Recall real analysis:

① $x_n \in \mathbb{R}$, $x \in \mathbb{R}$

$\lim_{n\to\infty} x_n = x \iff \forall \varepsilon > 0, \exists N > 0 : \forall m > N : (|x_m - x| < \varepsilon)$

② $\{x_n\}_{n\geq 1}$ is called Cauchy if

$\forall \varepsilon > 0 \ \exists N \ \forall n, m > N \ (|x_m - x_n| < \varepsilon)$

③ Completeness of $\mathbb{R}$

if $\{x_n\}$ is a Cauchy seq then

$\exists x \in \mathbb{R}$ s.t. $x_n \to x$

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claim. If $X_1 \supseteq X_2 \supseteq X_3 \cdots$ are closed subsets on $\mathbb{R}$

then $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall x, y \in X_n \ (|x - y| < \varepsilon)$ ⊛

then $\bigcap_{n=1}^{\infty} X_n = \{x\}$, $x \in \mathbb{R}$

e.g. $\forall x, y \in X_n$

$|x - y| < \frac{1}{n}$

Note: $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \phi$

proof. Take $x_n \in X_n$, then $\{x_n\}$ Cauchy

e.g. ⊛ says that $\forall n, m > N$, $x_n, x_m \in X_N$

in higher level means in lower level

So $|x_n - x_m| < \frac{1}{N}$

So $\exists x \in \mathbb{R}$ s.t. $\lim_{n\to\infty} x_n = x$

Given n, $\forall m > n$, $x_m \in X_n$

So $x \in X_n$ since X_n is closed.

$\forall n$: $x \in X_n$, so $x \in \bigcap_n X_n \Rightarrow |\bigcap_n X_n| \geq 1$

if $x, y \in \bigcap_n X_n$, then $\forall n$, $x, y \in X_n$

$\forall n \quad |x-y| < \frac{1}{n} \Rightarrow x = y$

so $|\bigcap_n X_n| = 1$ $\square$

Recall. Assume $X \subseteq \mathbb{R}$ is perfect

(X is closed and has no isolated pts, $X \neq \phi$)

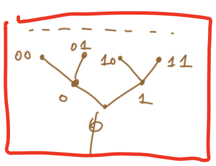
NTS. $|2^w| \leq |X|$

$a \in 2^w$

$n \in w$

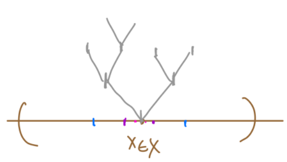
$a\restriction_n \in 2^n$

↑ infinite binary path



BINARY SEQUENCE

X is perfect, $X \neq \phi$



Reminder: $|w \times w| = |w|$

Define I_t for $t \in 2^n$, $n \in w$

I_t is an open interval

1) $I_t \cap X \neq \phi$, $\forall t \in 2^n$, $\forall n$

2) $I_t \cap I_s = \phi$, $\forall s, t \in 2^n$

(same level)

3) $\forall n < m$, $s \in 2^n$, $t \in 2^m$

if $s = t\restriction n$ then $I_t \subseteq I_s$

4) $\forall x, y \in I_t$, $t \in 2^n$, $|x-y| < \frac{1}{2^n}$

Given such $\{I_t \mid t \in 2^n, n \in \omega\}$

Define a map $f: 2^\omega \to X$ such that

$\forall b \in 2^\omega$
$$\bigcap_{n\in\omega} (\underbrace{\overline{I_{b \restriction n}} \cap X}_{\text{closed}}) = \{f(b)\}$$

Note (Ex.)

A, B closed

$\Rightarrow A \cap B$ closed

f is well-defined since $\forall b \in 2^\omega$,

$$\left| \bigcap_n \overline{I_{b\restriction n}} \cap X \right| = 1$$

☆ Claim. f is injective (last time)

How to construct $\{I_t \mid t \in 2^n, n \in \omega\}$

$n = 0 \Rightarrow$ take I_ϕ to be some open interval with $I_\phi \cap X \neq \phi$ (since $X \neq \phi$)

fix $x \in I_\phi \cap X$. By assumption, x is not isolated in X. So there is some $y \in I_\phi \cap X$, $y \neq x$. Take I_0, I_1 open interval such that $x \in I_0$, $y \in I_1$ and $I_0 \cap I_1 = \phi$

and $\overline{I_0}, \overline{I_1} \subseteq I_\phi$

Given I_t, $t \in 2^n$

By assumption, $I_t \cap X \neq \phi$ and no point in X is isolated. So there are $x, y \in I_t \cap X$, $x \neq y$. Find $I_{t^\frown 0}$, $I_{t^\frown 1}$ s.t. $I_{t^\frown 0} \cap I_{t^\frown 1} = \phi$

- $x \in I_{t^\frown 0}$, $y \in I_{t^\frown 1}$ (1)
- $I_{t^\frown 0} \cap I_{t^\frown n} = \phi$ (2)
- $\overline{I_{t^\frown 0}}, \overline{I_{t^\frown 1}} \subseteq I_t$ (4)
- $\text{diam}(I_{t^\frown 0}), \text{diam}(I_{t^\frown 1}) < \frac{1}{2^{n+1}}$ (3)

this finishes the construction. $\square$

Thm. (Cantor Bendixson)
If $X \subseteq \mathbb{R}$ closed, either
X is countable, or $|2^w| \leq |X|$

Remarks. If $\{C_i \mid i \in I\}$ where $C_i \subseteq \mathbb{R}$ closed
then $C = \bigcap_{i \in I} C_i$ is closed.

Ref. $X \subseteq \mathbb{R}$ (closed)

$X' = \{x \in X \mid x \text{ is not isolated in } X\}$

$\quad = X \setminus \{\text{isolated points in } X\}$

Cantor Bendixson Derivative

Note. If X is closed, then X' is also closed

why? Suppose $x_n \in X'$, $x_n \longrightarrow x$

limit points are contained

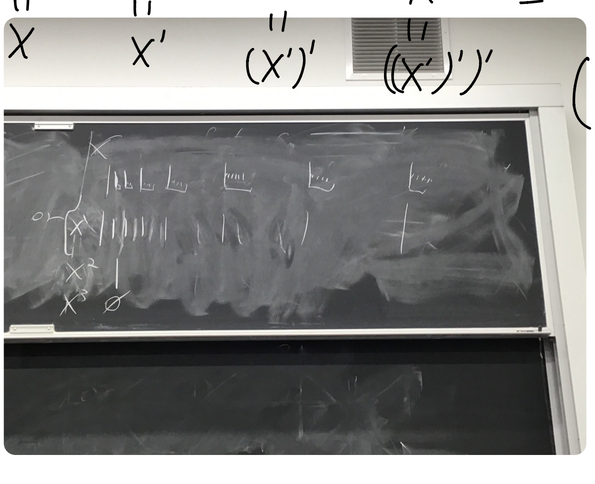
NTS. for $x \in X'$, if $\exists n$ such that $x = x_n \Rightarrow x = x_n \in X'$

So assume $x_n \neq x$, $\forall n$

$(\forall \varepsilon \, \exists N \, \forall n \geq N \; |x - x_n| < \varepsilon)$ for any open interval $I \ni x$
there is some $x_n \neq x$, $x_n \in I \Rightarrow |I \cap X| \geq 2$

Since $x_n \in X' \subseteq X$, also $x \in X$ by X closed,
we conclude x is <u>not</u> isolated in $X \Rightarrow x \in X'$

Hope:
$$\begin{array}{ccccccccc} X^0 & \supseteq & X^1 & \supseteq & X^2 & \supseteq & X^3 & \supseteq \cdots & X^\omega \\ \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ X & & X' & & (X')' & & ((X')')' & & \left(\bigcap_n X^n\right)' \end{array}$$



Note. If X is closed, X is perfect iff $X = X'$
(no isolated pts to throw away)

Recall. there is an ordinal α such that
α is not countable $(|\alpha| \not\leq |\omega|)$
(By Hartog)

Let ω_1 be the minimal $\beta \in \alpha$ (or $\beta = \alpha$)
such that β is not countable.

Given $\beta \in \omega_1$ let $\beta + 1$ be min $\omega_1 \setminus \beta$
$\beta + 1 = \beta \cup \{\beta\}$

Define recursively along ω_1
$X^{(\alpha)}$, $\alpha \in \omega_1$ so that
$X^{(\alpha)} = \left(\bigcap_{\beta<\alpha} X^{(\beta)}\right)'$ and $X^{(0)} = X$

e.g. $X^{(0)} = X$, $X^{(1)} = \left(X^{(0)}\right)' = X'$
$X^{(2)} = \left(X^{(0)} \cap X^{(1)}\right)' = X''$
$X^{(\alpha+1)} = \left(X^{(\alpha)}\right)'$

Lemma. Given $X \subseteq \mathbb{R}$ closed, $\exists \alpha$ such that

$$X^{\alpha+1} = X^{\alpha}$$

$\left(X^{(\alpha)}\right)'$

Note: the minimal such α is called the Cantor Bendixson rank of X

<u>And</u> If $X^{(\alpha)} = \phi \Rightarrow X$ is countable

If $X^{(\alpha)} \neq \phi$ then $X^{(\alpha)} \subseteq X$ is a nonempty perfect set.

Note: $X \setminus X' = \{$ isolated points of $X\}$ is countable

why? If $x \in X$ is isolated, there is open interval $x \in I$ s.t. $I \cap X = \{x\}$

may assume $I = (a, b)$, $a, b \in \mathbb{Q}$

$\mathbb{Q} \times \mathbb{Q}$ is countable

injective map: $x, y \in X \setminus X'$

$$x \neq y \Rightarrow I_x \cap X = \{x\} \neq \{y\} = I_y \cap X$$
$$\Rightarrow I_x \neq I_y$$

Recursively at each $\alpha < \omega_1$

$$X^{(\alpha)} = \Big(\bigcap_{\beta<\alpha} X^{\beta}\Big)' \qquad X^{(\alpha+1)} = \left(X^{(\alpha)}\right)'$$

Define I_α rational end point interval such that

$$\exists\, x \in \bigcap_{\beta<\alpha} X^{(\beta)} \setminus X^{(\alpha)}$$

with $I_\alpha \cap \bigcap_{\beta<\alpha} X^{(\beta)} = \{x\}$