Thm. If $(A,<)$ is a well-ordered set then $\exists!\,\alpha$ such that α is ordinal and $(A,<)$ is isomorphic to $(\alpha,\in)$

Proof. uniquess: by def of isomorphism

Existence Define a function F with domain A by recursion on A by

$F(x) = \{ F(y) \mid y < x \}$ such F exists by recursive func. Thm.

(Formally, $\Phi(x,y,f)$ is the statement that "$z \in y$ iff $\exists w \in x\,(z = f(x))$"

Claim1 $\forall x \in A$, If $w \in F(x)$, then there is $y < x$ with $w = F(y)$. By def $\Rightarrow$ image of F is some set

Claim2 $\forall a$: $F(a) \notin F(a)$

proof by induction.

Assume $\forall b < a$, $F(b) \notin b$

AFSOC, if $F(a) \in F(a) \Rightarrow \exists\, b < a : F(b) = F(a)$

but $F(b) \in F(a) \Rightarrow F(b) \in F(b)$ ↯

claim3 $\forall\, a, b \in A$, either $\mathcal{F}(a) = \mathcal{F}(b)$ or $\mathcal{F}(a) \in \mathcal{F}(b)$ or $\mathcal{F}(b) \in \mathcal{F}(a)$

proof. By $(A, <)$ well ordered:

either $a = b \Rightarrow \mathcal{F}(a) = \mathcal{F}(b)$ or

$a < b \Rightarrow \mathcal{F}(a) \in \mathcal{F}(b)$ or

$b < a \Rightarrow \mathcal{F}(b) \in \mathcal{F}(a)$ $\square$

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claim. Let $\alpha = \mathrm{Im}(\mathcal{F}) = \{ F(a) \mid a \in A \}$
then α is an ordinal

proof. α is transitive because

if $w \in \mathcal{F}(a) \in \alpha$, NTS $w \in \alpha$

indeed, $\exists\, b < a$ with $\mathcal{F}(b) = w$

so by def, $w = \mathcal{F}(b) \in \alpha$

$F : A \to \alpha$ is order preserving map
between $(A, <)$ and $(\alpha, \in)$

why? $b < a \Rightarrow F(b) \in F(a)$

since $F(a) \notin F(a)$, then F is one-to-one

So $(A, <)$ and $(\alpha, \in)$ are isomorphic

via $\mathcal{F}$. It follows that $(\alpha, \in)$ is
a well order.

Thm. (Hartog) If X is a set, there is an ordinal α such that there is no injective map from α to X.

e.g. for $X=\omega$ there is an ordinal α such that there exists no injective map $\alpha \hookrightarrow \omega$

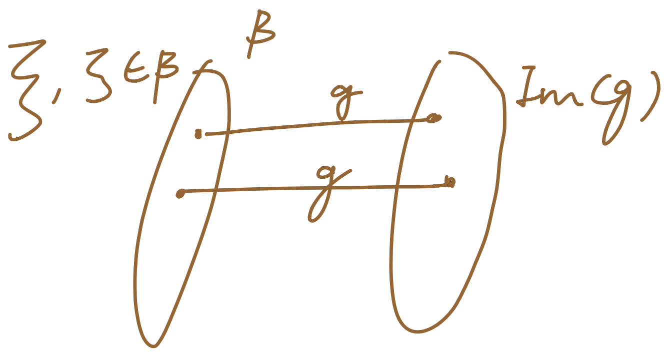
i.e. there is ALWAYS a BIGGER set.

proof. want to construct α recursively, ruling out potential injective map $\alpha \hookrightarrow X$.

Note Given β ordinal and $g: \beta \hookrightarrow X$ injective we may define an order $<$ on $\mathrm{Im}(g) \subseteq X$ by

want to rule out all injective map

$$\xi \in \zeta \Rightarrow g(\xi) < g(\zeta)$$



and $(\mathrm{Im}(g), <)$ is isomorphic to $(\beta, \in)$

(we proved that such β is unique;
want to construct α such that it's
"bigger" than any β)

Define $A = \{ \beta \mid \beta$ is an ordinal and there is a subset $C \subseteq X$ and a well order $<$ on C s.t. $(C, <)$ is isomorphic to $\beta \}$

why A a set?

first $\mathcal{P}(X) \times \mathcal{P}(X \times X)$

subset of X — orders on subsets of X

$(<$ order on C
$\Rightarrow\ < \subseteq C \times C \subseteq X \times X)$

$$D \subseteq \mathcal{P}(X) \times \mathcal{P}(X \times X)$$

$$= \{ (C, <) \mid C \subseteq X \text{ and } < \text{ is a well order on } C \}$$

by comprehension

Note:
(Replacement requires unique mapping,
otherwise could map to unbound objects)

By replacement, A is a set.

N.T.S. 1) A is an ordinal

2) there is no injective map $A \hookrightarrow X$

Assume 1), NTS 2)

AFSOC, let $g: A \hookrightarrow X$

use g to define a well order $<$ on

$C = \mathrm{Im}(g) \subseteq X$. Then by def, get $A \in A$

but A is ordinal $\Rightarrow\Leftarrow$

Lecture 12 - 2/10

Pf of 4 α is transitive by defn. If $\omega \in F(a) \in \alpha$, then $\omega = F(b), b < a$ and $F(b) \in \alpha$

~~$(\alpha, \in)$ is a well-ordering:~~

now $F: A \to \alpha$ is order preserving. Since $F(a) \notin F(a) \Rightarrow F$ is 1-to-1

So $(A, <)$ and $(\alpha, \in)$ are isomorphic via F. It follows that $(\alpha, \in)$ is a well-order (on HW).

Thm (Hartog): If X is a set, there is an ordinal α s.t. there is no injective map from α to X

e.g. for $X = \omega$ there is an ordinal α s.t. there is no injective map $\alpha \hookrightarrow \omega$ (α is not countable)

Pf Want to construct α recursively, ruling out potential injective maps.

Note: given β an ordinal and $g: \beta \to X$ injective.

We may define an order $<$ on $Im(g) \subseteq X$ by $\xi \in \zeta \Rightarrow g(\xi) < g(\zeta)$, $\xi, \zeta \in \beta$

and $(Img(g), <)$ is isomorphic $(B, \in)$

Define $A = \left\{ \beta \,\middle|\, \begin{array}{l} \beta \text{ is an ordinal and there is a subset } C \subseteq X \text{ and} \\ \text{well-order on } C \text{ s.t. } (C, <) \text{ is isomorphic to } (B, \in) \end{array} \right\}$

Why is A a set? First $\mathcal{P}(X) \times \mathcal{P}(X \times X)$

WTS A is an ordinal and there is no injective map $A \to X$.

Assume A an ordinal. If $g: A \to X$, g injective, use g to define a well-order $<$ on $Im(g)$. Then by defn ~~$Im(g)$~~ $A \in A$ $\times$ (shown in HW)

WTS A an ordinal. Since A is a set of ordinals, $\in$ is a wellordering.

This is b/c we saw for any 2 ordinals α, β either $\alpha = \beta$, $\alpha \in \beta$ or $\beta \in \alpha$.

in HW, if α is an ordinal then $\alpha \notin \alpha$, If α is an ordinal, then α is transitive by defn, $(\gamma \in \alpha \in \beta \Rightarrow \gamma \in \beta)$. So $\in$ is irreflexive, transitive, and total/linear

$\in$ is a well-order b/c if $Y \subseteq A$, $Y \neq \emptyset$ w/ some $\alpha \in Y$. If α is minimal then we're done. Otherwise, $\alpha \cap X \neq \emptyset$, $\alpha \cap X \subseteq \alpha$ nonempty and there is a minimal element $b \in X \cap \alpha$.

Note: If B is any set of ordinals, $\in$ is a wellordering of B. (justified above)

WTS A is transitive. If $\beta \in \alpha \in A$, $\beta \in A$. Let $\beta \in \alpha \in A$. $\alpha \in A \Rightarrow$ $\exists C \subseteq X$ and well-order on C s.t. $(C, <) \cong (\alpha, \in)$. Fix an $f: \alpha \to C$ order-pres. bij.

Then $f \restriction \beta$ will be order-preserving, bij and $Im(f \restriction \beta) \subseteq C \subseteq X \Leftrightarrow \beta \in A$

Lecture 13 - 2/12

defn $|A| \leq |B|$ if there exists $g: A \to B$ injective

Thm (Cantor-Schroder-Bernstein): $|A| \leq |B| \wedge |B| \geq |A| \Rightarrow |A| = |B|$

defn A is countable if $|A| \leq |\omega|$

note/fact: If $|A| \leq |\omega|$ and A is infinite $|A| = |\omega|$

Cantor: $\forall A, |P(A)| > |A|$ (there is no surjective map $A \to P(A)$)

- $|P(\omega)| = 2^{\omega} = |\mathbb{R}|$

Continuum Hypothesis (cantor): "$\mathbb{R}$" is the minimal uncountable size.

i.e. $\forall X \subseteq \mathbb{R}$ either $|X| \leq |\omega|$ or $|X| = |\mathbb{R}|$

defn a set $X \subseteq \mathbb{R}$ is closed if for any $x_1, x_2, \ldots \in X$ if $x = \lim_{n \to \infty} x_n$ exists, then $x \in X$.

ex $[0,1]$, $[0,1] \cup [3,5]$

$\{\frac{1}{n} \mid n \in \omega\} \cup \{0\}$

Thm If $X \subseteq \mathbb{R}$, X closed, then either $|X| \leq |\omega|$ or $|X| = |\mathbb{R}|$

Reminder a set $X \subseteq \mathbb{R}$ is closed if whenever $x_i \in X$, $\lim_{i\to\infty} x_i = x$ then $x \in X$

Goal (Cantor–Bendixson)

If $X \subseteq \mathbb{R}$ is closed then $|X| \leq |\mathbb{N}|$ or $|X| = |\mathbb{R}|$

Def A set $X \subseteq \mathbb{R}$ is *perfect* if it is closed and has no isolated points.

A point $x \in X$ is isolated if there is some interval I such that $I \cap X = \{x\}$

Examples

$X = \{0\} \cup \{\frac{1}{n}, n \in \mathbb{N}\}$

0 is isolated on X, but others aren't

$[0,1]$ is perfect.

$x \in X$ is NOT isolated if for any $N \in \mathbb{N}$, can find $y \in X \cap (x - \frac{1}{N}, x + \frac{1}{N})$ with $y \neq x$

$x \in X$ is NOT isolated if we can find $x_n \in X$, $x_n \neq x$ such that $\lim_{n\to\infty} x_n = X$

Lemma. If $X \subseteq \mathbb{R}$ is perfect, $X \neq \emptyset$, then

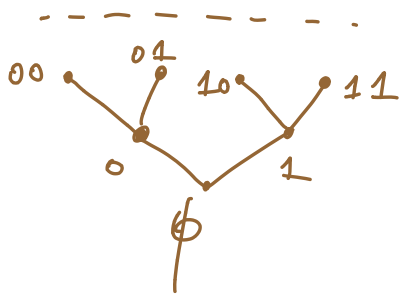
$$|2^\omega| \leq X$$ (Given $|2^\omega| = |\mathbb{R}|$, we conclude $|X| = |\mathbb{R}|$)

$a \in 2^\omega$

$n \in \omega$

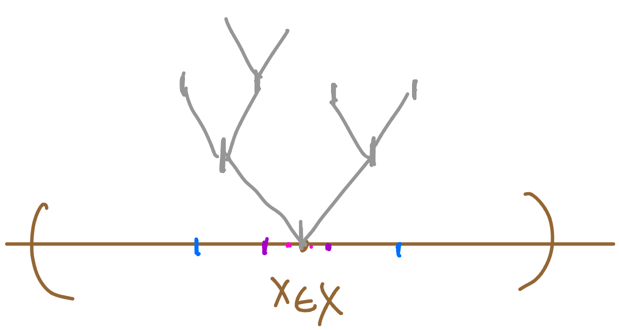
$a\restriction_n \in 2^n$

↑
infinite binary path



BINARY SEQUENCE

X is perfect, $X \neq \emptyset$



Reminder: $|\omega \times \omega| = |\omega|$

We will construct intervals I_t for $t \in 2^n$, $n \in \omega$, such that

1) $I_t \cap X \neq \phi$

disjoint intervals at same level

2) $\overline{I_t} \cap \overline{I_s} = \phi$, $s, t \in 2^n$

3) If $s \in 2^n$, $t \in 2^m$ and $s = t\restriction_n$, $n < m$

then $\overline{I_t} \subseteq I_s$ closed interval

4) $\forall n \ \forall t \in 2^n$ $\operatorname{diam}(I_t) < \frac{1}{2^n}$, $\operatorname{diam}(x,y) = |y-x|$

Why would this suffice?

Given such I_t, $t \in 2^n$, $n \in \omega$

Note: $\forall b \in 2^\omega, \forall n < m$ $\overline{I_{b\restriction_m}} \subseteq I_{b\restriction n}$

$$\bigcap_{n \in \omega} I_{b\restriction n} = \bigcap_{n \in \omega} \overline{I_{b\restriction n}} \neq \phi$$

Completeness of $\mathbb{R}$

Fact: If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$, $A_i \subseteq \mathbb{R}$

A_i is closed, then $\bigcap_{i=1}^{\infty} A_i \neq \phi$

By 4), $\left|\bigcap_{n \in \omega} I_{b\restriction n}\right| = 1$ $\qquad \bigcap_{n \in \omega} I_{b\restriction n} = \{x_b\}$

Define $f : 2^\omega$