CHAPTER 3

FOURIER ANALYSIS

The basic theory for the description of periodic signals was formulated by Jean-Baptiste Fourier (1768-1830) in the beginning of the 19th century. Fourier showed that an arbitrary periodic function could be written as a sum of sine and cosine functions. This is the basis for the transformation of time histories into the frequency domain and all kinds of digital frequency analysis.

Jean-Baptiste Fourier (1768-1830)

3.1 Fourier series

Let $x(t)$ be a periodic function with period T

$$x(t+nT)=x(t)\,, \tag{3-1}$$

where n is an integer. The functions

$$\begin{aligned} s_m(t) &= \sin\left(m\frac{2\pi t}{T}\right) \\ c_m(t) &= \cos\left(m\frac{2\pi t}{T}\right) \end{aligned}\,, \tag{3-2}$$

for an integer m are all periodic over the time T. The functions have m whole periods over time T. Fourier showed that the arbitrary periodic function $f(t)$ may be written as an infinite sum of those sine and cosine functions

$$x(t)=\sum_{m=0}^{\infty} a_m \cos\left(m\frac{2\pi t}{T}\right)+b_m \sin\left(m\frac{2\pi t}{T}\right). \tag{3-3}$$

To get simpler formulas let $\omega=\dfrac{2\pi}{T}$ and (3-3) becomes

$$x(t)=\sum_{m=0}^{\infty} a_m \cos(m\omega t)+b_m \sin(m\omega t)\,. \tag{3-4}$$

To calculate the coefficients both sides in (3-4) are multiplied by $\cos(n\omega t)$ with n an integer giving

$$\begin{aligned} &\cos(n\omega t)\cdot x(t)=\cos(n\omega t)\cdot\sum_{m=0}^{\infty} a_m \cos(m\omega t)+b_m \sin(m\omega t) \\ &\textit{or} \\ &\cos(n\omega t)\cdot x(t)= \\ &\sum \frac{a_m}{2}\left(\cos(n+m)\omega t+\cos(n-m)\omega t\right)+\frac{b_m}{2}\left(\sin(n+m)\omega t+\sin(n-m)\omega t\right) \end{aligned} \tag{3-5}$$

We now integrate both sides of (3-5) over the time interval (0, *T*). All trigonometric functions on the right hand side have an integer number of periods in the interval, and (nearly) all terms in the integral will become zero. But there is one exception. If we choose n = m, sin$((n\text{-}m)\omega t)$ = 0 in the whole interval, and the right side becomes

$$\int_0^T \frac{a_n}{2} \cdot 1dt = \frac{a_n T}{2}\ . \qquad (3\text{-}6)$$

From this

$$a_n = \frac{2}{T}\int_0^T \cos(n\omega t)\cdot x(t)dt \quad \text{with} \quad \omega = \frac{2\pi}{T}\ . \qquad (3\text{-}7)$$

In the same way:

$$b_n = \frac{2}{T}\int_0^T \sin(n\omega t)\cdot x(t)dt\ . \qquad (3\text{-}8)$$

For n = 0, which corresponds to the mean value of the signal, we will get a special case, and we write this term as $\frac{a_0}{2}$. We have then arrived at the formula for the Fourier series

$$
\begin{aligned}
&x(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + b_m \sin(m\omega t) \\
&\mathit{with} \\
&a_m = \frac{2}{T}\int_0^T \cos(m\omega t)\cdot x(t)dt \\
&b_m = \frac{2}{T}\int_0^T \sin(m\omega t)\cdot x(t)dt \\
&\omega = \frac{2\pi t}{T}
\end{aligned}
\quad . \qquad (3\text{-}9)
$$

We note, that the only thing we have used in the derivation of the formula is the fact that most of the terms in the integrals turn out to be zero. This is called orthogonality, the functions are said to be orthogonal. This is mathematically equivalent with the same concept for vectors, and the scalar product then corresponds to the integral of the multiplied functions. We may write

$$
\begin{aligned}
&x(t) = \sum a_m \phi_m(t) \\
&\mathit{with} \\
&\int \phi_n(t)\cdot\phi_m(t)dt = \begin{cases} 0 \ldots n \neq m \\ 1 \ldots n = m \end{cases} . \qquad (3\text{-}10) \\
&\mathit{and} \\
&a_m = \int \phi_m(t)\cdot x(t)dt
\end{aligned}
$$

This is called a generalised Fourier series, and is a widely used concept. In figure 3-1 the successive approximation with Fourier series to a square wave is shown.

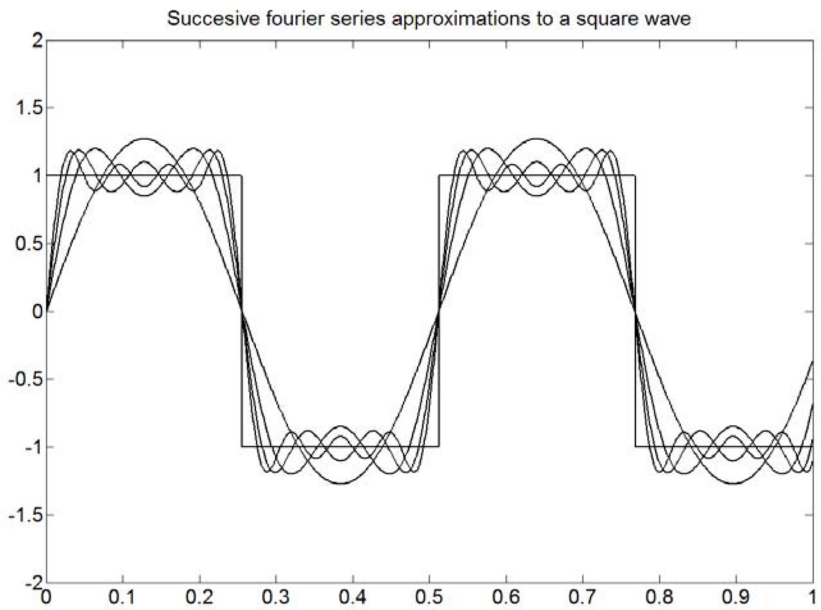

Figure 3-1 Successive Fourier series approximation to a square wave by adding terms.

Even if we let the number of terms in the Fourier series get very large we will not get a perfect agreement with the original square wave. We will still get the oscillations shown in figure 3-2. This is called Gibb's phenomenon and has to do with the non-uniform convergence of Fourier series.

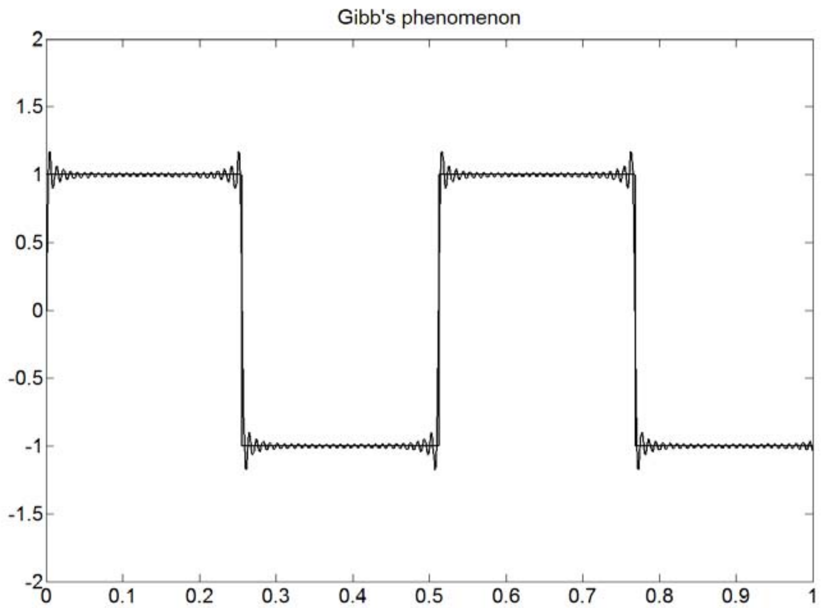

Figure 3-2 Gibb's phenomenon for Fourier series approximation with many terms.

We now know, that a periodic signal may be written as a Fourier series. In the sum there are terms of the type:

$$s(t) = a\cos(\omega t) + b\sin(\omega t) \tag{3-11}$$

We may write *s(t)* in a different way:

$$\begin{aligned} &s(t) \;=\; a\cos(\omega t) + b\sin(\omega t) \;=\; \sqrt{a^{2} + b^{2}}\,\cos(\omega t - \varphi) \\ &\textit{where} \\ &\tan(\varphi) \;=\; \frac{b}{a} \end{aligned} \tag{3-12}$$

and then introduce the amplitude and phase,

$$\begin{aligned} &\sqrt{a^2 + b^2} = \textit{amplitude} \\ &\varphi = \textit{phase} \end{aligned} \quad . \tag{3-13}$$

In the Fourier series we find, that the frequencies appear as multiplies of the basic frequency (1/*T*). The basic frequency is called the fundamental, while the multiples are called harmonics. Fourier analysis is often called harmonic analysis. A periodic signal may then be described with its fundamental and harmonics. For each frequency you give the frequency, the amplitude and the phase, alternatively, you give the sine and cosine components.

3.2 Complex Fourier series

By using Euler's formula giving the complex notation for sine and cosine functions

$$e^{i\phi} = \cos\phi + i\sin\phi \, , \tag{3-14}$$

or

$$\begin{aligned} \cos\phi &= \frac{e^{i\phi} + e^{-i\phi}}{2} \\ \sin\phi &= \frac{e^{i\phi} - e^{-i\phi}}{2} \end{aligned} \, , \tag{3-15}$$

we may write the Fourier series expression in a more compact way

$$\begin{aligned} & x(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + b_m \sin(m\omega t) = \\ & c_0 + \sum_{m=1}^{\infty} c_m e^{-im\omega t} + \sum_{m=1}^{\infty} d_m e^{im\omega t} \\ & c_m = \frac{a_m}{2} + i\frac{b_m}{2} \\ & d_m = \frac{a_m}{2} - i\frac{b_m}{2} \end{aligned} \, , \tag{3-16}$$

$$\begin{aligned} x(t) &= \sum_{m=-\infty}^{\infty} c_m e^{m\omega t} \\ c_m &= \frac{1}{T}\int_0^T x(t) \cdot e^{-im\omega t} dt \end{aligned} \, . \tag{3-17}$$

This is called the complex Fourier series. Please note, that the summation now also covers negative indices, we have "negative frequencies". We are not yet putting any physical meaning to those frequencies; we just use the compact mathematical notation.

3.3 Fourier Transforms

For a function *x(t)*, defined for all time t, we define the Fourier transform *X(ω)* by:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-i\omega t} dt \quad . \tag{3-18}$$

X(ω) is a complex-valued function of the variable f, frequency ($\omega = 2\pi f$), and is defined for all frequencies. As the function is complex, it may be described by a real and an imaginary part or with magnitude and phase, as with any complex number. The definition integral does not converge for all possible functions x(t), see the specialised literature on the subject. If the Fourier transform exists, there is an inverse Fourier transform formula:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{i\omega t} d\omega \quad . \tag{3-19}$$

The transform pair is very symmetric, the only difference being the sign of the exponent and the factor $1/2\pi$ in (3-19).

Example 3.1

A time series is given by

$$x(t) = \begin{matrix} 1 \ldots |t| \leq 1 \\ 0 \ldots |t| > 1 \end{matrix} . \tag{3-20}$$

The Fourier transform is then from direct calculation:

$$X(\omega) = \frac{2\sin(\omega a)}{\omega} = 2a \frac{\sin(\omega a)}{\omega a} \quad . \tag{3-21}$$

The transform pair is shown in figure 3-3.

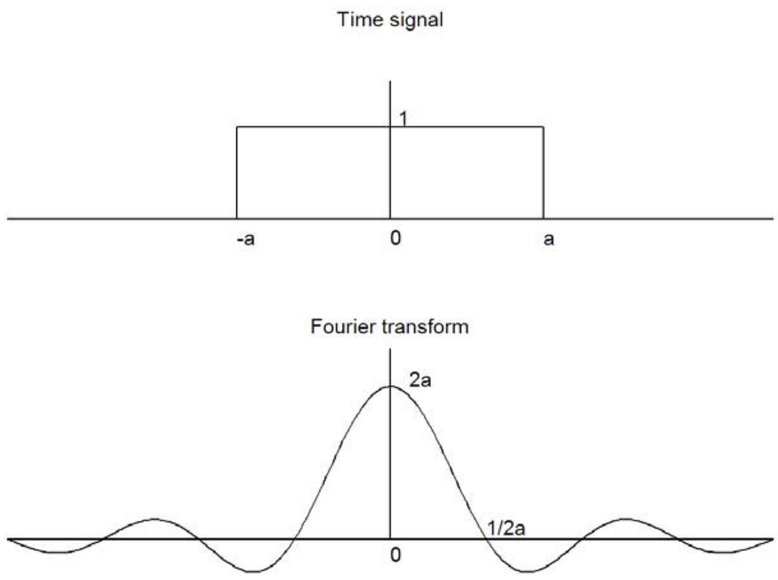

Figure 3-3 Time signal and corresponding Fourier transform.

The example given here result in a real Fourier transform, which stems from the fact that *x(t)* is symmetric around time zero.

Example 3.2

For the time series

$$x(t) = \begin{cases} e^{-at} \cdot \sin(2\pi bt) ... t \geq 0 \\ 0 .... t < 0 \end{cases}, \tag{3-22}$$

the Fourier transform becomes

$$X(\omega) = \frac{2\pi b}{-\omega^2 + 2i\omega a + a^2 + (2\pi b)^2}. \tag{3-23}$$

Note, that $X(\omega)$ now is complex. The real and imaginary parts are given in figure 3-4.

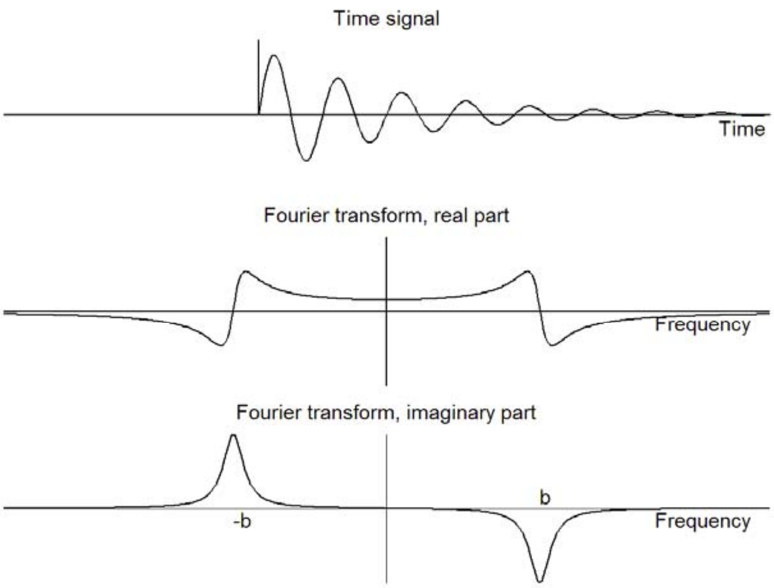

Figure 3-4 Time signal and corresponding Fourier transform.

For a real time signal *x(t)*, the real part of *X(ω)* is an even function of frequency, while the imaginary part is an odd function of frequency. This is illustrated in the figure. We also find, that the transform peaks around the ”frequency”, b, of the time signal. If we let the engineering units of *x(t)* be volts, [V], then the engineering units for *X(ω)* is [Vs] from the definition. From the definition we also find that

$$X(0) = \int_{-\infty}^{\infty} x(t)dt \quad . \tag{3-24}$$

If *x(t)* is a force, measured in N, applied to a structure, the integral of the force over time is called the impulse, with the engineering unit [Ns]. Equation (3-22) then tells us that the Fourier transform of the force, evaluated for frequency zero, is equal to the impulse. The Fourier transform is often denoted *F{x(t)}*. The transform is **linear**, which means:

$$F\{x_1(t) + x_2(t)\} = F\{x_1(t)\} + F\{x_2(t)\} \quad . \tag{3-25}$$

A **scaling in time** brings an inverse scaling in frequency:

$$F\{x(at)\} = \frac{1}{a} X\left(\frac{\omega}{a}\right). \tag{3-26}$$

A **time shift** of the signal introduces a phase factor:

$$F\{x(t-\tau)\} = X(\omega) \cdot e^{-i\omega\tau}. \tag{3-27}$$

The amplitude of F is not affected, since the amplitude of the exponential function is one for all frequencies.

Changing the direction of time introduces a complex conjugation:

$$F\{x(-t)\} = X^*(\omega). \tag{3-28}$$

A **multiplication with (a complex) sine** function introduces a **shift in frequency**:

$$F\{x(t) \cdot e^{i\omega_0 t}\} = X(\omega - \omega_0). \tag{3-29}$$

This is the basis for modulation. We will later find that the Fourier transform provides information about the distribution of signal power as a function of frequency. By multiplying the signal with a sine, we shift the signal power to another frequency region. This method is called modulation and has an important engineering applications, for instance in telecommunication.

The Fourier transform of the derivative of a time signal may be deduced from the definition

$$F\left\{\frac{dx(t)}{dt}\right\} = i\omega \cdot X(\omega). \tag{3-30}$$

So, to get the Fourier transform of the derivative, just multiply by $i\omega$. This may of course be used several times to get derivatives of higher order.

The Fourier transform of the integral of a time signal may also be deduced from the definition

$$F\left\{\int x(t)dt\right\} = \frac{X(\omega)}{i\omega} . \tag{3-31}$$

To get the Fourier transform of the integral we divide by *iω*.

The **convolution** *y(t)* between two time signals $x_1(t)$ and $x_2(t)$ is defined by

$$y(t) = \int_{-\infty}^{\infty} x_1(t-\tau)\cdot x_2(\tau)d\tau . \tag{3-32}$$

The Fourier transform of the convolution can be shown to be

$$F\{y(t)\} = F\left\{\int_{-\infty}^{\infty} x_1(t-\tau)\cdot x_2(\tau)d\tau\right\} = F\{x_1(t)\}\cdot F\{x_2(t)\} . \tag{3-33}$$

This relationship is extremely important, and is the basis for the analysis of linear systems.

3.4 Parseval's identity

Using the inversion formula (3-19) on (3-33) we find

$$\int_{-\infty}^{\infty} x_1(t-\tau)\cdot x_2(\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} F\{x_1(t)\}\cdot F\{x_2(t)\}\cdot e^{i\omega t}d\omega \quad . \tag{3-34}$$

Now put t = 0, and we get

$$\int_{-\infty}^{\infty} x_1(-\tau)\cdot x_2(\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} F\{x_1(t)\}\cdot F\{x_2(t)\}d\omega \quad , \tag{3-35}$$

and if we put $x(\tau) = x_1(-\tau)$ and $x(\tau) = x_2(\tau)$ and, using (3-26), we get

$$\int_{-\infty}^{\infty} x^2(\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} F\{x(t)\}\cdot F\{x(-t)\}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot X^*(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega \tag{3-33}$$

This is called **Parseval's identity** and is the very important connection between energy in the time domain and the frequency domain.