① Algebraic Structures ⇝ "sets with operations on them"

Examples: $(\mathbb{Z}, +, \cdot)$

② Reminder:

$\boxed{\text{Sets}}$

- $A \Delta B = (A/B) \cup (B/A)$ 
- $A \subsetneq B$ strict subset
- $|A|$: cardinality of A, # of elements
- $\mathcal{P}(A)$: powerset, set of all subsets

$\boxed{\text{function}}$

- $f: A \to B$, $\mathrm{dom}(f) = A$,

  $\mathrm{im}(f) = B = \{ f(a),\ a \in \mathrm{dom}(f) \}$

- B^A : the set of functions from A to B

  Ex. If A, B are finite, $|B^A| = |B|^{|A|}$

- f injective if $f(a) = f(b) \Rightarrow a = b$ OR

  $a \neq b \Rightarrow f(a) \neq f(b)$

  surjective if $\mathrm{im}(f) = B$

  bijective if injective & surjective

- Two functions f, g are equal if

  $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $f(x) = f(g)$, $\forall x \in \mathrm{dom}(f)$

③ Examples:

1) Let A be a set. Let $\{0\}$ be a 1-element set.

What is $A^{\{0\}}$?

The set of all functions $f: \{0\} \to A$

Claim: $A^{\{0\}} \cong A$ (exist a bijection)

↳ elements of A can be construed as functions from a 1-element set to A

$f \mapsto f(a)$

2) $A^{\{0,1\}} \cong A \times A = \{(a_1, a_2) : a_1, a_2 \in A\}$

$f \mapsto (f(0), f(1))$

Ex. 3) $A^{\phi} = ?$ $\qquad \phi^{A} = ?$

$\{\phi\}$

4) $\{0,1\}^{A} \cong \wp(A)$

each element is either included / excluded

④ Composition of functions

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & B \arrow[r, "f"] & C \end{tikzcd}$$

$\text{dom}(f \circ g) = \{ x : g(x) \in \text{dom}(f) \}$

for $x \in \text{dom}(f \circ g)$, define $(f \circ g)(x) = f(g(x))$

☆ Ex. composition is associative

Let f, g, h be functions, then $(f \circ g) \circ h = f \circ (g \circ h)$

Proof: To show equality, we need to show their domains are the same, and for each element in that domain, the element is mapped to identical values.

1) $\text{dom}(f \circ (g \circ h)) = \{ x : (g \circ h)(x) \in \text{dom}(f) \}$
$= \{ x : g(h(x)) \in \text{dom}(f) \}$

$\text{dom}((f \circ g) \circ h) = \{ y : h(y) \in \text{dom}(f \circ g) \}$

$\text{dom}(f \circ g) = \{ z : g(z) \in \text{dom}(f) \}$

Then $\text{dom}((f \circ g) \circ h) = \{ y : g(h(y)) \in \text{dom}(f) \}$
$= \text{dom}(f \circ (g \circ h))$

2) Let $x \in \text{dom}(f \circ (g \circ h))$, $(f \circ (g \circ h))(x) = f((g \circ h)(x))$
$= f(g(h(x)))$. On the other hand, $((f \circ g) \circ h)(x) =$
$(f \circ g)(h(x)) = f(g(h(x))) = (f \circ (g \circ h))(x)$ □

Example. Applying a function to an element can be encoded as a composition operation

$f : A \to B$

Define $g_a : \{0\} \to A : 0 \mapsto a$

Then $f \circ g_a : \{0\} \to B : 0 \mapsto f(a)$

⑤ Identity functions: A is a set

$Id_A : A \to A : a \mapsto a$

Ex. $f : A \to B$ then $f \circ Id_A = Id_B \circ f = f$

Def Let $f : A \to B$, $g : B \to A$

If $f \circ g = Id_B$, then f is a left inverse of g and g is a right inverse of f.

Ex. Give example of f, g such that f is left but not right inverse of g.

$g = Id_{\{1,2\}}$, $f = Id_{\mathbb{Z}}$, $f \circ g = Id_{\{1,2\}}$

$g \circ f$ undefined

If f is both a left & right inverse of g, then f is a two-sided inverse of g.

$f: A \to B$, $A \neq \emptyset$

Ex. • f has a left inverse $g: B \to A$

$\Leftrightarrow$ f is injective

$g(f(a)) \mapsto a \Leftrightarrow g \circ f = Id_A \Leftrightarrow$ left inverse

• f has a right inverse $g: B \to A$

$\Leftrightarrow$ f is surjective

use axiom of choice

$f(g(b)) \mapsto b \Leftrightarrow f \circ g = Id_B \Leftrightarrow$ right inverse