E- invariant
VS Equiv kernel

Def. Let E be an equivalence rel on X.
A function $f: X \to Y$ is E-invariant if
$x \, E \, y \Rightarrow f(x) = f(y)$ ← more general than equivalence kernel

Thm. E an eq. rel on X. If $f: X \to Y$ is E-inv
then there is a unique func $g: X/E \to Y$
such that $f = g \circ p_E$.

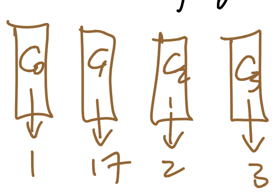
Proof. for each eq class $C \in X/E$ in quotient,
let $g(C) := f(x)$ for any $x \in C$. i.e. $g([x]_E) := f(x)$
(Since f is E-invariant, this does not depend on the choice of $x \in C$.)

$$\begin{tikzcd} X \arrow[r, "p_E"] \arrow[dr, "f"', "(E\text{-inv})"] & X/E \arrow[d, dashed, "\exists ! g"] \\ & Y \end{tikzcd}$$

▶ why $f = g \circ p_E$?
$(g \circ p_E)(x) = g(p_E(x))$
$= g([x]_E) = f(x)$

▶ why is such g unique?
$(g \circ p_E)(x) = g(p_E(x)) = g([x]_E) = f(x)$
Since f is E-invariant, g must be unique.

Ex. i) Suppose such g is injective $\Leftrightarrow$
the equivalence kernel of f is E
i.e. $x \, E_f \, y \Leftrightarrow f(x) = f(y)$



... like unique ID for each equivalence class proof. ($\Rightarrow$) g is injective, NTS $x E y \Leftrightarrow f(x)=f(y)$

$x E y \Leftrightarrow P_E(x)=P_E(y)$

$\Leftrightarrow g(P_E(x))=g(P_E(y)) \Leftrightarrow f(x)=f(y)$

($\Leftarrow$) $x E y \Leftrightarrow f(x)=f(y)$, NTS g is injective.

i.e. $\forall c_1, c_2 \in X/E$, $c_1 \neq c_2 \Rightarrow g(c_1) \neq g(c_2)$

AFSOC, suppose $\exists c_1, c_2 : c_1 \neq c_2 \Rightarrow g(c_1)=g(c_2)$

Then $\exists x, y: P_E(x)=c_1, P_E(y)=c_2$:

$f(x)=g(c_1)=g(c_2)=f(y)$ AND $P_E(x) \neq P_E(y)$

Contradiction $\square$

ii) g surjective $\Leftrightarrow$ f surjective

proof. ($\Rightarrow$) Given $\forall y \in Y: \exists c \in X/E: g(c)=y$

NTS $\forall y \in Y: \exists x \in X: f(x)=y$

choose $x: P_E(x)=c \Rightarrow f(x)=g(c)=y$

($\Leftarrow$) Given $\forall y \in Y: \exists x \in X: f(x)=y$

NTS $\forall y \in Y: \exists c \in X/E: g(c)=y$

choose $c=P_E(x)$ $\square$

iii) $\underbrace{(E_f=E) \& (\mathrm{im}(f)=Y)}_{\text{equivalence kernel AND } f \text{ surjective}} \Rightarrow g$ is a bijection

proof. by i), ii), g is surjective & injective $\square$

Corollary (canonical decomposition)

Let $f: X \to Y$. Then f decomposes as follows

$$\begin{tikzcd} X \arrow[r, twoheadrightarrow, "p"] \arrow[rrr, bend left, "f"] & X/E_f \arrow[r, leftrightarrow, "b", "\cong"'] & \operatorname{im}(f) \arrow[r, hookrightarrow, "i"] & Y \end{tikzcd}$$

\twoheadrightarrow \hookrightarrow

p is the quotient map (surjective)

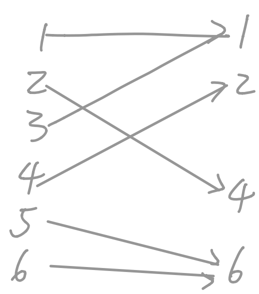
i is the inclusion map (injective)

b is the bijection

divide
↓
encode
↓
embed

e.g.

f



$X/E_f =$
$\{\ \{1,3\}, \{4\}, \{2\}, \{5,6\}\}$

$\operatorname{im}(f) =$
$\{1, 2, 4, 6\}$

Functors & universal elements

functor: $\begin{cases} \text{object map } \& \\ \text{function map} \end{cases}$

Def A functor (on sets to sets) is a rule that assigns to each set S a set $F(S)$ and to each function $f: S \to T$ a function

$$\mathcal{F}(f) : \mathcal{F}(S) \to \mathcal{F}(T) \text{ s.t.}$$

(i) for every set S, $F(id_S) = id_{F(S)}$

$$\begin{tikzcd} S \arrow[r, "id_S"] \arrow[d, squiggly, "F"] & S \\ F(S) \arrow[r, "id_{F(S)}"'] & F(S) \end{tikzcd}$$

$F(id_S)$

(ii) for all $f: S \to T$, $g: R \to S$

$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$$

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & S \arrow[r, "f"] & T \end{tikzcd}$$

$$\Big\downarrow \mathcal{F}$$

$$\begin{tikzcd} F(R) \arrow[r, "\mathcal{F}(g)"] \arrow[rr, bend right, "\mathcal{F}(f) \circ \mathcal{F}(g)"'] & F(S) \arrow[r, "\mathcal{F}(f)"] & F(T) \end{tikzcd}$$

$= \mathcal{F}(f \circ g)$

Examples:

(i) the identity functor $Id(S) := S$

$Id(f) := f$

(ii) Fix a set A

Let $F(S) := A$ for all S

Let $F(f) := id_A$ for all functions f

$f : S \to T \qquad F(f) : A \to A$

Ex. Let $A := \{0,1\}$

Any other functors with $F(S) = A$ for all S?

Proof. Suppose such functor exists.

Then $F(S) = A$. $F(id_S) = id_A$.

$\exists\, g : S \to T$ such that $F(g) \neq id$

Define $F(f) = \begin{cases} id_A, & \text{if } f = id_S \\ flip, & \text{if } Dom(f) = \{0\} \text{ XOR } Im(f) = \{0\} \\ id_A, & \text{otherwise} \end{cases}$

Then F is a functor $\square$

(iii) The powerset functor

Let $f_*(A)$ be the image of A under f

$P(S) :=$ the powerset of S

$f: S \to T$, $P(f): P(S) \to P(T)$

For each subset $A \subseteq S$, let

$$(P(f))(A) := \{ f(a) : a \in A \} = f_*(A)$$

the image of A under f

Ex. Why is it a functor?

▷ $P(id_S) : P(S) \to P(S)$

$(P(id_S))(A) = \{ id_S(a) : a \in A \} = A = id_{P(S)}(A)$

▷ Let $R \xrightarrow{g} S \xrightarrow{f} T$

Consider $P(f \circ g) \stackrel{?}{=} P(f) \circ P(g)$

$P(f \circ g)(A) = \{ (f \circ g)(a) : a \in A \} = \{ f(g(a)) : a \in A \}$

$(P(f) \circ P(g))A = (P(f))\{ g(a) : a \in A \} = \{ f(g(a)) : a \in A \}$

(iv) Fix a set A, Raising to the power A

$S \mapsto S^A$

$f \mapsto f^A$ (Recall $f : S \to T$, $f^A : S^A \to T^A$)

Ex. This defines a functor

hint: $(f \circ g)^A = f^A \circ g^A$

proof. ▷ $\mathcal{F}(id_A) = id_A{}^A : A^A \to A^A$

$$\begin{tikzcd} & A \arrow[dl, "h"'] \arrow[dr] & \\ S \arrow[rr, "f"] & & T \end{tikzcd}$$

$\mathcal{F}(id_A)(h) = id_A \circ h = h$

$\mathcal{F}(id_A) = id_{A^A}$

▷ $(\mathcal{F}(f\circ g))h = (f\circ g)\circ h$

$(\mathcal{F}(f)\circ\mathcal{F}(g))(h) = (\mathcal{F}(f))(g\circ h) = (f\circ g)\circ h$

$\mathrm{Dom}(f\circ g) = \mathrm{Dom}(g)$

$\mathrm{Dom}(\mathcal{F}(f\circ g)) = (\mathrm{Dom}(g))^A$

$\mathrm{Dom}(\mathcal{F}(f)\circ\mathcal{F}(g)) = (\mathrm{Dom}(g))^A$

(v) for each set S, let $[S]^{<\infty}$ be the set of all finite subsets of S. Let $\mathcal{F}(S) := [S]^{<\infty}$

Each $f: S \to T$ is mapped to $\mathcal{F}(f): [S]^{<\infty} \to [T]^{<\infty}$

Given by $(\mathcal{F}(f))(A) := f_*(A) = \{f(a) : a \in A\}$

$A \subseteq S$ still finite

proof. This is a functor.

▷ $\mathcal{F}(id_S) : [S]^{<\infty} \to [S]^{<\infty}$

$(\mathcal{F}(id_S))(A) = id_{S*}(A) = A = id_{[S]^{<\infty}}(A)$

▷ $\mathcal{F}(f\circ g) : [\mathrm{Dom}(g)]^{<\infty} \to [\mathrm{Im}(f)]^{<\infty}$

$(\mathcal{F}(f\circ g))(A) = (f\circ g)_*(A)$

$(\mathcal{F}(f)\circ\mathcal{F}(g))(A) = (\mathcal{F}(f))\, g_*(A) = (f\circ g)_*(A)$ □

A functor G is called a subfunctor of a functor F

if: for every set S, $G(S) \subseteq F(S)$

and for every $f: S \to T$, $G(f) = (F(f))\big|_{G(S)}$

restricted to

Example:

<u>powerset functor</u>

$S \to P(S)$

$f \mapsto f_*$

$f: S \to T$

$\boxed{f_*}: P(S) \to P(T)$

<u>finite subset functor</u>

$S \mapsto [S]^{<\infty} = \{$ finite subsets of $S\} \subseteq P(S)$

$f: S \to T$

$\boxed{f_*\big|_{[S]^{<\infty}}}: [S]^{<\infty} \to [T]^{<\infty}$

finite subset functor is a subfunctor of powerset functor

Observation: suppose F is a functor and let G_0 be a mapping that assigns to each set S a subset $G_0(S) \subseteq F(S)$. Then there is a (unique) subfunctor G of F whose object map is G_0 iff

$\forall f: S \to T$ and for all $s \in G_0(S)$, we have

$$(F(f))(s) \in G_0(T)$$

$$f : S \to T$$

$$\begin{array}{ccc} \mathcal{F}(f) : \mathcal{F}(S) & \to & \mathcal{F}(T) \\ \cup\!\mid & & \cup\!\mid \\ G_0(S) & & G_0(T) \end{array}$$

$$\mathcal{F}(f)\big|_{G_0(S)} : G_0(S) \to \mathcal{F}(T)$$
$$?G_0(T)$$

Example. Fix a set A, let $\mathcal{F}$ be the functor given by $\mathcal{F}(S) := S^A$, and for each $f : S \to T$, $\mathcal{F}(f) = f^A : S^A \to T^A$

$[f^A(g) = f \circ g]$

Let E be an equivalence relation on A.

Let $G_0(S) = \{ \text{E-inv functions} : A \to S \}$

constant on equiv class

This gives rise to a subfunctor of $\mathcal{F}$:

$$f : S \to T \qquad \begin{array}{ccc} f^A : S^A & \to & T^A \\ \cup\!\mid & & \cup\!\mid \\ f^A\big|_{G_0(S)} : G_0(S) & \overset{?}{\to} & G_0(T) \end{array}$$

$$\begin{tikzcd}
S \arrow[r, "f"] & T \\
A \arrow[u, "E\text{-inv}\ g"] \arrow[ur, "f^A(g) = f \circ g"'] &
\end{tikzcd}$$

is it invariant?

YES. Take $a, b \in A$ such that $a E b$

NTS $(f \circ g)(a) = (f \circ g)(b)$

Showing: by g is E-inv, $g(a) = g(b)$

Thus $f(g(a)) = f(g(b))$

Def A universal element for a functor F is a pair (R, u) where R is a set and $u \in F(R)$ such that:

for every set S and each $s \in F(S)$, there exist a unique function $h: R \to S$ satisfying $(F(h))(u) = s$

$$\begin{tikzcd}
R \arrow[d, dashed, "\exists!\ h"] & F(R) \arrow[d, "F(h)"] & \ni u \arrow[d, mapsto] \\
S & F(S) & \ni s
\end{tikzcd}$$

Example.

For a set A and eq rel on A, we have a functor $\mathcal{F}$ that maps each set S to

$$\mathcal{F}(S) \in \{ E\text{-inv } g : A \to S \}$$

and each $f : S \to T$ gives rise to

$$\mathcal{F}(f) : \mathcal{F}(S) \to \mathcal{F}(T)$$
$$(\mathcal{F}(f))g = f \circ g$$

Q. Universal element for $\mathcal{F}$?

Need a set R, and $u \in \mathcal{F}(R) = \{ E\text{-inv from } A \to R \}$

Universal property: For all S and $s : A \to S$, E-inv, there is a unique $h : R \to S$ s.t.

$$(\mathcal{F}(h))(u) = h \circ u$$

$$\begin{tikzcd}
A \arrow[rr, "{u,\ E\text{-inv}}"] \arrow[dr, "{s,\ E\text{-inv}}"'] & & R \arrow[dl, dashed, "\exists ! h"] \\
& S &
\end{tikzcd}$$

Take $R = A/E$ (quotient), $u = p_E$ (quotient map)

Then by the canonical decomposition theorem, (R, u) is the universal element.

Recall the theorem's idea:
what matters is how we map equivalence classes

Ex 2. Fix 2 sets A & B. Define a functor $\mathcal{F}$:

$$\mathcal{F}(S) = S^A \times S^B \quad \text{(cartesian product of functions)}$$

Given: $f: S \to T$, let

$$\mathcal{F}(f) : (S^A \times S^B) \to (T^A \times T^B)$$

$$(\mathcal{F}(f))(g, h) = (f \circ g, f \circ h)$$

a. Universal element for $\mathcal{F}$?

Need set R, and $u \in \mathcal{F}(R) = R^A \times R^B$

such that $\forall S,\ s \in \mathcal{F}(S)$: $\exists!\ h: R \to S$

such that $\mathcal{F}(h)(u) = s := (s_a, s_b)$

$$\begin{tikzcd}
R \arrow[d, "h"'] & & B \arrow[ll, bend right, "u_b"'] \arrow[dll, bend left, "s_b"] \\
S & A \arrow[ul, "u_a"'] \arrow[l, "s_a"] &
\end{tikzcd}$$

Suppose $u = (u_a, u_b)$

$\mathcal{F}(h)(u)$

$= (h \circ u_a, h \circ u_b)$

If $A \cap B = \phi$, i.e. A, B are disjoint,

Define $R = A \cup B$. $u = (id_A, id_B)$

$$h(x) = \begin{cases} s_a(x), & \text{if } x \in A \\ s_b(x), & \text{if } x \in B \end{cases}$$

▷ why unique? s_a, s_b fixed

If $A \cap B \neq \phi$, Let $A' = A \times \{0\}$, $B' = B \times \{1\}$

Note $A' \cap B' = \phi$.

Define $R = A' \cup B'$. $u = (id_{A'} \circ (\times\{0\}), id_{B'} \circ (\times\{1\}))$

$$h(x) = \begin{cases} S_a(x[0]), & \text{if } x[1] = \{0\} \\ S_b(x[0]), & \text{if } x[1] = \{1\} \end{cases}$$

can write as inverse.

This is known as the disjoint union.

$$\begin{tikzcd}
A \arrow[rr, bend left, "i"] \arrow[drr, "g"'] & \cong A' \hookrightarrow & A' \cup B' \arrow[d, dashed, "f", "\text{unique?}"'] & \hookleftarrow B' \cong & B \arrow[ll, bend right, "j"'] \arrow[dll, "h"] \\
 & & S & &
\end{tikzcd}$$

$$f(x) := \begin{cases} g(i^{-1}(x)) & \text{if } x \in A' \\ h(j^{-1}(x)) & \text{if } x \in B' \end{cases}$$

Claim: Let (R, u) be a universal element for a functor $\mathcal{F}$. Suppose $b: R \to R'$ is a bijection. Then (R', u') is also a universal element for $\mathcal{F}$, where $u' = \mathcal{F}(b)(u)$.

$$\begin{tikzcd} R \arrow[d, "b"] \\ R' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}(R) \arrow[d, "\mathcal{F}(b)"] & \ni u \arrow[d, mapsto, "\mathcal{F}(b)"] \\ \mathcal{F}(R') & \ni u' \end{tikzcd}$$

proof. Take any set S, $s \in \mathcal{F}(S)$. need to show that there is a unique $h' : R' \to S$ such that $(\mathcal{F}(h'))(u') = s$.

$$\begin{tikzcd} R \arrow[r, "b"] \arrow[dr, "h"'] & R' \arrow[d, dashed, "h' = h \circ b^{-1}"] \\ & S \end{tikzcd}$$

Existence Since (R, u) is universal, there is unique $h: R \to S$ with $\mathcal{F}(h)(u) = s$.

Take $h' = h \circ b^{-1}$

Then $\mathcal{F}(h')(u') = \mathcal{F}(h')(\mathcal{F}(b)(u))$

$= (\mathcal{F}(h') \circ \mathcal{F}(b))(u)$

$= \mathcal{F}((h' \circ b))(u)$

$= \mathcal{F}(h \circ b^{-1} \circ b)(u)$

$= \mathcal{F}(h)(u) = s$ ✓

Uniqueness: AFSOC suppose $\exists g: R' \to S$, $g \neq h'$, $(\mathcal{F}(g))(u') = s$.

Consider $m: R \to S = g \circ b \neq h' \circ b = h$

Then $\mathcal{F}(m)(u) = \mathcal{F}(g) \circ (\mathcal{F}(b)(u)) = (\mathcal{F}(g))(u') = s$

Then h is not unique. Contradiction. $\square$

Thm. (uniqueness of universal element)

Let $\mathcal{F}$ be a functor. If (R, u) and (R', u') are universal for $\mathcal{F}$, then there is a unique bijection $b: R \to R'$ s.t. $u' = \mathcal{F}(b)(u)$.

Proof. Since (R, u) is universal, there is a unique function $b: R \to R'$ such that $\mathcal{F}(b)(u) = u'$. (take $S = R'$, $u' \in \mathcal{F}(R')$)

NTS: b is bijective

Similarly, (R', u') is universal, so there is a unique $b': R' \to R$ such that $\mathcal{F}(b')(u') = u$.

want to argue that $b \circ b' = id_{R'}$ and $b' \circ b = id_R$ (they are inverse)

$b' \circ b = \mathrm{id}_R$?

$$\mathcal{F}(b' \circ b)(u) = (\mathcal{F}(b') \circ \mathcal{F}(b))(u)$$
$$= (\mathcal{F}(b'))((\mathcal{F}(b))(u))$$
$$= (\mathcal{F}(b'))(u') = u$$

(R, u) universal $\to$ $\exists!\, h : R \to R$ s.t. $\mathcal{F}(h)(u) = u$, namely $h = \mathrm{id}_R$.

But $\mathcal{F}(b \circ b')(u) = u$, so $b \circ b' = \mathrm{id}_R$, as desired.

$b' \circ b = \mathrm{id}_{R'}$ is proved the same. $\square$

Contravariant functors

A contravariant functor G assigns to each set S a set $G(S)$ and to each $f: S \to T$, a function

$$G(f): G(T) \to G(S) \quad \text{such that}$$

(i) $G(id_S) = id_{G(S)}$

(ii) $f: S \to T$, $g: R \to S$ then

to be legal

$$G(f \circ g) = G(g) \circ G(f)$$

Examples. ① Contravariant powerset functor

$$S \mapsto P(S)$$
$$f \mapsto f^*$$

$f: S \to T \quad f^*: P(T) \to P(S)$

for each $B \subseteq T$, $f^*(B) = \{a \in S : f(a) \in B\}$

↑ preimage of B under f

Proof: ▷ $G(id_S): P(S) \to P(S)$

$G(id_S)(B) = \{a \in S : id_A(a) \in B\} = B$

So $G(id_S) = id_{P(S)} = id_{G(S)}$

▷ Let $R \xrightarrow{g} S \xrightarrow{f} T$, then

$$G(f\circ g) : P(T) \to P(R)$$

$$G(f\circ g)(B) = \{a \in R : (f\circ g)(a) \in B\}$$

$$G(g) : P(S) \to P(R)$$

$$G(g)(B) = g^*(B) = \{a \in R : g(a) \in B\}$$

$$G(f) : P(T) \to P(S)$$

$$G(f)(B) = f^*(B) = \{a \in S : f(a) \in B\}$$

So $G(g) \circ G(f) : P(T) \to P(R)$

$$(G(g)\circ G(f))(B) = G(g)(f^*(B))$$

$$= g^*(f^*(B)) = g^*(\{a \in S : f(a) \in B\})$$

$$= \{b \in R : f(g(b)) \in B\} = (f\circ g)^*(B) \qquad \square$$

Note: ordinary functor $\rightsquigarrow$ covariant

Ex2. Fix a set A. The following is a contravariant functor:

$S \mapsto A^S$

$f \mapsto A^f \qquad f : S \to T \rightsquigarrow A^f : A^T \to A^S$

$$A^f(g) = g \circ f$$

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, dashed, "g\circ f"'] & T \arrow[d, "g"] \\ & A \end{tikzcd}$$

Proof: ▷ $\mathcal{F}(id_S) = A^{id_S} : A^S \to A^S$

$$A^{id_S}(h) = h \circ id_S = h$$

$$= id_{A^S \to A^S}(h)$$

$$\begin{tikzcd} & S \arrow[d, "f"] \\ A \arrow[ur] & T \arrow[l, "h"] \end{tikzcd}$$

▷ $\mathcal{F}(g \circ h) = A^{g \circ h} : A^{Im(g)} \to A^{Dom(h)}$

$$A^{g \circ h}(k) = k \circ (g \circ h)$$

$$(A^h \circ A^g)k = A^h(k \circ g) = (k \circ g) \circ h$$

$$= k \circ (g \circ h) \quad \square$$

An universal element for a contravariant functor G is a pair (R, u) where R is a set and $u \in G(R)$ such that for all set S and $s \in G(S)$, there is a unique function $h: S \to R$ s.t.

$$(G(h))(u) = s$$

Note: for ordinary, or covariant functor, $h: R \to S$.

Example.

Fix sets A, B. Define a contravariant functor G as follows: $G(S) := A^S \times B^S$

for $f: S \to T$, let $G(f): (A^T \times B^T) \to (A^S \times B^S)$

be given by $(G(f))(g, h) := (g \circ f, h \circ f)$

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "g \circ f"'] \arrow[dr, "h \circ f" near end] & T \arrow[d, "h"] \arrow[dl, "g" near start] \\
A & B
\end{tikzcd}$$

Universal Element?

Need R, $u \in G(R) = A^R \times B^R$

Let $u = (p, q)$ $\quad p: R \to A$, $q: R \to B$

$\forall$ set S and $s \in G(S) = A^S \times B^S$, let $s = (g, h)$

$\exists!\, f: S \to R$ such that

$$(G(f))(u) = (g, h) = (p \circ f, q \circ f)$$

$$\begin{tikzcd}
A & R \arrow[l, "p"'] \arrow[r, "q"] & B \\
& S \arrow[ul, "g"] \arrow[u, dashed, "\exists! f"'] \arrow[ur, "h"'] &
\end{tikzcd}$$

$R = A \times B$, $f(x) := (g(x), h(x))$

↳ disjoint union is "dual" to cartesian product

$p((a,b)) = a$

$q((a,b)) = b$

Thm. (Uniqueness of Universal Element for Contravariant Functors)

If (R, u), (R', u') are universal functors for a contravariant functor G, then there is a unique bijection $b': R' \to R$ s.t. $u' = G(b')(u)$.

proof. Since (R, u) universal,

$\exists h: R' \to R$ such that $F(h)(u) = u'$

Since (R', u') universal,

$\exists h': R \to R'$ such that $F(h')(u') = u$

want to show h, h' are inverse of each other.

That is NTS $h \circ h' = id_R$, $h' \circ h = id_{R'}$

$F(h \circ h')(u) = F(h')(F(h)(u))$

$= F(h')(u') = u$

Since $F(id_R)(u) = u$, we know $h \circ h' = id_R$.

Similarly, $F(h' \circ h)(u') = F(h)(F(h')(u'))$

$= (F(h))(u)) = u' \Rightarrow h' \circ h = id_{R'}$ □