Thm. Let $c > 0$, then $\lim_{n\to\infty} \sqrt[n]{c} = 1$

Q. $\lim_{n\to\infty} (\sqrt[n]{n} - 1)^n = ?$

Monotone Sequence

Def. we say $(x_n)_{n=1}^{\infty} \subset \mathbb{R}$ is increasing if $x_n \leq x_{n+1}$ for $\forall n \in \mathbb{N}$

.... strictly increasing if $x_n < x_{n+1}$ $\forall n \in \mathbb{N}$

decreasing if $x_n \geq x_{n+1}$, $\forall n \in \mathbb{N}$

strictly decreasing if $x_n > x_{n+1}$, $\forall n \in \mathbb{N}$

Thm. A monotone sequence $(x_n) \subset \mathbb{R}$ converges iff it is bounded

Proof. $(\Rightarrow)$ already proved

$(\Leftarrow)$ Suppose (x_n) is bounded

Set $E = \{x_n, n \in \mathbb{N}\}$, then E is bounded

Given (x_n) montone,

Assume (x_n) is increasing (decreasing similar)

By supremum property, $\exists x \in \mathbb{R}$: $x = \sup E$

Let $\varepsilon > 0$, $x - \varepsilon$ is not an upper bound for E

so $\exists n_0 \in \mathbb{N}$ such that $x - \varepsilon \leq x_{n_0}$

$\forall n \geq n_0$: $x - \varepsilon \leq x_{n_0} \leq x_n \leq x < x + \varepsilon$

So $|x_n - x| < \varepsilon$ if $n \geq n_0$ $\square$

example.

For $a>0, p\in\mathbb{N}$, let $x_1 = a^{\frac{1}{p}}$

Define $x_{n+1} = \frac{1}{p}\left((p-1)x_n + \frac{a}{x_n^{p-1}}\right)$

Recall we've proved that x_n is bounded below:

$$x_{n+1} = \frac{1}{p}\Big(\underbrace{x_n + \cdots x_n}^{p-1 \text{ terms}} + \frac{a}{x_n^{p-1}}\Big) \geq a^{\frac{1}{p}}$$

(using AM-GM)

claim: x_n bounded below & decreasing

To check that x_n is decreasing

$$x_{n+1} - x_n = \frac{p-1}{p} x_n - x_n + \frac{a}{p x_n^{p-1}}$$

$$= \frac{-1}{p} x_n + \frac{a}{p (x_n)^{p-1}}$$

(Application $x^{\frac{1}{n}}$ approximation $\Rightarrow$ Newton Method; See Rudin)

• Liminf and limsup

Let $(x_n) \subset \mathbb{R}$ be a sequence

For each $k \in \mathbb{N}$, let

$y_k = \inf\{x_n : n \geq k\}$ ⇒ increasing sequence (think about negative)

$z_k = \sup\{x_n : n \geq k\}$ ⇒ decreasing sequence

If y_k bounded above: define $\liminf\limits_{n\to\infty} x_n = \lim\limits_{k\to\infty} y_k = \sup\limits_{k\in\mathbb{N}} y_k$

If z_k bounded below: $\limsup\limits_{n\to\infty} x_n = \lim\limits_{k\to\infty} z_k = \inf\limits_{k\in\mathbb{N}} z_k$

If (z_k) is not bounded below, define $\limsup\limits_{n\to\infty} x_n = -\infty$

Example. $x_1 = -2018$ $x_2 = 2018$ $x_n = (-1)^n$

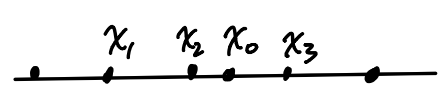
$\inf E = -2018$

$\sup E = 2018$

$\liminf = -1$

$\limsup = 1$

weaker & weaker oscillation



Thm. Characterization of liminf & limsup

$\underline{\lim}$ $\overline{\lim}$

Let $x_n \subset \mathbb{R}$ be a sequence. Then $a = \liminf_{n\to\infty} x_n$

if and only if $\forall \alpha < a$, the set

$\{n \in \mathbb{N} : x_n < \alpha\}$ is finite and

$\forall \beta > a$, the set $\{n \in \mathbb{N} : x_n < \beta\}$ is infinite.

Similarly, $b = \limsup_{n\to\infty} x_n$ iff

$\forall \alpha < b$, the set $\{n \in \mathbb{N} : x_n > \alpha\}$ is infinite

and $\forall \beta > b$, the set $\{n \in \mathbb{N}, x_n > \beta\}$ is finite

Proof. suppose $a = \liminf_{n\to\infty} x_n$, let $\alpha < a$

Then $\exists K_0$ such that $y_{K_0} > \alpha$.

By definition of y_{K_0}: $x_n \geq y_{K_0}$, $\forall n \geq K_0$

and so $\{n \in \mathbb{N} : x_n < \alpha\}$ has no more than K_0 elements.

Similarly, at $\beta > a$, $\forall k$: $y_k < \beta$,

by def of y_k, $\forall k$, $\exists\, n_k \geq k$ such that $y_k \leq x_{n_k} < \beta$

So $\{n_k : k \in \mathbb{N}\} \subset \underbrace{\{n \in \mathbb{N} : x_n < \beta\}}_{\text{is infinite}}$

Conversely, suppose a satisfies the "α-condition" and the "β-condition", we want to show

$a = \liminf_{n\to\infty} x_n$. Let $y = \liminf_{n\to\infty} x_n = \sup_k y_k$

want will show $y \leq a$ and $a \leq y$ ~ common strategy

If $a = -\infty$ then $y \geq a$.

If $a > -\infty$, let $\alpha < a$

Then $\{n \in \mathbb{N} : x_n < \alpha\}$ is finite, so

$\exists k_0$ s.t. $y_k \geq \alpha$, $k \geq k_0$

$\Rightarrow \sup_k y_k \geq \alpha$

$\Rightarrow y \geq \alpha \Rightarrow y \geq a$

by $\alpha < a$

Next if $a = \infty$, then $y \le a$

suppose $a < \infty$, then if $\beta > a$,

we have $y_k < \beta$ for every $k \in \mathbb{N}$

$$y = \sup_{k \in \mathbb{N}} y_k \le \beta$$ ~ ANY number greater than a

But $\beta > a$ is arbitrary, $y \le a$

Thus $y = a$ $\square$

Def. Let $a \in \mathbb{R}$. By a neighborhood of a,

we mean any open intervals in $\mathbb{R}$ containing a

Def. Let $(x_n) \subseteq \mathbb{R}$ be a sequence and let $c \in \mathbb{R}$

we say c is a cluster point for x_n if every neighborhood of c has infinitely many x_n

ex. (-1 and 1 is cluster point in prev ex, while ± 2018 is not)

Thm. Let $(x_n) \subset \mathbb{R}$ be a sequence
and let A denote the set of all cluster points of x_n

i) $\liminf_{n\to\infty} x_n$, $\limsup_{n\to\infty} \in A$

ii) $\forall c \in A$, $\liminf_{n\to\infty} x_n \le c \le \limsup_{n\to\infty} x_n$

Proof. Ex.

Thm. Let $(x_n) \subset \mathbb{R}$ be a sequence. Then x_n converges iff

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$$

when that happens:

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \lim_{n\to\infty} x_n$$

Proof. ($\Rightarrow$) Suppose $\lim_{n\to\infty} x_n$ exists

Let $a = \liminf_{n\to\infty} x_n$; $b = \limsup_{n\to\infty} x_n$. $a \le b$.

Suppose $a < b$, choose $a < \beta < \alpha < b$ Let $\epsilon = \frac{\alpha - \beta}{2}$.

The goal is to show that for $\forall \epsilon$,

there exist infinitely many n s.t. x_n lies outside $(x-\epsilon, x+\epsilon)$

The sets

(1) $\{n \in \mathbb{N}, x_n < \beta\}$ is infinite and

(2) $\{n \in \mathbb{N}; x_n > \alpha\}$ is infinite

If n satisfies (1), then if $x > \alpha$

$$x_n + \epsilon < \beta + \frac{\alpha - \beta}{2} = \frac{\alpha + \beta}{2} < \frac{\alpha + \alpha}{2} = \alpha < x$$

$$x_n < x - \epsilon$$

Similarly if n satisfies (2),

$$x_n - \epsilon > x$$

Both of the conclusions imply that there are infinitely many x_n's that lie outside $(x-\epsilon, x+\epsilon)$

contradiction □

($\Leftarrow$) if $\underset{\substack{\parallel \\ a}}{\underline{\lim}\, x_n} = \overline{\lim}\, x_n = b$, want to show x_n converges.

Let $\epsilon > 0$ so that $a - \epsilon < a$

Then $\{n \in \mathbb{N}: x_n < a - \epsilon\}^{*}$ is finite

$a + \epsilon = b + \epsilon > b = a$

So $\{n \in \mathbb{N}: x_n > a + \epsilon\}^{**}$ is finite

Then if $N_1 = \text{card}(*)$, $N_2 = \text{card}(**)$

then $\forall n \geq \max(N_1, N_2)$

$$|x_n - a| < \epsilon \Rightarrow x_n \to a \quad \square$$

Cauchy Sequence

Def. A sequence $(x_n) \subset \mathbb{C}$ is said to be a Cauchy sequence if $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ s.t. if $m, n \geq N$ then $|x_m - x_n| < \epsilon$

Thm. A sequence $(x_n) \subset \mathbb{C}$ converges iff it is Cauchy

proof. ($\Rightarrow$) Let x_n be convergent sequence

Then $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|x_n - x| < \frac{\varepsilon}{2}$

if $m, n \geq N$, then $|x_m - x_n| \leq |x_m - x| + |x_n - x| = \varepsilon$ $\square$

($\Leftarrow$) Suppose x_n is Cauchy

first want to show: x_n bounded

By x_n Cauchy, obtain $N_1 \in \mathbb{N}$ s.t. if $n \geq N_1$:

$$|x_n - x_{N_1}| < 1$$

Then if $n \geq N_1$

$$|x_n| \leq |x_n - x_{N_1}| + |x_{N_1}| < 1 + |x_{N_1}|$$

Let $\alpha = 1 + \max(|x_1|, |x_2|, \cdots, |x_n|)$, then $\forall n \in \mathbb{N}$, $|x_n| < \alpha$

In particular, if $a = \liminf x_n$, $b = \limsup x_n$, then

$$-\infty < -\alpha < a \leq b < \alpha < \infty$$

If $a < b$; define $\epsilon = \frac{b-a}{4} > 0$

By Cauchy $\Rightarrow \exists N$ s.t. $\forall m, n \geq N$, $|x_m - x_n| < \epsilon$

Since a is a cluster point, we can find

$m \geq N$ such that $|x_m - a| < \epsilon$

Similarly, b is a cluster point, we can find

$n \geq N$ s.t. $|x_n - b| < \epsilon$

By triangle inequality,

$$|a-b| \leq |a - x_m| + |x_m - x_n| + |x_n - b| < \epsilon + \epsilon + \epsilon$$

But $\epsilon = \frac{|b-a|}{4} \Rightarrow 4\epsilon < 3\epsilon$, contradiction

Thus $a = b \Rightarrow \liminf x_n = \limsup x_n \Rightarrow x_n$ convergent $\square$

for complex number, apply same proof twice

for both the real & imaginary part

Subsequence

Def. Let $(x_n) \subset \mathbb{C}$. If $n_1 < n_2 < n_3 < \cdots$

then (x_k) is said to be a subsequence of (x_n)

Thm. Let x_n be a convergent. Then $\forall$ subsequences of x_n, say (x_{n_j}), the limit of x_{n_j} exists and $\lim_{j\to\infty} x_{n_j} = \lim_{n\to\infty} x_n$.

Proof. Let $\epsilon > 0$, obtain N such that $|x_n - x| < \epsilon$ if $n \geq N$.

If n_j is a subsequence of $\mathbb{N}$,

then $\exists j_0$ s.t. $n_j > n_{j_0} \geq N$, $\forall j > j_0$

Then $|x_{n_j} - x| < \epsilon$ □

Thm. Every bounded sequence of complex numbers has a convergent subsequence

Proof. (for $\mathbb{R}$)

if $|x_n| < A$, $\forall n \in \mathbb{N}$ then $-A < \liminf x_n \leq \limsup x_n < N$

In perticular, they are both finite.

So pick any cluster point c of $(x_n)_{n=1}^{\infty}$

Then, obtain $n_1 \in \mathbb{N}$ such that $|x_{n_1} - c| < 1$

Obtain $n_2 > n_1$, $n_2 \in \mathbb{N}$, such that $|x_{n_2} - c| < \frac{1}{2}$

By induction, obtain a sequence $n_k > n_{k-1}$, $n_k \in \mathbb{N}$, s.t. $|x_{n_k} - c| < \frac{1}{k}$

Given $\epsilon > 0$, by Archimedean properties, $\exists k_0$ such that $\frac{1}{k_0} < \epsilon$

then $\forall k \geq k_0$, $|x_{n_k} - c| < \frac{1}{k} \leq \frac{1}{k_0} < \epsilon \implies \lim_{k\to\infty} x_{n_k} = c$ □