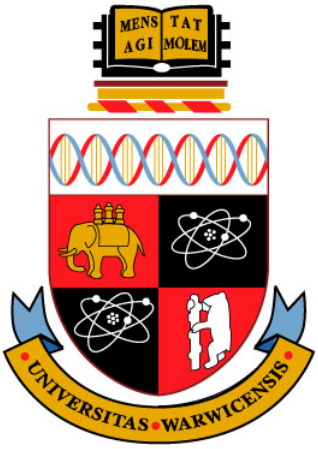

An analysis of strategies in the re-pairing game

CS344 Discrete Mathematics Project

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Abstract

The project expands on the work of Chistikov and Vyali, which introduced a simple one-player game; The re-pairing game can be played on any well-formed sequence of opening and closing brackets (a Dyck word). A move consists of "pairing" any opening bracket with any closing bracket to the right of it, and "erasing" the two. The process is repeated until we are left with no remaining brackets. Such a game can have many strategies, but the effectiveness of a strategy is measured by it's width, which is the maximum number of nonempty segments of symbols seen during a play of the game.

Keywords: *Dyck language, Re-pairing brackets, Combinatorics, Web application, Python, ReactJS*

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Chapter 1

Introduction

We start with a simple one-player game. Take any Dyck word (a balanced sequence of opening and closing brackets), for example:

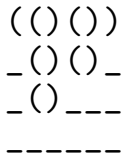
```
(()())
```

A **move** in this game consists of pairing any opening bracket with any closing bracket to its right, and replacing the two with a blank space (denoted with the _ symbol). This process is repeated until there are no more brackets to pair up, resulting in the empty string, and the sequence of moves made is called a **re-pairing**. An example of a re-pairing is as follows:

```
(()())
(_)(_)
___(_)
______
```

Note that our first move pairs up two brackets that are not matched to each other in the initial word. We allow such moves in a re-pairing.

Finding any re-pairing following these rules is a trivial task, but we want to measure how good a re-pairing is. For this we introduce a property called the **width**, defined as the maximum number of non-empty segments of brackets seen during the re-pairing. We can see that the previous re-pairing had width 2. A natural question then arises; can we obtain a re-pairing with a smaller width? The answer for this particular string is yes, and such a re-pairing is shown below:



Here we see this re-pairing has width 1. Our goal is to try and minimise this width as much as possible.

1.1 Objectives

This project aims to take both a theoretical and practical approach to this game. It is split into two parts; a literature review and software development.

The literature review aims to assess the current knowledge available on the game by understanding and expanding on the relevant background knowledge, context and results. This will also help gain a more intuitive grasp on the topic.

The software development aspect aims to implement strategies in the re-pairing game to allow for visualisation and experimentation. A subgoal of this is to attack the goal of minimising width with software, by analysing certain strategies and subcases of Dyck words.
More specifically, the objectives of the project can be outlined as follows:

1. An introduction to the relevant definitions and content to lay out background knowledge.

2. Expanding on and providing potential novel insights on well established results.

3. Establishing the research landscape on the re-pairing game and relevant topics.

4. Creating software to demonstrate re-pairing strategies from literature and allow manual experimentation on Dyck word re-pairings.

5. Using software to analyse subcases of Dyck words and exhaustively find their width in an attempt to pin down the formula of a general Dyck word.

1.2 Related work

This project builds on the work from Chistikov and Vyalyi [1], whose paper serves as a basis for this project. The paper introduces this game as a

way to prove a lower bound on translating one-counter automata (OCA) into Parikh-equivalent nondeterministic finite automata (NFA). It provides lower bounds on the width for simple and non-simple re-pairings, which is then used to prove lower bounds on these translations.

Due to the specialised nature of this game's origins, there are no other works on this game at the time of writing.

Chapter 2

Literature Review

This section aims to cover the relevant definitions, theorems, and results from literature. We will also attempt to provide insight on and formalise some well known ideas in the context of this project.

2.1 Background

2.1.1 Dyck Words

Definition 1. *A **Dyck Word** is a string that satisfies the following:*

1. *The only characters in the string are "(" or ")".*
2. *At any point in the string, there are no more "(" brackets than ")" brackets.*
3. *The string contains the same number of "(" and ")" brackets*

We can alternatively define a Dyck word through the convention "(" $= 1$ and ")" $= -1$:

Definition 2 ([1])**.** *A **Dyck word** is a sequence $D = (d_n)_{n\geq 1}$ such that $d_n \in \{1, -1\}$ and the following is satisfied:*

1. *For every $1 \leq k \leq n$, we have $\sum_{i=1}^{k} d_i \geq 0$*
2. *$\sum_{i=0}^{n} d_i = 0$*

2.1.2 North-East Lattice Walks

These words have interesting combinatorial properties, one of which is their unsurprising link to Lattice Walks:

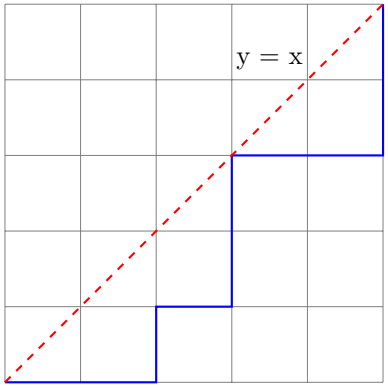


Figure 2.1: North-East lattice walk for `(()())(())`

Definition 3. *A **North-East lattice walk** is a sequence of vectors (a, b) starting at $(0, 0)$ and finishing at (n, n), such that every step on the walk goes from some (a, b) to either $(a + 1, b)$ or $(a, b + 1)$.*

In other words, every step is exactly one unit vector right or one unit vector up.

Theorem 1. *Every Dyck word of length $2n$ corresponds to a unique North-East lattice walk from $(0, 0)$ to (n, n) that does not cross above the line $y = x$.*

This is a trivial mapping, as we can take a move east $(1, 0)$ to be "`(`", and a move north $(0, 1)$ to be "`)`". This works out as expected since we have $2n$ brackets to match, and n moves east and north each during such lattice walks. Our restriction on Dyck words that ensures there are never more "`)`" brackets than "`(`" brackets corresponds to the restriction of the walk being below (or on) the line $y = x$. This connection gives us another result:

Theorem 2 ([2])**.** *The number of Dyck words of length $2n$ corresponds to the n^{th} Catalan number $C_n = \sum_{i=1}^{n-1} C_i C_{n-1-i}$.*

Proof. Let W_n be the number of North-East lattice walks from $(0, 0)$ to (n, n) that do not cross above the line $y = x$. Given any such walk, it may touch this line at any (i, i) where $0 \leq i \leq n - 1$. This gives us a natural recursion; every such North-East lattice walk can be broken down into a North-East lattice walk from $(0, 0)$ to (i, i), and from (i, i) to (n, n). Using this idea, we see that $W_n = \sum_{i=0}^{n-1} W_i W_{n-1-i}$. This is precisely the recursion for the n^{th} Catalan number. □

2.1.3 Binary Trees

However, we can also relate this result to the concept of binary trees as well. Suppose we had four factors of a number $x = x_1 x_2 x_3 x_4$. How many

ways can we *completely parenthesize* this expression of factors (i.e. place brackets around this to group factors)?

1. $(x_1(x_2x_3))x_4$
2. $((x_1x_2)x_3)x_4$
3. $x_1(x_2(x_3x_4))$
4. $x_1((x_2x_3)x_4)$
5. $(x_1x_2)(x_3x_4)$

There are 5 ways, but this is the same as $C_3 = \frac{1}{3+1}\binom{2(3)}{3} = 5$. To understand why, let G_n be the number of ways to place brackets around $x = x_1 \dots x_n$. Note that the outermost factor will not be within any pair of brackets.

We look at the leftmost "(", and its matching ")". Suppose this surrounds $0 \leq i \leq n-1$ factors. We can completely parenthesize this expression of factors in G_i ways. Since the outermost factor will not be within any pair of brackets, this leaves a further $n-i-1$ factors to completely parenthesize, which can be done in G_{n-i-1} ways. Therefore the number of ways to completely parenthesize an expression of n factors is $G_n = \sum_{i=0}^{n-1} G_i G_{n-1-i}$, which is precisely the n^{th} Catalan number.

Theorem 3. *The number of proper binary trees with $n+1$ leaves is C_n.*

Proof. The idea is to show that every proper binary tree of $n+1$ leaves corresponds to a unique completely parenthesized expression of n factors. Recall that a proper binary tree is a tree such that every non-leaf node has 2 nodes. □

Chapter 3

A web application for the game

This section discusses the process of implementing a web application that allows for visualisation and experimentation of the re-pairing game.

3.1 Design and Implementation

The development of the web application went through a lot of ideas before settling on the current choice of technologies used. The application uses the JavaScript library React for the frontend and user interface, and the Python web framework Flask for the backend algorithms. The initial designs, choices made and features implemented are expanded on below.

3.1.1 Software Requirements

Before any software was written, a list of requirements for the software was made. These are features that the software must have in order to fulfill the relevant objectives, and were subsequently tested throughout the development of the web application:

1. Allow a user to enter Dyck words.
2. Validate a given input and provide a relevant error message if required.
3. Display the given Dyck word with the ability for manual interaction via a mouse.
4. Verify that the characters selected are a valid move before allowing them to be paired.

5. Allow for the use of strategies from literature (Simple, Non-Simple recursive), as well as our own strategies (Simple Greedy, Brute Force).

6. If selecting a strategy, generate a list of moves before visualising the play.

7. Display the list of moves, and allow the user to jump to any move in the play.

8. Display a running width counter from the moves seen so far, and the width of the overall play if a strategy has been used.

9. (OPTIONAL) Allow for a mix between manual re-pairing and the use of pre-defined strategies (e.g. re-pairing the first half of a word using a strategy, and completing the second half using manual selections).

10. Allow the user to brute force a given list of Dyck Words, and return the optimal widths of each along with the re-pairing.

3.1.2 Initial Design

The project was initially to be a local Python only application, using Tkinter. This was because the author is most familiar with the Python programming language, and has some limited experience with the Tkinter library. However, early prototypes of this software using Tkinter proved to be cumbersome to work with and have a dated visual style. Alternative Python UI libraries, such as PyQt and Kivy, were also considered, but proved to have learning curves no steeper than using an entirely different language.

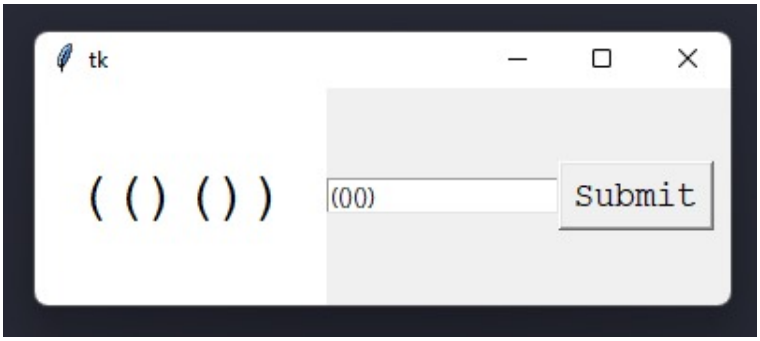


Figure 3.1: An early iteration of the software using Tkinter

We then turned our attention to the prospect of a web application; the author again had some limited experience with HTML/CSS but using these languages to control the layout and sizing of the UI seemed much more intuitive. This approach also naturally allowed for separation between the

UI and the strategies, and the author's familiarity with Python could still be leveraged. The author had also wished to learn the technologies used for web development, so this seemed like the natural approach to take.

If a web-application was to be used, some way of communicating between the code for the UI and Python was needed. This led to considering React, which is a JavaScript library used to create user interfaces [3].

A second prototype was then created to confirm this was indeed the right choice. This seemed promising; the implementation was not overly complicated, and the layout was easy to control.

3.2 Project Management

This project used the Agile methodology, as the goals and limitations of the project aligned closely with the principles of Agile [4]:

- Creating functional software was of a higher priority than comprehensive documentation. The faster that the software was made functional, the more time could be devoted to its refinement. Given that the author was also new to writing web applications and many of the libraries chosen at the start of this project, making such refinements proved to have a steeper learning curve.

- Responding to changes in requirements was of a higher priority than a strict plan and implementation. Throughout the process of absorbing the seminal work [1], more elegant and useful ideas arose and became the clear option to implement over the current option.

The libraries and languages the author was most familiar with were developed first. We begun with the backend

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