Recall.

Recursive definition along a well-ordered set

Example. can define $+, \cdot$ on ω recursively

(define $n+m$ recursively on m

$n+0=n$, $n+S(m) = S(n+m)$

Similarly, $n \cdot m$ can be defined:

$n \cdot 0 = 0$ and $n \cdot S(m) = n \cdot m + n$)

Recursion vs Induction

Thm. Given a well order $(A, <)$ and some property $\Phi(x, y, f)$ such that

$x \in A$, y some set, f function

$\forall x, f$ there is a unique y such that $\Phi(x, y, f)$ holds

function

e.g. $\Phi(m+1, y, f) \Leftrightarrow y = S(f(m))$

Φ some property

Then there exists a function F with domain A such that $\forall x \in A$:

$\Phi(x, F(x), F \upharpoonright A_x)$ holds e.g. if $A = \omega$, $\exists !\, y_0$ such that $\Phi(0, y_0, \phi)$ holds

$F(0) = y_0$ $\quad F\restriction_{A_0} = \phi$

$\exists !\, y_1$ such that $\Phi(1, y_1, 0 \mapsto F(0)) \to F(1) = y_1$

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proof. Last week we prove by induction on A that for any x there is a unique f_x such that

$\mathrm{dom}(f_x) = A_x \cup \{x\}$ and

$\forall z \in A_x \cup \{x\}$.

$\Phi(z, F(z), F\restriction_{A_z})$ this property is true

Existence Given this for all $z < x$

$$z \mapsto f_z \text{ (unique association)}$$

Using replacement, we can collect a set

$$\{ f_z \mid z < x \} = B$$

then $\bigcup B$ is a function with domain A_x

union of functions make sense when functions agree

Note that $y = \langle z, y \rangle$ is $\in \bigcup B$ iff

$\exists z < x$ with $\langle z, y \rangle \in f_z$

The set of f_z agree. i.e.

If $z' > z$, $f_{z'} \restriction_{A_z \cup \{z\}} = f_z$

By assumption, $\exists ! y$ with $\Phi(x, y, g)$

Define $f_x = g \cup \{\langle x, y \rangle\}$

(i.e. $f_x \restriction_{A_x} = g$, $f_x(x) = y$)

Now $f_x : A_x \cup \{x\} \to$

satisfies $\circledast$

So we have $x \to f_x$ definitely

By replacement, there exists a set

$B = \{ f_x \mid x \in A \}$. Let $F = \bigcup B$

then $F(x) = y \Leftrightarrow f_x(x) = y$

$\Rightarrow F$ satisfies $\circledast$

e.g. show $\{ P^n(\omega) \mid P^n(\omega)$ is the nth power set of $\omega \}$

is a set

Example of Application

Thm. Given $(A, <^A)$, $(B, <^B)$ well ordered sets, exactly one of the following holds

1) A, B are isomorphic ($\exists$ order preserving bijection $f: A \to B$)

2) A is isomorphic to an initial segment of B

3) B is isomorphic to an initial segment of A

meaning $\exists f: A \to B$, f order-preserving and $\mathrm{Im}(f)$ is an initial segment of B and is proper.

Def. Given sets A, B, say $|A| \leq |B|$ if there is an injective map $f: A \to B$

Q. Given $\underbrace{A, B}_{\text{potentially infinite!}}$, can we compare their "size"?

Say that A is well-orderable if there is some relation R s.t. (A, R) is a well order.

Corollary. If A, B are well-orderable then either $|A| \leq |B|$ or $|B| \leq |A|$

Proof. fix $<^A, <^B$ well orders of A, B

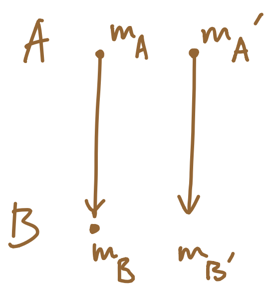
then either $\exists$ injective $f: A \to B$ OR

$\exists$ injective $f: B \to A$

Compatibility for well orders ☆

If $(A, <^A)$, $(B, <^B)$ are well orders, then exactly one of the following holds:

1) $\exists f: A \to B$ an order preserving bijection — isomorphic
2) $\exists f: A \to B$ order-preserving and $\mathrm{Im}(f)$ an initial segment of B
3) $\exists f: B \to A$ . . . . — Not mutually exclusive!



. . . keep mapping minimum until one set runs out

Proof. Fix a set $* \notin A \cup B$

Define recursively a map: $f: A \to B \cup \{*\}$

by $f(x) = \min\{b \in B \mid \forall y \overset{A}{<} x\ (f(y) < b)\}$

e.g. $f(a_{min}) = \min\{b \in B\} := b_{min}$

$\phi(x, b, f) \leftrightarrow$ "b is $\min\{f(y) \mid y \overset{A}{<} x\}$"

$\to$ If such b exists (i.e. if $B \setminus \{f(y) : y \in x\}\}$ is not empty)

otherwise $f(x) = *$

<u>Note:</u> Could be that $* \in Im(f)$ or $* \notin Im(f)$

Let $I \subset A$ be $I = \{a \in A \mid f(a) \in B\}$

This is an initial segment of A

<u>why</u> if $a < b \in I$, $b \in I \Rightarrow$

$$\{f(y) : y < b\} \text{ not onto } B$$
$$\cup$$
$$\{f(y) : y < a\}$$

$\Rightarrow f(a) \in B$

claim: $f : I \to B$ is order preserving

proof. If $a < b$, $a, b \in I \subseteq A$ then

$f(b) = \min\{w \in B \mid \forall y \overset{A}{<} b \; (f(y) \overset{B}{<} w)\} > f(a)$

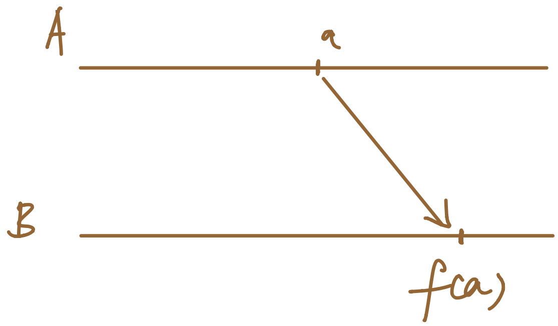
$\Downarrow$

$f(a) \overset{B}{<} w$

Claim: $\{f(a) : a \in I\}$ is an initial segment in B

Proof. NTS. if $w \overset{B}{<} f(a)$, $a \in I$

then $\exists a' < a$ with $f(a') = w$



want: $\forall a \in I$, if $w \overset{B}{<} f(a)$

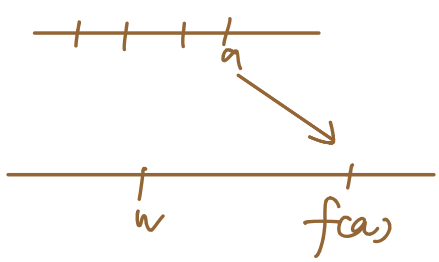
$\Rightarrow \exists a' < a$ with $f(a') = w$

prove by induction on I.

Assume this is true for all $\bar{a} < a$

Take $w \overset{B}{<} f(a)$

$$f(a) = \min \left\{ b \in B \;\middle|\; \begin{array}{l} \forall a' < a, \\ b \overset{B}{>} f(a') \end{array} \right\}$$



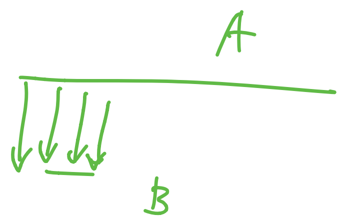
So $\exists \bar{a} \overset{A}{<} a$, with $w \overset{B}{<} f(\bar{a})$

by I.H. for $\bar{a}$,

$\exists a' \overset{A}{<} \bar{a}$ with $f(a') = w$ □

If $I = A$ (* never shows up), then
$f : A \to B$ is order preserving, Either onto 1)
or $\mathrm{Im}(f)$ is an initial segment 2)

If $I \subsetneq A$, let $a = \min A \setminus I$.



$a = \min\{a \in A \mid f(a) = *\}$

so $f\restriction_I : I \to B$ is onto & order preserving

so B is isomorphic to A_a, a proper initial segment of B. $\Rightarrow$ 3)

Def. An **ordinal** is a set α which is

1) transitive ($x \in y \wedge y \in \alpha \Rightarrow x \in \alpha$)
2) $(\alpha, \in)$ is a well ordered set

Example.

i) If $n \in \omega$ then n is an ordinal
ii) ω is an ordinal
iii) $\omega \cup \{\omega\}$ is an ordinal

Lemma. If x is an ordinal and $y \in X$ then y is an ordinal

proof.

(2) X transitive, so $y \subseteq X$. Since $\in$ is a well-ordering of X, it is also a well-ordering of y.

(1) Suppose $z \in y$ and $t \in z$, NTS $t \in y$

Since $\in$ is a strict ordering on X, and $t \in X$ and trans of X, either $t \in y$, $t = y$ or $y \in t$

If $t = y$, then $(y \in z \wedge z \in y) \Rightarrow y \in y$ contradicts irreflexivity of partial order $\in$

If $y \in t$ then $y \in t \in z \in y$. By transitivity, $y \in y$. Contradicts irreflexivity.

Thus $t \in y$. $\square$

Comparing ordinals

Suppose α, β are ordinals, g is an order-preserving map from the order $(\alpha, \in)$ onto an initial segment of $(\beta, \in)$. Then g is the identity map.

✓ unique order-preserving map

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proof. By induction on $y \in \alpha$.

I.H.

$\boxed{\text{Assume } \forall z \in y\, (g(z) = z)}$ NTS $g(y) = y$

Since g has the order property, we know

$$\forall z \in y,\ z = g(z) \in g(y)$$

So $\quad y \subseteq g(y)$

Claim. $\mathrm{Im}_g(y)$ is an initial seg of $(\beta, \in)$

proof. Suppose $z \in \mathrm{Im}_g(y)$ and $w \in z$

Fix some $t \in y$ such that $\underset{\overset{\shortparallel}{t}}{g(t)} = z$

$$w \in z = t \Rightarrow g(w) = w$$

so $w \in \mathrm{Im}_g(y)$ $\quad\square$

Now $\forall w \in g(y)$, since g is onto an initial segment, $\exists z$ such that $g(z) = w$

Since g is order preserving and $g(z) \in g(y)$

$\Rightarrow$ $z \in y$, so $w = g(z) \overset{I.H.}{=} z \in y$

Corollary. If α, β are orders, either

(1) $\alpha = \beta$. 2) $\alpha \in \beta$ or 3) $\beta \in \alpha$

proof. By comparability of well orders,

either 1) $\exists$ order preserving bij $\alpha \xrightarrow{g} \beta$

So $g = id$ and $\alpha = \beta$

2) $\exists$ order preserving $g : \alpha \to \beta$ onto a proper initial segment.

Let $\gamma \in \beta$ s.t. $Im(g) = \underbrace{\{x \in \beta \mid x \in \gamma\}}_{\gamma}$

Then $g : \alpha \cong \gamma$ and $g = id$.

So $\alpha = \gamma \in \beta$

3) $\exists$ order preserving $g : \beta \to \alpha$ onto a proper I.C.

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Thm. suppose $(A, <)$ is a well order, then there's a unique ordinal α such that $(A, <)$ is isomorphic to $(\alpha, \in)$

pf. <u>uniqueness</u> If $(A, <)$ is is isomorphic to $(\alpha, \in)$ and $(\beta, \in)$, then $\alpha = \beta$

(by the lemma of id as the unique order preserving map)

<u>Existence</u> Define a function F with domain A by recursion on A by

$$F(x) = \{ F(y) \mid y < x \}$$

(Formally, $\Phi(x, y, f)$ is the statement that "$z \in y$ iff $\exists w \in x (z = f(x))$"

<u>claim.</u> $\forall x \in A$, If $w \in F(x)$, then there is $y < x$ with $w = F(y)$. ~~By def~~