① Inverse

EX Given $f: A \to B$, the following are equivalent:

1) f is bijective
2) f has both left & right inverse
3) f has a two-sided inverse

curry func (150)

Proof.

If $|A| = 0$ OR $|B| = 0$ then the statements are all vacuously true. Otherwise:

▶ 1) to 2):

Given f is bijective, f is both injective and surjective

NTS $\exists\, g_r, g_l$ such that $\begin{cases} f \circ g_r = I_{\mathrm{dom}(g_r)}, \\ g_l \circ f = I_A \end{cases}$

By f injective, for $a, a' \in A$, $f(a) = f(a') \Rightarrow a = a'$

Define $g_l: B \to A : f(a) \mapsto a$, such function is valid because of injectivity, and $g_l(f(a)) = a$.

By f surjective, $\forall b \in B$, $\exists a \in A : f(a) = b$

Since we assume B not empty, choose $b' \in B$ such that $\exists a' \in A : f(a') = b'$.

Define $g_r : \{b'\} \to \{a'\} : b' \mapsto a'$, Then $f(g_r(b')) = b'$

▶ 2) to 3)

Known $\exists\, g_r, g_l : \overset{D \leftarrow A \;\; A \leftarrow D}{f \circ g_r} = I_{\mathrm{dom}(g_r)}$ AND $\overset{A \leftarrow B \;\; B \leftarrow A}{g_l \circ f} = I_A$

Since f has a left inverse, it is injective.
Since f has a right inverse, it is surjective.
So f is bijective and it has a two sided inverse.

▶ 3) to 1)

Known $\exists\, g: B \to A$ such that $f \circ g = I_B$, $g \circ f = I_A$

NTS f is bijective

AFSOC suppose f is not surjective, then $\exists\, b \in B$:
$\forall a \in A$, $f(a) \neq b$. But $(f \circ g)(b) = b \Rightarrow f(g(b)) = b$, $g(b) \in A$.
Contradiction.

Now suppose $f(a) = f(a')$, then $g(f(a)) = g(f(a'))$
Since $g \circ f = I_A$, we know $a = a'$. Thus f is injective.

The inverse of f is denoted f^{-1}!

Ex. $f: A \to B$, $g: B \to A$, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof. It suffices to show that $(f^{-1} \circ g^{-1})$ is the inverse for $(g \circ f)$

$$Id_B = \underbrace{g \circ g^{-1}}_{B \leftarrow A \leftarrow B} = \overbrace{g \circ Id_B \circ g^{-1}}^{A \leftarrow B \leftarrow B \leftarrow A}$$

$$= g \circ (f \circ f^{-1}) \circ g^{-1}$$

By associativity of function composition,

$$= (g \circ f) \circ (f^{-1} \circ g^{-1}) \Rightarrow (f^{-1} \circ g^{-1}) \text{ is the right inverse}$$

Similarly, $Id_A = \overbrace{f^{-1} \circ f}^{A \leftarrow B \leftarrow A} = f^{-1} \circ Id_B \circ f$

$$= f^{-1} \circ (\overbrace{g^{-1} \circ g}^{B \leftarrow A \leftarrow B}) \circ f$$

$$= (f^{-1} \circ g^{-1}) \circ (g \circ f)$$

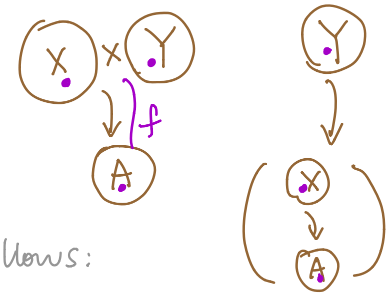
$\Rightarrow (f^{-1} \circ g^{-1})$ is the left inverse $\square$

② Higher-order Function

Functions can take other functions as inputs and output funcs.

e.g. (i) Let X, Y, A be sets

There is a bijection $A^{X \times Y} \cong (A^X)^Y$



Given $f \in A^{X \times Y}$, i.e. $\{f : X \times Y \to A\}$

we associate to f function $F : Y \to A^X$ as follows:

For each $y \in Y$, $x \in X$:

$$(F(y))(x) := f(x, y)$$

The inverse of this function is, for each $F \in (A^X)^Y$

assign $f : X \times Y \to A$ given by

$$f(x, y) := (F(y))(x)$$

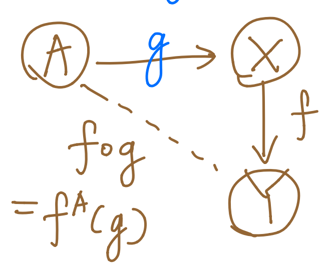
(ii) Let X, Y, A be sets   function raised to power of set

To each function $f : X \to Y$ corresponds a function

$f^A : X^A \to Y^A$ given by $f^A(g) = f \circ g$

We defined a function

$$Y^X \to (Y^A)^{(X^A)}$$



Prop. $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(g \circ f)^A = g^A \circ f^A$

Pf. compare the domains:

$g \circ f : X \to Z$, so $(g \circ f)^A : X^A \to Z^A$

$g^A : Y^A \to Z^A$, $f^A : X^A \to Y^A$, so $g^A \circ f^A : X^A \to Z^A$

Thus $\mathrm{dom}(g \circ f) = \mathrm{dom}(g^A \circ f^A)$

compare the values:

take any $h \in X^A$, so $h : A \to X$

$(g \circ f)^A (h) = (g \circ f) \circ h$

$(g^A \circ f^A)(h) = g^A(f^A(h)) = g^A(f \circ h)$

$= g \circ (f \circ h) = (g \circ f) \circ h$

e.g. X, Y, A sets Set raised to power of function

For each function $f : X \to Y$ corresponds a function

$$A^f : A^Y \to A^X$$

Given by

$$A^f(g) := g \circ f$$

$$\begin{tikzcd} A & X \arrow[l, dashed, "g \circ f = A^f(g)"'] \arrow[d, "f"] \\ & Y \arrow[ul, "g"] \end{tikzcd}$$

Ex. $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $A^{g \circ f} = A^f \circ A^g$

Pf. Compare the domain:

$g \circ f : X \to Z$, so $A^{g \circ f} : A^Z \to A^X$

$f : X \to Y$, $A^f : A^Y \to A^X$, $A^g : A^Z \to A^Y$

so $A^f \circ A^g = A^Z \to A^X$

Compare the values:

Let $h \in A^Z$, $A^{g \circ f}(h) = h \circ (g \circ f)$

$(A^f \circ A^g)(h) = A^f(A^g(h)) = A^f(h \circ g) = (h \circ g) \circ f$

(by associativity) $= h \circ (g \circ f)$ $\square$

③ Equivalence Relations & Quotients

Motivation: Consider the set $\mathcal{L}$ of all lines in $\mathbb{R}^2$

Given $k, l \in \mathcal{L}$, write $k \parallel l$ if k and l are parallel

want to talk about "direction" of lines so that
$\mathrm{dir}(l) = \mathrm{dir}(k) \Leftrightarrow l \parallel k$

geometric approach: define coord system & define slope

algebraic approach: use equivalence classes, each equivalence class stands for a particular direction

(i.e. k and l are parallel if they are equivalent)

Def A (binary) relation on a set X is a subset $R \subseteq X \times X$

Usually write $x R y$ instead of $(x, y) \in R$

Def An equivalence relation E on X is a relation s.t.

(i) E is reflexive $\quad x E x, \ \forall x \in X$

(ii) E is symmetric $\quad x E y \Rightarrow y E x, \ \forall x, y \in X$

(iii) E is transitive $\quad x E y, y E z \Rightarrow x E z, \ \forall x, y, z \in X$

Def. If $f: X \to Y$, the equivalence kernel of f is the relation E_f on X given by

equivalence relation

$$x E_f y \Leftrightarrow f(x) = f(y)$$

Given an equivalence relation E on X, for $x \in X$,

let $[x]_E = \{ y \in X : x E y \}$ ← the equivalence class of x

let $X/E := \{ [x]_E : x \in X \}$ be the quotient of X by E

the map $p_E : X \to X/E : x \mapsto [x]_E$ is called the quotient map

Prop. Let E be an eq. rel. on X. Then $p_E : X \to X/E$ is a surjection with equivalence kernel E.

Pf. Let $C \in X/E$, then $\exists x \in X : C = [x]_E$ equivalence class

Thus p_E is a surjection.

To show equivalence kernel, NTS $x E y \Leftrightarrow p_E(x) = p_E(y)$

($\Rightarrow$) Given $x E y$, we know $\forall z \in [x]_E : z \in [y]_E$

Similarly, $\forall z' \in [y]_E : z' \in [x]_E$. So $p_E(x) = p_E(y)$

($\Leftarrow$) Given $p_E(x) = p_E(y)$, we know $[x]_E = [y]_E$.

So $\exists z \in [x]_E$ (otherwise vacuously true) such that $z \in [y]_E$. It follows that $x E z$ and $y E z \Rightarrow x E y$ $\square$

quotient map