Definition: $\mathbb{Z}$ and $\mathbb{Q}$

define integers by $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup -\mathbb{N}$

A real number x is said to be rational if $x = \frac{p}{q}$, $p, q \in \mathbb{Z}$

Thm. $\mathbb{Q}$ is an ordered subfield of $\mathbb{R}$

order duality of the rationals in the real

Thm. Let $x, y \in \mathbb{R}$, $x < y$. Then $\exists\, z \in \mathbb{Q}$ such that $x < z < y$

proof. By the Archimedean property, obtain $b \in \mathbb{N}$ such that $b(y-x) > 1$

Set $a = [bx] + 1 \in \mathbb{Z}$, where $\forall p \in \mathbb{R}$, $[p]$ is the greatestest integer s.t. $[p] \leq p$.

In particular, $[p]$ satisfies the property that
$$[p] \leq p < [p] + 1$$

By , $a \geq [bx]$

Also, $bx < [bx] + 1 = a$

$$b \in \mathbb{N} \Rightarrow b > 0 \Rightarrow x < \frac{a}{b} \overbrace{\leq x +}^{\text{By } a = [bx]+1} \underbrace{\frac{1}{b} < y}_{\text{by choice, } b(y-x) > 1}$$

$$\frac{a}{b} = \frac{[bx]+1}{b} \leq x + \frac{1}{b}$$

$$b(y-x) > 1 \Rightarrow by - bx > 1 \Rightarrow bx < by - 1 \Rightarrow x < y - \frac{1}{b}$$
$$\Rightarrow y > x + \frac{1}{b}$$

choose $z = \frac{a}{b} \in \mathbb{Q}$ $\square$