Def. Let X be nonempty set

A function $f: \mathbb{N} \to X$ is referred to a sequence in X

If $f(n) = x_n \in X$, we say x_n is the nth item of the sequence

Notation $(x_n)_{n=1}^{\infty}$ or $(x_1, x_2, \dots)$

In place of $\mathbb{N}$, we can index the sequence by any set $\{p, p+1, p+2, \dots\}$ for some $p \in \mathbb{Z}$

Def. A sequence (x_n) in $\mathbb{R}$ or $\mathbb{C}$ is said to converge to x in $\mathbb{R}$ or $\mathbb{C}$ if

$\forall\, \varepsilon > 0$, $\exists\, N_\varepsilon \in \mathbb{N}$, such that if $n \geq N_\varepsilon$ then

$$|x_n - x| < \varepsilon$$

A sequence that does not converge is said to diverge.

Example. $x_n = \frac{1}{n}$

Let $\varepsilon > 0$, by the Archimedean property of $\mathbb{R}$, let $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$.

Then $\forall\, n \in \mathbb{N}$, $n > N$, $\frac{1}{n} < \frac{1}{N} \Rightarrow |x_n - 0| = \frac{1}{n} < \frac{1}{N} < \varepsilon$ □

Notation: $\lim_{n \to \infty} x_n = 0$ or $x_n \to 0$

Example. $x_n = (-1)^n$

This sequence does not converge.

Want to show: $\exists\, \varepsilon > 0$, $\forall\, N \in \mathbb{N}$, $\forall\, x \in \mathbb{R}$, there exist $n > N$ such that $|x_n - x| > \varepsilon$.

(uniqueness of limits) Hint for proof: triangle inequality

Thm. Let $x_n \subseteq \mathbb{R}$ (or $\mathbb{C}$) be a sequence.
If x_n converges to x and x_n converges to y
then $x=y$.

Proof. Let $\varepsilon>0$, since x_n converges to x,
obtain N_ε^1 such that if $n \geq N_\varepsilon^1$ then $|x_n - x| < \frac{\varepsilon}{2}$. (1)

Similarly, since x_n converges to y,
obtain N_ε^2 such that if $n \geq N_\varepsilon^2$ then $|x_n - y| < \frac{\varepsilon}{2}$. (2)

So if $n \geq \max(N_\varepsilon^1, N_\varepsilon^2)$ then both (1), (2) holds

By triangle inequality,

$$|x-y| \leq |x_n - x| + |x_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (3)$$

If $x=y$, we're done. If $x \neq y$, choose $\varepsilon = \frac{|x-y|}{2} > 0$

Then equality from (3) says that

$$|x-y| < \frac{|x-y|}{2} \Rightarrow 1 < \frac{1}{2} \Rightarrow \text{Contradiction} \ \square$$

Common limits

Thm. Let $z \in \mathbb{C}$ and $k \in \mathbb{N}$ then

i) if $|z|<1$ then $\lim_{n\to\infty} z^n = 0$

ii) if $|z|>1$ then z^n does not converge

iii) $\lim_{n\to\infty} \frac{1}{n^k} = 0$

proof.

magnitude is real

i) Let $\alpha = \inf\{|z|^n, n \in \mathbb{N}\}$. Then $\alpha \geq 0$

Want to show $\alpha = 0$.

Suppose for contradiction that $\alpha > 0$

We may also assume $z \neq 0$, then $\frac{\alpha}{|z|} > \alpha$ (by $|z|<1$)

Since α is the infimum, $\frac{\alpha}{|z|}$ is not a lower bound for $(z^n)_{n=1}^{\infty}$.

So there exists $n \in \mathbb{N}$ s.t. $\frac{\alpha}{|z|} > |z|^n \Rightarrow \alpha > |z|^{n+1}$

But α is the infimum of such number, contradiction

Then $\alpha = 0$.

By definition of infimum:

using def of limit $|z^n - 0| = |z^n|$

Given $\varepsilon > 0$, $\exists N_\varepsilon$ such that $|z|^{N_\varepsilon} < \varepsilon$, if $n \geq N_\varepsilon$

$|z^n| = |z|^n = |z|^{n-N_\varepsilon} |z|^{N_\varepsilon} < 1 \cdot \varepsilon = \varepsilon$

So $(z_n) \to 0$ $\square$

ii) Exercise

iii) Exercise (Archimedean Prop.)

Limit properties

Thm. Let (z_n), (w_m) be sequences of complex numbers such that $\lim_{n\to\infty} z_n = z$ and $\lim_{m\to\infty} w_m = w$

Then (i) $\lim_{n\to\infty} (z_n + w_n) = z + w$

(ii) $\lim_{n\to\infty} z_n w_n = zw$

(iii) $\lim_{n\to\infty} c z_n = cz$, $\forall c \in \mathbb{C}$

(iv) if $w \neq 0$ then $\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{z}{w}$

Remark: It's possible that the sum of two sequences converges, while neither of the sequences converge
e.g. $z_n = (-1)^n$, $w_n = (-1)^{n+1}$

Proof. i) Let $\varepsilon > 0$, obtain N_ε^1, N_ε^2 such that if $n \geq N_\varepsilon^1$ (similarly for $n \geq N_\varepsilon^2$) then $|z_n - z| < \frac{\varepsilon}{2}$

The triangle inequality says that
$$|z_n + w_n - (z+w)| = |(z_n - z) + (w_n - w)| \leq \varepsilon$$

So $\lim_{n\to\infty} z_n + w_n = z + w$ ◻

ii) Let $\varepsilon > 0$, obtain N_ε^1 (similarly for N_ε^2)

such that if $n \geq N_\varepsilon^1$ then

$$|z_n - z| < \varepsilon \frac{1}{2b_2}$$
$$|w_n - w| < \varepsilon \frac{1}{2b}$$

By Lemma: Let $(x_n) \to x$, then x_n is bounded.
i.e. $\exists b: |x_n| < b, \forall n \in \mathbb{N}$

we know $|z_n| < b_1$, $|w_n| < b_2$, $\forall n \in \mathbb{N}$

Then
$$\begin{aligned} |z_n w_n - zw| &= |z_n(w_n - w) + w(z_n - z)| \\ &\leq |z_n||w_n - w| + |w||z_n - z| \\ &< b_1 \frac{\varepsilon}{2b_1} + b_2 \frac{\varepsilon}{2b_2} = \varepsilon \quad \square \end{aligned}$$