Last time

For $n \in \omega$, if $y \subsetneq n$, then $y \not\approx n$

Corollary: If x is finite, $y \subsetneq x$, then $x \not\approx y$

proof. by assumption, $\exists\, n \in \omega$ and a bijection $f: x \to n$

AFSOC suppose $\exists$ bijective $g: y \to x$.

Let $y' = \{f(l) \mid l \in y\}$, $y' \subsetneq n$

Define $g': y' \to n$ by

$$g'(f(l)) = f(g(l))$$

Then g' is a bijection. Contradiction $\lightning$

$$\begin{tikzcd}
y \arrow[r, "g"] & x \arrow[d, "f"] \\
y' \arrow[r, dashed, "g'"] & n
\end{tikzcd}$$

Thm. ω is infinite

proof. Define $S: \omega \to \omega \setminus \{\phi\}$

$S(x) = x \cup \{x\}$

S is a bijection

$\omega \setminus \{\phi\} \subsetneq \omega$ → by Corollary, ω is infinite

Lemma. If X is set and $\exists$ injective $f: \omega \to X$ then X is infinite

proof. Define $g: X \to X \setminus \{f(\phi)\}$

$$g(x) = \begin{cases} x, & \text{if } x \neq f(n),\ \forall n \in \omega \\ f(S(n)), & \text{if } x = f(n) \end{cases}$$

g is a bijection,

$X \setminus \{f(\phi)\} \subsetneq X$ $\square$

$(A, <)$ is an ordered set

$<$ linear order

Def. $<$ is a well ordering if

existence of smallest element

for $\forall Y \subseteq A$, there is $y_0 \in Y$ such that

$Y \neq \phi$

$\forall y \in Y \, (\underbrace{y = y_0 \text{ or } y > y_0}_{y \geq y_0})$

Proposition. $(\omega, \in)$ is a well ordering

proof. Suppose $Y \subseteq \omega$. $Y \neq 0$

AFSOC Assume Y does not have a min element

then $\phi \notin Y$ and

$\forall n \in \omega$, if $\forall m \in n \, (m \notin y)$ then $n \notin y$

Define $B := \{ n \in \omega \mid n \notin Y \}$

Note $\phi \in B$ and $n \in B \Rightarrow S(n) \in B$

So $B = \omega$ by induction

and $Y = \phi \Rightarrow (\omega, \in)$ is well-ordered $\square$

In fact, "well order" $\Rightarrow$ inductive

Suppose $(W, \in)$ is well ordered

Suppose $B \in W$ s.t. $\phi \in B$ and

$\forall n (n \in B \Rightarrow S(n) \in B)$ inductive

NTS. $B = W$

Take $Y = W \setminus B$

NTS $\underline{Y = \phi}$

by well-ordered

AFSOC, $\exists$ minimal $y_0 \in Y$

That is: $y_0 \in Y \leftrightarrow y_0 \notin B$

$\forall y \in y_0 (y \notin y) \Rightarrow \forall y \in y_0 (y \in B)$

$y_0 \neq 0 \Rightarrow y_0 = S(y') \Rightarrow y' \in B \Rightarrow$
$S(y') \notin B$

contradict I.H.

Example. $(\mathbb{Q}, <)$ not a well ordered

$(\mathbb{Z}, <)$

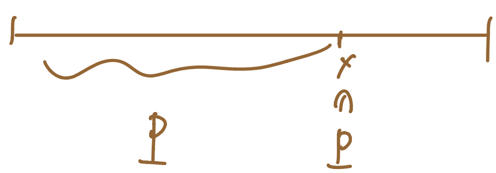
$([0,1), <)$ $\quad Y = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$ has no minimal element

Example. If X is finite and $(X, <)$ order then $(X, <)$ is a well ordering

Lemma. (proof by ind. for a well order)

Suppose $(X, <)$ is a well order.

$P \subseteq X$ such that $\forall x \in X ($ if $\forall y < x (y \in P)$ then $x \in P)$ $\quad \phi$ for $x = x_0$



like $P(k) \to P(k+1)$

Then

Note. If $x_0 :=$ min element in X then $x_0 \in P \Rightarrow \boxed{P = X}$

$\exists y$ satisfy

proof. AFSOC, assume $P \neq X$

Let $Y = X \setminus P$, then $Y \subsetneq X$, $Y \neq \phi$

By well-ordering, let y_0 be minimal in Y

Let $x \in X$. If $x < y_0$, then $x \in P$

but by induction hypothesis

$(\forall y < x (y \in p \Rightarrow x \in p))$ then $y_0 \in p$ $\lightning$

Recall principle of induction:

If $\Phi(x)$ is a property s.t. $\forall x \in A$

always true for base case

if $(\forall y (y < x \Rightarrow \Phi(y)) \to \Phi(x))$

then $\forall x \in A$, $\Phi(x)$ holds

Note. If $A = \mathbb{N} = \omega$, this is standard induction

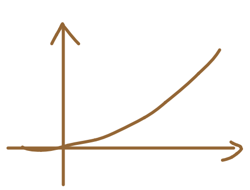
---

Def. Suppose $(A, <^A)$, $(B, <^B)$ are linear ordered sets,

a map $F: A \to B$ is order-preserving (monotonic) if

$\forall a_1, a_2 \in A$, $a_1 <^A a_2 \Rightarrow F(a_1) <^B F(a_2)$

Examples: 1) monotonic function on $(\mathbb{R}, <)$



$F: \mathbb{R} \to \mathbb{R}$

2) $F: \mathbb{N} \to \mathbb{N} : n \mapsto 2n$

order preserving for $\begin{pmatrix} (\omega, \in) \\ (\mathbb{N}, <) \end{pmatrix}$

Note. If $F: A \to B$ order preserving
$(A, <^A)$ $(B, <^B)$ then
F is injective.

proof. if $\underline{F(a_1) = F(a_2)} \Rightarrow (F(a_1) \not<^B F(a_2))$
$\Rightarrow (F(a_1) \not>^B F(a_2))$

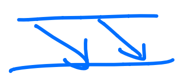
either $a_1 = a_2$ ✓

or $a_1 < a_2 \Rightarrow F(a_1) < F(a_2)$ ↯

or $a_2 < a_1 \Rightarrow F(a_2) < F(a_1)$ ↯

3) $(\mathbb{Q}, <)$ natural order

$F: \mathbb{Q} \to \mathbb{Q}$

$F(x) = x+1$, 

$F(x) = x-1$ 

Def. $(A, <^A), (B, <^B)$ are **isomorphic**

if $\exists$ bijective order-preserving map

$F: A \to B$

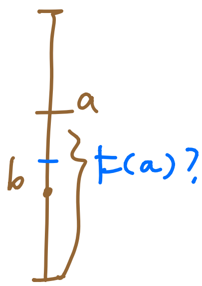
e.g. $(\omega, <)$ and $(\omega \setminus \{0\}, <)$

are isomorphic via $n \mapsto n+1$

Lemma. $(A,<)$ is a well order and $F: A \to A$ is order preserving then $\forall a \in A : F(a) \geq a$

Proof. By induction

Fix $a \in A$ and assume $\forall b < a,\ F(b) \geq b$



AFSOC, suppose $F(a) < a$

then $\exists\, b = F(a) < a$

By assumption: $F(b) \geq b$

but $F(b) = F(F(a)) < F(a) = b$

Contradiction $\lightning$

Def. $(A,<)$ an ordered set.

- an initial segment in $(A,<)$ is a set $I \subseteq A$ such that

$$\forall a \in I \ \underbrace{(\forall b \in A\ (b < a \Rightarrow b \in I))}_{\text{closed downwards}}$$

e.g. Given $a \in A$ let

$A_a = \{ b \in A \mid b < a \}$ is an initial segment

- a final segment $I \subseteq A$ is a set s.t.
$\underbrace{\forall a \in I \ \forall b \in A}_{\text{closed upward}} (a < b \Rightarrow b \in I)$

Lemma. 1) suppose $(A, <)$ is well ordering and $F: A \to A$ is order preserving such that $\mathrm{Im}(F) = \{F(a) \mid a \in A\}$ is an initial segment Then F is $\mathrm{id}: A \to A$

In particular, if $(A, <)$ is a well order, then it is not isomorphic to $(I, <)$ whenever $I \subsetneq A$ is an initial segment.

2) If $F: A \to A$ is surjective and order-preserving then $F = \mathrm{id}$

e.g. - $(\mathbb{Q}, <)$ and $((-\infty, 0) \cap \mathbb{Q}, <)$ are isomorphic

- $w = \mathbb{N}$ and $w+1 = \mathbb{N} \cup \{\boxed{\infty}\}$ are not isomorphic

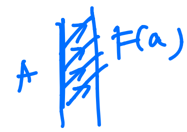
both are well-ordered

define to be max element

proof. AFSOC, $\exists\, a \in A : F(a) \neq a$

by previous lemma $\Rightarrow F(a) > a$

order-preserving



Let a be minimal such that $F(a) \neq a$

by well-order

So $\forall b < a$, $F(b) = b$

Now since $Im(F)$ is an initial segment,

$a < F(a) \in Im(F)$

then $a \in Im(F)$ by initial segment

So $\exists\, b$ such that $F(b) = a$

since $F(b) = a < F(a) \Rightarrow b < a$

$F(b) = b < a$ ↯ $\square$

<u>Recall</u> $(A, <)$ is a well-ordered set, then

A is NOT similar (isomorphic) to any proper initial segment $I \subset A$

---

similar to $\mathbb{N}$, recursive definition along a well-ordered set.

$X_0 = \omega, \quad X_{n+1} = P(X_n)$

Let $X = \bigcup_n X_n$ $\Leftarrow$ not supported by axioms to be set

---

Recall axioms so far

i) Extensionality
ii) empty set
iii) iv) pairing, union
v) power set
vi) comprehension (reduction)

$\Big\}$ zermelo's

vii) Replacement Axiom (Fraenkel)

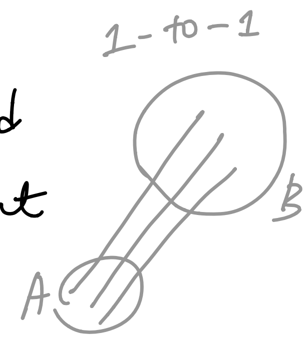
suppose $\phi(x, y)$ is "a property"

e.g. $y = P(x)$

suppose A is a set and $\forall a \in A$, there $\exists! y$ such that $\phi(a,y)$ hold

Then there is a set B such that

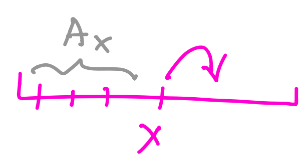
$$b \in B \iff \exists\, a \in A\, (\Phi(a,b))$$



e.g. $A = \omega$, $\Phi(n,y) =$ "y is the nth powerset of ω"

Replacement gives us $B = \bigcup_n \mathcal{P}(\omega)$

Thm. suppose $(A,<)$ is a well order and $\phi(x,y,f)$ such that for $\forall a \in A$ and f there $\exists! y$ with $\overbrace{\phi(a,y,f)}$ property of 3 variables



Then $\exists$ function F with domain A such that $\forall x \in A$. $\Phi(x, F(x), F\restriction_{A_x})$ holds

recursive

function with restricted domain

$A_x = \{y \in A \mid y \in x\}$

initial segment up to x

e.g. $\phi(n, y, f)$

if $n = m+1$ then $y = P(f(m))$

if $n = 0$, then $y = \omega$; $n=1$, $y = P(\omega)$

$n=2$, $y = P(P(\omega))$

this example gives a function $F: \omega \to$

$F(n) =$ nth power set of ω

First note: such F is unique

<u>why</u> suppose F_1, F_2 satisfy $\circledast$

we prove by induction that $F_1 = F_2$

Assume $x \in A$ and $\forall y < x$, $F_1(y) = F_2(y)$

now $\phi(x, F_1(x), F_1 \upharpoonright A_x)$ and

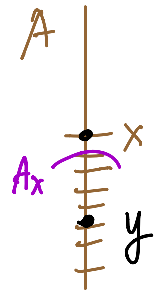
$\phi(x, F_2(x), F_2 \upharpoonright A_x)$ both hold

note: $F_1 \upharpoonright A_x = F_2 \upharpoonright A_x$

So by assumption on ϕ: $F_1(x) = F_2(x)$

Thus such F is unique

proof of thm. (existence)



$g_x(y) = ?$

claim. $\forall x \in A$, $\exists !\, g_x$ with $\mathrm{dom}(g_x) = A_x$ such that

$\forall y < x$: $\phi(y, g_x(y), g_x \restriction A_y)$

this is unique as we proved before.

Consider $\psi(s,w)$ ["$w = g_s$"]

w is the unique function with A_s such that $\forall y < s$: $\phi(y; w(y), w\restriction_y)$ holds

Idea: $\forall s \ \exists !\, w$

by replacement: $G : A \to$

$G(x) = g_x$ ← unique recursive def up to x

Note: if $y < x$, then $g_x \restriction A_y = g_y$
(by uniqueness)

Now we can define $F : A \to$

by $F(x) = z \leftrightarrow \exists x'$ with $g_{x'}(x) = z$