


① Algebraic Structures  $\leadsto$  "sets with operations on them"

Examples:  $(\mathbb{Z}, +, \cdot)$

② Reminder:  $\left\{ \begin{array}{l} A \Delta B = (A \setminus B) \cup (B \setminus A) \\ \text{Sets } A \subsetneq B \text{ strict subset} \\ |A|: \text{cardinality of } A, \# \text{ of elements} \\ \mathcal{P}(A): \text{powerset, set of all subsets} \end{array} \right.$  

### Function

- $f: A \rightarrow B$ ,  $\text{dom}(f) = A$ ,  
 $\text{im}(f) = B = \{ f(a), a \in \text{dom}(f) \}$
- $B^A$ : the set of functions from  $A$  to  $B$   
ex. If  $A, B$  are finite,  $|B^A| = |B|^{|A|}$
- $f$  injective if  $f(a) = f(b) \Rightarrow a = b$  OR  
 $a \neq b \Rightarrow f(a) \neq f(b)$   
surjective if  $\text{im}(f) = B$   
bijective if injective & surjective
- Two functions  $f, g$  are equal if  
 $\text{dom}(f) = \text{dom}(g)$  and  $f(x) = g(x), \forall x \in \text{dom}(f)$

### ③ Examples:

1) Let  $A$  be a set. Let  $\{0\}$  be a 1-element set.

What is  $A^{\{0\}}$ ?

The set of all functions  $f: \{0\} \rightarrow A$

Claim:  $A^{\{0\}} \cong A$  (exist a bijection)

$\hookrightarrow$  elements of  $A$  can be construed as functions from a 1-element set to  $A$

$$f \mapsto f(0)$$

$$2) A^{\{0,1\}} \cong A \times A = \{(a_1, a_2) : a_1, a_2 \in A\}$$

$$f \mapsto (f(0), f(1))$$

Ex. 3)  $A^\emptyset = ?$        $\emptyset^A = ?$

$\emptyset$

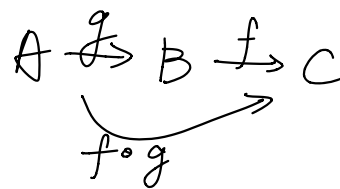
$$4) \{0,1\}^A \cong \mathcal{P}(A)$$

each element is either included / excluded

④ Composition of functions

$$\text{dom}(f \circ g) = \{ x : g(x) \in \text{dom}(f) \}$$

for  $x \in \text{dom}(f \circ g)$ , define  $(f \circ g)(x) = f(g(x))$



★ Ex. composition is associative

Let  $f, g, h$  be functions, then  $(f \circ g) \circ h = f \circ (g \circ h)$

Proof: To show equality, we need to show their domains are the same, and for each element in that domain, the element is mapped to identical values.

$$\begin{aligned} 1) \text{ dom}(f \circ (g \circ h)) &= \{ x : (g \circ h)(x) \in \text{dom}(f) \} \\ &= \{ x : g(h(x)) \in \text{dom}(f) \} \end{aligned}$$

$$\text{dom}((f \circ g) \circ h) = \{ y : h(y) \in \text{dom}(f \circ g) \}$$

$$\text{dom}(f \circ g) = \{ z : g(z) \in \text{dom}(f) \}$$

$$\begin{aligned} \text{Then } \text{dom}((f \circ g) \circ h) &= \{ y : g(h(y)) \in \text{dom}(f) \} \\ &= \text{dom}(f \circ (g \circ h)) \end{aligned}$$

$$\begin{aligned} 2) \text{ Let } x \in \text{dom}(f \circ (g \circ h)), & (f \circ (g \circ h))(x) = f((g \circ h)(x)) \\ &= f(g(h(x))). \text{ On the other hand, } ((f \circ g) \circ h)(x) = \\ & (f \circ g)(h(x)) = f(g(h(x))) = (f \circ (g \circ h))(x) \quad \square \end{aligned}$$

Example. Applying a function to an element can be encoded as a composition operation

$$f : A \rightarrow B$$

$$\text{Define } g_a : \{0\} \rightarrow A : 0 \mapsto a$$

$$\text{Then } f \circ g_a : \{0\} \rightarrow B : 0 \mapsto f(a)$$

⑤ Identity functions:  $A$  is a set

$$\text{Id}_A : A \rightarrow A : a \mapsto a$$

Ex.  $f : A \rightarrow B$  then  $f \circ \text{Id}_A = \text{Id}_B \circ f = f$

Def Let  $f : A \rightarrow B$ ,  $g : B \rightarrow A$

If  $f \circ g = \text{Id}_B$ , then  $f$  is a left inverse of  $g$   
and  $g$  is a right inverse of  $f$ .

Ex. Give example of  $f, g$  such that  $f$  is left but not right inverse of  $g$ .

$$g = \text{Id}_{\{1,2,3\}}, f = \text{Id}_{\mathbb{Z}}, f \circ g = \text{Id}_{\{1,2,3\}}$$

$g \circ f$  undefined

If  $f$  is both a left & right inverse of  $g$ , then  
 $f$  is a two-sided inverse of  $g$ .

$$f: A \rightarrow B, \text{ } \underline{A \neq \emptyset}$$

Ex. •  $f$  has a left inverse  $g: B \rightarrow A$

$\Leftrightarrow f$  is injective

$$g(f(a)) \mapsto a \Leftrightarrow g \circ f = I_A \Leftrightarrow \text{left inverse}$$

•  $f$  has a right inverse  $g: B \rightarrow A$  use axiom of choice

$\Leftrightarrow f$  is surjective

$$f(g(b)) \mapsto b \Leftrightarrow f \circ g = I_B \Leftrightarrow \text{right inverse}$$