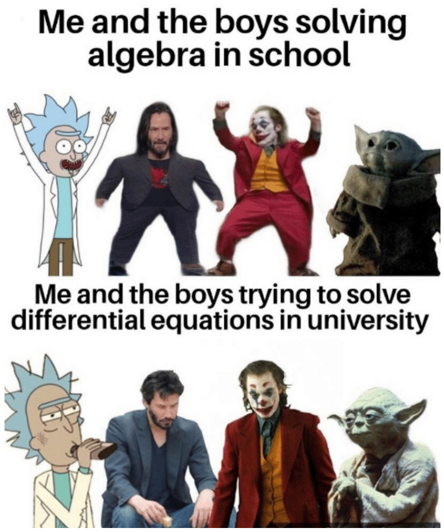

TODAY:

3:40 - 4:20 Sys of diff Eq & Linear Algebra

(include important example for HW12 Q4)

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1. Linear Algebra (Review, assume 241)

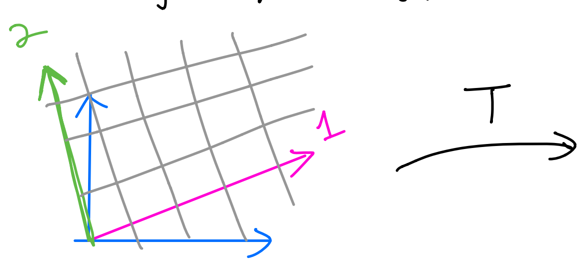
- what is the dimension of a system of equations?

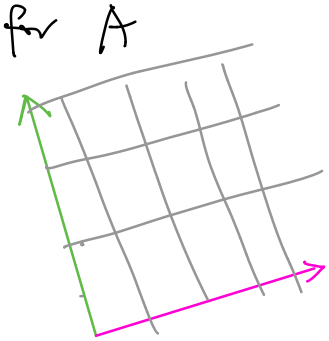
  # of dependent variable

- what are eigenvalues/eigenvector of a matrix? How do we find the eigenpairs?

  v is a eigenvector for A

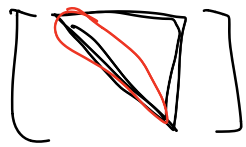
  if $Av = \lambda v$





- What is a diagonal matrix?
  Why are they nice?

$$D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$



- What is diagonalization?

$$A = P D P^{-1}$$

change of basis

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2. System of Diff Eq

- How can we solve $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = D \begin{bmatrix} x \\ y \end{bmatrix}$, where D is diagonal?

$$\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$$

→ no LA:

$$x' = d_1 x \Rightarrow e^{d_1 t}$$

$$y' = d_2 y \Rightarrow e^{d_2 t}$$

→ LA: $\lambda_1 = d_1, \ \lambda_2 = d_2$

$$\left\{ e^{d_1 t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ e^{d_2 t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

- How can we solve $\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix} = A\begin{bmatrix}x\\y\end{bmatrix}$,

  where A is not diagonal BUT diagonalizable?

  Remark: diagonalizable $\Leftarrow$ distinct eigenvalues

$$\{e^{\lambda_1 t} v_1,\ e^{\lambda_2 t} v_2\}$$

- How can we solve $\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix} = A\begin{bmatrix}x\\y\end{bmatrix}$,

  where A is not diagonal and not diagonalizable?

  (i.e. A has repeated eigenvalue)

find. set of sln. $\{e^{\lambda t} v_1,\ e^{\lambda t} w + t e^{\lambda t} v_1\}$

$(x-\lambda)^2$ $\quad\lambda \to$ eigenvalue

$(A-\lambda I)v$

$=(A-\lambda I)^2 w = 0$

$(A-\lambda I)\begin{bmatrix}x\\y\end{bmatrix} = 0 \Rightarrow v_1$

① $(A-\lambda I)w = v_1$ ② $(A-\lambda I)^2 w = 0$ Jordan decomposition

3. Example

$(A-\lambda I)v = 0 \hookrightarrow v \in Nul(A-\lambda I)$

$\ker(A-\lambda I)$

$w \in \ker\left((A-\lambda I)^2\right)$

Consider the 3-D sys of lin. equations

$$X' = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} X$$

Find a fundamental set of sln.

$$A - \lambda I = \begin{vmatrix} -2-\lambda & 0 & 1 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix}$$

$$\det|A-\lambda I| = (-2-\lambda)(1-\lambda)(-2-\lambda)$$
$$= (-2-\lambda)^2(1-\lambda)$$

$\lambda_1 = 1$, $\lambda_2 = -2$

v_1 $\quad v_2$

$$(A+2I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$A - 1I = \begin{bmatrix} -2-1 & 0 & 1 \\ 1 & 1-1 & 0 \\ 0 & 0 & -2-1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$(A-1I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3x+z \\ x \\ -3z \end{bmatrix} = 0$$

$x=0, \quad 3x = z = 0 \qquad y$

$v_1 \in \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix}$ pick $v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$A+2I = \begin{bmatrix} -2+2 & 0 & 1 \\ 1 & 1+2 & 0 \\ 0 & 0 & -2+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A+2I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ x+3y \\ 0 \end{bmatrix}$$

$z = 0$

$x = -3y$

$$\begin{bmatrix} -3y \\ y \\ 0 \end{bmatrix} = y\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

find set of sln. $e^{t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e^{-2t}\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{cases} z = -3 \\ x+3y = 1 \Rightarrow x = 1-3y \end{cases}$$

$$\begin{bmatrix} 1-3y \\ y \\ -3 \end{bmatrix} \quad y=0 \downarrow \quad w = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

find set of sln. $\left\{ e^{t}\begin{bmatrix}0\\1\\0\end{bmatrix},\ e^{-2t}\begin{bmatrix}-3\\1\\0\end{bmatrix}\right.$

$$\left. te^{-2t}\begin{bmatrix}-3\\1\\0\end{bmatrix}+e^{-2t}\begin{bmatrix}1\\0\\-3\end{bmatrix} \right\}$$

Sln: $c_1 e^{t}\begin{bmatrix}0\\1\\0\end{bmatrix}+c_2e^{-2t}\begin{bmatrix}-3\\1\\0\end{bmatrix}+$

$$c_3\left(te^{-2t}\begin{bmatrix}-3\\1\\0\end{bmatrix}+e^{-2t}\begin{bmatrix}1\\0\\-3\end{bmatrix}\right)$$