A while-loop and a sum formula in pseudocode

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Proving a sum formula by induction

- **Problem:** Prove that for $n = 1, 2, 3, \ldots$

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

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- **Solution:** Define the predicate $P(n)$ as the equation above and use induction:
  - **Basis:** $P(1)$ holds because for $n = 1$ we have $\sum_{i=1}^{n} i = \sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2} = \frac{n(n+1)}{2}$

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Note: If we instead want to prove that the formula holds for $n = 0, 1, 2, \ldots$ then we would use $P(0)$ as the basis.

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By inductive hypothesis

$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n^2+n+2n+2}{2} = \frac{n^2+3n+2}{2} = \frac{(n+1)(n+2)}{2}$ which is what we needed to prove.

General while-loop

Sauðakóði

meðan C

S

Pseudocode

while C

S

Java

```
while( C )
{
    S
}
```

General while-loop

$\boxed{\{\ F\ \}}$

while C

$\quad\boxed{\{\ I\ \}}$

$\quad$S

$\boxed{\{\ E\ \}}$

The precondition of the loop describes the state just before the loop starts

The loop invariant makes it possible to verify the loop

The postcondition of the loop describes the state just after the loop finishes

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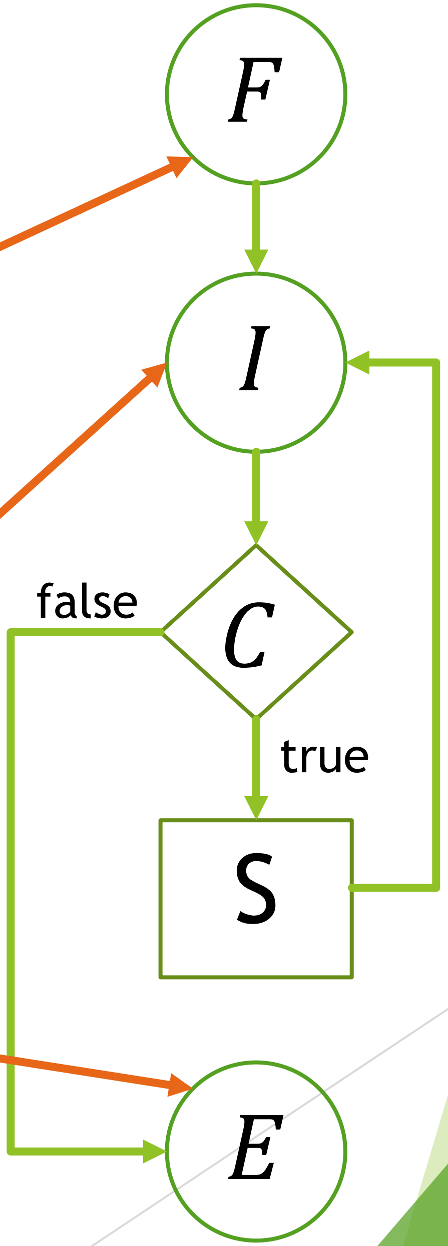


General while-loop

$\{ F \}$

while C

$\{ I \}$

S

$\{ E \}$

The precondition of the loop describes the state just before the loop starts

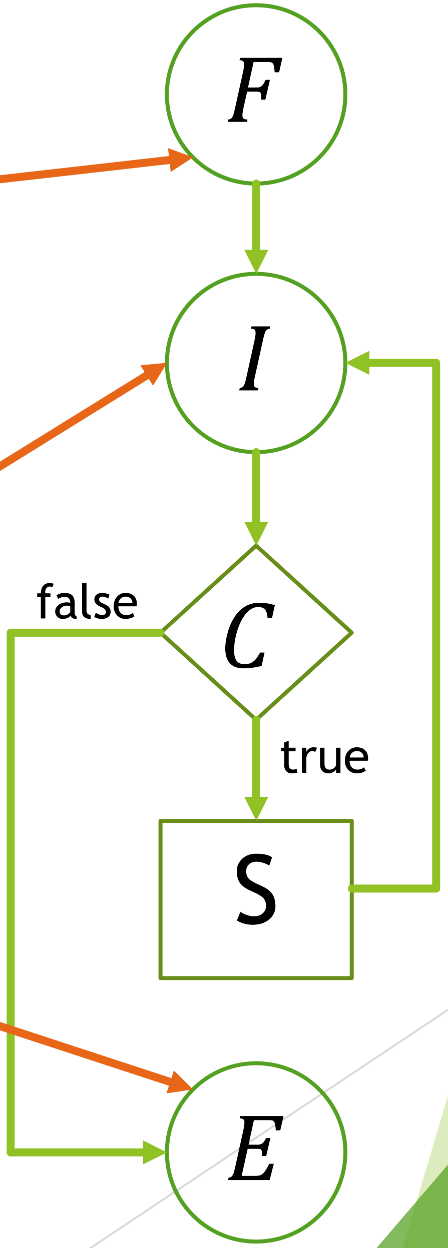
Rule 1: $F \rightarrow I$

The loop invariant makes it possible to verify the loop

Rule 2: $\{I \wedge C\}\mathbf{S}\{I\}$

The postcondition of the loop describes the state just after the loop finishes

Rule 3: $I \wedge \neg C \rightarrow E$



Prove the sum formula using program verification

$\{\ n \geq 0\ \}$

An executable and verified sequence of code where n does not change, but s gets a new value

$\{\ s = 1 + 2 + \cdots + n\ \}$
$\{\ s = n(n+1)/2\ \}$

- Here we prove the formula for $n \geq 0$, not just $n \geq 1$, but that is a minor point
- Is equivalent to proof by induction if and only if the termination of the sequence of code is also proven

Prove the sum formula using program verification

$\{ n \geq 0 \}$

Initialize s and k

while $k \neq n$

$\{ 0 \leq k \leq n \}$

$\{ s = 1 + 2 + \cdots + k \}$

$\{ s = k(k+1)/2 \}$

Increase k and preserve the loop invariant

$\{ s = 1 + 2 + \cdots + n \}$

$\{ s = n(n+1)/2 \}$

- Here we prove the formula for $n \geq 0$, not just $n \geq 1$, but that is a minor point
- Is equivalent to proof by induction if and only if the termination of the sequence of code is also proven

Prove the sum formula using program verification

$\{ n \geq 0 \}$
$k \coloneqq 0$
$s \coloneqq 0$
while $k \neq n$
$\quad \{ 0 \leq k \leq n \}$
$\quad \{ s = 1 + 2 + \cdots + k \}$
$\quad \{ s = k(k+1)/2 \}$
$\quad k \coloneqq \mathrm{k} + 1$
$\quad s \coloneqq s + k$
$\{ s = 1 + 2 + \cdots + n \}$
$\{ s = n(n+1)/2 \}$

- Here we prove the formula for $n \geq 0$, not just $n \geq 1$, but that is a minor point
- This is equivalent to proof by induction if and only if the termination of the sequence of code is also proven

Prove the sum formula using program verification

$\{ n \geq 0 \}$
$k := 0$
$s := 0$
while $k \neq n$
$\quad \{ 0 \leq k \leq n \}$
$\quad \{ s = 1 + 2 + \cdots + k \}$
$\quad \{ s = k(k+1)/2 \}$
$\quad k := \mathrm{k} + 1$
$\quad s := s + k$
$\{ s = 1 + 2 + \cdots + n \}$
$\{ s = n(n+1)/2 \}$

- Here we prove the formula for $n \geq 0$, not just $n \geq 1$, but that is a minor point
- This is equivalent to proof by induction if and only if the termination of the sequence of code is also proven
- In this case termination is obvious: we will go through the loop exactly n times
- The value of the formula $n - k$ is originally n and decreases by 1 each pass through the loop and cannot decrease below 0 because then the loop terminates

Arguments

- Remember that the code
  $\{ F \}$
  while C
  $\quad \{ I \}$
  $\quad$ S
  $\{ E \}$
- needs to fulfil the following:

1. $F \rightarrow I$
2. $\{C \wedge I\}\mathrm{S}\{I\}$
3. $I \wedge \neg C \rightarrow E$

- Clearly rules 1 and 3 are fulfilled
- Rule 2 is also fulfilled since if we let k' and s' stand for the new values in the variables k and s after each general pass through the loop,we get $k' = k + 1$ and we get

$$\begin{aligned} s' = \quad & s + k' \\ = \quad & 1 + 2 + \cdots + k + k' \\ = \quad & 1 + 2 + \cdots + k + (k + 1) \\ = \quad & 1 + 2 + \cdots + k' \end{aligned}$$

and we also get

$$\begin{aligned} s' = \quad & s + k' \\ = \quad & k(k + 1)/2 + k' \\ = \quad & (k^2 + k)/2 + (k + 1) \\ = \quad & (k^2 + 3k + 2)/2 \\ = \quad & (k + 1)(k + 2)/2 \\ = \quad & k'(k' + 1)/2 \end{aligned}$$

which is what we needed to prove that the loop invariant is preserved (in addition to $0 \le k' \le n$, which is obvious)