Def. A set $\subset \mathbb{R}$ is said to be inductive if
1) $1 \in I$
2) $x \in I$ then $x+1 \in I$

Def. Let ϕ denote the set of all inductive subsets of $\mathbb{R}$.
We define the natural numbers to be the smallest inductive set of $\mathbb{R}$
That is $\mathbb{N} = \bigcap_{I \in \phi} I$

Thm. finite induction principle
Let $S \subseteq \mathbb{N}$ be an inductive set, then $S = \mathbb{N}$

Proof. Let $S \subseteq \mathbb{N}$, but $\mathbb{N} \subseteq S$, so $S = \mathbb{N}$ □

Archimedean Property of $\mathbb{N}$

Let $a > 0$ be a real number. Let $b \in \mathbb{R}$.
Then $\exists\ n \in \mathbb{N}$ such that $na > b$

Proof. Suppose not. Let $S = \{na,\ n \in \mathbb{N}\}$. by contradition we have
$na \le b$ for every $n \in \mathbb{N}$

The supremum property implies that exist $s \in \mathbb{R}$ s.t. $s = \sup S$.
This implies that $s-a$ is not an upper bound for S.
then $\exists n \in \mathbb{N}$ s.t. $s-a < na \Rightarrow s < (n+1)a$
But $(n+1)a \in S$ and $s = \sup S$. Contradiction □

Application. Give $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ $(n\varepsilon > 1,\ b=1)$

Thm Properties of $\mathbb{N}$. Let $n \in \mathbb{N}$

i) $n \geq 1$

ii) if $n > 1$ then $n-1 \in \mathbb{N}$

iii) if $x \in \mathbb{R}$, $x \geq 0$ and $x+n \in \mathbb{N}$, then $x \in \mathbb{N}$

iv) if $m \geq 0$, $m+n \in \mathbb{N}$ then $m \in \mathbb{N}$

v) if $a \in \mathbb{R}$, $n < a < n+1$, then $a \notin \mathbb{N}$

Proof. i) $\{x \in \mathbb{R} : x \geq 1\}$ is inductive

which means $\mathbb{N} \subset \{x \in \mathbb{R} : x \geq 1\}$ $\square$

ii) $S = \{1\} \cup \{n-1 : n \in \mathbb{N}\}$?

Thm. closure: If $m, n \in \mathbb{N}$, then $m+n \in \mathbb{N}$ and $mn \in \mathbb{N}$

Proof. By induction, fix $m \in \mathbb{N}$, write $S = \{n \in \mathbb{N} : m+n \in \mathbb{N}\}$

and induct on n to show $S = \mathbb{N}$

Similar for mn

Thm. (well Ordering Property)

Let A be a nonempty subset of $\mathbb{N}$. Then A has a smallest argument.

Proof. Suppose A does not have a smallest element.

Define $S = \{n \in \mathbb{N} : n < a \text{ for every } a \in A\}$ S and A are disjoint

Base Case: $1 \in S$: because otherwise A would have a smallest element

Inductive step: Let $n \in S$, if $n+1 \notin S$ then $n+1 \geq a$ for some $a \in A$

But $n \in S$ means $n < a$, $\forall a \in A$.

So $n+1$ is the smallest of A, contradicting .

Essentially we show $n \in S \Rightarrow n+1 \in S$ by induction.

To finish, $A \neq \emptyset \Rightarrow \exists a \in A \subset S = \mathbb{N}$

Then by definition of $S \Rightarrow a < a$, Contradiction $\square$