Recall Thm.

$A \subseteq \mathbb{R}$ closed

Thm. Either A is countable $(|A| = |\omega|)$ or there is a perfect set $C \subseteq A$

(in fact $A = C \,\underline{\cup}\, D$ where C perfect & D countable)

Cantor Bendixson Derivative

$A^{(0)}$ $\quad A^{(1)}$ $\quad \cdots\cdots \quad A^{(\alpha)}$

$A^{(1)} = A \setminus \{\text{isolated pts of } A\}$

$$A^{(\alpha+1)} = A^{(\alpha)} / \{\text{isolated pts in } A^{(\alpha)}\}$$

α limit $\quad A^{(\alpha)} = \bigcap_{\beta<\alpha} A^{(\beta)}$

An ordinal α is limit if $\alpha \neq \beta \cup \{\beta\}$ for any $\beta \in \alpha$

(e.g. ω is limit $\quad \omega+1$ successor $\quad \omega+\omega$ is limit)

We saw that there is $\alpha < \omega_1$ s.t. $A^{\alpha+1} = A^{\alpha}$

perfect

first uncountable ordinal

we'll show $A \setminus A^{(\alpha)}$ is countable.

So if $A^{(\alpha)} \neq \phi$ ✓

if $A^{(\alpha)} = \phi \Rightarrow A^{(\alpha)}$ is countable ✓

Let $J = \{ \underbrace{(p,q) \in \mathbb{Q} \times \mathbb{Q} \mid}_{\text{all open intervals with rational endpoints}} p < q \}$

Since $\mathbb{Q} \times \mathbb{Q}$ is countable, fix $c : J \to \omega$ bijection

Recursively build functions

$$h^{\alpha} : \underbrace{A^{(\alpha)} \setminus A^{(\alpha+1)}}_{\text{isolated pts}} \to J$$

by $h^{\alpha}(x) = (p,q) \iff A^{(\alpha)} \cap (p,q) = \{x\}$

and $c((p,q))$ is minimal among

all $c((p',q'))$ with $(p',q') \cap A = \{x\}$

Let n_α be $\min\{\underbrace{c(h^\alpha(x)) \mid x \in A^{(\alpha)} \setminus A^{(\alpha+1)}}_{\text{if the set is nonempty}}\}$

Last time, we showed that $\alpha \mapsto n_\alpha$ is injective (as long as defined)

If $\beta < \alpha$, there is $x \in A^{(\beta)} \setminus A^{(\beta+1)}$

things lost at stage β

$$h^\beta(x) = (p,q), \quad c(p,q) = n,$$

$$A^\beta \cap (p,q) = \{x\}$$

$$A^{\beta+1} \cap (p,q) = \phi \Rightarrow \alpha^\beta \cap (p,q) = \phi$$

and $n_\alpha = c(p',q')$

where $(p',q') \cap A^\alpha = \{x\}$, some $x \in A^{(\alpha)}$

$$(p',q') \neq (p,q) \Rightarrow c(p',q') \neq c(p,q)$$

Since there is no injective map $w_1 \to w$ there must be α such that

$$A^{(\alpha)} = A^{(\alpha+1)}$$

Cantor-Bendixson rank of A is the min such α. Define $h: A \setminus A^{(\alpha)}$

$h(x) = h^\beta(x)$ for the unique $\beta < \alpha$ such that $x \in \operatorname{dom} h^\beta$

claim. $h: A \setminus A^{(\alpha)} \to J$ is injective

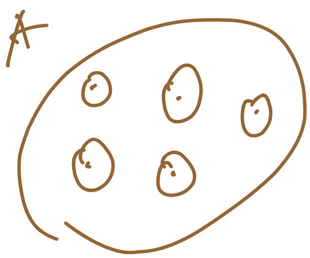
countable.

Ex. $|A \setminus A^{(\alpha)}| \leq |\omega|$

$$A = \underbrace{A^{(\alpha)}}_{\substack{\text{either empty} \\ \text{or perfect \& nonempty}}} \,\underline{\cup}\, \underbrace{A \setminus A^{(\alpha)}}_{\text{countable}}$$

Axiom of Choice

If A is a set of non-empty sets ($\emptyset \notin A$)
then there is a function f with
domain A such that $f(a) \in a, \ \forall a \in A$



e.g. If A_1, A_2, A_3 are non-empty sets then
there is $a_1 \in A_1, a_2 \in A_2, a_3 \in A_3$
we can "choose" $\langle a_1, a_2, a_3 \rangle$
with $a_i \in A_i$

non-trivial even for $A = \{A_n \mid n \in \omega\}$

Example. infinite pairs of shoes
can "choose left shoes"
doesn't require choice

for inf. seq of pairs of sock
-- do need AC

Note If there is a well order $<$ on
$\bigcup A$, then we can define (w/o AC)

$f(a) = <$-minimal element of a

(f exists by comprehension)

Thm. Given the axioms w/o AC, the following are equivalent.

1) AC

2) for any set A, there is a well ordering of A

(e.g. $\mathbb{R}$ are well-orderable)

(equiv: $\forall A \; \exists$ ordinal α with $|A| = |\alpha|$)

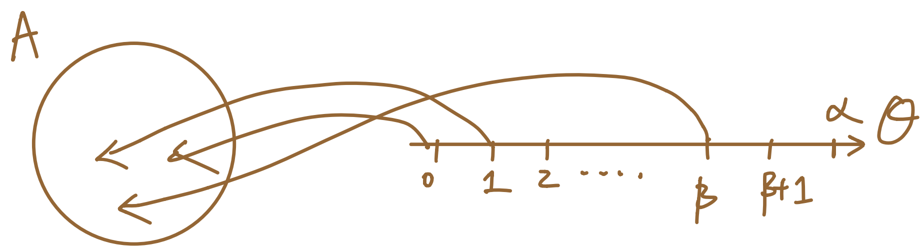
3) for any sets A, B either $\exists f : A \hookrightarrow B$ or $\exists g : B \hookrightarrow A$ $\quad |A| \leq |B|$ or $|B| \leq |A|$

comparable

2) ⇒ 1) Fix a well order on $\bigcup A$ and "choose smallest element"

1) ⇒ 2) Enough to find an ordinal α and an injective map $f : A \hookrightarrow \alpha$

(then define $a < b \Leftrightarrow f(a) \in f(b)$, this is a well order)



Fix θ the Hartog ordinal of A

(there is no injective map from $\theta \hookrightarrow A$)

Let $B = \mathcal{P}(A) \setminus \{\phi\}$

By Axiom of choice, fix a choice function c

with dom $C = B$ and $C(X) \in X$,

for any nonempty $X \subseteq A$

Fix some set w such that $w \notin A$

Define recursively a function

$$f : \theta \to A \cup \{w\}$$

By
$$f(\alpha) = \begin{cases} c(A \setminus \{f(\beta) ; \beta < \alpha\}), & \text{if } A \setminus \{f(\beta) \mid \beta < \alpha\} \neq \phi \\ w, & \text{otherwise} \end{cases} \quad ⊛$$

Fix $f : \theta \to A \cup \{w\}$ satisfying ⊛

Note: if $f(\beta) = w$ and $\beta < \alpha < w$ then $f(\alpha) = w$

why? $f(\beta) = w \Rightarrow A \setminus \{f(\xi) ; \beta < \alpha\} = \phi$

$\Rightarrow A \subseteq \{f(\xi) : \xi < \beta\} \subseteq \{f(\xi) : \xi < \alpha\}$

$\Rightarrow A \setminus \{f(\xi) \mid \xi < \alpha\} = \phi \Rightarrow f(\alpha) = w$

Note: $I = \{ a < \theta \mid f(a) \in A\}$ is an initial seg

---

claim. If $\beta < \alpha$, $\alpha, \beta \in I$ then $f(\alpha) \neq f(\beta)$

proof. since $\beta < \alpha$, then

$$\left.\begin{array}{l} f(a) = c(A \setminus \{f(\xi) \mid \xi < \alpha\} \\ \quad \in A \setminus \{f(\xi) \mid \xi < \alpha\} \end{array}\right\} f(\alpha) \neq f(\beta)$$

Since θ is the Hartog ordinal of A

$\Rightarrow I \neq \theta \Rightarrow I$ is a proper initial segment

$I = \{a \in \theta \mid \alpha < \gamma\} = \gamma$, for some $\gamma < \theta$

so $\gamma \notin I$

$f(\gamma) = w \Rightarrow A \setminus \{f(\xi) \mid \xi < \gamma\} = \phi$

$\Rightarrow f : \underset{I}{\underset{\shortparallel}{\gamma}} \to A$ is surjective

(f is a bijection)

(2) $\Rightarrow$ (3) Fix sets A, B. By (2) Fix some well orders $<^A$, $<^B$ on A and B respectively.
Apply comparison for well orders.

either $\exists$ injective o.p. map $A \to B$

or $\exists$ injective o.p. map $B \to A$

3) $\Rightarrow$ 2) Fix A. want to find a well order on A
suffices to find an injective map: $f : A \hookrightarrow \alpha$
for some ordinal α.

Let θ be the Hartog ordinal of $A \Rightarrow |\theta| \nleq |A|$

By (3), either $|A| \leq |\theta|$ or $\cancel{|\theta| \leq |A|}$

So $|A| \leq |\theta| \Rightarrow$ there exists injective map.

Lemma. If $\exists$ surjective map $f: A \twoheadrightarrow B$
then $|B| \leq |A|$ (with AC)

where did we use it?

proof. Fix $<$ a well ordering on A
Define $g: B \to A$

$$g(b) = \min\{a \in A \mid f(a) = b\}$$

g is injective because

$$b_1 \neq b_2 \Rightarrow f(g(b_1)) = b_1 \neq b_2 = f(g(b_2))$$
$$\Rightarrow g(b_1) \neq g(b_2)$$

Corollary. The Following Are Equiv

1) $|A| < |B|$ ($|A| \leq |B|$ and $|B| \nleq |A|$)
2) there is no surjective map from A to B

1) $\Rightarrow$ 2) if $|B| \nleq |A|$, then $\nexists\, A \twoheadrightarrow B$

2) $\Rightarrow$ 1) if $|B| \leq |A|$ there there is surjective map from A to B

Lemma. (without AC) if there's an injective map $B \hookrightarrow A$ then there's a surjective map $A \twoheadrightarrow B$

proof. Fix some $b_0 \in B$

Assume $f: B \hookrightarrow A$ injective

Define $g: A \to B$ by

$$g(a) = \begin{cases} b, & \text{if } f(b) = a \\ b_0, & \text{if } a \notin \operatorname{Im} f \end{cases}$$

g is surjective since $\forall b \in B$

$g(f(b)) = b$ $\square$

Corollary: $|A| < |B|$ iff there's no surjective map $A \twoheadrightarrow B$

Pf. $(\Rightarrow)$ Assume $|A| < |B|$, $|A| \leq |B|$ and

By AC $\rightarrow$ $\boxed{|B| \not\leq |A|}$, then there's no map $A \twoheadrightarrow B$

$(\Leftarrow)$ if $\neg\exists\, A \twoheadrightarrow B$ then $\neg\exists\, B \hookrightarrow B$

So $|B| \not\leq |A|$ so $|A| \leq |B| \Rightarrow |A| < |B|$

Why does AC imply comparison?

$$\begin{array}{l} \vdots \\ \omega+2 \\ \omega+1 \\ \omega \\ \vdots \\ n \\ \vdots \\ 2 \\ 1 \\ 0 \end{array}$$

countable ordinals

$\omega+1 = \omega \cup \{\omega\}$

Def. α is <u>cardinal</u> if α is an ordinal and $\forall$ ordinal $\beta < \alpha$: $|\beta| < |\alpha|$ (there is no injective map from α to β)

Example. $n \in \omega \Rightarrow n$ is cardinal

ω is cardinal

$\omega+1, \omega+2$ not cardinals ← because $|\omega+1| = |\omega|$

ω_1 (first uncountable ordinal) is a cardinal

Alt. Def. α is a cardinal if α is the minimal ordinal in $\{\beta \mid |\beta| = |\alpha|\}$

Def. For a set A its <u>cardinality</u> $|A|$ is the minimal unique α s.t. $|\alpha| = |A|$

e.g. $|\omega| = \omega = \aleph_0$ $|\omega+1| = \omega$ $|\mathbb{Q}| = \omega$

$|\omega_1| = \omega_1 = \aleph_1$ $|\omega_1+1| = \omega_1$

Note: $\underbrace{|A|=|B|}_{\substack{\text{prev notation}\\ \exists \text{ bijection}}} \Leftrightarrow \underbrace{|A|=|B|}$

$|A| = \kappa$
$|B| = \kappa$ } cardinals

with A.C.

Cantor's Continuum Hypothesis:

$|\mathbb{R}| = \omega_1 = \aleph_1$

the first uncountable cardinal

$(\forall X \subseteq \mathbb{R}$ either $|X| = |\mathbb{R}|$ or $|X| = |\omega|)$

By A.C. there is some ordinal that has a bijection with $\mathbb{R}$

Recall $|\omega \times \omega| = |\omega|$

Thm. (not using AC) TFAE

1) for any infinite set A, $|A \times A| = |A|$

2) AC

Lemma. (without AC)

For any ordinal α, $|\alpha \times \alpha| = |\alpha|$ (bijection)
infinite

NTS $|\alpha \times \alpha| \leq |\alpha|$

(We could show, with AC, that any infinite set has a bijection with an infinite ordinal)

Proof by induction on Cardinals

(Note. If Lemma true for α and $|\beta| = |\alpha|$ then Lemma is true)

(Note on Induction. Say $\Phi(\alpha)$ is a property.
$\forall$ inf. card α if $[(\forall \beta < \alpha \; \Phi(\beta)) \Rightarrow \Phi(\alpha)]$
then $\forall$ inf card α, $\Phi(\alpha)$)

"Proof". otherwise $\exists$ cardinal α s.t. $\phi(\alpha)$ fails. Let $\bar{\alpha}$ be the minimal inf cardinal s.t. $\phi(\bar{\alpha})$ fails. Contradicts I.H.