An action of G on X is a homomorphism

$\alpha : G \to \mathrm{Sym}(X)$

$\alpha : G \curvearrowright X$

A right action of G on X is a function

$\beta : G \to \mathrm{Sym}(X)$ s.t. $\forall g_1, g_2 \in G$:

$$\beta(g_1 g_2) = \beta(g_2) \cdot \beta(g_1)$$

Usually write $\beta : X \curvearrowleft G$

$x \cdot_\beta g$ or $x \cdot g$ for $\beta(g)(x)$

$$\boxed{(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)}$$

(dihedral groups)

The shift action

g acts on powerset of G

$\sigma : G \curvearrowright \mathcal{P}(G)$

$g \cdot_\sigma S := gS := \{gh : h \in S\}$ left coset

Ex. convince yourself that this is an action

$\sigma : G \to (\mathcal{P}(G) \to \mathcal{P}(G))$

$\sigma(g_e)(S) = \{h : h \in S\} = S \Rightarrow \sigma(g_e) = id_{\mathcal{P}(G)}$

$$\sigma(xy)(S) = \{xyh : h \in S\}$$

$$\sigma(x)(\sigma(y)(S)) = \sigma(x)(\{yh : h \in S\}) = \{xyh : h \in S\}$$

So σ is an action $\square$

Notice: $\mathcal{P}(G) \cong \{0,1\}^G$, so $G \curvearrowright \{0,1\}^G$

More generally, there is a shift action $G \curvearrowright X^G$, for any given set X_i.

Take $g \in G$, $f \in X^G$, $g \cdot f \in X^G$ is the function $G \to X$ given by $(g \cdot f)(h) := f(g^{-1}h)$

Consider $f : G \to \{0,1\}$ s.t. $f(h) = 1 \Leftrightarrow h \in S$

So $(g \cdot f)h = 1 \Leftrightarrow h \in gS \Leftrightarrow f(g^{-1}h) = 1$

Even more generally, given any action $G \curvearrowright Y$, can "lift" it to an action $G \curvearrowright X^Y$. Ex. explain how to do this

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Actions of $(\mathbb{Z}, +)$? $\alpha : \mathbb{Z} \curvearrowright X$

$\alpha(0) = \mathrm{id}_X$

$\alpha(1) =: f \in \mathrm{Sym}(X)$

$\alpha(n) = f^n, \ \forall n \in \mathbb{Z}$

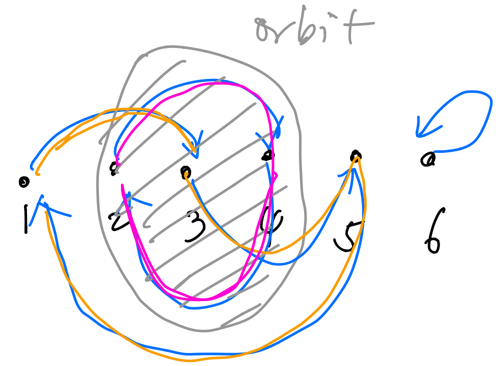
there is a natural bijection:

$$\{\text{actions } \mathbb{Z} \curvearrowright X\} \longleftrightarrow \mathrm{Sym}(X)$$

$$\alpha \longmapsto \alpha(1)$$

Eg. $a: \mathbb{Z} \curvearrowright [6]$

$$a(1) = \begin{pmatrix} 1 \mapsto 3 \\ 2 \mapsto 4 \\ 3 \mapsto 5 \\ 4 \mapsto 2 \\ 5 \mapsto 1 \\ 6 \mapsto 6 \end{pmatrix}$$



Def: $\alpha: G \curvearrowright X$, $x \in X$. The orbit of x under α is the set

$$G \cdot_\alpha x = G \cdot x = O_\alpha(x) = O_x := \{g \cdot x : g \in G\}$$

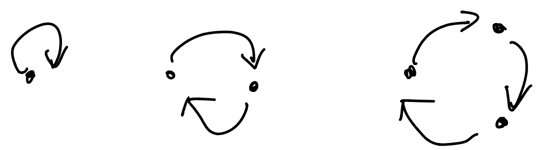
The orbit equivalence relation E_α is the binary relation on X given by $x E_\alpha y$ $\Leftrightarrow y \in G_\alpha x$ (if y is in the orbit)

Ex. This is indeed an equivalence relation, whose classes are precisely the orbits of α.

Ex. For actions of $(\mathbb{Z}, +)$, orbits are of the following "types":

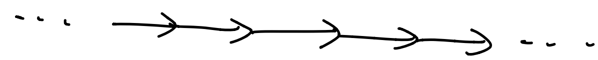
$\alpha(2) = \alpha(1+1) = \alpha(1) \circ \alpha(1)$

cycle



$\alpha(0) = \mathrm{Id}_{X \to X}$

OR

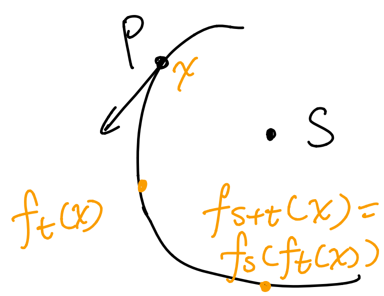


$\alpha(1) = f$

$\alpha(2) = f^2$

infinite orbit

example. want to calc the trajectory of a planet going around the sun



The state of the planet is described by a point in $\mathbb{R}^6$: 3 spatial coords + 3 velocity coordinates

For each $x \in \mathbb{R}^6$ and $t \in \mathbb{R}$, let $f_t(x)$ be the state of the planet at time t if the starting state is x.

This defines an action

$$\mathbb{R} \curvearrowright \mathbb{R}^6 : t \mapsto f_t$$

$$f_0(x) = x \ , \ f_t(f_s(x)) = f_{t+s}(x)$$

The orbits under this action are "orbits" in normal sense

The Conjugate action

$$\alpha_{conj} : G \curvearrowright G : g \cdot h := ghg^{-1}$$

This is an action:

$$\alpha_{conj}(g_1 g_2)(h) = g_1 g_2 h (g_1 g_2)^{-1}$$

$$= g_1 g_2 h g_2^{-1} g_1^{-1} = g_1 \, \alpha_{conj}(g_2)(h) \, g_1^{-1}$$

$$= \alpha_{conj}(g_1) \, \alpha_{conj}(g_2) \, h$$

Remember: $Aut(G) = \{$ automorphisms of $G\}$

$$\alpha_{conj} : G \to Aut(G) \leq Sym(G)$$

NTS: $\forall g \in G$: $\alpha_{conj}(g) : G \to G$ is a homomorphism, i.e.:

$\forall\, h_1, h_2 \in G$

$$\alpha_{conj}(g)(h_1, h_2) \stackrel{?}{=} \alpha_{conj}(g)(h_1) \cdot \alpha_{conj}(g)(h_2)$$

$$\alpha_{conj}(g)(h_1, h_2) = g h_1 h_2 g^{-1}$$

$$\alpha_{conj}(g)(h_1) \cdot \alpha_{conj}(g)(h_2) = g h_1 g^{-1} g h_2 g^{-1} = g h_1 h_2 g^{-1}$$

$$g h_1 h_2 g^{-1} = g h_1 g^{-1} g h_2 g^{-1}$$

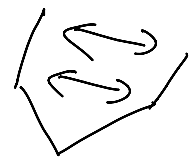
If G is a concrete group on X

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "g"'] & X \arrow[d, "g"] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

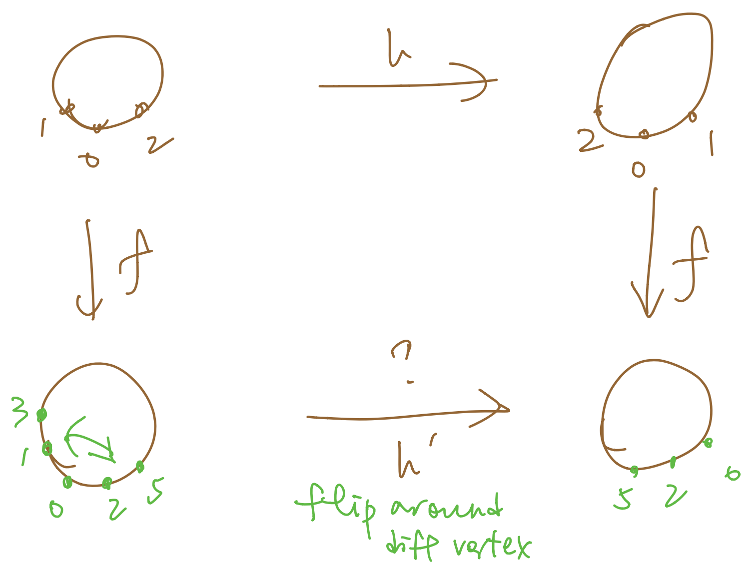
g is a homomorphism on graph

The dihedral group D_{2n}:

$g =$  rotate

$h =$ flip 

What is the orbit of h under the conjugate action?



$f \in D_{2n}$, so f is an automorphism of the cycle. h' "looks like" h, modulo changing the labels of the vertices

E_{conj} = the orbit equivalence relation on G. g and h are conjugate if $g \, E_{conj} \, h$.

Note: G is Abelian $\Leftrightarrow$ the conjugation action is trivial

$\Rightarrow$: $\alpha_{conj}(g)(h) = ghg^{-1} = gg^{-1}h = h$

$\Leftarrow$: Exercise

$Sub(G) = \{ H \subseteq G : H \text{ is a subgroup} \}$

Instead of conjugating elements, we can conjugate entire subgroup

$\alpha_{conj} : G \curvearrowright Sub(G)$

$g \cdot H = gHg^{-1} = \{ ghg^{-1} : h \in H \}$

Notice $g \cdot H$ is actually a subgroup of G.

E_{conj} : the orbit equivalance relation on $Sub(G)$

Two subgroups H_1 and H_2 are conjugate if $\underbrace{H_1 \, E_{conj} \, H_2}_{\text{i.e. } gH_1g^{-1} = H_2 \text{ for some } g \in G}$

H_1, H_2 conjugate $\Rightarrow H_1 \cong H_2$ ← isomorphism

A subgroup $H \subseteq G$ is normal if

$O_{conj}(H) = \{H\}$ (H only conjugate to itself)

Denote $H \trianglelefteq G$

Examples: • $\{e\}$, G are normal subgroups of G

- if G is Abelian, every subgroup is normal
- $G = D_{2n}$ the dihedral group

g: shift $\langle h \rangle = \{e, h\}$ is not a normal subgroup of D_{2n}

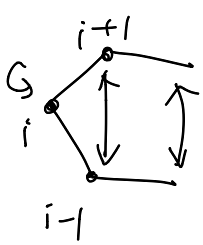
h: flip

why?

h is conjugate to h_i, $\forall i$?

so $\langle h \rangle$ is conjugate to $\langle h_i \rangle$

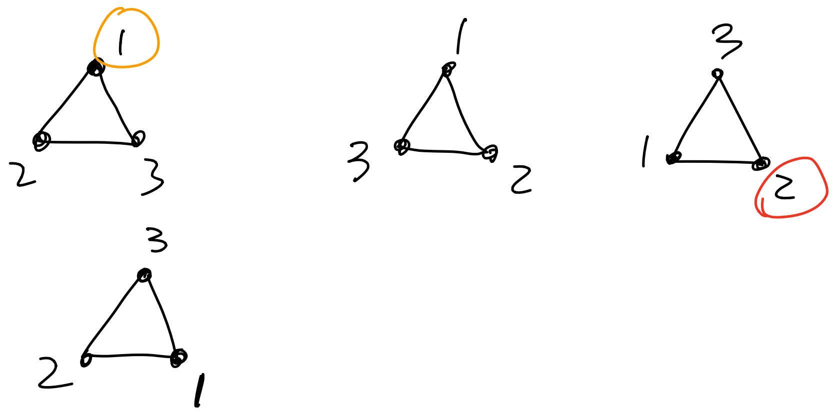
but $\langle h \rangle \neq \langle h_i \rangle$ for $i \neq 0$

$h_i =$ 

$h_i = g^i h \Rightarrow h_i \in G$

$h_i h h_i^{-1} =$

$\langle g \rangle = \{e, g, g^2, \dots g^{n-1}\} \trianglelefteq D_{2n}$ is normal



$\alpha : G \curvearrowright X \qquad \beta : G \curvearrowright Y$

A homomorphism from α to β is a function $\varphi : X \to Y$ s.t. for all $g \in G$ and $x \in X$: $\varphi(g \cdot_\alpha x) = g \cdot_\beta \varphi(x)$

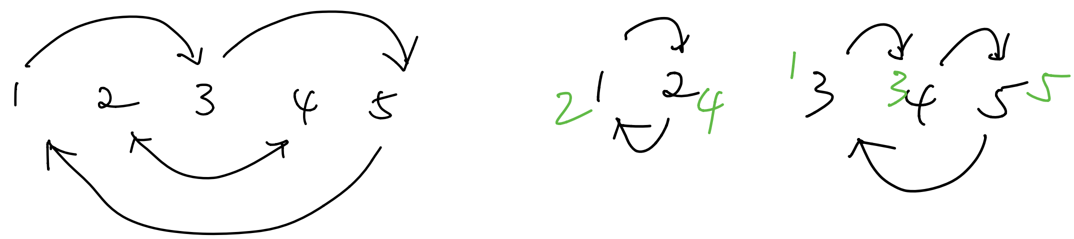
Another term: φ is G-equivariant

$$\begin{tikzcd}
X \arrow[d, "\varphi"'] & X \arrow[l, "\alpha(g)"'] \arrow[d, "\varphi"] \arrow[dl] \\
Y & Y \arrow[l, "\beta(g)"]
\end{tikzcd}$$

$\varphi \circ \alpha(g) = \beta(g) \circ \varphi$

A bijective homomorphism $\varphi: X \to Y$ of group actions is an isomorphism

Example: $\alpha: \mathbb{Z} \curvearrowright X$, $\beta: \mathbb{Z} \curvearrowright Y$



renaming the elements would make the picture the same.

$\Rightarrow$ These two $\mathbb{Z}$ actions are isomorphic (same type of orbit)

α, β are isomorphic iff $\exists$ bijection $b: X/E_\alpha \to X/E_\beta$

such that the type of each orbit $O \in X/E_\alpha$ is the same as the type of $b(O)$. i.e. $|O| = |b(O)|$

In general, $\alpha : G \curvearrowright X$ and $\beta : G \curvearrowright Y$ are isomorphic iff there is bijection:

$$X/E_\alpha \to Y/E_\beta$$

s.t. for each orbit $O \in X/E_\alpha$ the induced actions $G \curvearrowright O$ and $G \curvearrowright b(O)$ are isomorphic.

Reduce the problem to studying actions with just one orbit.
(these are called transitive)

Let $\alpha : G \curvearrowright X$. Pick $x \in X$.

Define $f_x : G \to X$. $f_x(g) = g \cdot x$