21-329: SET THEORY LECTURES

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1. Friday March 20

Without the axiom of choice, we cannot prove that any set can be well ordered. For example, the reals $\mathbb{R}$ cannot be proven to have a well order. The reals $\mathbb{R}$ do have a linear order (the natural order). Without choice, cannot prove that any set admits some linear order. For example, for $\mathcal{P}(\mathbb{R})$ we cannot prove that there is a linear order on it, without using some form of choice.

The selection principle: For any set X such that for any $A \in X$, $|A| \geq 2$, then there is a selection function g with domain X such that for $A \in X$, $g(A) \subsetneq A$ (strict subset), $g(A) \neq \emptyset$.

Theorem 1.1. T.F.A.E

(1) The selection principle.
(2) For any set X there is an ordinal α and an injective map $g\colon X \hookrightarrow \mathcal{P}(\alpha)$.

Proof of (2) implies (1). Reminder: We identify $\mathcal{P}(\alpha)$ with 2^α, functions from α to $\{0,1\}$, which we identify as binary sequences of length α, which we think of as branches in a binary tree of height α.

Fix a set X such that for all $A \in X$, $|A| \geq 2$. Let $Y = \bigcup X$. By (2) there is an injective map $h\colon Y \to \mathcal{P}(\alpha)$ for some ordinal α. Want to define g, $g(A) \subsetneq A$. Given $A \in X$, let $\beta < \alpha$ be minimal such that there are $a, b \in A$ with β the first coordinate where $h(a)$ and $h(b)$ disagree. Now let $g(A)$ be all $a \in A$ such that $h(a)(\beta) = 0$. By choice of β, $g(A)$ and $A \setminus g(A)$ are not empty. So g is a selection function as desired.

Back to (1) implies (2). Fix some set X. Let C be the set of all subsets of X with at least 2 elements. By the selection principle, there is a selection map g with domain C. Choose α a Hartog ordinal for $\mathcal{P}(X)$ (that is, there is not injective map $\alpha \hookrightarrow \mathcal{P}(X)$. We defined X^t for $t \in 2^{<\alpha}$ recursively. ($t \in 2^\beta$ for some β, recursion along β.) If X^t has at least two elements, $X^{t^\frown 0} = g(X^t)$, $X^{t^\frown 1} = X^t \setminus X^{t^\frown 0}$. If X^t has 1 element or less let $X^{t^\frown i} = \emptyset$ for $i = 0, 1$. For limit stages ($t \in 2^\beta$, β limit ordinal (e.g. ω), $X^t = \bigcap_{\gamma<\beta} X^{t\restriction\gamma}$.

Finally, we defined a map $X \to 2^\alpha$ as follows. Given $x \in X$, "follow the sets where x is in". We will define a map $\psi\colon X \to 2^\alpha$. $\psi(x) = h$ defined recursively on β so that: $x \in X^{h\restriction\beta}$ (as long as $X^{h\restriction\beta}$ is not empty.) Once $X^{h\restriction\beta}$ is empty or is $\{x\}$, just define $h(\beta) = 0$.

Show that $x \mapsto h$ is injective.

Date: Spring 2020.

2. Monday March 23

A clarification:

Axiom of choice is equivalent to: Let I be a set and $< A_i; i \in I >$ an I-indexed sequence of sets (this is just a function f with domain I such that $f(i) = A_i$.) If each A_i is not empty, then the product $\prod_i A_i$ is not empty, that is, there is an indexed sequence $< a_i; i \in I >$ such that $a_i \in A_i$.

Given AC, and let $< A_i; i \in I >$ be such that A_i is not empty for each $i \in I$. Let Y be such that $C \in Y$ if and only if there is some $i \in I$ such that $C = \{< i, x > ; x \in A_i\}$. ($Y$ is a set by comprehension.) (for each $C \in Y$, C is not empty) By AC, there is some choice function f whose domain is Y and for any $C \in Y$, $f(C) \in C$. Define a_i to be $f(C)$ for the unique $C \in Y$ where $C = \{< i, x >, x \in A_i\}$. Now $< a_i; i \in I$¿ is a choice function in $\prod_{i \in I} A_i$.

To see these statements are equivalent: Suppose we have choice for indexed sets. Let X be a set of non empty sets. Define $g \colon X \to X$, $g(A) = A$. (so g is an indexed sequence of sets indexed by X), $< A_A; A \in X >$. Now by assumption $\prod_{A \in X} A$ is not empty, so there is f with domain X such that $f(A) \in A_A = A$, so f is a choice function.

Theorem 2.1 (Using AC)**.** : If I is a countable set and A_i for $i \in I$ are countable sets, then $\bigcup_{i \in I} A_i$ is countable.

Recall (w/o AC) Suppose $I = \omega$, suppose $< f_i; i \in I >$ is a sequence of bijections, f_i a bijection from A_i to ω. Then $\bigcup_i A_i$ is countable.

Now given $< A_i; i \in I >$ as above, define C_i to be the set of all bijections from A_i to ω. By assumption, $< C_i; i \in I >$ is an I-indexed sequence of non-empty sets. So there is $< f_i; i \in I >$ such that f_i is a bijection from A_i to ω. (Also can fix some bijection from I to ω, so w.l.o.g. I is ω.

Question 2.2. Is there a natural weakening of AC that implies the theorem above?

Yes, many in fact. One of the most natural and commonly used is the following:

Definition 2.3. DC (the axiom of dependent choices) given a relation R (on some set) such that for any x in the domain of R there is some y such that xRy ($(x, y) \in R$). Then: there is an infinite sequence $< x_n; n \in \omega >$ satisfying $x_n R x_{n+1}$.

Fact: DC follows AC, it is strictly weaker than AC, and it implies that a countable union of countable sets is countable.

A Hierarchy of sets. We define by recursion on the ordinal sets V_α for an ordinal α. (one way to formalize is, given any ordinal γ we define V_α for $\alpha < \gamma$) as follows:

- Define $V_0 = \emptyset$.
- Given V_α define $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.
- If α is a limit and V_β was defined for any $\beta < \alpha$ then $V_\alpha = \bigcup_{\beta < \alpha} V_\beta$.

Note: V_n is a finite set for each $n \in \omega$. $V_\omega = \bigcup_{n \in \omega} V_n$ is infinite, contains finite sets of finite sets.... In $V_{\omega+1}$ we have ω. 2^ω is in $V_{\omega+2}$. (In $V_{\omega+1}$ can talk about $\mathbb{Q}$. so in $V_{\omega+2}$ can talk about subsets of rationals, so can talk about reals numbers.)

Proposition 2.4. For each α, V_α is transitive. (if $a \in V_\alpha$ and $b \in a$ then $b \in V_\alpha$).

Proof. By induction on the ordinals. (nothing to prove for V_0.) We saw: if V_α is transitive then $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ is transitive. For limit stage: if α is a limit ordinal,

V_β is transitive for each $\beta < \alpha$. Then $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ is transitive (Check: a union of transitive sets is transitive). □

Exercise 2.5. if $\alpha < \beta$, $V_\alpha \subseteq V_\beta$.

Proposition 2.6. For each ordinal α, $\alpha \subseteq V_\alpha$ and $\alpha \notin V_\alpha$.

Proof. We prove by induction on ordinals. (nothing to prove for V_0). Successor stage: suppose this is true for V_α. Want to show this for $\alpha + 1$: That is, want that $\alpha + 1 \subseteq V_{\alpha+1}$, that is, $\alpha \in V_{\alpha+1}$. By assumption $\alpha \subseteq V_\alpha$ and so $\alpha \in V_{\alpha+1}$. We show $\alpha + 1 \notin V_{\alpha+1}$. assume for contradiction that $\alpha + 1 \in V_{\alpha+1}$, then by definition $\alpha + 1 \subseteq V_\alpha$, which implies $\alpha \in V_\alpha$ (since $\alpha \in \alpha + 1$), a contradiction.

If α is a limit ordinal and we know the proposition for $\beta < \alpha$. Then $\beta \subseteq V_\beta$ and so $\alpha = \bigcup_{\beta<\alpha} \beta \subseteq V_\alpha$. And if $\alpha \in V_\alpha$, then by definition $\alpha \in V_\beta$ for some $\beta < \alpha$. Now we get $\beta \in \alpha \in V_\beta$ so since V_β is transitive, $\beta \in V_\beta$ a contradiction. □

3. Wednesday March 25

We defined V_α for ordinals α. V_α is transitive, $V_\alpha \subseteq V_\beta$ for $\alpha < \beta$. $V_0 = \emptyset$ and $V_{\alpha+1} = \mathcal{P}(V_\alpha)$. We also saw: $\alpha \subseteq V_\alpha$ and $\alpha \notin V_\alpha$.

Lemma 3.1. Let α be an ordinal. Then V_α satisfies the axioms of extensionality, comprehension and union. That is,

- For any $A, B \in V_\alpha$, $A = B$ if and only if for all $x \in V_\alpha$, $x \in A \iff x \in B$.
- If $A \in V_\alpha$ and $P(x)$ is a property (of a variable x), then $\{x \in A; P(X) \text{ holds}\} \in V_\alpha$.
- If $A \in V_\alpha$ then $\bigcup A \in V_\alpha$.

Proof. First, we saw that V_α is transitive, that is, for $x \in A \in V_\alpha$, $x \in V_\alpha$. If $A = B$ then for any x, $x \in A \iff x \in B$ and so this is true for $x \in V_\alpha$. If for all $x \in V_\alpha$, $x \in A \iff x \in B$ then in fact for any x, $x \in A \iff x \in B$. So by extensionality we get that $A = B$.

For second part , let $A \in V_\alpha$ and P a property. Note that $A \subseteq V_\beta$ for some $\beta < \alpha$. (If α is a successor $\alpha = \beta + 1$, then by definition $A \subseteq V_\beta$. If α is a limit, then by definition $A \in V_\beta$ for some $\beta < \alpha$ and so $A \subseteq V_\beta$.) Now $B = \{x \in A; P(x) holds\} \subseteq V_\beta$ (and it is a set by comprehension). So $B \in \mathcal{P}(V_\beta) = V_{\beta+1} \subseteq V_\alpha$.

Lastly, if $A \in V_\alpha$ then $A \subseteq V_\beta$ for some $\beta < \alpha$, then $\bigcup A \subseteq V_\beta$ (since V_β is transitive). So $\bigcup A \in V_{\beta+1} \subseteq V_\alpha$. So $\bigcup A \in V_\alpha$.

□

Note: for $\alpha > \omega$ then $\omega \in V_\alpha$, so the axiom of infinity holds in V_α.

Exercise 3.2. Let α be a limit ordinal (e.g. ω, $\omega + \omega$, ω_1). Then V_α satisfies pairing and powerset. That is,

- If $A, B \in V_\alpha$ then $\{A, B\} \in V_\alpha$.
- $A \in V_\alpha$ then $\mathcal{P}(A) \in V_\alpha$.

Proposition 3.3. (Assuming the axiom of choice) Let α be a limit ordinal, then V_α satisfies the axiom of choice. That is, $X \in V_\alpha$, $\emptyset \notin X$, then there is a function $f \in V_\alpha$ with domain X and such that $f(A) \in A$ for any $A \in X$.

Proof. Let $X \in V_\alpha = \bigcup_{\beta<\alpha} V_\beta$. There is $\beta < \alpha$ such that $X \in V_\beta$. By the axiom of choice there is a choice function f. So $f \colon X \to \bigcup X$, so $f \subseteq X \times \bigcup X$. We know that $\bigcup X \in V_\alpha$ and $X \in V_\alpha$. By the exercise above, $X \times \bigcup X \in V_\alpha$. So there is some $\gamma < \alpha$ with $X \times \bigcup X \in V_\gamma$, now $f \in V_{\gamma+1} \subseteq V_\alpha$ so $f \in V_\alpha$. □

Theorem 3.4. In $V_{\omega+\omega}$: the axioms of extensionality, pairing, union, comprehension, infinity, powerset and the axiom of choice all hold. However, the axiom of replacement fails in $V_{\omega+\omega}$.

So the axioms without replacement cannot directly prove the axiom of replacement. So the axiom of replacement is *independent* from the rest of the axioms. (Note: we used the axiom of replacement to define the set $V_{\omega+\omega}$.)

Proof of the theorem. Since $\omega+\omega$ is a limit ordinal above ω, all these axioms hold. To see that replacement fails: note that $\omega+\omega \notin V_{\omega+\omega}$, so it cannot be that all the axioms hold. (more directly, we have the property $P(n,y)$ defined by $P(n,y)$ holds if and only if $n \in \omega$ and $y = \omega + n$. Then for any $n \in \omega$ there is a unique $y \in V_{\omega+\omega}$ such that $P(n,y)$ holds. Yet there is no $Y \in V_{\omega+\omega}$ containing $\omega+n$ for all n.) □

4. The axiom of foundation.

Can there be a set x such that $x \in x$? (we will soon assume this cannot happen) Note: we proved that for an ordinal α, $\alpha \notin \alpha$.

Definition 4.1. The axiom of foundations (sometimes called the axiom of regularity) is the following statement:

For any non-empty set x there is an element $y \in x$ such that y and x contain no common elements (that is $x \cap y = \emptyset$).

Remark 4.2. The axiom of foundation implies that $z \notin z$ for any set z. Otherwise, assume that there is a z with $z \in z$. Define $x = \{z\}$. Now for any $y \in x$, $y = z$. And since $z \in z$ then $z \cap x \ni z$. So $y \cap x$ is not empty for any $y \in x$, contradicting the axiom of foundation.

We will see that in V_α the axiom of foundation always holds, for any α.

5. Friday March 27

Recall: The axiom of foundation: For any set X there is some $y \in X$ such that $y \cap x = \emptyset$. This implies there is not set z such that $z \in z$.

Theorem 1. (not assuming the axiom of foundation) The follow are equivalent:

(1) The axiom of foundation.
(2) For any set x there is an ordinal α such that $x \in V_\alpha$.

Proof. (2) implies (1). Assume (2), take any set x and let β be minimal such that $x \cap V_\beta \neq \emptyset$. Let $y \in x \cap V_\beta$. $y \in V_\beta$ then for any $z \in y$, there is some $\gamma < \beta$ with $z \in V_\gamma$. By definition of β, for any $\gamma < \beta$, $x \cap V_\gamma = \emptyset$. So $x \cap y = \emptyset$.

(1) implies (2) Assume (1). Let x be a set, we want to find α with $x \in V_\alpha$. Let T be a transitive set such that $x \in T$. ($a \in b \in T$ then $a \in T$). (For example, T can be chosen as the transitive closure of $\{x\}$. [tc(Y) is minimal transitive set containing Y.] [$T = \{x\} \cup x \cup \bigcup x \cup \bigcup\bigcup x \cup ...$]

Remark: If z is a set and for any $y \in z$ there is some α such that $y \in V_\alpha$, then there is some β such that $z \in V_\beta$.

Why? Consider the definable map sending $y \in z$ to the minimal α_y such that $y \in V_{\alpha_y}$. By Replacement there is a set $B = \{\alpha_y; y \in z\}$, then $\beta' = \bigcup B$ is an ordinal such that $z \subseteq V_{\beta'}$, and so $z \in V_{\beta'+1}$. So take $\beta = \beta' + 1$. So if T satisfies that for any $y \in T$ there is some α with $y \in V_\alpha$ we are done. (since $x \in T$). Assume otherwise, let C be the set of all $y \in T$ such that there is no α for which $y \in V_\alpha$. Take any $y \in C$. By the remark, there must be some $z \in y$ such that $z \notin V_\alpha$ for any ordinal α. Since T is transitive, $z \in y$ implies $z \in T$. And so $z \in C$. So for any $y \in C$ there is some $z \in y \cap C$. So C contradicts the axiom of foundation. □

Axioms of ZFC:

- Extensionality
- Comprehension
- Pairing
- Union
- Power set
- Infinity
- Replacement
- Choice
- Foundation

Remark 5.1. (without assuming foundation). We defined a collection "$\bigcup_\alpha V_\alpha$ " (just the collection of all sets x for which there is some α such that $x \in V_\alpha$. This collection always satisfies the axiom of foundation and the rest of the axioms. (all axioms but replacement are true in V_α whenever α is a limit.) But in this non-set collection $\bigcup_\alpha V_\alpha$, replacement holds: in the sense that Given a property $P(x, y)$ and a set $X \in V_\alpha$, if for any $x \in X$ there is a unique y such that $y \in V_\gamma$ for some γ and $P(x, y)$ holds. Then there is a set B in some V_β such that for any $x \in X$ there is a $y \in B$ such that $P(x, y)$ holds. We essentially proved it today: by replacement there is some set B with $y \in B$ if and only if $y \in V_\alpha$ for some α and $P(x, y)$ holds for some $x \in X$. Any $y \in B$ is in some V_α, so B is in some V_β by a previous remark.

6. Monday March 30

Recall: Axioms of ZFC (Zermelo, Fraenkel with choice):

- Extensionality
- Comprehension
- Pairing
- Union
- Power set
- Infinity
- Replacement
- Choice
- Foundation

Foundation = the collection $\bigcup_\alpha V_\alpha$ is the entire universe.

Cofinality: Given ordinals α and β, a map $f : \alpha \to \beta$ is said to be cofinal if for any $\delta < \beta$ there is $\gamma < \alpha$ with $f(\gamma) \geq \delta$. (equivalently: $\sup\{f(\gamma); \gamma < \alpha\} = \beta$) (Let's assume both α and β are limit ordinals.)

Definition 6.1. Given a limit ordinal β, its **cofinality**, $\mathrm{cf}(\beta)$ is the minimal ordinal α such that there is a cofinal map from α to β. Examples: $\mathrm{cf}(\omega) = \omega$. $\mathrm{cf}(\omega+\omega) = \omega$.

Lemma 6.2. If $f : \alpha \to \beta$ is a cofinal then there is an ordinal $\bar{\alpha} \leq \alpha$ and a cofinal increasing map $g : \bar{\alpha} \to \beta$. That is: for $\gamma < \delta < \bar{\alpha}$, $g(\gamma) < g(\delta)$, and g is cofinal.

Proof. Define g recursively by: $g(0) = f(0)$. $g(\gamma) = \max\{\sup\{g(\delta); \delta < \gamma\}, f(\gamma)\}+1$ - as long as this is defined (that is, as long as $\max\{\sup\{g(\delta); \delta < \gamma\}, f(\gamma)\} < \beta$. Let $\bar{\alpha}$ be the first ordinal such that this construction stops. (It is possible that $\bar{\alpha} = \alpha$.) Check: g is cofinal. □

Claim 6.3. $\mathrm{cf}(\omega_1) = \omega_1$. (Using AC!)

Proof. Need to show for any $\alpha < \omega_1$ (that is for any countable ordinal), there is no cofinal map from α to ω_1. Suppose α is a countable ordinal and $f: \alpha \to \omega_1$, then $\beta = \sup\{f(\gamma); \gamma < \alpha\} = \bigcup\{f(\gamma); \gamma < \alpha\}$ which is countable! (it is a countable union of countable sets). So β is a countable ordinal, and so $\beta < \omega_1$. [$f(\gamma) \in \omega_1$, which is a countable ordinal] □

Examples: $\mathrm{cf}(\omega_1 + \omega) = \omega$. $\mathrm{cf}(\omega_1 \cdot \omega) = \omega$.

Definition 6.4. An ordinal α is **regular** if it is equal to its cofinality, $\mathrm{cf}(\alpha) = \alpha$.

Examples: ω, ω_1 are regular. $\omega + \omega, \omega_1 \cdot \omega$, $\omega_1 + \omega_1$ are not regular. (α is not regular if and only if $\mathrm{cf}(\alpha) < \alpha$.

Definition 6.5. Let κ is a cardinal. Let κ^+ be the smallest $\alpha > \kappa$ such that α is a cardinal. Equivalently: the smallest $\alpha > \kappa$ such that there is no injective map from α to κ.

For example, $\omega^+ = \omega_1$.

For a cardinal θ, say that θ is a **successor cardinal** if there exists a cardinal κ such that $\theta = \kappa^+$. Examples: ω is not a successor cardinal. ω_1 is a successor.

Theorem 6.6 (Using AC)**.** If θ is a successor cardinal then it is regular, $\mathrm{cf}(\theta) = \theta$.

First note: for any cardinal κ, a union of κ-many sets of size κ has size κ. Given A_α for $\alpha < \kappa$, $|A_\alpha| \leq \kappa$, we can choose f_α for $\alpha < \kappa$, $f_\alpha : A_\alpha \hookrightarrow \kappa$ injective. Now given $x \in \bigcup A_\alpha$, let γ be the minimal such that $x \in A_\gamma$ and define $g(x) = (\gamma, f_\gamma(x))$. Check: $g : \bigcup_{\alpha<\kappa} A_\alpha \to \kappa \times \kappa$ is injective. Similarly, if A_α for $\alpha < \delta$ and $|\delta| = \kappa$ can do similar trick.

Proof of theorem: If $\theta = \kappa^+$, then for any $\gamma < \kappa^+$ and any map $f : \gamma \to \kappa^+$.. As before, $\sup\{f(\zeta); \zeta < \gamma\}$ has size κ and so is strictly below κ^+.

7. Wednesday April 1st

In Q2 in HW, can use ZFC.

Last time: we showed that if θ is a successor cardinal (that is, $\theta = \kappa^+$ for some cardinal κ), then θ is a regular cardinal.

[We used the axiom of choice: to show that a union of κ many sets of size κ, has size κ . (even for ω_1 is regular, need AC.)]

[[Recall: ω_1 was defined as the minimal ordinal α such that α is not countable. [Recall: we proved (without using AC) – Hartog's theorem – that there is an uncountable ordinal.] Note: $\omega_1 = \omega^+$.]]

Recall: $V_{\omega+\omega}$ satisfies all of ZFC but replacement.

We will see that (ZFC - powerset) does not imply the powerset axiom.

Definition 7.1. A set X is called hereditarily countable (HC) if the transitive closure of X , $\mathrm{tc}(X)$ is countable.

[X is countable, $\bigcup X$ is countable, $\bigcup(\bigcup X)$ is countable,...]

Note: if X is transitive, then X is hereditarily countable if and only if it is countable.

Examples: ω is HC, V_ω is HC.

ω_1, $\mathbb{R}$ are not countable, so not HC.

$\{\omega_1\}$ is countable, but not HC.

Definition 7.2. HC is the collection of all sets x such that x is hereditarily countable.

Claim 7.3 (Using the axiom of foundation)**.** HC $\subseteq V_{\omega_1}$.

Corollary 7.4. HC is a set (by comprehension).

Proof of the claim. By the axiom of foundation, any set x is in some V_α for some ordinal α . For a set x , let $\rho(x)$ be the minimal ordinal α such that $x \in V_{\alpha+1}$. (minimal α such that $x \subseteq V_\alpha$).

Subclaim: If x is in HC, and $y \in x$ then y is in HC.

proof: since $y \in x$ then $\mathrm{tc}(y) \subseteq \mathrm{tc}(x)$. Since tc(x) is countable, so is $\mathrm{tc}(y)$, and so y is in HC.

Back to the proof: Suppose for contradiction that there is some x in HC but not in V_{ω_1} and take such x with minimal $\rho(x) = \alpha$. Now $x \subseteq V_\alpha$ and $x \subseteq HC$. By minimality of $\rho(x)$, $x \subseteq V_{\omega_1}$. [all $y \in x$ have smaller degree, $\rho(y) < \rho(x)$ and by minimality, there can be no $y \in x$ with $y \in HC \setminus V_{\omega_1}$ So if $y \in$ HC then $y \in V_{\omega_1}$]

Since $x \subseteq V_{\omega_1}$ and x is countable then there is a countable ordinal $\beta < \omega_1$ with $x \subseteq V_\beta$ and so $x \in V_{\beta+1} \subseteq V_{\omega_1}$. So $x \in V_{\omega_1}$, a contradiction. □

Theorem 7.5. HC satisfies all axioms of ZFC but the powerset axiom.

First, $\omega \in$ HC and $\mathcal{P}(\omega)$ is not countable, and so is not in HC, so powerset fails.

Extensionality follows since HC is transitive. (if A and B are in HC, then $A = B$ if and only if for any $x \in HC$, $x \in A \iff x \in B$.)

Pairing: $\mathrm{tc}(\{A, B\}) = \{A, B\} \cup \mathrm{tc}(A) \cup \mathrm{tc}(B)$. So if both tc(A) and tc(B) are countable, so is $\mathrm{tc}(\{A, B\})$.

Union: follows since countable union of countable sets is countable.

Infinity: $\omega \in$ HC.

Note: if $x \subseteq$ HC and x is countable then $x \in$ HC . (proof in picture)

Replacement: Suppose P(x,y) is a property and A in HC is a set such that for any $x \in A$ there is a unique $y \in$ HC such that $P(x, y)$ holds. Need to show that there is a set B in HC such that for any $x \in A$ there is a $y \in B$ such that P(x,y) holds. Let B be the set of all y such that there is $x \in A$ with $P(x, y)$. B is a set by comprehension and $B \subseteq$ HC by assumption. So $B \in$ HC. (See picture.)