Inequalities

Thm 1. (Arithmetic Geometric Mean)

Let $a_1, \cdots a_n$ be positive real numbers

Then arithmetic mean $= \dfrac{a_1 + \cdots + a_n}{n}$ geometric mean $= \sqrt[n]{a_1 \cdot \cdots a_n}$

arithmetic mean $\geq$ geometric mean

equality holds iff $a_1 = \cdots = a_n$

proof. (Cauchy Induction)

① powers of 2

B.C. $\begin{cases} n=1: & a_1 = a_1 \ \checkmark \\ n=2: & \dfrac{a_1+a_2}{2} \geq \sqrt{a_1 a_2} \iff a_1 + a_2 \geq 2\sqrt{a_1 a_2} \iff (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0 \ \checkmark \end{cases}$

I.H. If $n = 2^k$, suppose the statement is true

want to prove: statement true for $n = 2^{k+1} = 2 \cdot 2^k$

showing:

$$\frac{a_1 + \cdots + a_{2^k} + a_{2^k+1} + \cdots a_{2^{k+1}}}{2^{k+1}} = \frac{\left(\overbrace{\dfrac{a_1 + \cdots a_{2^k}}{2^k}}^{x} + \overbrace{\dfrac{a_{2^k+1} + \cdots a_{2^{k+1}}}{2^k}}^{y}\right)}{2}$$

By I.H., $\dfrac{x+y}{2} \geq \sqrt{xy}$. So $\bigcirc \geq \left(\dfrac{(a_1 + \cdots a_{2^k})(a_{2^k+1} + \cdots a_{2^{k+1}})}{(2^k)^2}\right)^{\frac{1}{2}}$

Iteratively apply I.H. for $n = k, \frac{k}{2}, \frac{k}{4}, \cdots, 1$

we eventually get $\bigcirc \geq \left((a_1 \cdots a_{2^k})^{\frac{1}{2^k}} (a_{2^k+1} \cdots a_{2^{k+1}})^{\frac{1}{2^k}}\right)^{\frac{1}{2}} = (a_1 \cdots a_{2^{k+1}})^{\frac{1}{2^{k+1}}}$ $\square$

② general $n \in \mathbb{N}$

obtain k s.t. $2^k < n < 2^{k+1}$

Big idea: Apply AM-GM corresponding to 2^{k+1} by "filling the gap"

want to show: $\dfrac{a_1+a_2+\cdots+a_n}{n} \geq (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$

arithmetic mean

Start with: $\dfrac{a_1+a_2+\cdots+a_n+\overbrace{A+\cdots A}^{2^{k+1}-n \text{ many}}}{2^{k+1}}$ where $A = \dfrac{a_1+a_2+\cdots+a_n}{n}$

then by the special case, $\dfrac{a_1+\cdots+a_n+A+\cdots+A}{2^{k+1}} \leq \left(a_1\cdots a_n A^{2^{k+1}-n}\right)^{\frac{1}{2^{k+1}}}$

$$\frac{nA+(2^{k+1}-n)A}{2^{k+1}} \leq (a_1\cdots a_n)^{\frac{1}{2^{k+1}}} A^{-\frac{n}{2^{k+1}}} A$$

$$\frac{\cancel{2^{k+1}}\cancel{A}}{\cancel{2^{k+1}}} \leq (a_1\cdots a_n)^{\frac{1}{2^{k+1}}} A^{-\frac{n}{2^{k+1}}} \cancel{A}$$

$$A^n \leq (a_1\cdots a_n)$$

$$A \leq (a_1\cdots a_n)^{\frac{1}{n}} \quad \square$$

Recall: triangle inequality $\forall a,b \in \mathbb{R}, |a+b| \leq |a|+|b|$

Proof. want to show $|a+b|^2 \leq (|a|+|b|)^2$

Note: $\forall x \in \mathbb{R}, |x|^2 = x^2$

showing: $(|a|+|b|)^2 = |a|^2+2|a||b|+|b|^2$

$= a^2+b^2+2|a||b| \geq a^2+b^2+2ab = (a+b)^2 = |a+b|^2 \quad \square$

Thm. Cauchy-Schwarz

Let $a_1, \cdots, a_n$ and $b_1, \cdots b_n$ be complex numbers

Then $\left| \sum_{j=1}^{n} a_j b_j^* \right|^2 \leq \left(\sum_{j=1}^{n} |a_j|^2 \right) \left(\sum_{j=1}^{n} |b_j|^2 \right)$

with equality iff $a_j = z b_j$, for $z \in \mathbb{C}$

if we think in terms of vectors, the special case is saying that for $u, v \in \mathbb{R}^n$,

$|u \cdot v| \leq \|u\| \|v\|$

$\Downarrow$

$|u \cdot v|^2 \leq \|u\|^2 \|v\|^2$

Proof.

In special case, $n=2$, $a_1, a_2, b_1, b_2 \in \mathbb{R}$

$a = (a_1, a_2) \in \mathbb{R}^2 \qquad b = (b_1, b_2) \in \mathbb{R}^2$

$a \cdot b = a_1 b_1 + a_2 b_2 = |a||b| \cos\theta$

$a \times b = a_1 b_2 - a_2 b_1 = |a||b| \sin\theta$

$|a \cdot b|^2 + |a \times b|^2 = |a|^2 |b|^2 (\cos^2\theta + \sin^2\theta) = |a|^2 |b|^2$

$|a \cdot b|^2 \leq |a|^2 |b|^2$

equality iff $|a \times b| = 0 \Rightarrow a = \lambda b$, $\lambda \in \mathbb{R}$

General case

Define $f(z) = \sum_{j=1}^{n} |a_j - z b_j|^2$

$0 \leq f(z) = \sum_{j=1}^{n} (a_j - z b_j)(\overline{a_j} - \overline{z b_j})$

$= \sum_{j=1}^{n} \left(|a_j|^2 - z b_j \overline{a_j} - \overline{z b_j} a_j + |z|^2 |b_j|^2 \right)$

Define $\sum_{j=1}^{n} |a_j|^2 = A \qquad \sum |b_j|^2 = B \qquad \sum a_j \overline{b_j} = C$

$0 \leq A - \overline{z} C - z \overline{C} + |z|^2 B = f(z)$

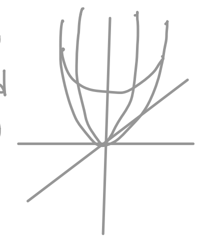
Two cases: if $B=0$, then $b_j=0$, $\forall j \Rightarrow$ inequality is trivial

Then consider $f\left(\frac{C}{B}\right)$

$$0 \le A - \frac{|C|^2}{B} - \frac{|C|^2}{B} + \frac{|C|}{B^2} B$$

$$\Rightarrow |C|^2 \le AB$$

Graphically, a paraboloid (complex number)



equality holds iff $f(z)=0$ for some $z \in \mathbb{C}$

i.e. $a_j = z b_j$, $\forall j = 1, \dots, n$ $\square$

Thm. Bernoulli Inequality

Let $x \in \mathbb{R}$, $x > -1$, $x \neq 0$.
Let $n \in \mathbb{N}$, $n > 1$, then $(1+x)^n > 1+nx$

Proof. B.C. $n=2$

$$(1+x)^2 = 1+2x+x^2 > 1+2x$$

Suppose the inequality true for $n \geq k$,
then $(1+x)^k > 1+kx$

we have
$$(1+x)^{k+1} = (1+x)(1+x)^k$$
$$> (1+x)(1+kx) = 1+kx+x+kx^2$$
$$> 1+(k+1)x \quad \square$$

Young Inequality

Let $u,v>0$ and $p,q\in\mathbb{Q}^+$: $\frac{1}{p}+\frac{1}{q}=1$

then $uv \le \frac{u^p}{p}+\frac{v^q}{q}$ (Remark: when $p=q=2$, generalization for AM-GM)

proof. Let $m,n\in\mathbb{N}$ s.t. $p=\frac{m+n}{m}$, $q=\frac{m+n}{n}$

(By $\frac{1}{p}+\frac{1}{q} \Rightarrow p,q>1$)

Set $u^p=x \Rightarrow u=x^{\frac{1}{p}}$

Set $v^q=y \Rightarrow v=y^{\frac{1}{q}}$

We have

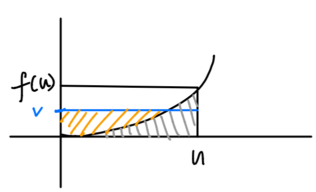
$$\frac{u^p}{p}+\frac{v^q}{q}=\frac{mx}{m+n}+\frac{ny}{m+n}=\frac{mx+ny}{m+n}$$

$$=(x+\overbrace{x+\cdots x}^{m\text{ times}}+\overbrace{y+\cdots y}^{n\text{ times}})\frac{1}{m+n}$$

By AM-GM

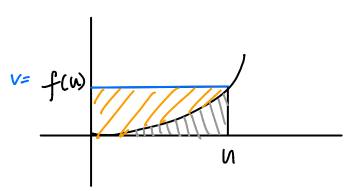
$$\geq (x^m y^n)^{\frac{1}{m+n}}=(u^{pm}v^{qn})^{\frac{1}{m+n}}=uv \quad \square$$

Graphically, define $f(x)=x^{p-1}$



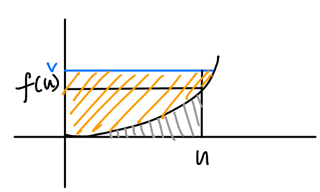
① $v<f(u)$

$uv<$ //// + ////



② $v=f(u)$

$uv=$ //// + ////



③ $v>f(u)$

$uv<$ //// + ////

By calculus, //// $=\frac{v^q}{q}$

//// $=\frac{u^p}{p}$

Minkowski Inequality

Let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$

Let $p \geq 1$, $p \in \mathbb{Q}$

$$\left(\sum_{j=1}^{n} |a_j + b_j|^p \right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{n} |a_j|^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} |b_j|^p \right)^{\frac{1}{p}}$$

(special case: $p = 1, 2 \Rightarrow$ triangle inequality)

proof.
$$\sum_{j=1}^{n} (a_j + b_j)^p = \sum_{j=1}^{n} a_j (a_j + b_j)^{p-1} + b_j (a_j + b_j)^{p-1}$$

(idea: use Young's inequality twice)

$u = a_j$
$V = (a_j + b_j)^{p-1}$
$\frac{1}{p} + \frac{1}{q} = 1$

$$\leq \sum_{j=1}^{n} \left\{ \frac{(a_j)^p}{p} + \frac{\left[(a_j + b_j)^{p-1} \right]^{\frac{p}{p-1}} (p-1)}{p} + \frac{(b_j)^p}{p} + \frac{\left((a_j + b_j)^{p-1} \right)^{\frac{p}{p-1}}}{p} (p-1) \right\}$$

$$= \sum_{j=1}^{n} \frac{a_j^p}{p} + \frac{b_j^p}{p} + \frac{2(p-1)}{p} (a_j + b_j)^p$$