Logistics:

WEH 7122

OH TUE 3:40 ~ 5 pm

THUR 9:30 ~ 10:30 am

HW due WED

Real Numbers

The real number system ($\mathbb{R}$) consist of

- a set $\mathbb{R}$
- A subset P of $\mathbb{R}$ (the set of positive reals)
- two binary operations $+ : (x,y) \mapsto x+y$
  $\cdot : (x,y) \mapsto x \cdot y$

ANY set that satisfy these properties is $\mathbb{R}$ (uniquely identify)

Properties:

(1) $(\mathbb{R}, +)$ is a commutative abelian group

(1.1) $\forall a, b \in \mathbb{R}$, $a+b = b+a$

(1.2) $(a+b)+c = a+(b+c)$

(1.3) $\exists$ unique $0 \in \mathbb{R}$ s.t. $0+a = a+0 = a$

(1.4) $\forall r \in \mathbb{R}$, $\exists$ unique element $\in \mathbb{R}$, s.t. $a+(-a) = 0$

(2) $\left((\mathbb{R} \setminus \{0\}), \cdot \right)$ is a commutative group

Satisfy commutativity, associativity, zero and inverse.
(2.1) (2.2) (2.3) (2.4)

(3) distributive law

$\forall a, b, c \in \mathbb{R}$, $a(b+c) = ab + ac = (b+c)a$

(4) $\forall a \in \mathbb{R}$, exactly one of the following holds:

$a = 0$ OR $a \in P$ or $-a \in P$

(5) P is a semigroup for $+$ and $\cdot$, that is

$\forall a, b \in P$, $a+b \in P$ and $ab \in P$

Def The set P is called the positive numbers.
$P \cup \{0\}$ is called the nonnegative numbers.
$-P = \{-a, a \in P\}$ is the negative numbers.
write $x < y$ if $y - x \in P$; $x \leq y$ if $y - x \in P \cup \{0\}$
(Note: $y - x = y + (-x)$)

Def A set with properties (1) ~ (3) is called a field
A set with properties (1) ~ (5) is called an order field
can compare

Def of ordered field 1.17
basic comparison for elements in the field
(add, multi, etc.)

examples of a field that is NOT orderded:
Complex number, $\mathbb{Z}$ mod p, $p \in \mathbb{Z}^+$

Polynomial is a ring?

Thm 1 If $x, y, z, w \in \mathbb{R}$, then
(i) $x + z = y + z$ implies $x = y$
(ii) If $w \neq 0$, $xw = yw$ implies $x = y$

Proof. (i) $x \overset{(1.3)}{=} x + 0 \overset{(1.4)}{=} x + z + (-z) \overset{\text{Given}}{=} y + z + (-z)$
$\overset{(1.4)}{=} y + 0 \overset{(1.3)}{=} y$ □

(ii) similar

Thm 2. If $x, y, z, w \in \mathbb{R}$, $z \neq 0$, $w \neq 0$, then
(i) $x \cdot 0 = 0$
(ii) $-(-x) = x$
(iii) $(w^{-1})^{-1} = w$
(iv) $(-1)x = -x$
(v) $x(-y) = -(xy) = (-x)y$
(vi) $-x + -y = -(x+y)$
(vii) $(-x)(-y) = xy$
(viii) $\frac{x}{z} \frac{y}{w} = \frac{xy}{zw}$
(ix) $\frac{x}{z} + \frac{y}{w} = \frac{xw + yz}{zw}$

Proof. (i) $x \cdot 0 + x \cdot 0 \overset{(3)}{=} x \cdot (0 + 0) \overset{(1.3)}{=} x \cdot 0$
By (1.3), $x \cdot 0 = 0$

(iv) $(-1)x + x = (-1)x + 1 \cdot x = (-1 + 1)x$
$= 0 \cdot x = 0$
By (1.4), $(-1)x = -x$

Lecture 2 Supremum Property

So far $(\mathbb{R}, P, +, \cdot)$ constitutes an ordered field

Def Let E be an nonempty subset of $\mathbb{R}$, we say that $b \in \mathbb{R}$ is an upper bound for E if $x \leq b$, $\forall x \in E$

Similarly, if F is an nonempty subset of $\mathbb{R}$, we say $\ell \in \mathbb{R}$ is a lower bound of F if $x \geq \ell$ for $\forall x \in F$

Let $E \subset \mathbb{R}$ be bounded above, E has an upper bound.
We say that $s \in \mathbb{R}$ is the least upper bound or supremum of E if
$\begin{cases} \text{i) } s \text{ is an upper bound for } E \\ \text{ii) if } t \text{ is an upper bound of } E \text{, then } t \geq s \end{cases}$

Notation: $s = \sup E$
Remark: it is possible that $s \notin E$. if $s \in E$, then we call s the maximum of E.

Similarly, let $F \subset \mathbb{R}$ be nonempty and bounded below (i.e it has a lower bound), ... greatest lower bound or infimum.

Notation: $m = \inf F$

Supremum property: Let S be an ordered field, S is said to have supremum property if for every nonempty subset $E \subset S$ which is bounded above, $\sup E$ exists in S.

Thm. there exists an ordered field with the supremum property. It is unique up to* field isomorphism

This field is called the real numbers $\mathbb{R}$

↳ This thm also implies about infimum

* bijection

example.

1) Let $A = \{ p \in \mathbb{Q}, p > 0, p^2 < 2 \}$

$B = \{ p \in \mathbb{Q}, p > 0, p^2 > 2 \}$

Claim. A has no longest element
B has no smallest element

($\mathbb{Q}$ is an ordered field without supremum property.
Intuitively, "holes" exist)

Proof. Let $p > 0$, $p \in \mathbb{Q}$. Set $p \in A$, $q > p$
and if $p \in B$ then $q < p$.

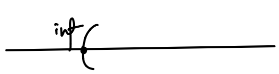
pos or neg depending on where p belongs

$$q = p - \frac{p^2 - 2}{p+2} = \frac{p^2 + 2p - p^2 + 2}{p+2} = \frac{2p+2}{p+2}$$

we calculate $q^2 - 2 = \frac{(2p+2)^2}{p+2} = \frac{2(p^2-2)}{(p+2)^2}$

If $p \in A$, then $q > p$, $q^2 < 2$, so $q \in A$
If $p \in B$, then $q < p$, $q^2 > 2$, so $q \in B$ □

Thm. Let S be an ordered field with the supremum property,
Let B be the subset of S which is nonempty and bounded below. Then $\inf B$ exists in S.



Proof. Let L be the set of all lower bounds of B
Then $L \neq \phi$ and bounded above by any ele of B
By the supremum property, $\sup L = \alpha$ exist in S.
Claim: $\alpha = \inf B$
need to check i) α is a lower bound
ii) if β is a lower bound, $\alpha \geq \beta$

i) Contrapositive   Let $y < \alpha$,   ...   then $y \notin B$

ii) If $\beta > \alpha$

then β not a lower bound

Absolute value   Let $x \in \mathbb{R}$, then we define $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Thm. Let $x, y \in \mathbb{R}$

i) $|xy| = |x||y|$

ii) $|x+y| \leq |x| + |y|$ triangle inequality

iii) $|x-y| \geq \big||x| - |y|\big|$

Proof. ii) $\left.\begin{array}{l} -|x| \leq x \leq |x| \\ -|y| \leq y \leq |y| \end{array}\right\} -|x| - |y| = -(|x| + |y|) \leq x+y \leq |x| + |y|$ $\square$