02_combinatorics

Wednesday, January 8, 2020 9:03 AM

02_combin...

a fancy way to say counting

02: Combinatorics

David Varodayan
January 8, 2020
Adapted from slides by Lisa Yan

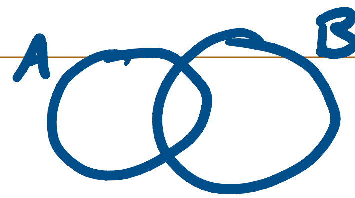
Takeaways from last time

Review

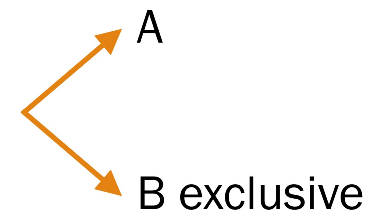
Inclusion-Exclusion Principle (generalized Sum Rule)

If the outcome of an experiment can be either from Set A **or** set B, where A and B may overlap, then the total number of outcomes of the experiment is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$



One-step experiment



General Principle of Counting (generalized Product Rule)

If an experiment has r **steps**, such that step i has n_i outcomes for all $i = 1, \ldots, r$, then the total number of outcomes of the experiment is

$$n_1 \times n_2 \times \cdots \times n_r = \prod_{i=1}^{r} n_i .$$

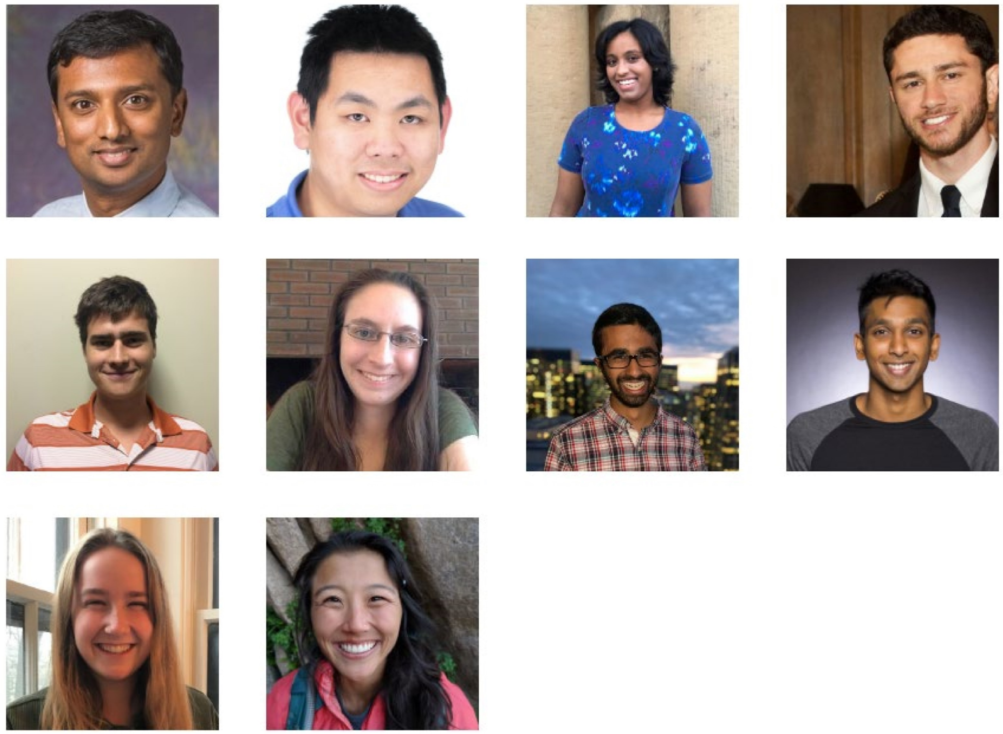
Multi-step experiment

→ 1 → 2 → ...

Essential information

Website cs109.stanford.edu

Teaching Staff



Today's plan

Permutations (sort objects)

Combinations (choose objects)

Put objects into buckets

Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct: r^n
  - Indistinct: $\dfrac{(n+r-1)!}{n!\,(r-1)!}$

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Put objects in r buckets

Distinct (distinguishable)

Sort n indistinct objects

DOESN'T WORK

2 choices 2 choices depends

Sort n distinct objects

Ayesha Tim Irina Joey Waddie

$\underline{\quad 5 \quad} \times \underline{\quad 4 \quad} \times \underline{\quad 3 \quad} \times \underline{\quad 2 \quad} \times \underline{\quad 1 \quad}$

Permutations

A permutation is an ordered arrangement of distinct objects.

The number of unique orderings (permutations) of n distinct objects is

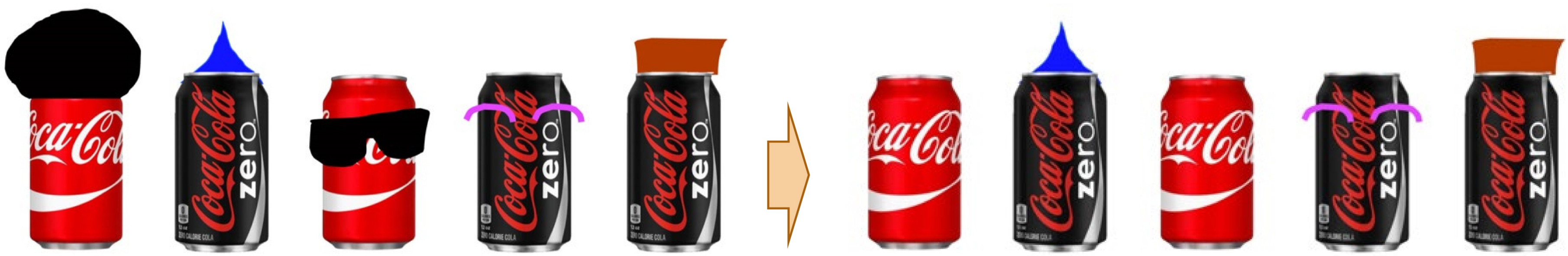
$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

Sort semi-distinct objects

Order n distinct objects	$n!$

All distinct

Some indistinct



Ayesha Tim Irina Joey Waddie

Coke Tim Coke Joey Waddie

$$5! = 120$$

$$\frac{5!}{2!} = \frac{120}{2} = 60$$

Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

permutations of distinct objects $=$ permutations considering some objects are indistinct $\times$ Permutations of just the indistinct objects

Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

$$\frac{\text{permutations of distinct objects}}{\text{Permutations of just the indistinct objects}} = \text{permutations considering some objects are indistinct}$$

General approach to counting permutations

When there are n objects such that

n_1 are the same (indistinguishable or indistinct), and
n_2 are the same, and
...
n_r are the same,

The number of unique orderings (permutations) is

$$\frac{n!}{n_1!\, n_2! \cdots n_r!}$$

For each group of indistinct objects,
Divide by the overcounted permutations

Sort semi-distinct objects

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many permutations?

Coke Coke0 Coke Coke0 Coke0

$$\frac{5!}{2!\,3!} = 10$$

Strings

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many orderings of letters are possible for the following strings?

1. BOBA

BAOB
BOBA
⋮
ABOB

$$\frac{4!}{2!\,1!\,1!} = 12$$

2. MISSISSIPPI

$$\frac{11!}{1!\,4!\,4!\,2!} = 34650$$

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$
- Choose k objects (combinations)
- Put objects in r buckets

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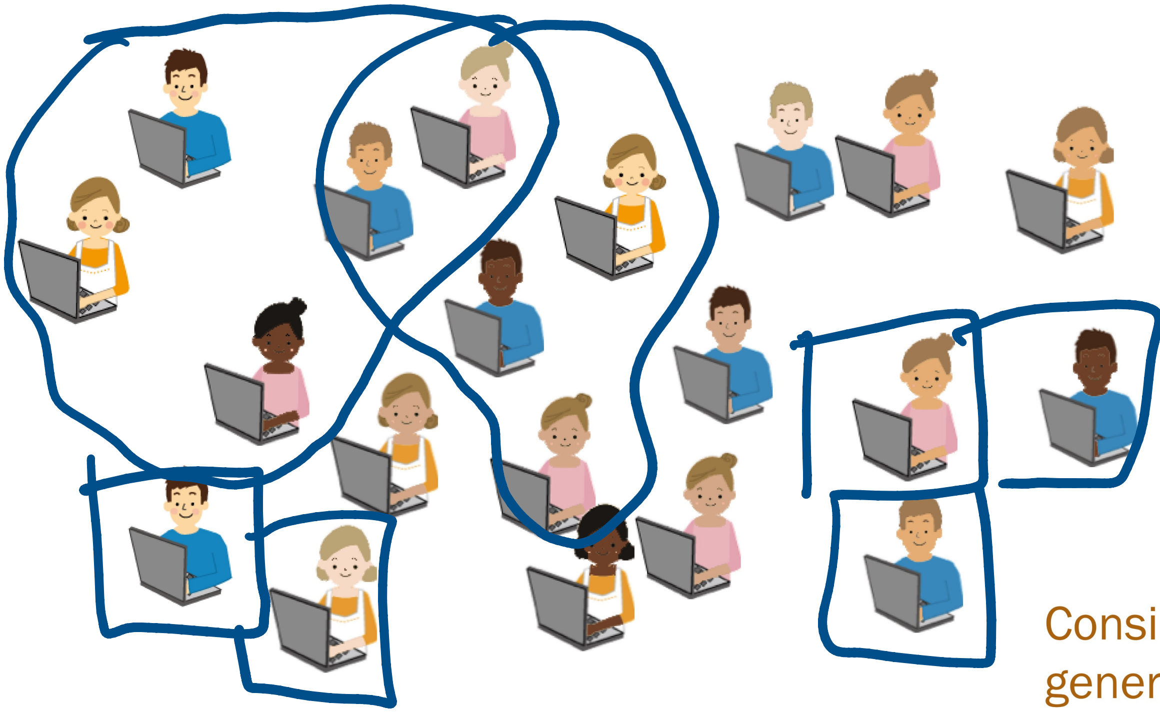
  - Distinct
- Put objects in r buckets

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



Consider the following generative process...

Combinations with cake

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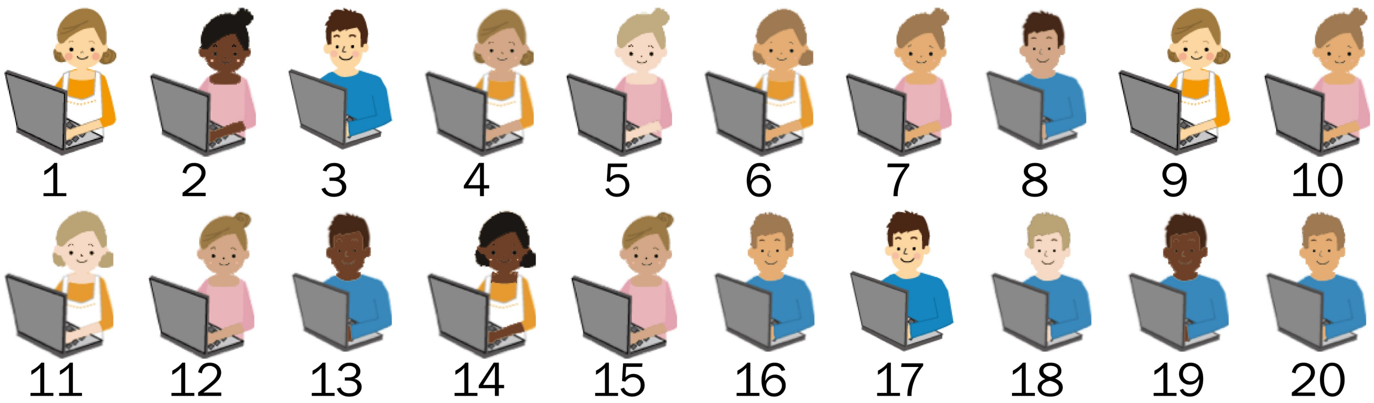
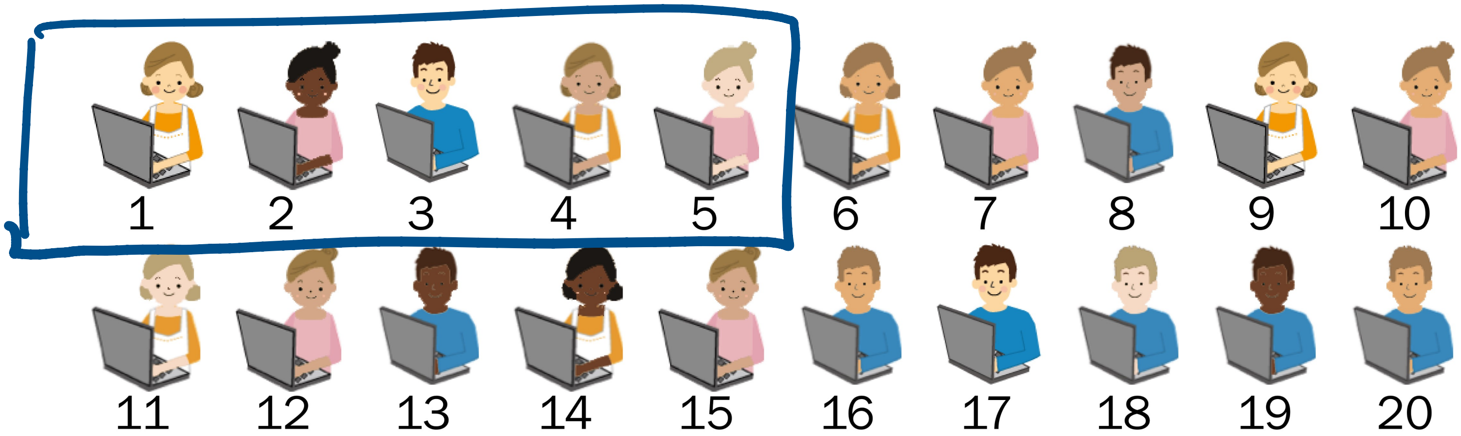


1. n people get in line

   $n!$ ways

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line
2. Put first k in cake room

$n!$ ways

1 way

Combinations with cake

There are $n = 20$ people.
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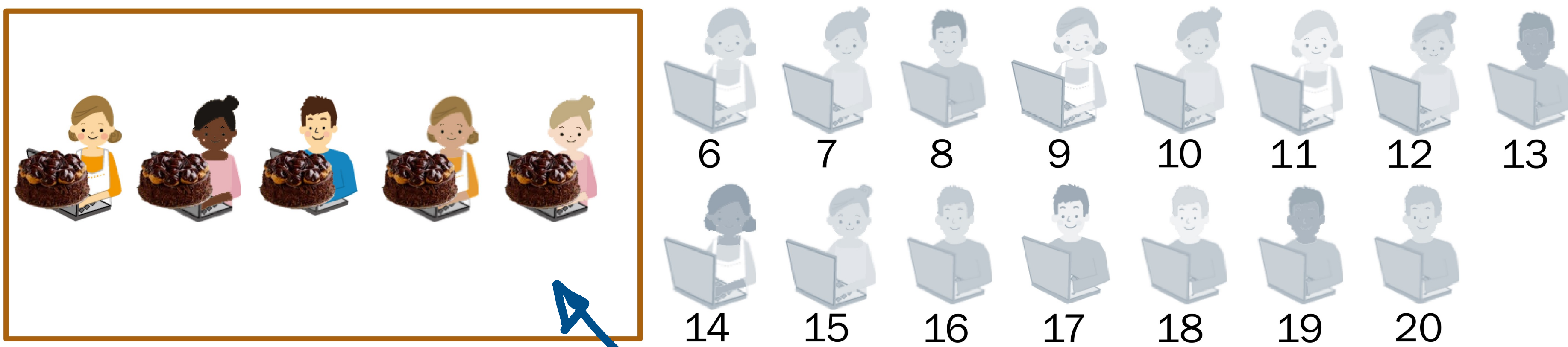

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

overcounting
$5!$

Combinations with cake

There are $n = 20$ people.
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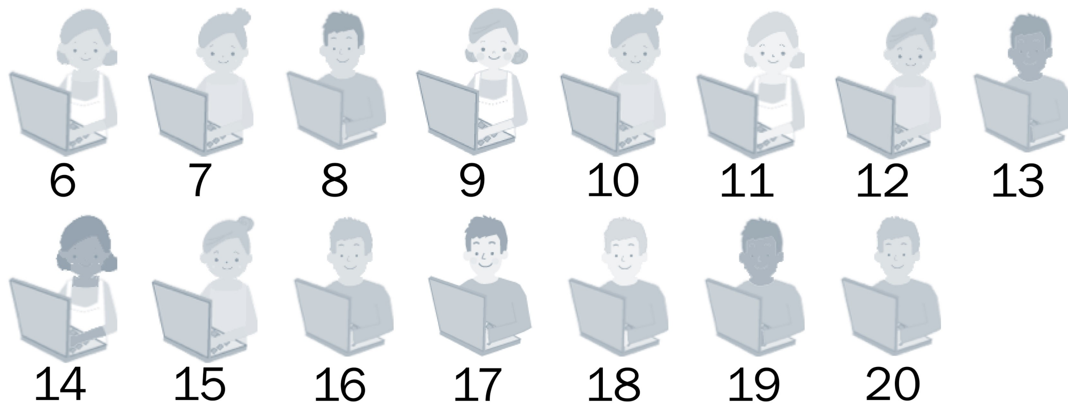


1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

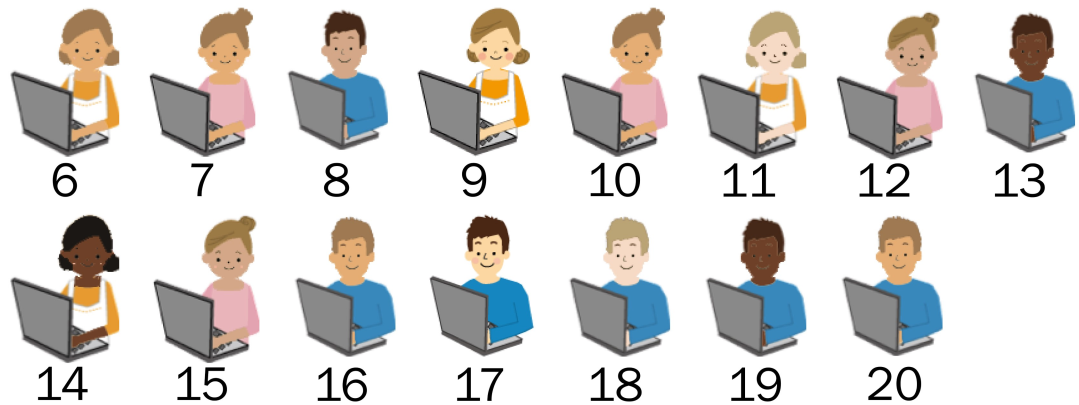
3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

Combinations with cake

overcounting by 15!

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

   $(n-k)!$ different permutations lead to the same mingle

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

The number of ways of making this selection is

$$n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!} = \frac{n!}{k!(n-k)!} = \boxed{\binom{n}{k}}$$

Binomial Coefficient

Interesting

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!3!} = 20$$

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

2. What if we do not want to read both the 9th and 10th edition of Ross?

Case 1 : 9th ed + 2 other of 4 $\quad \binom{4}{2} = 6$

Case 2 : 10th ed + 2 other of 4 $\quad \binom{4}{2} = 6$

Case 3 : 3 other of 4 $\quad \binom{4}{3} = 4$

$\boxed{16}$

Probability textbooks (solution 2)

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

2. What if we do not want to read both the 9th and 10th edition of Ross?

Forbidden case: 9th ed + 10th ed + 1 other book of 4

$$\binom{4}{1} = 4$$

So answer $= 20 - 4 = 16$

Sometimes easier to exclude forbidden cases

Break

Announcements

PS#1

Out:	today
Due:	Friday 1/17, 1:00pm
Covers:	through Friday

Staff help

Piazza policy:	student discussion
Office hours:	start today

cs109.stanford.edu/staff.html

Python tutorial

When:	Friday 3:30-4:20pm
Location:	420-041
Recorded?	maybe
Notes:	to be posted online

Section sign-ups

Preference form:	today
Due:	Saturday 1/11
Results:	latest Monday

Handout: Calculation Reference

Week	Monday	Wednesday	Friday
1	JAN 6 1: Counting Slides Lecture Notes Administrivia Read: Ch 1.1-1.2	JAN 8 2: Permutations and Combinations Lecture Notes Calculation Ref Read: Ch 1.3-1.6 Out: PSet #1	JAN 10 3: Axioms of Probability Lecture Notes Python for Probability Serendipity Demo Read: Ch 2.1-2.5, 2.7
2	JAN 13 4: Conditional Probability and Bayes Lecture Notes Medical Bayes Demo	JAN 15 5: Independence Lecture Notes	JAN 17 6: Random Variables and Expectation Lecture Notes

Geometric series:

$$\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$$

$$\sum_{i=m}^{n} x^i = \frac{x^{n+1}-x^m}{x-1}$$

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \text{ if } |x| < 1$$

Integration by parts (everyone's favorite!):

Choose a suitable u and dv to decompose the integral of interest:

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

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- Choose k objects (combinations)
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups
- Put objects in r buckets

General approach to combinations

The number of ways to choose r groups of n distinct objects such that

For all $i = 1, \dots, r$, group i has size n_i, and

$\sum_{i=1}^{r} n_i = n$ (all objects are assigned), is

$$\frac{n!}{n_1!\, n_2! \cdots n_r!} = \binom{n}{n_1, n_2, \dots, n_r}$$

Multinomial Coefficient

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines
A	6
B	4
C	3

$$\binom{13}{6,4,3} = \frac{13!}{6!\,4!\,3!} = 60\,060$$

Datacenters (solution 2)

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines
A	6
B	4
C	3

Steps:

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose ~~2~~ 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

$$\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60060$$

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Sort objects (permutations)

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Choose k objects (combinations)

- Distinct
  - 1 group: $\binom{n}{k}$
  - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- Indistinct?

same

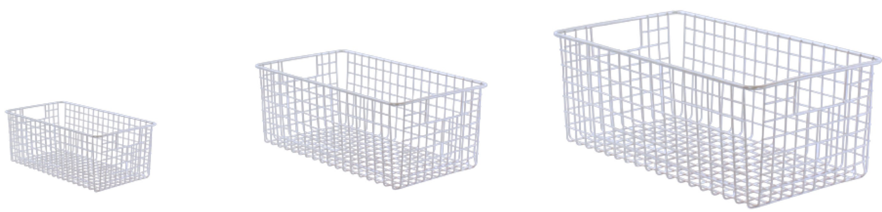
Put objects in r buckets

A trick question

Choose k of n distinct objects into r groups of size $n_1, \dots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

How many ways are there to group 6 indistinct (indistinguishable) objects into 3 groups, where group A, B, and C have size 1, 2, and 3, respectively?

Only 1



A (fits 1) B (fits 2) C (fits 3)

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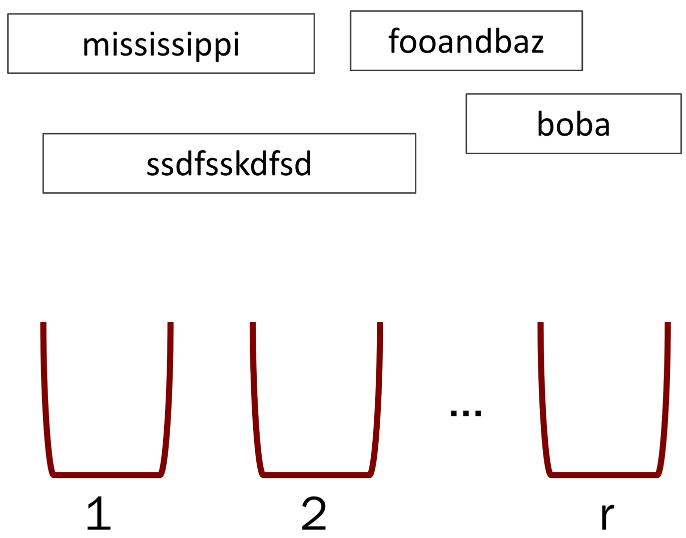
- **Put objects in r buckets**
  - Distinct
  - Indistinct

Hash tables and distinct strings

How many ways are there to hash n distinct strings to r buckets?



Steps:

1. Bucket 1st string $\quad r$
2. Bucket 2nd string $\quad r$

...

n. Bucket nth string $\quad r$

Total = r^n

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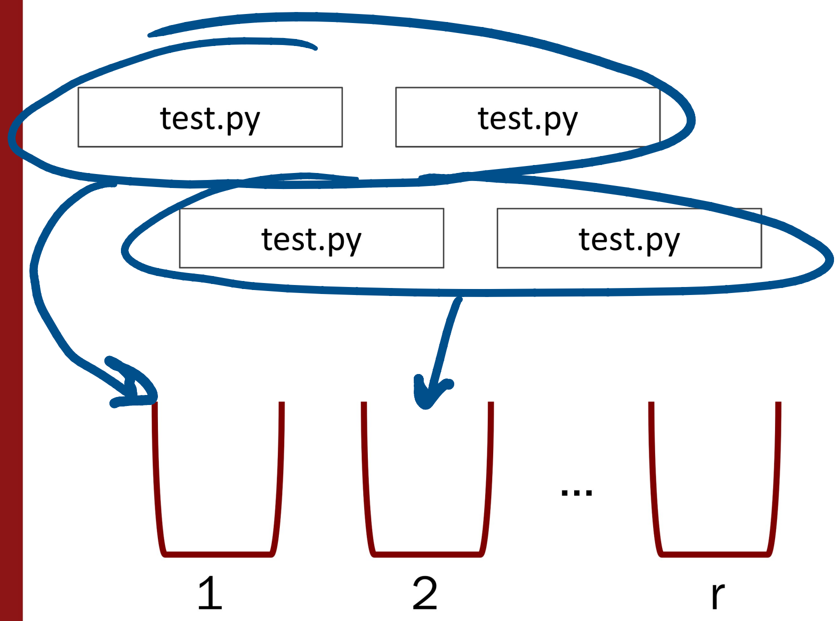
- **Put objects in r buckets**
  - Distinct: r^n
  - Indistinct

Hash tables and indistinct strings

How many ways are there to distribute n indistinct strings to r buckets?

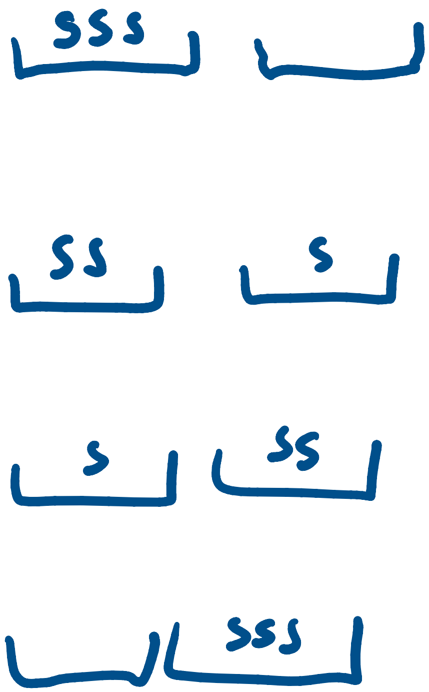


Goal

Bucket 1 has x_1 strings,
Bucket 2 has x_2 strings,
...
Bucket r has x_r strings (the rest)

How many different sets of counts are possible?

Simple example: $n = 3$ strings and $r = 2$ buckets



SIMPLIFY DRAWING

sss|

ss|s

s|ss

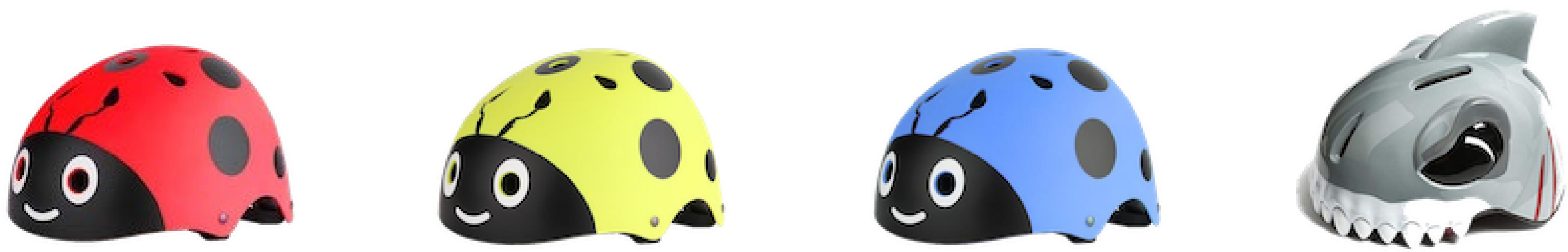
|sss

All permutations of three S and one divider

$$\frac{4!}{3!\,1!} = \binom{4}{1} = 4$$

Bicycle helmet sales

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?



Consider the following generative process…

Bicycle helmet sales: 1 possible assignment outcome

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

$n = 5$ indistinct objects $\quad$ $r = 4$ distinct buckets

Bicycle helmet sales: 1 possible assignment outcome

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

$n = 5$ indistinct objects $\qquad$ $r = 4$ distinct buckets

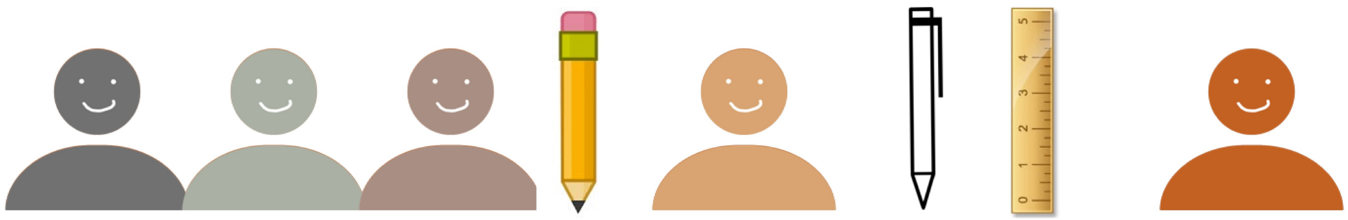
Goal Order n indistinct objects and $r - 1$ indistinct dividers.

Bicycle helmet sales: A generative proof

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

Goal Order n indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct



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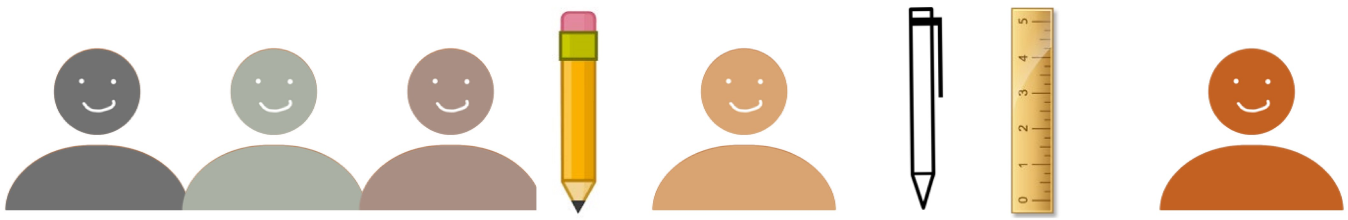

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

Bicycle helmet sales: A generative proof

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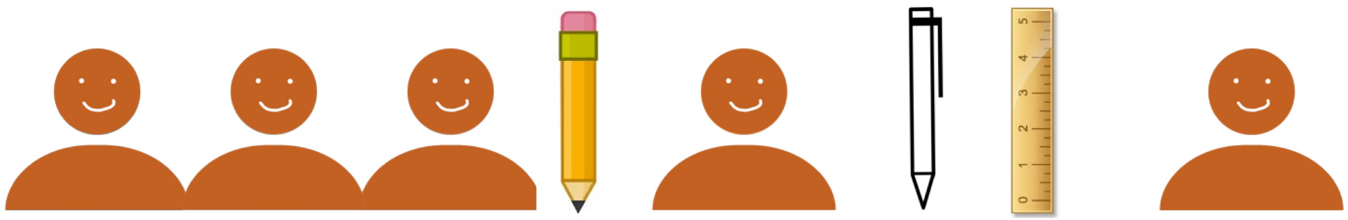

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

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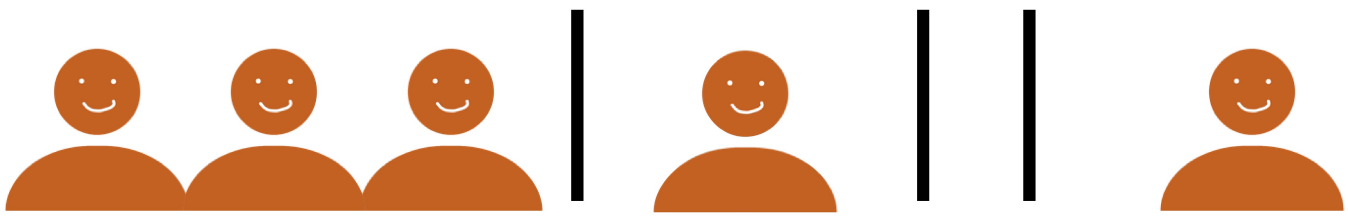

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

3. Make $r - 1$ dividers indistinct

$$\frac{1}{(r-1)!}$$

Divider method

The number of ways to distribute n indistinct objects into r buckets is equivalent to the number of ways to permute $n + r - 1$ objects such that

n are indistinct objects, and

$r - 1$ are indistinct dividers:

$$\text{Total} = (n+r-1)! \times \frac{1}{n!} \times \frac{1}{(r-1)!} = \binom{n+r-1}{r-1}$$

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- Put objects in r buckets
  - Distinct: r^n
  - Indistinct: $\frac{(n+r-1)!}{n!\,(r-1)!}$