Planning & Scheduling
Assignment 2

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1 Graded Exercise 2.2

Minimize the total tardiness where each job has common due dates.
Show that the schedule given by the ordering Smallest Processing Time first, is optimum.

Tardiness T_j is in general defined as:

$$T_j = max\{C_j(S) - d_j, 0\} \tag{1}$$

In our case every job has the same due date d, hence

$$T_j = max\{C_j(S) - d, 0\} \tag{2}$$

Moreover, the optimal schedule is supposed to be given by ordering the jobs by processing times from smallest to largest. It follows that

Theorem: The ordering of jobs such that

$$p_1 \leq p_2 \leq ... \leq p_n \tag{3}$$

gives

$$min \sum_{1}^{n} T_j$$

The problem could be considered as the following:
There exist two adjacent jobs i and k where i precedes k such that $p_i < p_k$ which leads to 3 cases:

- **Case 1** $C_i < C_k \leq d$ which implies
  $T_i = max\{C_i - d, 0\} = 0$
  $T_k = max\{C_k - d, 0\} = 0$
- **Case 2** $C_i \leq d \leq C_k$ which implies
  $T_i = max\{C_i - d, 0\} = 0$
  $T_k = max\{C_k - d, 0\} = C_k - d$
- **Case 3** $d \leq C_i < C_k$ which implies
  $T_i = max\{C_i - d, 0\} = C_i - d$
  $T_k = max\{C_k - d, 0\} = C_k - d$

In case 1 the schedule is always optimum.
In case 2 by exchanging jobs i and k with $p_k > p_i$ the due date would proceed both the completion times C_k and C_i such as in the following picture:

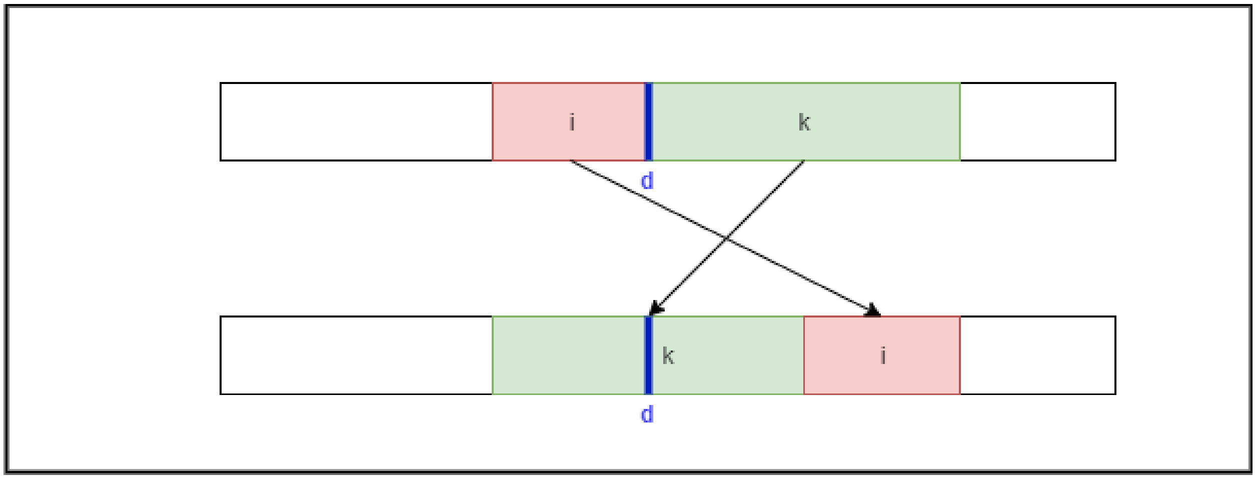

Figure 1: Caption

which leads to a setting similar to Case 3.
Hence, proving Case 3 by the exchange argument allows to prove the theorem.
Consider a schedule OPT that doesn't have property (3) i.e. it's not equal to our schedule and thus cannot be optimum.
Assume that the optimum schedule OPT and our schedule OPT' don't agree on the position of the two adjacent jobs i and k.
We now also assume that there exist two jobs i^* and k^* such that $p_{i^*} > p_{k^*}$ in OPT i.e.

$$OPT = (\, , \, , \, , i^*, k^*, \, , \, , \,)$$

while our schedule OPT' (which follows property (3) is such that

$$OPT' = (\, , \, , \, , k^*, i^*, \, , \, , \,)$$

All jobs before and after the pair have the same ordering in both schedules.The completion time for the last job of the pair is the same in both schedules. What changes are the completion times of the job that comes first in the pair - either $i*$ or $k*$ - due to their reordering. Thus, having the same due date d the change in completion times leads to a change in the tardiness for each job.
Furthermore, if both jobs are late with respect to the due date d it's clear that now the problem is equivalent to minimizing the completion time. Thus, our schedule OPT' is better than OPT if

$$\sum C_j(OPT') < \sum C_j(OPT)$$

which is true since it leads to

$$p_{k^*} - p_{i^*} < 0$$

by our assumption $p_{i^*} > p_{k^*}$. Thus, OPT cannot be optimum and thus our algorithm for minimizing tardiness selecting jobs following the SPT (shortest processing time first) rule is optimum.