03_probability

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03: Intro to Probability

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January 10, 2020

Adapted from slides by Lisa Yan

Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\dfrac{n!}{n_1!\, n_2! \cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct: r^n
  - Indistinct: $\dfrac{(n+r-1)!}{n!\,(r-1)!}$

Divider method

n objects

r buckets (ss) (ss) ( ) $\Longleftrightarrow$ ss|ss| permutations

n+r-1 objects

The number of ways to distribute n indistinct objects into r buckets is equivalent to the number of ways to permute $n + r - 1$ objects such that

n are indistinct objects, and

$r - 1$ are indistinct dividers:

$$\text{Total} = (n + r - 1)! \times \frac{1}{n!} \times \frac{1}{(r-1)!}$$

$$= \binom{n + r - 1}{r - 1}$$

Integer solutions to equations

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

How many integer solutions are there to the following equation:

$$x_1 + x_2 + \cdots + x_r = n,$$

where for all i, x_i is an integer such that $0 \le x_i \le n$?

Objects : n units of 1

Buckets : x_i

Answer = $\binom{n+r-1}{r-1}$

Positive integer equations can be solved with the divider method.

Venture capitalists

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have $10 million to invest in 4 companies (in $1 million increments).

1. How many ways can you fully allocate your $10 million?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

$$x_i \geq 0$$

$n = 10$

$r = 4$

Answer $= \binom{10+4-1}{4-1} = \binom{13}{3} = 286$

Venture capitalists

Divider method (n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have $10 million to invest in 4 companies (in $1 million increments).

1. How many ways can you fully allocate your $10 million?
2. What if you want to invest at least $3 million in company 1?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

⚠ $3 \le x_1$

$x_i \geq 0$ for $i = 2, 3, 4$

Give $3M to Co. 1

$$\longrightarrow \quad x_1 + x_2 + x_3 + x_4 = 7$$

$$x_i \geq 0$$

$$\text{Ans} = \binom{7+4-1}{4-1} = \binom{10}{3} = 120$$

Venture capitalists

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have \$10 million to invest in 4 companies (in \$1 million increments).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?
3. What if you don't invest all your money?

Set up

$$x_1 + x_2 + x_3 + x_4 \leq 10$$

x_i: amount invested in company i

$$x_i \geq 0$$

Add a bucket for yourself $\longrightarrow$ $x_1 + x_2 + x_3 + x_4 + x_5 = 10$

$x_i \geq 0$

Ans: $\binom{10+5-1}{5-1} = \binom{14}{4} = 1001$

Today's plan

Key definitions: sample spaces and events

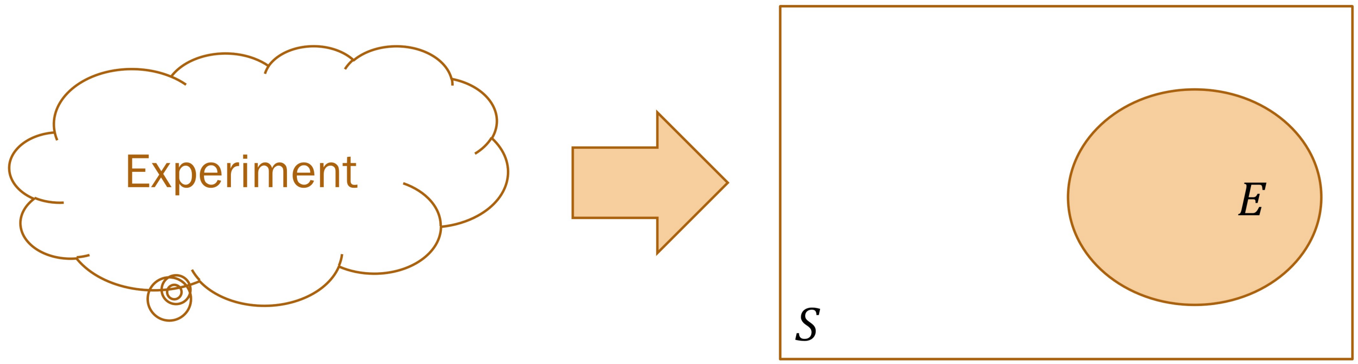
Axioms of Probability

Equally likely outcomes (counting)

Corollaries of Axioms of Probability

Key definitions

An experiment in probability:



Sample Space, S: The set of all possible outcomes of an experiment

Event, E: Some subset of S ($E \subseteq S$).

Key definitions

Sample Space, S

- Coin flip
  $S = \{\text{Heads}, \text{Tails}\}$
- Flipping two coins
  $S = \{(\text{H,H}), (\text{H,T}), (\text{T,H}), (\text{T,T})\}$
- Roll of 6-sided die
  $S = \{1, 2, 3, 4, 5, 6\}$
- # emails in a day
  $S = \{x \mid x \in \mathbb{Z}, \ x \geq 0\}$
- YouTube hours in a day
  $S = \{x \mid x \in \mathbb{R}, 0 \leq x \leq 24\}$

Event, E

- Flip lands heads
  $E = \{\text{Heads}\}$
- ≥ 1 head on 2 coin flips
  $E = \{(\text{H,H}), (\text{H,T}), (\text{T,H})\}$
- Roll is 3 or less:
  $E = \{1, 2, 3\}$
- Low email day (≤ 20 emails)
  $E = \{x \mid x \in \mathbb{Z}, \ 0 \leq x \leq 20\}$
- Wasted day (≥ 5 YT hours):
  $E = \{x \mid x \in \mathbb{R}, 5 \leq x \leq 24\}$

What is a probability?

A number between 0 and 1
to which we ascribe meaning.*

*our belief that an event E occurs.

What is a probability?

Frequentist definition

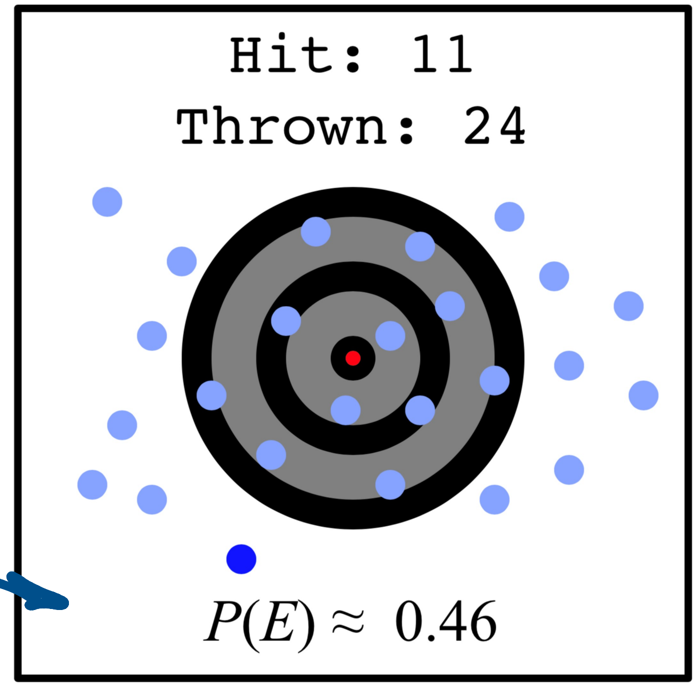
$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

n = # of total trials

$n(E)$ = # trials where E occurs

Let E = the set of outcomes where you hit the target.

$P(E) \approx \frac{11}{24}$



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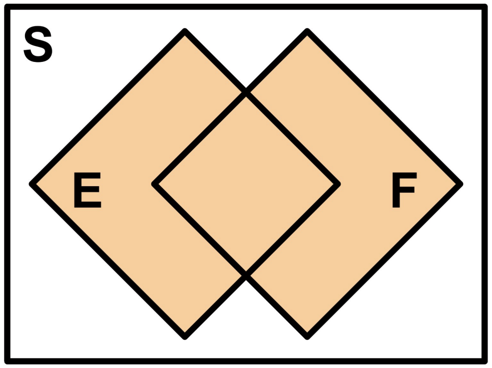
Quick review of sets



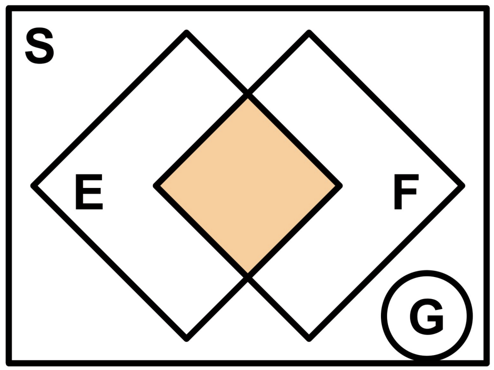
E and F are events in S.
Experiment:
Dice roll
$S = \{1, 2, 3, 4, 5, 6\}$
Let $E = \{1, 2\}$, and $F = \{2,3\}$

def Union of events, $E \cup F$

The event containing all outcomes in E *or* F.

$$E \cup F = \{1,2,3\}$$

Quick review of sets



E and F are events in S.
Experiment:
Dice roll
$S = \{1, 2, 3, 4, 5, 6\}$
Let $E = \{1, 2\}$, and $F = \{2,3\}$

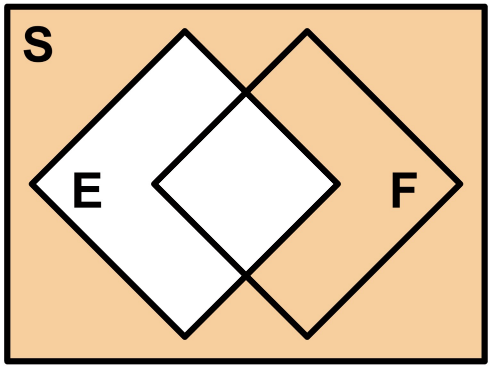
def Intersection of events, $E \cap F$

The event containing all outcomes in E *and* F.

def Mutually exclusive events F and G means that $F \cap G = \emptyset$

$$E \cap F = EF = \{2\}$$

Quick review of sets



E and F are events in S.
Experiment:
Dice roll
$S = \{1, 2, 3, 4, 5, 6\}$
Let $E = \{1, 2\}$, and $F = \{2,3\}$

<u>def</u> Complement of event E, E^C

The event containing all outcomes in S that are ***<u>not</u>*** in E.

$$E^C = \{3, 4, 5, 6\}$$

3 Axioms of Probability

Definition of probability: $P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$

Axiom 1: $0 \leq P(E) \leq 1$

Axiom 2: $P(S) = 1$

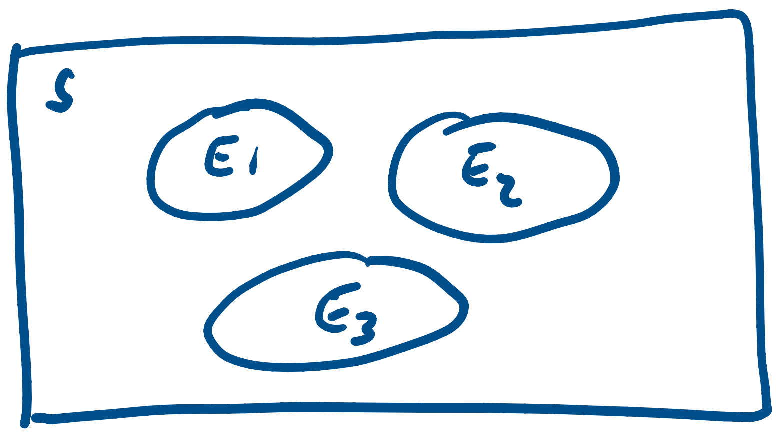
Axiom 3: If $E \cap F = \emptyset$, then $P(E \cup F) = P(E) + P(F)$

Axiom 3 is the (analytically) useful Axiom

Axiom 3: If E and F are mutually exclusive ($E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$

More generally, for any sequence of mutually exclusive events $E_1, E_2, \ldots$:

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$



(like the Sum Rule of Counting, but for probabilities)

Today's plan

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Equally likely outcomes (counting)

Corollaries of Axioms of Probability

Equally Likely Outcomes

Some sample spaces have equally likely outcomes.

- Coin flip: S = {Head, Tails}
- Flipping two coins: S = {(H, H), (H, T), (T, H), (T, T)}
- Roll of 6-sided die: S = {1, 2, 3, 4, 5, 6}

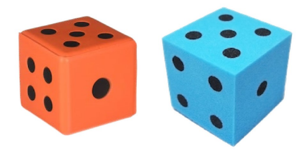
$$P(\text{each outcome}) = \frac{1}{|S|}$$

Then $P(E) = \frac{|E|}{|S|}$ by Axiom 3

Roll two dice

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Roll two 6-sided dice. What is P(sum = 7)?



$S = \{(1,1), (1,2), \ldots, (6,6)\}$ $|S| = 36$

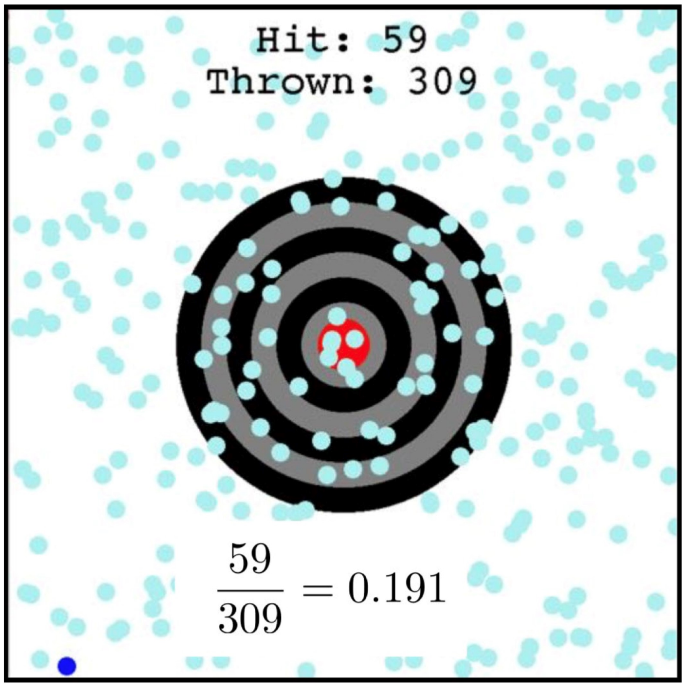
$E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$ $|E| = 6$

$P(E) = \frac{6}{36} = \frac{1}{6}$

Target revisited

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Let E = the set of outcomes where you hit the target.



The dart is equally likely to land anywhere on the screen.

What is $P(E)$, the probability of hitting the target?

Screen size = 800×800 $|S| = 800^2$

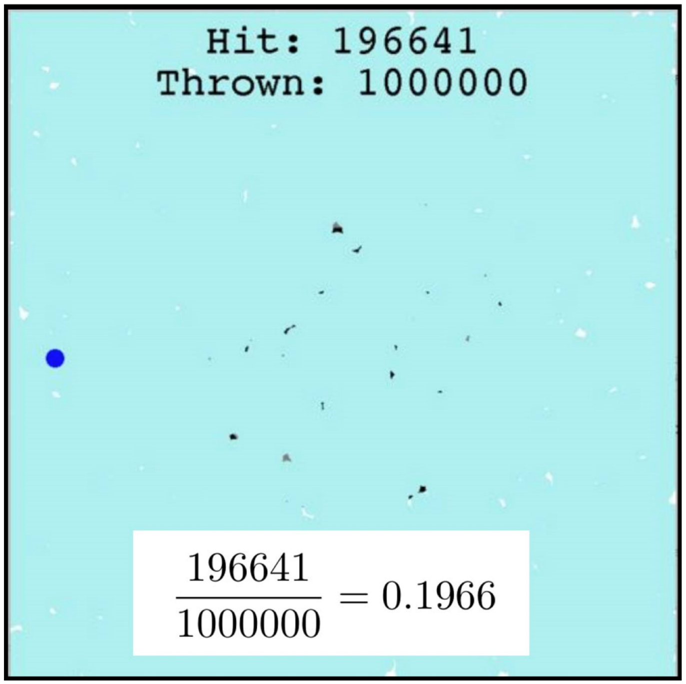
Radius of target: 200 $|E| = \pi \cdot 200^2$

$$P(E) = \frac{|E|}{|S|} = \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$

Target revisited

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Let E = the set of outcomes where you hit the target.



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Not equally likely outcomes

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Play the lottery.

What is $P(\text{win})$?

Not equally likely

$S = \{\text{Lose, Win}\}$

$E = \{\text{Win}\}$

$$P(E) = \frac{|E|}{|S|} = \frac{1}{2} = 50\%?$$ No!

The hard part: defining equally likely outcomes ***consistently*** across sample space and events

Cats and carrots

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 carrots in a bag. 3 drawn.

What is P(1 cat and 2 carrots drawn)?

Note: Do indistinct objects give you an equally likely sample space?

Make indistinct items distinct

$\Rightarrow$ equally likely

Cats and carrots (ordered solution)

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 carrots in a bag. 3 drawn.

What is P(1 cat and 2 carrots drawn)?

Define

- S = Pick 3 distinct items
- E = 1 distinct cat,
  2 distinct carrots → Pick cat 1st

$|S| = 7 \cdot 6 \cdot 5 = 210$

Pick cat 1st $\quad 4 \cdot 3 \cdot 2 = 24$

2nd $\quad 3 \cdot 4 \cdot 2 = 24$

3rd $\quad 3 \cdot 2 \cdot 4 = 24$

$|E| = 72$

$P(E) = \frac{72}{210} = \frac{12}{35}$

Cats and carrots (unordered solution)

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 carrots in a bag. 3 drawn.

What is P(1 cat and 2 carrots drawn)?

Define

- S = Pick 3 distinct items
- E = 1 distinct cat,
  2 distinct carrots

$|S| = \binom{7}{3} = 35$

$|E| = \binom{4}{1}\binom{3}{2} = 12$

$P(E) = \frac{12}{35}$

Announcements

Section sign-ups

Preference form:	out
Due:	Saturday 1/11
Results:	latest Monday

Python tutorial

When:	Friday 3:30-4:20pm
Location:	420-040
Notes:	to be posted online
Installation:	On Piazza

Any Poker Straight

Consider 5-card poker hands.

- “straight” is 5 consecutive rank cards of any suit

What is P(Poker straight)?

A♡ 2◇ 3◇ 4♣ 5◇

$$\left.\begin{array}{l} A \to 5 \\ 2 \to 6 \\ \vdots \\ 9 \to K \\ 10 \to A \end{array}\right\} 10$$

- What is an example of an outcome?
- Is each outcome equally likely?
- Should objects be ordered or unordered?

Any Poker Straight

Consider 5-card poker hands.

- “straight” is 5 consecutive rank cards of any suit

What is P(Poker straight)?

Define

- S (unordered)
- E (unordered, consistent with S)

$|S| = \binom{52}{5}$

# of starting ranks

$|E| = 10 \binom{4}{1}^5$ — # of choices of suits

$$\frac{10\binom{4}{1}^5}{\binom{52}{5}} = 0.00394$$

“Official” Poker Straight

Consider 5-card poker hands.

- “straight” is 5 consecutive rank cards of any suit
- “straight flush” is 5 consecutive rank cards of same suit

What is P(Poker straight, but not straight flush)?

Define

- S (unordered) $|S| = \binom{52}{5}$
- E (unordered, consistent with S) $|E| = 10\binom{4}{1}^5 - 10\binom{4}{1}$

$$\frac{10\binom{4}{1}^5 - 10\binom{4}{1}}{\binom{52}{5}} \simeq 0.00392$$

Chip defect detection

n chips are manufactured, 1 of which is defective.
k chips are randomly selected from n for testing.

What is P(defective chip is in k selected chips?)

Define

- S (unordered) $\quad |S| = \binom{n}{k}$
- E (unordered, consistent with S) $\quad |E| = \binom{1}{1}\binom{n-1}{k-1}$

$$P(E) = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$$

Chip defect detection (solution 2)

n chips are manufactured, 1 of which is defective.
k chips are randomly selected from n for testing.

What is P(defective chip is in k selected chips?)

Redefine experiment

1. Choose k indistinct chips (1 way)
2. Wave a wand and make one defective

Define

- S (unordered)
- E (unordered, consistent with S)

$|S| = 1 \cdot n$

$|E| = 1 \cdot k$

$P(E) = \frac{k}{n}$

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Axioms of Probability

Review

Definition of probability: $P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$

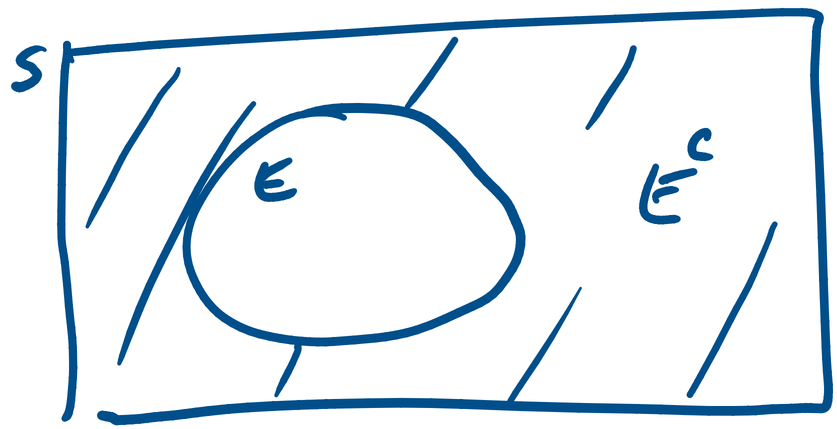
Axiom 1: $0 \le P(E) \le 1$

Axiom 2: $P(S) = 1$

Axiom 3: If E and F are mutually exclusive ($E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$

3 Corollaries of Axioms of Probability

Corollary 1: $P(E^C) = 1 - P(E)$



Proof of Corollary 1

Corollary 1: $P(E^C) = 1 - P(E)$

Proof:

E, E^C are mutually exclusive	Definition of E^C
$P(E \cup E^C) = P(E) + P(E^C)$	Axiom 3
$S = E \cup E^C$	Everything must either be in E or E^C, by definition
$1 = P(S) = P(E) + P(E^C)$	Axiom 2
$P(E^C) = 1 - P(E)$	Rearrange

3 Corollaries of Axioms of Probability

Corollary 1: $P(E^C) = 1 - P(E)$

Corollary 2: If $E \subseteq F$, then $P(E) \leq P(F)$

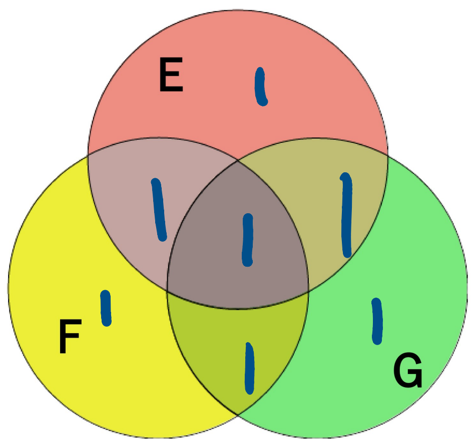
Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$
(Inclusion-Exclusion Principle for Probability)

Inclusion-Exclusion Principle (Corollary 3)

Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$

(Inclusion-Exclusion Principle for Probability)

General form: $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} (-1)^{(r+1)} \sum_{i_1 < \cdots < i_r} P\left(\bigcap_{j=1}^{r} E_{i_j}\right)$



$P(E \cup F \cup G) =$

r = 1: $P(E) + P(F) + P(G)$

r = 2: $- P(E \cap F) - P(E \cap G) - P(F \cap G)$

r = 3: $+ P(E \cap F \cap G)$

Selecting Programmers

$P(E \cup F) = P(E) + P(F) - P(EF)$ Inclusion-Exclusion

- P(student programs in Java) = 0.28 $= P(E)$
- P(student programs in Python) = 0.07 $= P(F)$
- P(student programs in Java and Python) = 0.05. $= P(EF)$

What is P(student does not program in (Java or Python))?

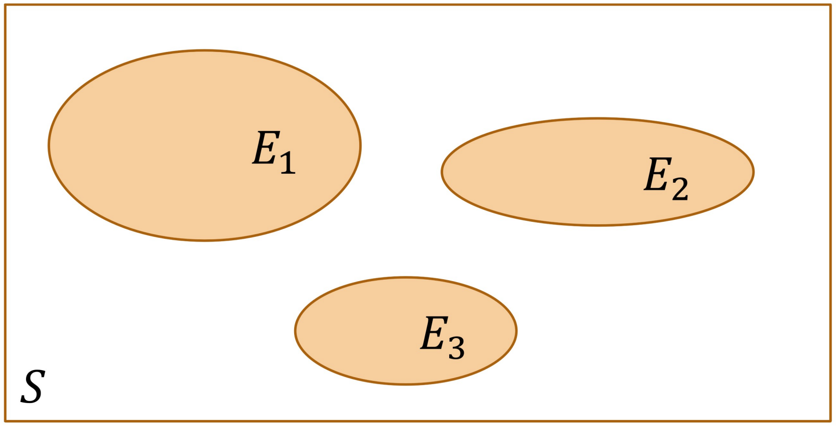
1. Define events & state goal
2. Identify <u>known</u> probabilities
3. Solve

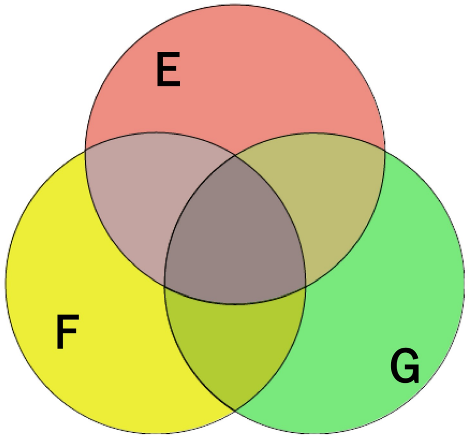
E: Student programs in Java

F: " " Python

$$P((E \cup F)^c) = 1 - P(E \cup F)$$
$$= 1 - (P(E) + P(F) - P(EF))$$
$$= 1 - (0.28 + 0.07 - 0.05)$$
$$= 0.7$$

Takeaway: Mutually exclusive events





Axiom 3,
Mutually exclusive events

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Inclusion-Exclusion Principle

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} (-1)^{(r+1)} \sum_{i_1<\cdots<i_r} P\left(\bigcap_{j=1}^{r} E_{i_j}\right)$$

Design your experiment to compute easier probabilities.

Serendipity

- The population of Stanford is $n = 17{,}000$ people.
- You are friends with $r = 100$ people.
- Walk into a room, see $k = 268$ random people.
- Assume you are equally likely to see each person at Stanford.

What is the probability that you see someone you know?

Serendipity

- The population of Stanford is $n = 17{,}000$ people.
- You are friends with $r = 100$ people.
- Walk into a room, see $k = 268$ random people.
- Assume you are equally likely to see each person at Stanford.

What is the probability that you see someone you know?

$|E^c| = \binom{16900}{268}$

Define

- S (unordered)
- E: see ≥ 1 friend in the room

$$|S| = \binom{n}{k} = \binom{17000}{268}$$

$$P(E) = 1 - \frac{\binom{16900}{268}}{\binom{17000}{268}}$$

How should we compute $P(E)$?

A. $P(\text{exactly } 1) + P(\text{exactly } 2)$ $P(\text{exactly } 3) + \cdots$

B. $1 - P(\text{see no friends})$

It is often much easier to compute $P(E^c)$.

The Birthday ~~Paradox~~ Problem

What is the probability that in a set of *n* people, <u>at least one</u> pair of them will share the same birthday?

For you to think about (and discuss in section!)