minimal inductive set

: $\phi \in \omega$ & $x \in \omega \Rightarrow x \cup \{x\} \in \omega$

$\mathbb{N} = \omega =$ minimal inductive set

Successor function: $S(x) = x \cup \{x\}$

e.g. $\overset{0}{\phi} \in \overset{1}{\{\phi\}} = \phi \cup \{\phi\} \in \overset{2}{\{\phi, \{\phi\}\}} \in$

$\overset{3}{\{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}}$ - - - - .

membership relation ~ $\leq$ relation

Inductive Scheme:

$P \subseteq \boxed{\omega}$ is a property

ω is the minimal inductive set. i.e. if X inductive then $\omega \subseteq X$

$\phi \in P$ and $\forall x: x \in P \Rightarrow S(x) = x \cup \{x\} \in P$

Then $\forall x \in \omega$, $x \in P$ (i.e. $P = \omega$)

proof. Fix $P \subseteq \omega$ s.t. $\phi \in P$ and
$\forall x: x \in P \Rightarrow x \cup \{x\} \in P$

So by def, P is inductive

by ω is minimal inductive set, $\omega \subseteq P$

Also by assumption: $P \subseteq \omega$

then $P = \omega$ $\square$

Axiom of Comprehension?

$(\omega, 0, S)$ $(0 \leftrightarrow \phi)$

Lemma.

$\forall x \in \omega$, x is $\boxed{\text{transitive}}$

set membership rel is transitive

(meaning $\forall z\, \forall u\, (u \in z \wedge z \in x)$ then $u \in x$)

Proof. By induction

B.C. ϕ is transitive (vacuously true)

Assume $x \in \omega$ is transitive

NTS $x \cup \{x\}$ is transitive

Take any $z \in x \cup \{x\}$ and $u \in z$, NTS $u \in x \cup \{x\}$

either 1) $z \in x$ OR 2) $z \in \{x\} \Rightarrow z = x$

In case 1) then since x is transitive, $u \in x$

So $u \in x \cup \{x\}$

In case 2) $z = x$ so $u \in x$

$\square$

Notation $\langle x, y \rangle = \{ x, \{x, y\} \}$

$\langle x, y, z \rangle = \langle x, \langle y, z \rangle \rangle$

Def. Given set X and Y

a relation R on X is $R \subseteq X \times Y$

a function $f : X \to Y$ is a relation $f \subseteq X \times Y$
such that $\forall x \in X$ there $\exists ! y \in Y$
such that $(x, y) \in f$

f is injective if $\forall x_1, x_2 \in X$:
$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

f is surjective if $\forall y \in Y$:
$\exists x \in X : f(x) = y$

f is bijective if f injective & surjective

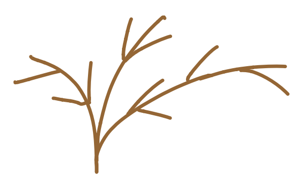
$Dom(f) = X \quad Im(f) = \{ y \in Y : \exists x \in X : f(x) = y \}$

partial order R on a set X is
a relation $R \subseteq X \times X$ s.t.

1) transitive: $\forall x, y, z \in X$: $xRy,\ yRz \Rightarrow xRz$
2) irreflexive: $\forall x \in X$: $\neg(xRx)$
3) antisymmetric: $\forall x, y \in X$: if $x<y$ then $\neg(y<x)$

Notation: Say (X, R) is a partially ordered set
for $R \in X \times X$. Say $\operatorname{dom} R = X$

Example: $\mathbb{N}$ $R: <$
$\mathbb{Q}$ $R: <$



Def. R is a linear order if
R is a partial order and
$\forall x, y \in X$: either xRy or
$x = y$ or
yRx

$$\begin{pmatrix} x>y \\ x=y \\ x<y \end{pmatrix}$$

Proposition. (ω, R) is a linearly ordered set

$$R = \{(x, y) \in \omega \times \omega \mid x \in y\} :=$$

$$R = \in \mid \omega \times \omega$$

Let's prove irreflexive property by induction

(i.e. no set is a member of itself)

B.C. ϕ has no member ✓

I.H. $x \notin x$

N.T.S. $x \cup \{x\} \notin x \cup \{x\}$

AFSOC Suppose $x \cup \{x\} \in x \cup \{x\}$

then either $x \cup \underline{\{x\} \in \{x\}} \Rightarrow x \in x$ ↯

OR $x \cup \{x\} \in x$ $\quad$ $a \in \{x\} \Leftrightarrow a = x$

now $x \in x \cup \{x\} \in x$, since $x \in \omega$,

x is transitive $\Rightarrow x \in x$ ↯

Recall $\langle x,y\rangle := \{\{x\},\{x, y\}\}$

1st element of $\langle x,y\rangle$:

unique z such that $\forall t \in \langle x,y\rangle\ (z \in t)$

2nd element of $\langle x,y\rangle$:

$\begin{cases} \text{either } \langle x,y\rangle \text{ has only 1 member} \Rightarrow y = x \\ \text{or } \langle x,y\rangle \text{ has 2 members} \Rightarrow \end{cases}$

y is unique s.t. $\{x,y\} \in \langle x,y\rangle$

and $\{y\} \notin \langle x,y\rangle$

Lemma. If $x,y \in \omega$ and $x \in y$

then either $S(x) = y$ or

$S(x) \in y$

successor

Proof. By induction on y

B.C. $y = \emptyset \Rightarrow$ vacuously true

I.H. for y: $\forall x \in y$

either $S(x) = y$ or $S(x) \in y$

N.T.S. holds for $S(y)$

i.e. $\forall x \in S(y)$

either $S(x) = S(y)$ or $S(x) \in S(y)$

By $x \in S(y)$, we know $x \in y \cup \{y\}$

if $x \in y$, then $S(x) = y \in S(y)$

otherwise $x \in \{y\}$, then $x = y \Rightarrow S(x) = S(y)$

$\square$

minimal inductive set

Lemma. $x, y \in \omega$

$\phi \in \omega$ and inductive

either $x \in y$ or $x = y$ or $y \in x$

(verify order of natural numbers)

Proof. by induction on x

if $x = \phi$

proof by ind. on y:

if $y = \phi \Rightarrow y = x$ ✓

nothing is member of empty set

Assume true for y (either $y = \phi$ OR $\phi \in y$)

$\phi \in S(y) = S(\phi)$

$\{y, \{y\}\}$

$\phi \in y \in S(y)$

by transitity:

$\phi \in S(y)$

Continue ind. on x

Assume: $\forall y \in \omega \,(y = x \;\vee\; y \in x \;\vee\; x \in y)$

$y \in S(x)$ $\quad$ $y \in x \in S(x) \xrightarrow{\text{trans.}} y \in S(x)$ $\quad$ $S(x) \in y$

So $S(x)$ can be compared with any y $\square$

e.g. $X = \{0,1\}$

$R = \phi$ ← no order!

(like < or >)

Recall. R a relation on X is a (strict) partial order if

cannot compare all

1) transitive, 2) irreflexive, 3) antisymmetric

Def. R a linear order if

$\forall x, y \in X$ either $x R y$

$y R x$

$y = x$

can compare all elements

↳ we proved $(\omega, \in)$ is a linear order

$R = \{ (x,y) \mid x \in y \}$ is a linear order on ω

Def. 1) Let A, B be sets

A, B are equinumerous

(have the same cardinality)

denoted $A \approx B$ if $\exists$ bijective $f: A \to B$

2) A is <u>finite</u> if there is $n \in \omega$ such that $A \approx n$.

Lemma. If $x \in \omega$, $y \subsetneq x$, then $y \not\approx x$

proof. By induction on x.

If $x = \phi$: nothing to prove.

Assume for x, to prove for $S(x)$ $= x \cup \{x\}$

AFSOC Assume $\exists\, y \subsetneq S(x)$ and bijective $f: y \to S(x)$

<u>Case 1</u> $x \notin y$, then $y \subseteq x$

fix $z \in y$ such that $f(z) = x$

then $y' = y \setminus \{z\}$

Define $f': y' \to x$ by

$f'(u) = f(u)$

Note f' is onto. But $y' \subsetneq X$, ↯

<u>Case 2:</u> $X \in Y$

Reminder: $Y \subsetneq S(x) = x \cup \{x\}$

Take u s.t. $f(u) = x$,
w s.t. $f(x) = w$

Define $g: Y \to S(x)$ by $g(x) = x$

Let $y' = y \cap X$ $(y \subsetneq X)$

then $g': Y' \to X' : z \mapsto g(z)$
is a bijection ↯

↪ By this lemma, w is infinite.