A series of complex number (a_n) is denoted by the symbol $\sum_{n=1}^{\infty} a_n$

Consider "partial sums" of the series $S_n = \sum_{k=1}^{n} a_k$, $k \in \mathbb{N}$ ← nth partial sum

Then $(S_n)_{n=1}^{\infty}$ forms a sequence of complex numbers.

Def the series $\sum_{n=1}^{\infty} a_n$ converges to its sum s if
the sequence $(S_n)_{n=1}^{\infty}$ of its partial sums converges to s

(Cauchy criterion for Series)

Thm. Suppose a series $\sum_{n=1}^{\infty} a_n$ converges. Then $\forall \varepsilon > 0$,
$\exists N \in \mathbb{N}$ such that if $p, q \geq N$, then $\left| \sum_{n=p}^{q} a_n \right| < \varepsilon$
(Sequence of partial sum is Cauchy)

Proof. Let $S_n = \sum_{k=1}^{n} a_k \in \mathbb{C}$. Then given (S_n) converges,
(S_n) is Cauchy. Thus, given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that
if $q, p > N$ then $|S_q - S_p| < \varepsilon \Rightarrow \left| \sum_{n=p}^{q} a_n \right| < \varepsilon$ □

<u>Special case</u> If a series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$

proof. Let $\varepsilon > 0$. From previous theorem, let $q = p+1 > N$
Then $\forall p \geq N$, $|a_p| < \varepsilon$. So $\lim_{p \to \infty} a_p = 0$ □

Remark. the converse of the special case is NOT true:

Thm. the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

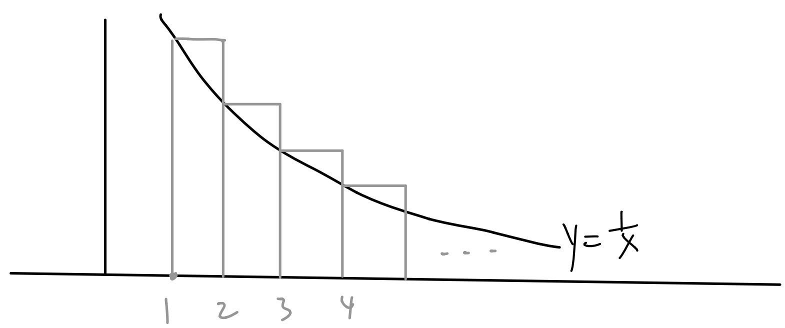
proof. let $S_n = \sum_{k=1}^{n} \frac{1}{k}$ and suppose S_n converges $\Rightarrow S_n$ is Cauchy

obtain $N \in \mathbb{N}$ s.t. $S_{2N} - S_N < \frac{1}{3}$

$$S_{2N} - S_N = \frac{1}{2N} + \frac{1}{2N-1} + \cdots + \frac{1}{N} > \frac{1}{2N} + \frac{1}{2N} + \cdots + \frac{1}{2N} = \frac{N}{2N} = \frac{1}{2}$$

But $S_{2N} - S_N < \frac{1}{3}$. Contradiction □

Graphically,



$$S_N = \text{sum of } \square > \int_1^n \frac{1}{x}\,dx = \log n \to \infty \text{ as } n \to \infty$$

Similar proof: $(n!)^{\frac{1}{n}}$ diverges

Thm. Let $a, z \in \mathbb{C}$.

Then the geometric series $\sum_{n=0}^{\infty} az^n$ converges if $|z| < 1$

proof. Let $S_n = a + az + \cdots + az^{n-1} \Rightarrow zS_n = az + \cdots + az^n$

Thus $(z-1)S_n = a(z^n - 1) \Rightarrow S_n = \dfrac{a(1-z^n)}{1-z}$, $|z| \neq 1$

if $|z| < 1$, $\lim_{n\to\infty} S_n = \frac{a}{1-z}$

if $|z| > 1$, $(S_n) \to \infty$ □

Remark:

Here's another useful situation to eval infinite series $\sum_{n=1}^{\infty} a_n$

suppose we can write $a_n = b_n - b_{n+1}$ and the sequence $(b_n) \to 0$.

Then $S_n = b_1 - b_2 + b_2 - b_3 + \cdots + b_n - b_{n+1} = b_1 - b_{n+1}$

Since $b_{n+1} \to \infty$, $S_n \to b_1$

Such a series is called a telescoping series.

Thm. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n + b_n$ converges

proof. (with triangle inequality)

Let $\varepsilon > 0$, obtain $N_1, N_2 \in \mathbb{N}$ such that

$$\left| \sum_{n=p}^{q} a_n \right| < \frac{\varepsilon}{2}, \ \forall p, q \geq N_1 ; \ \left| \sum_{n=p}^{q} b_n \right| < \frac{\varepsilon}{2}, \ \forall p, q \geq N_2$$

choose $N \geq \max(N_1, N_2)$

then $\left| \sum_{n=p}^{q} a_n + b_n \right| \leq \left| \sum_{n=p}^{q} a_n \right| + \left| \sum_{n=p}^{q} b_n \right| < \varepsilon$ □