Axioms of Set Theory

(i) Axiom of union:
for all set S, there exists a set A such that $\forall s \in S : s \in A$

(ii) Axiom of pairing:
for all set A, B, there exists a set C such that $C = \{A, B\}$

(iii) Axiom of infinity
there exists a set A such that $\phi \in A$ and $\forall a \in A : a \cup \{a\} \in A$

(iv) Axiom of Comprehension
Let $p(a)$ be a property of $a \in A$, where A is a set. then there exists a set B such that $B = \{a \in A : p(a)\}$

(v) Axiom of powerset
for set A, $P(A) := \{S : S \subseteq A\}$ is a set

(vi) Axiom of Extensionality
Let A, B be sets
$\forall a, b \in A : (a \in A \Leftrightarrow a \in B) \Rightarrow A = B$

Induction

(i) A set P is inductive if $\phi \in P$ and
$p \in P \Rightarrow S(p) = p \cup \{p\} \in P$

(ii) A set T is transitive if $t \in T \Rightarrow \forall x \in t \, (x \in T)$
Note: $\forall x \in \omega : x$ is transitive.

(iii) Given set X, Y. A relation R on X is $R \subseteq X \times Y$
A partial order on X is a relation $R \subseteq X \times X$ such that R is transitive, irreflexive and antisymmetric. (strict, no equality involved)

R is a linear order if $\forall x, y \in X$ either
xRy or $x = y$ or yRx.

(iv) Say sets A, B equinumerous if there is
a bijective function $f: A \to B$.
Say A finite if $\exists n \in \omega$ such that
$A \approx n$.

Lemma. Let $n \in \omega$. if $A \subsetneq n$ then $A \not\approx n$
Corollary. Let x be finite. if $y \subsetneq x$ then $y \not\approx x$
Corollary. ω is infinite

AFSOC finite, then bijection $S(x) = x \cup \{x\}$
$S: \omega \to \omega / \{\phi\}$. So ω not finite.

Lemma. If X is a set and $\exists$ injective $f:$
$\omega \to X$ then X is infinite

Define $g: X \to X / f(\phi)$: $x \mapsto x$, if $x \neq f(n)$
$\mapsto f(S(x))$, otherwise

g bijection. $X / f(\phi) \subsetneq X \Rightarrow X$ infinite.

Well orders

(i) $<$ is a well ordering if $\forall Y \subseteq X, Y \neq \phi$. $\exists y_0 \in Y$
such that $\forall y \in Y$ ($y = y_0$ or $y_0 < y$)
(minimum element)
$\hookrightarrow (\omega, \in)$ is a well order

(ii) well orders $\Rightarrow$ inductive

suppose $(\omega, \in)$ well ordered and

$B \in \omega$ s.t. $\phi \in B \wedge \forall n (n \in B \Rightarrow S(n) \in B)$
then $B = \omega$.

Take away: can perform induction on well-ordered set

Suppose X well ordered. $P \subseteq X$ such that
$\forall x \in X$ (if $\forall y < x$ ($y \in P$) then $x \in P$)
then $P = X$. (Principles of weak induction)

(iii) Let $(A, \overset{A}{<})$, $(B, \overset{B}{<})$ be linearly ordered set. Then a map f is order-preserving iff $\forall a_1, a_2 \in A$, $a_1 \overset{A}{<} a_2 \Rightarrow f(a_1) \overset{B}{<} f(a_2)$

↳ order preserving $\Rightarrow$ injective

↳ if $(A, <)$ well-ordered, $f: A \to A$ order preserving then $f(a) \geq a$, $\forall a \in A$

↳ A, B isomorphic if there exists order-preserving bijective $F: A \to B$

(iv) $I \subseteq A$ is an initial segment on $(A, <)$ if
$\forall i \in I \, (\forall a \in A : (a < i \Rightarrow a \in I))$

↳ If $f: A \to A$ order-preserving and $\mathrm{Im}(f)$ is an initial segment of A then $f = id$

↳ If $f: A \to A$ order-preserving and surjective then $f = id$

↳ Corollary: $(A, <)$ well ordered, then A is not isomorphic to any proper initial segment.

Replacement

i) Frankel's Axiom of replacement

Suppose $\phi(x,y)$ a property, A a set and $\forall a \in A$ $\exists! y$ such that $\phi(x,y)$ holds. Then there is a set B such that $b \in B \Leftrightarrow \exists a \in A\ (\phi(a,b))$

ii) recursive definition

Given a well order $(A,<)$ and some property $\phi(x,y,f)$ such that $\forall x, f,\ \exists! y\ (\phi(x,y,f))$

Then there exists a function F with domain A such that $\forall x \in A$: $\Phi(x, F(x), F \upharpoonright \underbrace{A_x}_{\text{initial segment}})$ holds

(iii) Application (compari bility of well orders)

not proved

Given $(A, \overset{A}{<})$, $(B, \overset{B}{<})$ well-ordered set.

(exactly) one of the following holds

a) A, B are isomorphic

b) A isomorphic to an initial segment of B

c) B isomorphic to an initial segment of A

Ordinals

(i) An ordinal is a well-ordered set $(\alpha, \in)$ which is transitive $(x \in y \wedge y \in \alpha \Rightarrow x \in \alpha)$

(Motivation: natural number is not LONG enough)

(ii) ↳ x an ordinal, $y \in X$ then y is an ordinal

proof. well ordered because $\forall z \in y : y \in x \Rightarrow z \in X$

So $y \subseteq x \Rightarrow y$ well ordered with $\in$

Transitive?

Let $z \in y$, $m \in z$. By linear order:

either $m = y \Rightarrow z \in y$ and $y \in z$ $\lightning$

or $y \in m \Rightarrow y \in m \in z \in y \Rightarrow y \in y$ $\lightning$

Thus it must be $m \in y$.

↳ $(A, \in)$ a well ordered set. Then it is isomorphic to a unique ordinal.

Claim. $(\omega, \in)$ is a linearly ordred set

Proof. (i) $\in$ is a partial order on ω

① irreflexive

Proof by induction

Assume $x \notin x$. NTS $S(x) \notin S(x)$

AFSOC $S(x) \in x \cup \{x\}$

if $S(x) \in x$, then $x \cup \{x\} \in x$

$\Rightarrow x \in x \cup \{x\} \in x \Rightarrow x \in x$, contradiction

if $S(x) \in \{x\}$ then $x \cup \{x\} = x$

$\Rightarrow \{x\} = \phi \Rightarrow$ contradiction

② antisymmetric

AFSOC $S(S(x)) \in S(x)$

then $S(x) \cup \{S(x)\} \in S(x)$

$S(x) \in S(x) \cup \{S(x)\} \in S(x) \Rightarrow S(x) \in S(x)$

Contradiction □

③ transitive

By definition of successor function

$S(x) = x \cup \{x\}$

So $\forall y \in x: y \in S(x)$

(ii) $\forall x, y \in \omega$: $x = y$ or $x \in y$ or $y \in x$

Proof. by induction on ω

Suppose $\forall x \in y$, either $S(x) = y$ or

$S(x) \in y$

NTS true $\forall x \in S(y)$

$x \in y \cup \{y\}$

if $x = y$, then $S(x) = S(y)$

else $x \in y$, then $S(x) \in y \in S(y)$

claim. $x \in \omega$, $y \subsetneq x$ then $y \not\approx x$

proof. by induction on x

B.C. $x = \phi$, nothing to prove.

Assume true for x, NTS true for $S(x)$

AFSOC $\exists y \subsetneq S(x)$ $(y \approx S(x))$

that is there exists bijective $f: y \to S(x)$

Case 1: $x \notin y$ (use I.H.)

then $y \subseteq x$

fix $z \in y$ such that $f(z) = x$

Define $y' := y / \{z\}$

then $f\restriction_{y'} : y' \to x$ is a bijection

but $y' \subsetneq x$, contradiction ↯

Case 2. $x \in y$ (Reminder, $y \subsetneq x \cup \{x\}$

then $\exists z \in x$ such that $z \notin y$

Take $y' := y \cap x$, Note $y' \subsetneq x$

Let $k \in x$ such that $f(k) = x$

Let $u \in S(x)$ such that $f(x) = u$

if $u = x$, then $f\restriction_{y'}$ is a bijection,

Contradiction.

if $u \neq x$, then

Define $g: y' \to x$ such that

$$g(z) = \begin{cases} f(z), & \text{if } f(z) \neq x \\ u, & \text{otherwise} \end{cases}$$

then g is a bijection Σ

claim. $(\omega, \in)$ is a well order.

Proof. AFSOC, suppose $\exists A \subseteq \omega, A \neq \phi$ such that

A contains no minimum element.

Then obviously $\phi \notin A$ and

$\forall n \in \omega$, if $(\forall m \in n\ (m \notin A))$ then $n \notin A$

Define $B := \{b \in \omega \mid b \notin A\}$

Note $\phi \in B$ and $n \in B \Rightarrow S(n) \in B$

By induction $B = \omega \Rightarrow A = \phi$ Contra Σ

claim. (Principle of Weak Induction)

Let $(X, <)$ be well ordered.

Suppose $P \subseteq X$ satisfies $\forall x \in X : (\forall y < x : y \in P) \Rightarrow x \in P$ then $P = X$.

Proof. AFSOC suppose $P \neq X$. then

let $B := P / X$, $B \neq \phi$

By well ordering, $\exists b_0 \in B : b_0 \leq b, \forall b \in B$

Let $x \in X$. if $x < b_0$, then $x \in P$

but then $b_0 \in P$. Contradiction. Σ

claim. Identity map & order preserving for well orders

$I \subseteq A$ is an initial segment on $(A, <)$ if $\forall i \in I \, (\forall a \in A : (a < i \Rightarrow a \in I))$

(i) $\hookrightarrow$ If $f : A \to A$ order-preserving and $Im(f)$ is an initial segment of A then $f = id$

(ii) $\hookrightarrow$ If $f : A \to A$ order-preserving and surjective then $f = id$

Proof (i) AFSOC, suppose $\mathcal{F} \neq id$ then $\exists a \in A$ such that $\mathcal{F}(a) \neq a$.

By $\mathcal{F}$ order-preserving, $\mathcal{F}(a) \geq a \Rightarrow \mathcal{F}(a) > a$

Let $B := \{a \in A : \mathcal{F}(a) > a\}$

by well-order, let $b_0 := \min B$.

Note $Im(\mathcal{F})$ is an initial segment and

$b_0 < \mathcal{F}(b_0) \Rightarrow b_0 \in Im(\mathcal{F})$

So $\exists a \in A : \mathcal{F}(a) = b_0 < \mathcal{F}(b_0)$

$\Rightarrow \mathcal{F}(a) < \mathcal{F}(b_0)$

$\Rightarrow a < b_0 \Rightarrow a \neq b_0$, contradiction $\Large\Sigma$

(ii) AFSOC, suppose $\exists a \in A : \mathcal{F}(a) \neq a$

By order-preserving, $\mathcal{F}(a) > a$

Let $B := \{a \in A : \mathcal{F}(a) \neq a\}$

By well orders, let $b_0 := \min B$

By surjective, $\exists d \in A : \mathcal{F}(d) = b_0 < \mathcal{F}(b_0)$

$\Rightarrow d < b_0 \Rightarrow \mathcal{F}(d) = b_0 \neq d$ $\Large\Sigma$

Claim.

Given a well order $(A, <)$ and some property $\phi(x, y, f)$ such that $\forall x, f, \exists! y\, (\phi(x, y, f))$

Then there exists a function F with domain A such that $\forall x \in A : \Phi(x, F(x), F\restriction_{A_x})$ holds

initial segment

proof. (i) uniqueness

Suppose $\mathcal{F}_1, \mathcal{F}_2$ satisfy claim.

Proof by induction to show $\mathcal{F}_1 = \mathcal{F}_2$.

Assume $x \in A$ and $\forall y \in x : \mathcal{F}_1(y) = \mathcal{F}_2(y)$

then $\phi(x, \mathcal{F}_1(x), \mathcal{F}_1 \upharpoonright A_x)$

$\phi(x, \mathcal{F}_2(x), \mathcal{F}_2 \upharpoonright A_x)$ both holds

Since $\mathcal{F}_1 \upharpoonright A_x = \mathcal{F}_2 \upharpoonright A_x$, by assumption about unique y, then $\mathcal{F}_1(x) = \mathcal{F}_2(x)$ $\square$

(ii) Existence (use replacement axioms)

$\forall x \in A$, $\exists ! g_x$ with $\mathrm{dom}(g_x) = A_x$

such that $\forall y < x : \phi(y, g_x(y), g_x \upharpoonright A_y)$

Consider the property $\psi(s, w) \Leftrightarrow w = g_s$

then $\forall s, \exists ! w$ such that $\varphi(s, w)$

By replacement axiom,

we can define $G : A \to$

$: x \mapsto g_x$

Note: by uniqueness, if $y < x$, then $g_x \upharpoonright A_y = g_y$

So we can define $F : A \to$

by $F(x) = z \Leftrightarrow \exists x'$ with $g_{x'}(x) = z$

Claim. (Comparibility of well orders)

not proved

Given $(A, \overset{A}{\leq})$, $(B, \overset{B}{\leq})$ well-ordered set.

(exactly) one of the following holds

a) A, B are isomorphic

b) A isomorphic to an initial segment of B

c) B isomorphic to an initial segment of A

Proof. (Application of recursive def & replacement)

Fix a set $* \notin A \cup B$

Define recursively map $f: A \to B \cup \{*\}$

by $f(x) = \begin{cases} \min\{b \in B \mid \forall y \overset{A}{<} x \, (f(y) < b)\}, & \text{if such } b \text{ exists}, \\ *, & \text{otherwise} \end{cases}$

$\phi(x, b, f) \Leftrightarrow b = \min\{f(y) \mid y \overset{A}{<} x\}$

Let $I \subset A$ be $I = \{a \in A : f(a) \in B\}$

Then I is an initial segment of A

since $\forall a \in A \ \forall i \in I: a \in i \Rightarrow f(a) \in B \Rightarrow a \in I$

Then $f: I \to B$ is order-preserving

why? Let $a_1 < a_2$, $a_1, a_2 \in I$

then $f(a_1) = \min\{b \in B \mid \forall y \overset{A}{<} a_1 (f(y) < b)\}$

$f(a_2) = \min\{b \in B \mid \forall y \overset{A}{<} a_2 (f(y) < b)\}$

$f(a_1) \in f(a_2)$

Also $\{f(a) : a \in I\}$ is an initial segment of B

why? NTS $\forall a \in I$, $\forall w \in f(a)$

$\exists a' \in I$ such that $f(a') = w$

proof by induction on I.

Assume true for all $\bar{a} < a$, NTS true for a

I.H. $\forall \bar{a} < a$: $\forall w \in f(\bar{a})$, $\exists a' \in I$ $(f(a') = w)$

Let $w < f(a)$. then

$w < \min\{b \in B \mid \forall y \overset{A}{<} a \ (f(y) \overset{B}{<} b)\}$

so $\exists \bar{a} \overset{A}{<} a$ such that $w \overset{B}{<} f(\bar{a})$

By I.H., $\exists a' \in I$: $f(a') = w$. $\square$

Now if $I = A$, then $f: A \to B$ is order preserving, either onto (isomorphic) or onto a proper initial segment $(|A| < |B|)$

If $I \subsetneq A$, let $a = \min A \setminus I$

$a = \min\{a \in A \mid f(a) = *\}$

so $f \restriction A_a : A_a \to B$ is onto and order-preserving

then B is isomorphic to a proper initial segment of A. $\square$