Part I Number System

1. Real Numbers

① group properties
② ordered field properties } arithmetic theorems

③ supremum property
→ upper/lower bound, supremum/infimum

④ Inequalities
→ triangle inequality $|a|+|b| \geq |a+b|$

2. Natural Numbers

→ definition of $\mathbb{N}$ ⇒ smallest inductive set of $\mathbb{R}$

→ Archimedean Properties of $\mathbb{N}$ ⇒ $\forall a \in \mathbb{R}^+, \forall b \in \mathbb{R}, \exists n \in \mathbb{N}: na > b$
Application: $\exists n \in \mathbb{N}: \varepsilon > \frac{1}{n}$

→ Properties of $\mathbb{N}$

→ Well Ordering Properties ⇒ any subset of $\mathbb{N}$ has a smallest element

3. Rationals

→ definition of integers & rational (ordered subfield of $\mathbb{R}$)

4. Complex Numbers

→ Properties

Part II Sequence & Series

1. Inequalities

① AM-GM $\frac{a_1+\cdots+a_n}{n} \geqslant \sqrt[n]{a_1\cdots a_n}$ proof: 2^N case + general case

② Cauchy-Schwarz (dot product VS product of magnitude)

③ Bernoulli's Inequality $\forall a>-1, n\in\mathbb{N}: (1+a)^n > 1+na$

④ Young's Inequality if $\frac{1}{p}+\frac{1}{q}=1$ then $uv \leq \frac{u^p}{p}+\frac{v^q}{q}$

⑤ Minkowski Inequality $\left(\sum_{j=1}^{n}(a_j+b_j)^p\right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{n}(a_j)^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n}(b_j)^p\right)^{\frac{1}{p}}$

2. Sequences

① definition: seq, convergence

↳ Thm: uniqueness of limit

↳ Thm: convergence of common sequences

(a_n) converges to ℓ if $\forall \varepsilon > 0, \exists N \in \mathbb{N}: \forall n \geq N: |a_n - \ell| < \varepsilon$.

(a_n) diverges if $\exists \varepsilon > 0: \forall \ell \in \mathbb{R}, \forall N \in \mathbb{N}: \exists n \geq N: |a_n - \ell| \geq \varepsilon$.

② Properties of limit

↳ examples: proving limit of sequence

proving limit: ε definition, Cauchy, monotone & bounded
subsequences converge to same

③ Montone Sequence

↳ liminf & limsup, characterization

↳ neighborhood & cluster point

Let $a_n \subset \mathbb{R}$, define $y_n = \sup\{a_k, k \geq n\}$ (Note y_n decreasing)
then $\limsup\{a_n\} = \lim_{n\to\infty} y_n$

Similarly, define $z_n = \inf\{a_k, k \geq n\}$ (Note z_n increasing)
then $\liminf\{a_n\} = \lim_{n\to\infty} z_n$

If a_n converges then $\limsup\{a_n\} = \lim_{n\to\infty} a_n = \liminf\{a_n\}$
Also, let c be the set of cluster points of a_n,
$\sup c = \limsup a_n$, $\inf c = \liminf a_n$.

④ Cauchy Sequence

a sequence converges iff it is Cauchy. (a_n) is Cauchy if
$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N: |x_m - x_n| < \varepsilon$.

⑤ Subsequence

Consider $(a_n)_{n=1}^{\infty} = n$. The sequence $(b_n) = 2n$ is a subsequence of a_n
But the sequence $(c_n) = \begin{cases} n-3, & \text{if } n \geq 4 \\ n+3, & \text{if } n=0 \end{cases}$ is NOT.

3. Series

① Def

A series of complex number a_n is denoted by $\sum_{n=1}^{\infty} a_n$

A series converges if the sequence of its partial sum $S_n = \sum_{k=1}^{n} a_k$ converges.

② Cauchy Criterion for sequence

(proving that S_n is Cauchy)

Suppose a series $\sum_{n=1}^{\infty} a_n$ converges, then $\forall \varepsilon > 0$,

$\exists N \in \mathbb{N}$ such that $\forall p, q \geq N$, $p \leq q$: $\left| \sum_{n=p}^{q} a_n \right| < \varepsilon$

Note: if a series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.

③ Common Series Eval

harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

geometric series $\sum_{n=1}^{\infty} a z^n$ converges if $|z| < 1$, diverges if $|z| > 1$.

④ Series Properties

if $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n + b_n$ converges

Part III HW Review

HW1

Q3 decide if the sets are bounded below/above
In case they are, find infima and suprema.

d) $E_4 = \{ 2^{-k} + 3^{-m} + 5^{-n},\ \{k,m,n\} \subset \mathbb{N} \}$

To prove inf: ① show $\forall a \in A$: $a \geq \inf$
② show $\forall b$: lower bound for A: $\inf \geq b$

Note: it's possible that $\inf \notin A$. ex: $x_n = \frac{1}{n}$, $\inf x_n = 0 \notin x_n$

Proof: E_4 is bounded above by $\frac{1}{2} + \frac{1}{3} + \frac{1}{5}$ and bounded below by 0.

Showing:

$0 < 2^{-k} \leq 2^{-1}$, $0 < 3^{-m} \leq 3^{-1}$, $0 < 5^{-n} \leq 5^{-1}$

So $0 < 2^{-k} + 3^{-m} + 5^{-n} \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{5}$

Q1 Using only the field axioms of $\mathbb{R}$, prove the following:

(a) $\forall x, y, z, w \in \mathbb{R}$, $z, w \neq 0$, show $x/z + y/w = (xw + yz)/zw$

(b) $\forall a \in \mathbb{R}$, it holds $(a^{-1})^{-1} = a$

(c) $1 > 0$

Proof. (a) $x/z + y/w = x/z \cdot 1 + y/w \cdot 1$ by identity

$= (x/z) \cdot (w/w) + (y/w) \cdot (z/z)$ by inverse

$= (xw)/(zw) + (yz)/(zw)$ by associativity & commutativity

$= (xw + yz)/(zw)$ by distributive law $\square$

(b) By definition of multiplicative inverse, it suffices to show that

$(a^{-1})\, a = 1$

$\Rightarrow a(a^{-1}) = 1$ by commutativity

is true by definition of multiplicative inverse □

(c) By contradiction, suppose $1 = 0$ OR $1 < 0$

if $1 = 0$, then let $a \in P$. Then $1 \cdot a = a$ (by identity)

and $0 \cdot a = 0 = a$ (by zero & above)

So $a \in P$ and $a = 0$, contradiction.

Now, if $1 < 0$, Note that $1 = 1 \cdot 1$

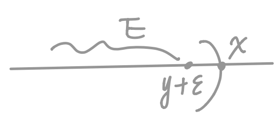
if $1 > 0$, then $1 = 1 \cdot 1 > 0$, contradiction

if $1 = 0$, then $1 = 1 \cdot 1 = 0 \cdot 0 = 0$, contradiction

if $1 < 0$, then $1 \cdot 1 = (-1)(-1)(1)(1) = (-1)(-1) > 0$, contradiction □

Q2. Let $E \subset \mathbb{R}$ be a non-empty set that is bounded from above. Prove that $x \in \mathbb{R}$ is an upper bound satisfies $x = \sup E$ iff $\forall \varepsilon > 0$, $\exists\, y \in E$ with $x - \varepsilon \le y$

Proof.



($\Rightarrow$) $x = \sup E$, so ① $\forall e \in E$, $e \le x$

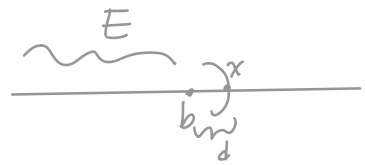
② $\forall b$: b is upper bound for E: $x \le b$

For contradiction, suppose $\exists\, \varepsilon > 0$: $\forall y \in E$, $x - \varepsilon > y$

then by definition, $x - \varepsilon$ is upper bound for E,

$x - \varepsilon < x$. Contradiction

$(\Leftarrow)$ $\forall \varepsilon > 0$, $\exists y \in E$ with $x - \varepsilon \leq y$



x is an upper bound.

For contradiction, suppose $\exists b$: b is an upper bound for E such that $b < x$.

Then $\forall y \in E$: $b \geq y$

Does there $\exists \varepsilon > 0$ such that $\forall y \in E$: $x - \varepsilon > y$?

Yes, let $d = x - b$, then $\forall y \in E$:

$x - y \geq x - b = d > \frac{d}{2} \Rightarrow x - \frac{d}{2} > y, \forall y \in E$ $\square$

HW2

Q1d) Prove by induction that $(1+\varepsilon)^n \leq 1 + 3^n \varepsilon$, $\forall \varepsilon \in (0,1)$ $\forall n \in \mathbb{N}$

proof. B.C. $n = 1$: $1+\varepsilon \leq 1+3\varepsilon$ ✓

I.H. $(1+\varepsilon)^n \leq 1 + 3^n \varepsilon$

want to show: $(1+\varepsilon)^{n+1} \leq 1 + 3^{n+1} \varepsilon$

By I.H. $(1+\varepsilon)^{n+1} \leq (1+3^n\varepsilon)(1+\varepsilon) = 1 + 3^n\varepsilon + \varepsilon + 3^n\varepsilon^2$

$= 1 + \varepsilon(3^n + 1 + 3^n\varepsilon)$

So it suffices to show that $3^n + 1 + 3^n\varepsilon \leq 3^{n+1}$

$\Leftrightarrow 3^n(1 + \varepsilon + \frac{1}{3^n}) \leq 3^n(3)$

$\Leftrightarrow 1 + \varepsilon + \frac{1}{3^n} \leq 3$ $\square$

Q5. (a) prove if $n \in \mathbb{N}$, then $n! \leq \left(\frac{n+1}{2}\right)^n$

(b) if $a_1, a_2, \dots, a_n$ all positive, then

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} \frac{1}{a_j}\right) \geq n^2$$

proof. (a)

By AM-GM: $\dfrac{x_1+x_2+\dots+x_n}{n} \geq (x_1 x_2 \dots x_n)^{\frac{1}{n}}$

$$\Rightarrow \left(\frac{x_1+x_2+\dots+x_n}{n}\right)^n \geq (x_1 x_2 \dots x_n) \quad \text{for } \forall x_i \geq 0$$

So $n! = n(n-1)\dots 2 \leq \left(\dfrac{1+2+\dots+n}{n}\right)^n = \left(\dfrac{1+n}{2}\right)^n$ $\square$

(b) By AM-GM: $\displaystyle\sum_{j=1}^{n} a_j \geq n(a_1 \dots a_n)^{\frac{1}{n}}$

$$\sum_{j=1}^{n} \frac{1}{a_j} \geq n\left(\frac{1}{a_1} \dots \frac{1}{a_n}\right)^{\frac{1}{n}}$$

So $\left(\displaystyle\sum_{j=1}^{n} a_j\right)\left(\displaystyle\sum_{j=1}^{n} \frac{1}{a_j}\right) \geq n^2 \left(\dfrac{a_1}{a_1} \dots \dfrac{a_n}{a_n}\right)^{\frac{1}{n}} = n^2$ $\square$

Q6. Prove that $\mathbb{C}$ cannot be ordered

proof. Consider i where $i^2 = 0$

if $i = 0$, then $5i = 0 = 3i$, contradiction.

if $i \neq 0$, then $i^2 > 0$. But $i^2 = -1 < 0$, contradiction $\square$

HW3

Q3. Evaluate the following limits

(a) $\lim_{n\to\infty} \left(\sqrt{n^2+2n} - n\right)$

(b) $\lim_{n\to\infty} \sqrt[n]{n!}$

Proof.

(a)

Note: $\left.\begin{array}{l} \sqrt{n^2+2n} - n < \sqrt{(n+1)^2} - n = 1 \\ \sqrt{n^2+2n} - n \geq \sqrt{n^2} - n = 0 \end{array}\right\}$ $0 \leq a_n \leq 1$

Guess: $\lim = 1$ by experiment

want to show: $\forall \varepsilon > 0, \exists N \in \mathbb{N}: \forall n \geq N: |a_n - 1| < \varepsilon$

by $a_n < 1$, $|a_n - 1| = 1 - a_n = 1 - \sqrt{n^2+2n} + n$

$|a_n - 1| < \varepsilon \Leftrightarrow \sqrt{n^2+2n} > 1 + n - \varepsilon \Leftrightarrow \cancel{n^2} + \cancel{2n} > 1 + \cancel{2n} + \cancel{n^2} + \varepsilon^2 - 2\varepsilon - 2\varepsilon n$

$\Leftrightarrow 2\varepsilon n > 1 + \varepsilon^2 - 2\varepsilon \Leftrightarrow n > \frac{(1-\varepsilon)^2}{2\varepsilon}$

Choose $n = \left\lceil \frac{(1-\varepsilon)^2}{2\varepsilon} \right\rceil + 1$.

(b) $(n!)^{\frac{1}{n}} = (n(n-1)(n-2)\cdots 1)^{\frac{1}{n}} \geq \left(\left(\frac{n}{2}\right)^{\frac{n}{2}}\right)^{\frac{1}{n}} = \sqrt{\frac{n}{2}}$

So it suffices to show that $x_n = n$ diverges.

Note that $\forall n \in \mathbb{N}: |x_n - x_{n+1}| = 1$

For contradiction, suppose x_n converges $\Rightarrow$ x_n Cauchy

then for $\varepsilon = \frac{1}{2}$, $\exists N \in \mathbb{N}: \forall n, m \geq N: |x_n - x_m| < \frac{1}{2}$

But $|x_N - x_{N+1}| = 1 > \frac{1}{2}$. Contradiction. $\square$

More Practice

- Liminf Limsup

Define $a_1 = 2018$, $a_2 = -2018$, $a_n = (-1)^n$

evaluate $\liminf a_n$, $\limsup a_n$, $\inf a_n$, $\sup a_n$

Proof. ① $\limsup a_n = 1$

Define $y_n = \sup\{a_k : k \geq n\}$

By definition, $\limsup a_n = \lim_{n \to \infty} y_n$

Note that $y_1 = 2018$, $y_k = 1$, $\forall k \geq 2$

then let $\varepsilon > 0$, want to show $\exists N \in \mathbb{N}: \forall n \geq N: |y_n - 1| < \varepsilon$

choose $N = 2$, then $|y_n - 1| = 0 < \varepsilon$ $\square$

- Series

evaluate $\sum_{n=0}^{\infty} x^n$

experiment: $x = 1$, $1 + 1 + \cdots = \infty$

$|x| < 1$, geometric series

$|x| > 1$, diverge

partial sum $S_n = \sum_{k=0}^{n} x^k$

Not that $xS_n - S_n = (x-1)S_n = x^{k+1} - 1 \Rightarrow S_n = \frac{1-x^{k+1}}{1-x}$

① the sequence $a_k = -x^{k+1}$ diverge if $|x| \geq 1$

proof. Note that $|a_{k+1} - a_k| = |-x^k(x+1)| \geq 2$

For contradiction suppose (a_k) converges then (a_k) is Cauchy:

for $\varepsilon = 1$, $\exists N \in \mathbb{N}: \forall m,n \geq N: |a_m - a_n| < \varepsilon$

But $|a_{N+1} - a_N| \geq 2 > 1$, contradiction □

② the sequence $a_k = x^k$ converge to 0 if $|x| < 1$

proof. Want to show:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}: \forall n \geq N, |a_n| < \varepsilon$$

if $\varepsilon \geq 1$, then choose $n = 1$

if $\varepsilon < 1$, want $|x|^n < \varepsilon \Leftrightarrow n > \log_{|x|} \varepsilon$

choose $n = \lceil \log_{|x|} \varepsilon \rceil + 1$ □

By ①, ②, S_n diverges if $|x| \geq 1$,

$\lim_{n\to\infty} S_n = \frac{1}{1-x}$ if $|x| < 1$ □