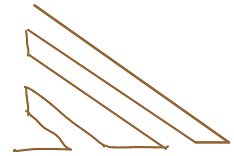
Lemma. (no AC)

for any infinite ordinal α, $|\alpha \times \alpha| = |\alpha|$

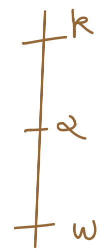


Proof. Know for $\alpha = \omega$

if true for α then true for any ordinal β with $|\beta| = |\alpha|$

$$\begin{tikzcd} \alpha \arrow[r, leftrightarrow, "h"] \arrow[d, leftrightarrow] & \beta \arrow[d, leftrightarrow, dashed] \\ \alpha\times\alpha \arrow[r, leftrightarrow] & \beta\times\beta \end{tikzcd}$$

Recall an ordinal is cardinal if $\forall \beta < \alpha$: $|\beta| < |\alpha|$



So if we prove the lemma for a cardinal α, then we know the lemma for all $\alpha \le \beta < k$, where k is the minimal cardinal above α.

$\to$ enough to prove lemma for all cardinals α.

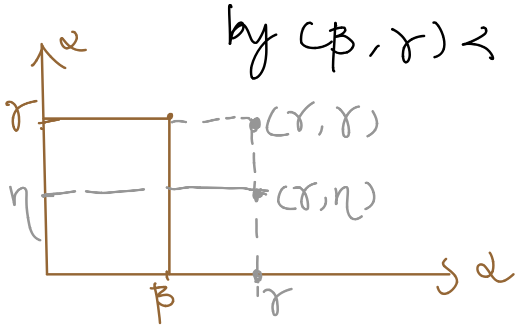
AFSOC, the lemma fails and let α be the minimal cardinal for which it fails

(for ordinal $\beta < \alpha$, $|\beta \times \beta| = |\beta|$) Induction Hypothesis

<u>want</u> find $\alpha \times \alpha \leftrightarrow \alpha$ Define an order $\triangleleft$ on $\alpha \times \alpha$ by $(\beta, \gamma) \triangleleft (\delta, \eta)$ iff $\max(\beta,\gamma) < \max(\delta,\eta)$ or

$\max(\beta,\gamma) = \max(\delta,\eta)$ and $\beta < \delta$ or

$\max(\beta,\gamma) = \max(\delta,\eta)$, $\beta = \delta$ and $\gamma < \eta$



claim 1. is a well order on $\alpha \times \alpha$

claim 2. there is an order-preserving map from $(\alpha\times\alpha, \prec)$ to $(\alpha, <)$

Note: Claim $\Rightarrow |\alpha \times \alpha| \leq |\alpha|$

proof of claim 2 assume claim 1

Otherwise, by Comparison for well orders, conclude $(\alpha, <)$ is isomorphic to a strict initial segment of $(\alpha\times\alpha, \prec)$

Let I be this initial segment.

$I = \{(\delta,\eta) \in \alpha\times\alpha \mid (\delta,\eta) \prec (\beta,\gamma)\}$

for a fixed $(\beta,\gamma) \in \alpha\times\alpha$ $\leftarrow \beta,\gamma<\alpha$

Suppose $\gamma = \max\{\beta,\gamma\}$

So $I \subseteq (\gamma+1)\times(\gamma+1)$ $(\gamma+1 := \gamma \cup \{\gamma\})$

✱ $|\alpha| \leq |(\gamma+1)\times(\gamma+1)| = |\gamma\times\gamma| = |\gamma|$

So $|\alpha| \leq |\gamma|$ for an ordinal $\gamma < \alpha$.

Contradicting the fact that α is a cardinal

✱ we have an order preserving bijection from $\alpha \hookrightarrow I \subseteq (\gamma+1)\times(\gamma+1)$

sketch of proof for claim 1

Sps $X \subseteq \alpha \times \alpha$ nonempty,

want to find min element.

Note (by def) $<$ is a strict linear order by $\alpha \times \alpha$

- first minimize $\max(\beta, \gamma)$ for $(\beta, \gamma) \in X$
  (Let $X' \subseteq X$ with the same max)

- next minimize β among all $(\beta, \gamma) \in X'$
  (Let $X'' \subseteq X$ with the same min)

- next minimize γ among $(\beta, \gamma) \in X''$

$\downarrow$

get minimal (β, γ) $\leftarrow$ $\gamma \in \alpha$ well ordered

Thm. (w/o AC) TFAE

1) AC 2) $|A \times A| = |A|$

for any set A

proof. 1) $\Rightarrow$ 2) AC: $\exists$ ordinal α with $|\alpha| = |A|$

since A is infinite, α is infinite

$|\alpha \times \alpha| = |\alpha|$

2) $\Rightarrow$ 1) $\forall$ set A, $\exists f: A \hookrightarrow \alpha$, α ordinal

Fix A. let α be the Hartog ordinal of A

Let $B = A \cup \alpha$. By 2): get injective map $h: B \times B \to B$

By h injective, $\forall a \in A$, $\exists \beta < \alpha$ with

$h(a, \beta) \notin A$

Now define $g: A \to B$

$g(a)$ is $h(a, \beta)$ for minimal β with $h(a, \beta) \notin A$. h is injective $\Rightarrow g$ is injective $\square$