Q1. Let $C := \bigcap \{ X \in \mathcal{P}(A) : f(X) \subseteq X \}$

Claim. C is a fixed point

NTS. $f(C) = C$

proof. Let $B := \{ X \in \mathcal{P}(A) : f(X) \subseteq X \}$

By def. $\forall X \in B : C \subseteq X$

By monotone: $\forall X \in B : f(C) \subseteq f(X) \subseteq X$ (1)

So $f(C) \subseteq \bigcap_{X \in B} X = C$

either $f(C) = C$, then we're done

or $f(C) \subsetneq C$. then $\exists x \in C : x \notin f(C)$

then $\exists X \in B : f(C) \nsubseteq X$

This contradicts (1), thus $f(C) = C$ $\square$

Q2. (a) f is monotone

let $X, Y \in P(A)$ such that $X \subseteq Y$ NTS $f(X) \subseteq f(Y)$

Proof. $f(X) = A \setminus h[B \setminus g(X)]$

$f(Y) = A \setminus h[B \setminus g(Y)]$

By g injective, $X \subseteq Y$

$\Rightarrow \quad g(X) \subseteq g(Y)$

$\Rightarrow \quad B \setminus g(X) \supseteq B \setminus g(Y)$

By h injective:

$h[B \setminus g(X)] \supseteq h[B \setminus g(Y)]$

$\Rightarrow A \setminus h[B \setminus g(X)] \subseteq A \setminus h[B \setminus g(Y)]$

$\Rightarrow f(X) \subseteq f(Y)$, as desired. $\square$

(b) By Q1, f has a fixed point $C \in P(A)$

$f(C) = A \setminus h[B \setminus g(C)] = C$

$A \setminus C = A \setminus (A \setminus h[B \setminus g(C)]) = h[B \setminus g(C)]$

So $A \setminus C \subseteq \operatorname{Im}(h)$

$\Rightarrow h^{-1}$ well-defined on $A \setminus C$

Define $l: A \to B$ by

$$l(a) = \begin{cases} g(a), & \text{if } a \in C \\ h^{-1}(a), & \text{if } a \in A \setminus C \end{cases}$$

claim: l is a bijection

proof.

$$h^{-1}(A \setminus C) = B \setminus g(C)$$

$$g(C) = g(C)$$

So $l(a)|_{a \in C}$ is a bijection $C \cong g(C)$

$l(a)|_{a \in A \setminus C}$ is a bijection $A \setminus C \cong B \setminus g(C)$

Since $g(C) \cap (B \setminus g(C)) = \phi$ and h, g injective, we can define $\xi: B \to A$ by

$$\xi(b) = \begin{cases} g^{-1}(b), & \text{if } b \in g(C) \\ h(b), & \text{if } b \in B \setminus g(C) \end{cases}$$

then $\forall b \in B$: $l(\xi(b)) = b \Rightarrow \xi = l^{-1} \Rightarrow l$ bijective $\square$

Q3. A infinite and

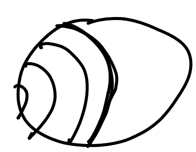
$\exists <$ such that $(A, <)$ is well order

NTS: $\exists f: \omega \to A$

proof. Define $f: \omega \to A$ such that

$$f(n) = \begin{cases} \min A, & \text{if } n = 0 \\ \min\{A / f(0)\}, & \text{if } n = 1 \\ \min\{A / \bigcup_{m<n} f(m)\}, & \text{otherwise} \end{cases}$$

claim: f is injective



Suppose $\forall k < n$, $\forall m \in \omega$,

(I.H.) $f(k) = f(m) \Rightarrow k = m$

NTS $\forall k' < n+1$, $\forall m \in \omega$

$f(k') = f(m) \Rightarrow k' = m$

showing. if $k' < n$, we're done by I.H.

otherwise $k' = n$,

$$f(k') = \min\{A / \bigcup_{\ell < n} f(\ell)\}$$

$$f(m) = \min\{A / \bigcup_{\ell < m} f(\ell)\}$$

By $f(k') = f(m)$, we know

$$\bigcup_{\ell < k'} f(\ell) = \bigcup_{\ell' < m} f(\ell')$$

Since $f(\ell) = f(\ell') \Rightarrow \ell = \ell'$, $\forall \ell < n$
it follows that $k' = m$

claim: f is order-preserving

Let $n \in \omega$, consider $f(n)$ and $f(n+k)$, $k >$

$f(n) = \min\{A / \bigcup_{m<n} f(m)\}$

$f(n+k) = \min\{A / \bigcup_{m<n+k} f(m)\}$

By f injective:

$$\bigcup_{m<n} f(m) \subset \bigcup_{m<n+k} f(m)$$

$$\Rightarrow \quad A / \bigcup_{m<n} f(m) \supset A / \bigcup_{m<n+k} f(m)$$

So $\min\{A / \bigcup_{m<n} f(m)\} \leq \min\{A / \bigcup_{m<n+k} f(m)\}$

if $\min\{A / \bigcup_{m<n} f(m)\} = \min\{A / \bigcup_{m<n+k} f(m)\}$

then $\bigcup_{m<n} f(m) = \bigcup_{m<n+k} f(m)$

$$\Rightarrow \bigcup_{n \leq m < n+k} f(m) = \phi$$

for $k \geq 2$, contradict that f is injective

if $k = 1$: $f(n) = \phi \Rightarrow$ there is an nonempty subset of A with no minimum element

contradict that A is well order.

A is infinite

So the inequality must be strict

$$\min\{A / \bigcup_{m<n} f(m)\} < \min\{A / \bigcup_{m<n+k} f(m)\}$$

$$f(n) < f(n+k)$$

claim. If A is infinite and $|A| \leq |\omega|$ then $|A| = |\omega|$

proof. we proved that there exists an injective map $f : \omega \hookrightarrow A$, so $|\omega| \leq |A|$. If $|A| \leq |\omega|$, it follows that $|A| = |\omega|$

Q4. ($\Rightarrow$) Let $X \subseteq \mathbb{R}$ be closed

NTS $\mathbb{R} \setminus X$ is open

i.e. $\forall x \in \mathbb{R}\setminus X$ there is an open interval I such that $x \in I \subseteq \mathbb{R}\setminus X$

proof. ① Suppose X is an interval,

let $a = \inf(X)$, $b = \sup(X)$

by X closed, $a, b \in X$

let $y \in \mathbb{R}\setminus X$

then either $y < a \Rightarrow y \in (-\infty, a)$

or $y > b \Rightarrow y \in (b, \infty)$

② Suppose X is not an interval

then X is the disjoint union of a family $\mathcal{A}$ of intervals. Formally,

$$X = \bigsqcup_{A \in \mathcal{A}} A$$

then $\forall A \in \mathcal{A}$, A is closed.

Why? Suppose not, then there exists a sequence $x_n \subseteq A$ such that $\lim_{n\to\infty} x_n \notin A$

Also $\lim_{n\to\infty} x_n \notin B$, $\forall B \in \mathcal{A}$ because $A \cap B = \phi$

so $\lim_{n\to\infty} x_n \notin X \Rightarrow X$ is not closed $\Rightarrow\!\Leftarrow$

let $y \in \mathbb{R} \setminus X$,

Define $a := \sup(s : s = \sup(A), A \in \mathcal{A}, s < y)$
$b := \inf(i : i = \inf(A), A \in \mathcal{A}, i > y)$

sup inf exist?

$a, b \in X$ because $\forall A \in \mathcal{A}$, by A is closed, $\sup(A), \inf(A) \in A \Rightarrow \sup(A), \inf(A) \in X$

then $y \in (a, b)$ and $(a, b) \subseteq \mathbb{R} \setminus X$

why? AFSOC $\exists k \in (a, b) : k \notin \mathbb{R} \setminus X$
then $k \in X \Rightarrow \exists A \in \mathcal{A} : k \in A$
if $k < y$, then a is not supremum ↯
if $k > y$, then b is not infimum ↯
if $k = y$, then $k \notin X$ ↯

Thus $\mathbb{R} \setminus X$ must be closed $\square$

Q5. Let $X \subseteq \mathbb{R}$

(1) Let $A := \bigcap \{C \subseteq \mathbb{R} : X \subseteq C \text{ and } C \text{ closed}\}$
Then $\overline{X} = A$

Proof. $(\Rightarrow)$ $\overline{X} \subseteq A$

Let $x \in \overline{X}$, if $x \in X$ then by def, $x \in A$
if $x \notin X$, then there exists a sequence
$(x_n)_{n \in \omega} \to x$.

AFSOC, $x \notin A$ then there exists a closed
set $C \subseteq \mathbb{R}$ such that $X \subseteq C$ and $x \notin C$.
but then C is not closed since $x \notin C$ ↯
Thus $x \in A$

$(\Leftarrow)$ $A \subseteq \overline{X}$

let $a \in A$. if $a \in X$ then $\{a\}_\omega \to a$
$\Rightarrow a \in \overline{X}$

if $a \notin X$, by lemma, a must be
a limit point of $X \Rightarrow a \in \overline{X}$

Lemma. $A = X \cup \{x : \exists (x_n)_{n\in\omega} \subseteq X : x_n \to x\}$

($\Rightarrow$) let $a \in A$, if $a \in X$ then we're done

if $a \notin X$ then a belongs to all closed set containing X.

a must be a limit point of X otherwise A is not closed

($\Leftarrow$) let $x \in X$, then $x \in A$ by def

let x be a limit point of X, then $x \in A$

otherwise, there exists C such that C does not contain limit point of a sequence in $X \Rightarrow C$ is not closed $\lightning$

(2) ($\Rightarrow$) let $x \in \bar{X}$, if $x \in X$ then $x \in I \cap X$

$$\Downarrow$$

$$I \cap X \neq \phi$$

if $x \notin X$ then x is a limit point of X

let $I = (x - \varepsilon_1, x + \varepsilon_2)$ be an open interval containing x. let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$

Then $\exists N \in \mathbb{N} : \forall m \geq N : |x_m - x| < \frac{\varepsilon}{2}$

So $x_m \in I \cap X \Rightarrow I \cap X \neq \phi$

($\Leftarrow$) If $x \in X$ then $(x)_{n \in \mathbb{N}} \to x$

and $x \in \overline{X}$

If $x \notin X$ then NTS x is a limit point of X

let $\varepsilon > 0$, then $(x-\varepsilon, x+\varepsilon)$ is an open interval containing x. By assumption,

$(x-\varepsilon, x+\varepsilon) \cap X \neq \phi \Rightarrow$

$\exists\, y \in X : \ |y - x| < \varepsilon$

$\Rightarrow$ x is a limit point of X $\square$