_4(3)\, p(3|2)\, q(0|2\to 3) + \beta_3(1)\, \gamma(1|2) + \beta_3(3)\, \gamma(3|2) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{6} + 0 \\ \frac{1}{3} \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{1}{3} + 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{4} \\ \frac{13}{36} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.75 \\ 0.361 \\ 0 \end{pmatrix}$$

$$\beta_2 = \begin{pmatrix} \beta_3(2)\, p(2|1)\, q(1|1\to 2) + \beta_2(2)\, r(3|1) \\ \beta_2(1)\, r(1|2) + \beta_2(3)\, r(3|2) + \beta_3(3)\, p(3|2)\, q(1|2\to 3) \\ \beta_3(1)\, p(1|3)\, q(1|3\to 1) + \beta_3(2)\, p(2|3)\, q(1|3\to 2) \end{pmatrix}$$

$$= \begin{pmatrix} 13/36 \cdot 1/6 \cdot 1/2 + 47/72 \cdot 1/6 \\ 5/36 \cdot 1/3 + 47/72 \cdot 1/3 + 0 \\ 3/4 \cdot 3/4 + 1/4 \cdot 13/36 \end{pmatrix}$$

$$= \begin{pmatrix} 5/36 \\ 19/72 \\ 47/72 \end{pmatrix} = \begin{pmatrix} 0.1388 \\ 0.2638 \\ 0.6528 \end{pmatrix}$$

$$\beta_1 = \begin{pmatrix} \beta_2(2)\, p(2|1)\, q(1|1\to 2) + \beta_1(3)\, r(3|1) \\ \beta_1(1)\, r(1|2) + \beta_1(3)\, r(3|2) + \beta_2(3)\, p(3|2)\, q(1|2\to 3) \\ \beta_2(1)\, p(1|3)\, q(1|3\to 1) + \beta_2(2)\, p(2|3)\, q(1|3\to 2) \end{pmatrix}$$

$$= \begin{pmatrix} 13/72 \cdot 1/2 \cdot 1/6 + 1/6 \cdot 49/288 \\ 85/1728 \cdot 1/3 + 49/288 \cdot 1/3 + 47/72 \cdot 1/3 \cdot 2/3 \\ 5/36 \cdot 3/4 + 19/72 \cdot 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} 85/1728 \\ 1133/5184 \\ 49/288 \end{pmatrix} \approx \begin{pmatrix} 0.049199 \\ 0.218558 \\ 0.171 \end{pmatrix}$$

$$\beta_0 = \begin{pmatrix} \beta_1(1)\,p(1|1)\,q(0|1\to 1) + \beta_1(2)\,p(1|2)\,q(0|2\to 1) + \beta_1(3)\,p(1|3)\,q(0|3\to 1) \\ 0 \\ 0 \end{pmatrix}$$

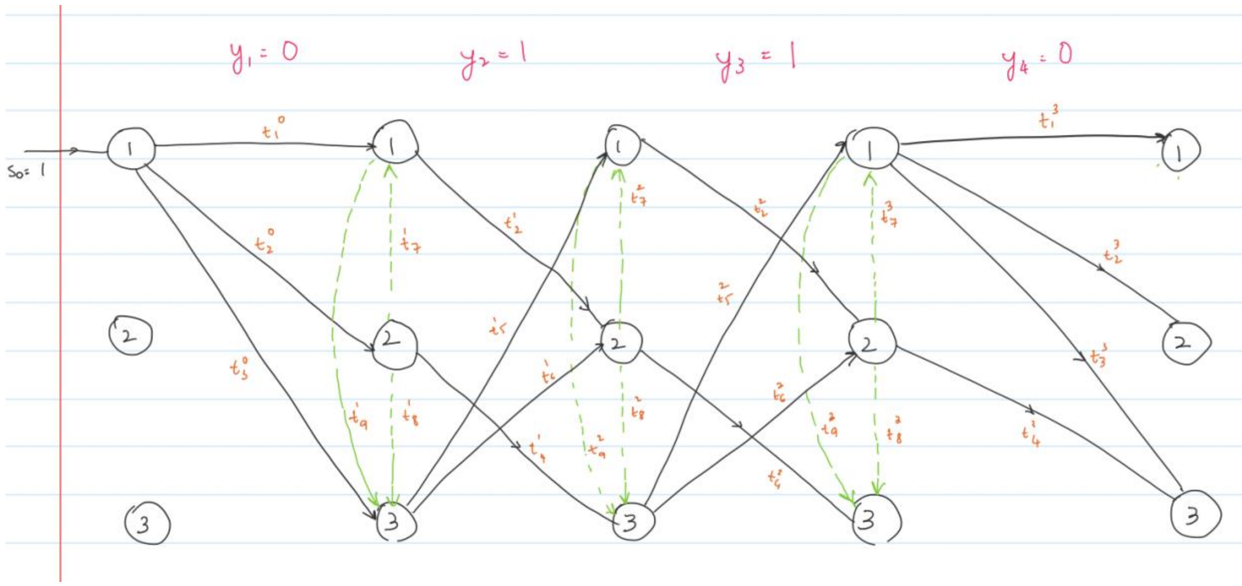
$$= \begin{pmatrix} \frac{29}{576} \cdot \frac{1}{2} + \frac{1133}{5184} \cdot \frac{1}{6} \cdot \frac{1}{2} + \frac{49}{288} \cdot \frac{1}{6} \cdot 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4463/62208 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.7174 \\ 0 \\ 0 \end{pmatrix}$$

posterior probability :-

Marginal prob = 0.07174318



$y_1 = 0$
$y_2 = 1$
$y_3 = 1$
$y_4 = 0$
$s_0 = 1$
t_1^0
t_2^0
t_3^0
t_2^1
t_5^1
t_6^1
t_4^1
t_7^1
t_8^1
t_9^1
t_2^2
t_5^2
t_6^2
t_4^2
t_7^2
t_8^2
t_9^2
t_1^3
t_2^3
t_3^3
t_4^3
t_7^3

formula

$$p(t_1^0 \mid y_1^N) = \frac{\alpha_0(1)\, p(1 \mid 1)\, q(0 \mid 1 \to 1)\, \beta_1(1)}{marg}$$

$$= \frac{1 \times 1/2 \times 1 \times 29/576}{0.07174}$$

$$= 0.35$$

$$p(t_2^0 \mid y_1^N) = \frac{\alpha_0(1)\, P(2 \mid 1)\, q(0 \mid 1 \to 2)\, \beta_1(2)}{marg}$$

$$= \frac{1 \cdot 1/6 \cdot 1/2 \cdot 1133/5184}{0.07174}$$

$$= 0.2539$$

$$p(t_3^0 \mid y_1^N) = \frac{\alpha_0(1)\, P(3 \mid 1)\, q(0 \mid 1 \to 3)\, \beta_1(3)}{marg}$$

$$= \frac{1/6 \cdot 49/288}{0.07174}$$

$$= 0.4$$

$$p(t_2^1 \mid y_1^N) = \frac{\alpha_1(1)\, P(2 \mid 1)\, q(1 \mid 1 \to 2)\, \beta_2(2)}{marg}$$

$$= \frac{0.5277 \times \frac{1}{6} \cdot \frac{1}{3} \cdot \frac{19}{72}}{0.07174}$$

$$= 0.1618$$

$$P(t_4^1 \mid y_1^N) = \frac{\alpha_1(2)\, p(3 \mid 2)\, q(1 \mid 2 \to 3)\, \beta_2(3)}{marg}$$

$$= 0.16$$

$$P(t_5^1 \mid y_1^N) = \frac{\alpha_1(3)\, p(1 \mid 3)\, q(1 \mid 3 \to 1)\, \beta_2(1)}{marg}$$

$$= 0.410$$

$$P(t_6^1 \mid y_1^N) = \frac{\alpha_1(3)\, p(2 \mid 3)\, q(1 \mid 3 \to 2)\, \beta_2(2)}{marg}$$

$$= 0.250$$

$$P(t_7^1 \mid y_1^N) = \frac{\alpha_1(2)\, r(1 \mid 2)\, \beta_1(1)}{marg}$$

$$= 0.0194$$

$$P(t_8^1 \mid y_1^N) = \frac{\alpha_1(2)\, r(3 \mid 2)\, \beta_1(1)}{marg}$$

$$= 0.06587$$

$$P(t_9^1 \mid y, w) = \frac{\alpha_1(1)\ r(3 \mid 1)\ \beta_1(3)}{\text{marg}}$$

$$= 0.20864$$

$$P(t_2^2 \mid y, w) = \frac{\alpha_2(1)\ p(2 \mid 1)\ q(1 \mid 1 \to 2)\ \beta_3(2)}{\text{marg}}$$

$$= 0.1048$$

$$P(t_4^2 \mid y, w) = 0$$

$$P(t_5^2 \mid y, w) = \frac{\alpha_2(3)\ P(1 \mid 3)\ q(1 \mid 3 \to 1)\ \beta_2(1)}{\text{marg}}$$

$$= 0.771$$

$$P(t_6^2 \mid y, w) = \frac{\alpha_2(3)\ p(2 \mid 3)\ q(1 \mid 3 \to 2)\ \beta_3(2)}{\text{marg}}$$

$$= 0.12$$

$$P(t_7^2 \mid y, w) = \frac{\alpha_2(2)\ r(1 \mid 2)\ \beta_2(1)}{\text{marg}}$$

$$= 0.14$$

$$p(t_8^2 \mid y_1^N) = \frac{\alpha_2(2)\, r(3 \mid 2)\, \beta_2(3)}{\text{marg}}$$

$$= 0.347$$

$$p(t_9^2 \mid y_1^N) = \frac{\alpha_2(1)\, r(3 \mid 1)\, \beta_2(3)}{\text{marg}}$$

$$= 0.4$$

$$p(t_1^3 \mid y_1^N) = \frac{\alpha_3(1)\, p(1 \mid 1)\, q(0 \mid 1 \to 1)\, \beta_4(1)}{\text{marg}}$$

$$= 0.61976$$

$$p(t_2^3 \mid y_1^N) = \frac{\alpha_3(1)\, p(2 \mid 1)\, q(0 \mid 1 \to 2)\, \beta_4(2)}{\text{marg}}$$

$$= 0.09$$

$$p(t_3^3 \mid y_1^N) = \frac{\alpha_3(1)\, p(3 \mid 1)\, q(0 \mid 1 \to 3)\, \beta_4(3)}{\text{marg}}$$

$$= 0.21$$

$$p(t_4^3 \mid y_1^N) = \frac{\alpha_3(2)\, p(3 \mid 2)\, q(0 \mid 2 \to 3)\, \beta_4(3)}{\text{marg}}$$

$$= 0.06$$

$$P(t_7^3 \mid y, ^N) = \frac{\alpha_3(2)\ \gamma(1 \mid 2)\ \beta_3(1)}{\text{marg}}$$

$$= 0.2$$

(c) Based on your calculations in (b), compute the *expected* counts $c(y,t)$ of each non-null arc, and *reestimate* the emission probability matrices $q(\mathbf{0}|s \to s')$ and $q(\mathbf{1}|s \to s')$.

$$q(0 \mid s \to s') = \begin{bmatrix} \frac{t_1^0 + t_1^3}{t_1^0 + t_1^3} & \frac{t_2^0 + t_2^3}{t_1^0 + t_2^1 + t_2^2 + t_2^3} & \frac{t_3^0 + t_3^3}{t_3^0 + t_3^3} \\ 0 & 0 & \frac{t_4^3}{t_4^3 + t_4^1} \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.55 & 1 \\ 0 & 0 & 0.3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$q(1 \mid s \to s') = \begin{bmatrix} 0 & \frac{t_2^1 + t_2^2}{t_2^0 + t_2^1 + t_2^2 + t_2^3} & 0 \\ 1 & 1 & \frac{t_4^1 + t_4^2}{t_4^0 + t_4^1 + t_4^2 + t_4^3} \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.45 & 0 \\ 1 & 1 & 0.7 \\ 1 & 1 & 1 \end{bmatrix}$$

(d) Based on your calculations in (b), compute the *expected* counts $c(t)$ of each transition, and *reestimate* the transition probability matrices $p(s'|s)$ and $q(s'|s)$.

$$c^*(t) = \sum_{i=0}^{k-1} p^*\{t^i = t\}$$

$$p(s|s') = \begin{bmatrix} \dfrac{t^0_1 + t^3_1}{\left(\begin{array}{c} t^0_1 + t^0_2 + t^0_3 + t^1_2 + t^1_9 + t^2_2 \\ + t^2_9 + t^3_1 + t^3_9 + t^3_2 + t^3_3 \end{array}\right)} & \dfrac{t^0_2 + t^1_2 + t^2_2 + t^3_2}{\left(\begin{array}{c} t^0_1 + t^0_2 + t^0_3 + t^1_2 + t^1_9 + t^2_2 \\ + t^2_9 + t^3_1 + t^3_9 + t^3_2 + t^3_3 \end{array}\right)} & \dfrac{t^0_3 + t^3_3}{\left(\begin{array}{c} t^0_1 + t^0_2 + t^0_3 + t^1_2 + t^1_9 + t^2_2 \\ + t^2_9 + t^3_1 + t^3_9 + t^3_2 + t^3_3 \end{array}\right)} \\ 0 & 0 & \dfrac{t^1_4 + t^2_4 + t^3_4}{\left(\begin{array}{c} t^1_4 + t^1_8 + t^1_7 + t^2_6 + t^2_7 + t^2_8 + t^3_4 \\ + t^3_7 + t^3_8 \end{array}\right)} \\ \dfrac{t^1_5 + t^2_5}{(t^1_5 + t^1_6 + t^2_5 + t^2_6)} & \dfrac{t^1_6 + t^2_6}{(t^1_5 + t^1_6 + t^2_5 + t^2_6)} & 0 \end{bmatrix}$$

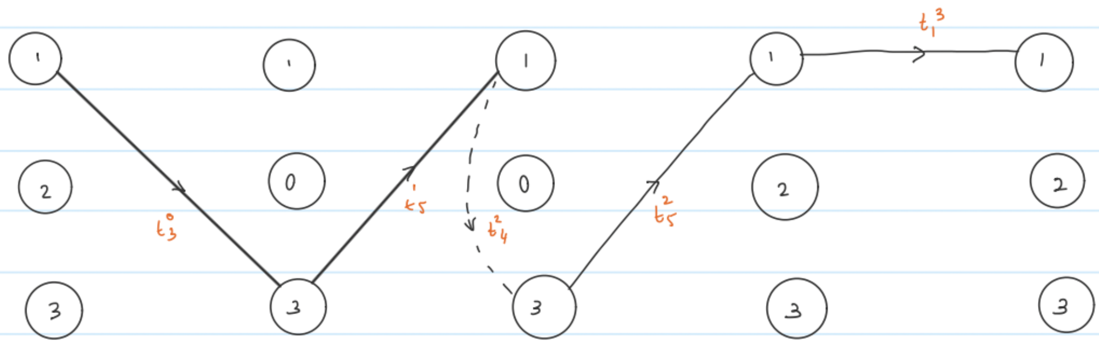
$$= \begin{bmatrix} 0.35 & 0.21 & 0.21 \\ 0 & 0 & 0.25 \\ 0.76 & 0.25 & 0 \end{bmatrix}$$

$$q(s \mid s') = \begin{bmatrix} 0 & 0 & \dfrac{t_9^1 + t_9^2}{\left(\begin{array}{c} t_1^0 + t_2^0 + t_3^0 + t_2^1 + t_9^1 + t_2^2 \\ + t_9^2 + t_1^3 + t_9^3 + t_2^3 + t_3^3 \end{array}\right)} \\ \dfrac{t_7^1 + t_7^2 + t_7^3}{\left(\begin{array}{c} t_4^1 + t_8^1 + t_7^1 + t_4^2 + t_7^2 + t_8^2 + t_4^3 \\ + t_7^3 + t_8^3 \end{array}\right)} & 0 & \dfrac{t_8^1 + t_8^2}{\left(\begin{array}{c} t_4^1 + t_8^1 + t_7^1 + t_4^2 + t_7^2 + t_8^2 + t_4^3 \\ + t_7^3 + t_8^3 \end{array}\right)} \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0.211 \\ 0.3 & 0 & 0.4 \\ 0 & 0 & 0 \end{bmatrix}$$

2. Consider the HMM of Homework #2 once again (!) and the Viterbi path you calculated in Homework #2 (f) for the output **0110**.

   (a) Compute the *Viterbi* or "hard" counts $\hat{c}(y, t)$ of each arc of the HMM. How many $\hat{c}(y, t)$ are nonzero?



$\hat{c}(y, t_3^0)$, $c(y, t_5^1)$, $c(y, t_5^2)$, $c(y, t_1^3) = p^*(t_1^3)$

$= p^*(t_3^0) = p^*(t_5^1) = p^*(t_5^2)$

so 4 $\hat{c}(y,t)$ are non-zero

(b) Replace $c^*(y,t)$ in Equations (36) on p33 with these $\hat{c}(y,t)$ and *reestimate* the HMM emission probability matrices $q(\mathbf{0}|s \to s')$ and $q(\mathbf{1}|s \to s')$.

$$q[0|s \to s'] = \begin{bmatrix} t_1^3/t_1^3 & 0 & t_5^0/t_3^0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$q(1 \mid s \to s') = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(c) Compute the *Viterbi* or "hard" counts $\hat{c}(t)$ of each arc of the HMM. How many $\hat{c}(t)$ are nonzero?

$$c^*(t) = \sum_{i=0}^{K-1} p^* \{ t_i = t \}$$

$$= 3 \text{ are non zeros}$$

(d) Replace $c^*(t)$ in Equations (37) on p33 with these $\hat{c}(t)$ and *reestimate* the HMM transition probability matrices $p(s'|s)$ and $q(s'|s)$. (See the Note above.)

$$p[s|s'] = \begin{bmatrix} \dfrac{t_1^3}{t_3^0 + t_1^3 + t_4^2} & 0 & \dfrac{t_3^0}{t_3^0 + t_1^3 + t_4^2} \\ 0 & 0 & 0 \\ \dfrac{t_5^1 + t_5^2}{t_5^1 + t_5^2} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.211 & 0 & 0.402 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$q[s|s'] = \begin{bmatrix} 0 & 0 & \dfrac{t_9^2}{t_3^6 + t_1^3 + t_4^2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0.4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$