STRINGS

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The wave equation for an ideal string

We consider an *ideal string* rigidly terminated at both ends (see Fig. 1). The string performs transversal vibrations with the direction of string displacement $u(x)$ normal to the length direction x. The concept of an ideal string means that the string is perfectly flexible, it has no *stiffness.* Consequently it does not offer any resistance to bending and can be curved to an arbitrarily small bending radius. Real strings show relatively large deviations from this idealization. Further, an ideal string is *homogenous*, which means that it has a constant cross section A [m^2] and linear density ρ_ℓ [kg/m] at all points. Modern strings meet this criterion fairly well.

We assume that the string performs vibrations with small amplitude around the equilibrium position, so that the slope $\partial u/\partial x$ is small. Further we assume that no energy is lost at the terminations or as internal losses in the string. This is clearly a simplification as the purpose of real strings is to deliver energy to a vibrating instrument body which in turn will radiate sound. Also, when terminating the string by pressing it against a fingerboard a substantial damping is introduced. Nevertheless, the treatment of an ideal string will allow us to derive fundamental properties of strings which apply with minor corrections to real strings.

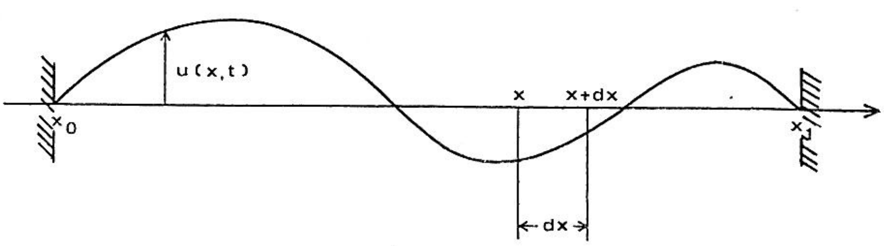


Fig. 1. An ideal string rigidly terminated at both ends.

The assumption that the vibration amplitudes are small is essential in several aspects. One regards the tension in the string. If the displacements are large the string will stretch and tension increase. According to Hooke's law an elongation $\Delta\ell$ gives an increase in tension ΔS

$$\Delta S = EA\frac{\Delta\ell}{L} \tag{1a}$$

where E is Young's modulus and A the cross section of the string.

The change in length $\Delta\ell$ due to the string motion can be obtained as a summation of the differences in length between the hypotenuse of a deflected string segment (Pythagorean theorem) and its undisturbed length dx

$$\Delta\ell = \int_{x_o}^{x_1}\left(\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}\right)dx - [x_1 - x_o] \tag{1b}$$

As the slope $\partial u/\partial x$ is assumed to be small, the overall change in length $\Delta\ell$ as the string vibrates is small and the tension S in the string can consequently be considered constant.

The equation of motion of the string can be derived by applying basic laws of physics to a short piece *ds* of the deflected string (see Fig. 2).

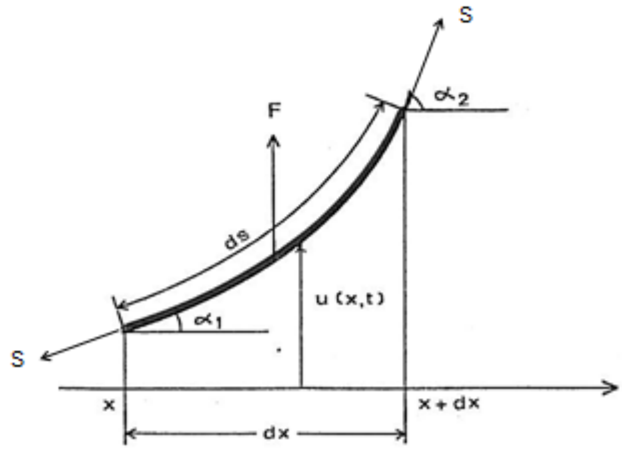


Fig. 2. A short piece *ds* of the deflected string.

Applying Newton's second law $F = ma$ on the string segment *ds* where F is the net external force and m the mass gives

$$S\sin\alpha_2 - S\sin\alpha_1 = \rho_\ell\, ds\frac{\partial^2 u}{dt^2} \tag{2}$$

The string displacement and slopes are small so the angles α are small and $\sin\alpha$ can be replaced by $\tan\alpha$ and the length of the string segment *ds* by *dx*

$$S\,(\tan\alpha_2 - \tan\alpha_1) = \rho_\ell\, ds\frac{\partial^2 u}{dt^2} \tag{3}$$

The string slope at coordinate x is

$$\tan\alpha_1 = \frac{\partial u}{\partial x} \tag{4}$$

A short distance away at $x + dx$ the slope has changed to

$$\tan\alpha_2 = \frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)dx \tag{5}$$

where the second term is the change in slope per unit length from x to $x + dx$.

Substituting in Eq. (3) gives

$$S\,\frac{\partial^2 u}{dx^2}dx = \rho_\ell\, dx\frac{\partial^2 u}{dt^2} \tag{6}$$

and rearranging

$$\frac{\partial^2 u}{dx^2} = \frac{1}{c^2}\frac{\partial^2 u}{dt^2} \tag{7}$$

This is the wave equation for an ideal string.

The constant c is the propagation velocity of a transversal wave on the string.

$$c = \sqrt{\frac{S}{\rho_\ell}} \tag{8}$$

Wave propagation on an ideal string

A general solution to the wave equation (d'Alembert's solution) is

$$u(x,t) = f_1(ct - x) + f_2(ct + x) \tag{9}$$

which easily can be verified

$$\frac{\partial u}{\partial x} = -f_1' + f_2' \quad \text{and} \quad \frac{\partial^2 u}{dx^2} = f_1'' + f_2'' \tag{10}$$

$$\frac{\partial^2 u}{dt^2} = c^2(f_1'' + f_2'') \tag{11}$$

A general property of all functions of variable $ct - x$ is that the function has the same value for a change Δt in time as for a change $\Delta x = c\Delta t$ in the spatial coordinate. It is simply a consequence of keeping the argument $ct - x$ constant. This property of the solution describes a propagating wave.

When observing a wave which propagates on the string in the positive x-direction we can see the same point on the wave at different positions at different times (see Fig. 3). We can wait at a certain position x_1 and at a certain moment the peak of the wave (circle) passes by to the right. Instead of waiting at x_1 we could have taken a step $c\Delta t$ to the left and observed the same peak Δt earlier (dot). The wave propagates to the right with a constant velocity without changing shape, described by the function $f(ct - x)$. A left-going wave is described by the function $f(ct + x)$.

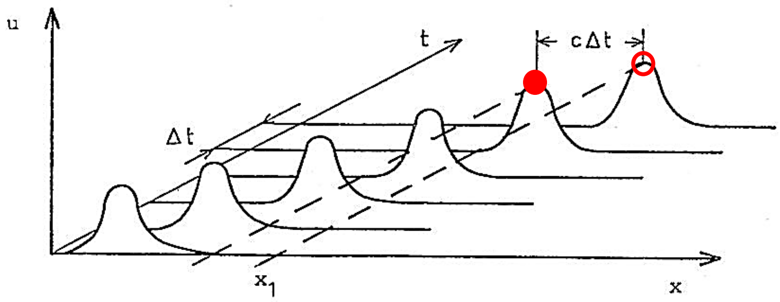


Fig. 3. A wave $f(ct\text{-}x)$ propagates along the string. The peak of the wave passes position x_1 at a certain time (*circle*) A little earlier (Δt) the same peak passed a point at a distance $c\Delta t$ to the left of x_1 (*dot*).

Real strings are terminated at both ends which means that the functions f_1 and f_2 cannot be arbitrary. The boundary conditions impose constraints. For a rigidly terminated string with lengh L the conditions are

$$u(0,t) = u(L,t) = 0 \tag{12}$$

The boundary condition

$$u(0,t) = f_1(ct - 0) + f_2(ct + 0) = 0 \tag{13}$$

implies that

$$f_2(ct + x) = -f_1(ct - x) \tag{14}$$

and $u(x,t)$ becomes

$$u(x,t) = f_1(ct - x) - f_1(ct + x) \tag{15}$$

The interpretation of Eq. (15) is a wave $f_1(ct - x)$ which propagates in the positive x-direction and another wave $-f_1(ct + x)$ with the same shape but reversed sign (inverted) propagating in the opposite direction (see Fig. 4a).

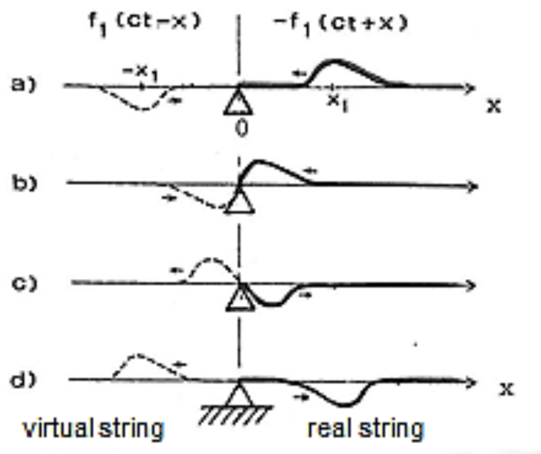


Fig. 4. Reflection of a left-going wave at a rigid string termination at $x = 0$.

Suppose that the left-going wave $-f_1(ct+x)$ passes the point x_1 at $t = 0$ (see Fig. 4a). The right-going wave $f_1(ct-x)$ must at the same time be at $-x_1$ according to Eq. (14). This can be interpreted as it is located on a virtual string to the left of the termination. An observer sees only the left-going wave $-f_1(ct+x)$. The real and virtual waves meet at $x = 0$ and are superpositioned, with the result that the displacement at the termination is zero at all times $u(0,t) = 0$ (see Fig. 4 b,c). The right-going wave $f_1(ct-x)$continues out on the real string and the left-going wave 'disappears' out on the virtual string (see Fig. 4 c,d). The observer sees a process where the left-going wave is reflected at the termination and converted to an inverted copy propagating back to the right.

Another way of describing the process is to say that when the left-going wave arrives at the termination a force is generated by the termination which counteracts the force from the string so that the termination remains at rest at all times $u(0,t) = 0$.

At the opposite end of the string the boundary condition $u(L,t) = 0$ gives rise to a similar reflection of the wave. A wave which starts at $t = 0$ at the left termination at $x = 0$ will arrive at the right termination at time $t = L/c$ and turn back inverted. After $t = 2L/c$ it will be reflected and inverted again at $x = 0$ and resume its initial shape. All wave motion on an ideal string with rigid terminations will consequently be periodic with period time $T = 2L/c$ independently of the shape of the wave $f(ct \pm x)$ and how and where the string is set in motion (plucked, struck).

Standing waves or modes of vibration

An alternative, and very powerful, solution of the wave equation describes the motion of the string by a set of generic vibration states. By superimposing a large number of such 'building blocks' of vibrations an equivalent description of the travelling waves reflected back and forth on the string can be obtained. The vibrational states are formally called *normal modes of vibration* or *eigenmodes*, but often referred to as '*string modes*,' '*standing waves*' or '*resonances*.' The frequencies of these modes are directly related to the periodicity of the round trip time of the reflected waves $T = 2L/c$.

The solution of the wave equation is approached by assuming that all free motion of the string can be descried by summation of functions where the dependence of x and t is not combined in a single argument ('separation of variables,' Bernoulli's solution). The displacement of the string $u(x,t)$ is described by

$$u(x,t) = \sum_{n}^{\infty} \Phi_n(x) \cdot q_n(t) \tag{16}$$

where $\Phi_n(x)$ describes a stationary distribution of amplitudes along the string (independent of t), see Fig. 5, and $q_n(t)$ the time dependence (oscillatory motion, independent of x). The mode number is denoted by n.

By derivation of Eq. 16 twice and substituting into the wave equation (Eq. 7) two normal differential equations are obtained with x and t as variables, respectively, which have simple *sin* and *cos* solutions. For the rigidly terminated string with boundary conditions according to Eq. 12 the solution is

$$\Phi_n(x) = \sin\frac{n\pi}{L}x \tag{17}$$

and

$$q_n(t) = A_n \cos\omega_n t + B_n \sin\omega_n t \tag{18}$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{L}x \cdot (A_n \cos\omega_n t + B_n \sin\omega_n t) \tag{19}$$

with the coefficients A_n and B_n set set by the initial conditions.

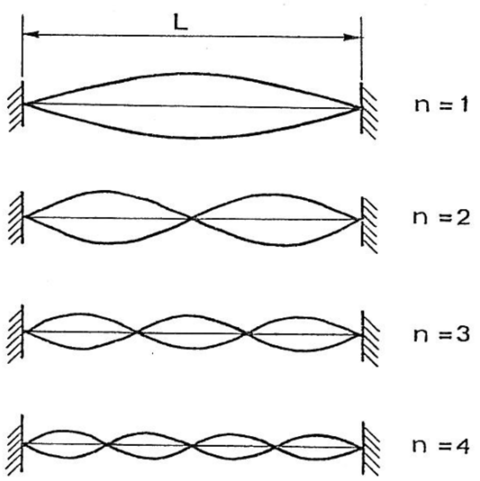


Fig. 5. The five lowest mode shapes for a rigidly terminated ideal string.

The mode frequencies are given by

$$\omega_n = 2\pi f_n = \frac{n\pi c}{L} \qquad n = 1,2,3 \ldots \tag{20}$$

$$f_n = \frac{nc}{2L} \tag{21}$$

Using the expression for the wave propagation velocity *c* (Eq. 8) the frequencies of the normal modes can be written

$$f_n = \frac{n}{2L}\sqrt{\frac{S}{\rho_\ell}} = \frac{n}{2L}\sqrt{\frac{A\sigma}{A\rho}} = \frac{n}{2L}\sqrt{\frac{\sigma}{\rho}} \tag{22}$$

with

$A = \frac{\pi d^2}{4}$ cross section area of a string with diameter *d*

$\sigma = \frac{S}{A}$ tensile stress [N/m^2]

ρ density of the string material [kg/m^3]

The mode frequencies correspond to the frequencies of the *partials* in the radiated sound from the string. The component corresponding to the lowest mode is often called the *fundamental* (labeled f_o) and the higher components overtones. Alternatively all spectral components are referred to as *partials*, with the fundamental as partial one. Sometimes the *fundamental frequency of the ideal string* is indicated by f_1^o in order to distinguish it from the fundamental frequency of a real string

$$f_1^o = \frac{1}{2L}\sqrt{\frac{S}{\rho_\ell}} \tag{23}$$

For the rigidly terminated ideal string the mode frequencies are exactly integer multiples of the fundamental. Such systems with *harmonic* partials are rare. The only vibrating system with harmonically related mode frequencies, besides the ideal string, are pipes with constant or conically varying cross section. Strings and pipes are the basic elements of all musical instruments with a clear pitch.

The mode frequencies f_n,, or eigenfrequencies, are often referred to as the 'resonance frequencies' of the string. Actually, the term *resonance* should be reserved to the case when a vibrating system is driven by an external periodic force. When the driving frequency is close to a mode frequency of the string the vibration amplitude will be high and much energy is stored in the string, and the system is in resonance. If the string is allowed to vibrate freely, the only possible frequencies of the oscillations are the mode frequencies.

The fixed amplitude distribution along the string $\Phi_n(x)$ given by Eq. (17) are called *mode shapes* (or standing wave patterns). For the ideal string they are parts of a sine functions, including an integer number of half wavelengths across the string length (see Fig. 5). The distance between two nodes or antinodes in each mode shape is half a wavelength $\lambda_n/2$. If one of the mode frequencies is known the propagation velocity $c = f_n \lambda_n$ can be determined as well as the string tension *S*.

The fact that the mode frequencies for ideal strings and pipes with uniform cross section are harmonic and that their mode shapes are described by sine functions is a rare exception. These two systems are the only cases with such simple solutions. The mode shapes for other vibrating structures like bars and membranes are described by more complex functions (sinh, Bessel functions) and the mode frequencies are strongly inharmonic.

The relation between the amplitudes of the string modes are determined by the details of the excitation process (pluck, strike). Knowing the initial conditions, including the displacement along he string $u(x,0)$ and velocity distribution $\partial u/\partial t(x,0)$, the coefficients A_n and B_n in Eq. 18 can be computed.

Before closing the discussion of ideal strings it is worth noting the diameter of the string does not enter as a design parameter when choosing the length of string for a certain pitch (fundamental frequency), see Eq. (22). If the diameter is increased the string becomes heavier (ρ_ℓ increases) and the fundamental frequency is lowered. But the change in frequency can be compensated by a higher tension. The critical parameter is the tensile stress σ. If the tensile strength is exceeded the string will break so a certain safety factor is needed. The ratio σ/ρ is thus the primary design parameter which determines the length of a string with a given fundamental frequency. Other factors, primarily stiffness and characteristic impedance, put restrictions on the diameter.

Stiff strings

Real strings differ from the ideal case in several respects. Two important differences are the string stiffness and the conditions at the terminations. Real strings do have a resistance to bending and will oppose the deformation of the material inside the string when it is deflected. The material will resist the internal shear forces and the string is said to have a certain *stiffness*. The stiffness influences the frequencies of all partials.

For the ideal string, the only restoring forces which act to pull it back to its equilibrium state, are the components of the string tension in the transversal direction (see Eq. 2). For a stiff string the bending stiffness will contribute with a restoring internal moment in the string and the string will return to equilibrium slightly faster than the ideal string. This means that the period times are shortened somewhat and the mode frequencies are raised. The contribution from stiffness will be larger the sharper the bends of the string. This means that the frequency shifts will be larger for the higher modes with shorter wavelengths. As a result the partial frequencies of a stiff string are not strictly harmonic. The phenomenon, which is called *inharmonicity*, is visible in the spectrum as successive larger frequency distances between the partials, so called 'stretched spectrum' (see Fig. 6).

In the time domain, stiffness is visible as *dispersion*, which reflects that the resistance to bending results in a propagation velocity $c(\omega)$ which is not constant but increases with frequency. The result is that the shape of a wave on the string changes continuously during the propagation. The travelling wave, which can be viewed as a superposition of many string modes, is distorted because the components with higher frequencies travel faster ('precursors').

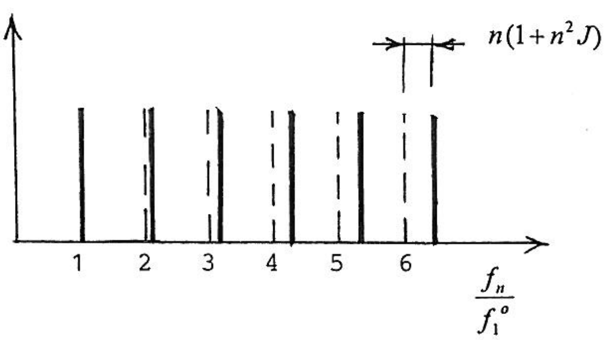
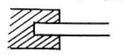
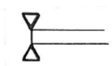


Fig. 6. Comparison between a harmonic (dashed) and stretched spectrum.

The magnitude of the frequency shift depends to some extent also on the conditions at the string terminations. Two important cases are clamped and hinged supports (see Fig. 7)

The boundary conditions at $x = 0$ and $x = L$ are

clamped $u = 0; \quad \frac{\partial u}{\partial x} = 0$  (24a)

hinged $u = 0; \quad \frac{\partial u}{\partial x}$ not restricted; $\frac{\partial^2 u}{\partial t^2} = 0$  (24b)

For real strings the conditions are often somewhere in between the two cases (string crossing over a bridge of a violin, guitar or piano, string depressed by a finger against a fingerboard).

A good approximation of the mode frequencies for a stiff string which applies for clamped as well as hinged conditions is given by

$$f_n = n f_1^0 \sqrt{1 + n^2 B} \tag{25}$$

where the frequency deviation from the ideal string is given by a correction term n^2B.

This correction is normally small and the square root may be replaced by $1 + \frac{1}{2}n^2B = 1 + n^2J$ where $B = 2J$ is the *inharmonicity coefficient.*

The frequencies for the higher modes can be related to the stretched frequency of the fundamental by

$$f_n \approx n f_1 [1 + (n^2 - 1)J] \tag{26}$$

The inharmonicity coefficient B is determined by the string dimensions and material properties of the string (E is Young's modulus)

$$B = \frac{\pi^3 d^4 E}{64 S L^2} = \frac{\pi^2 d^2 E}{16 \sigma L^2} \tag{27}$$

B will increase with larger diameter (d) and stiffer material (E), but can be reduced by making the string long (L) and kept under high tension (S). For piano strings a certain level of inharmonicity is desired in order to get the characteristic timbre. A typical value is $B = 400 \cdot 10^{-6}$ in the middle register. In contrast, for bowed string instruments like the violin even a much lower inharmonicity coefficient is detrimental to the playing properties.

It is worth noting that if a string instrument is detuned to a lower pitch (e.g. by not tuning the piano regularly), the tension S and in turn the stress σ will be reduced. As a consequence inharmonicity increases. The unpleasant sound of a severely out-of-tune piano is thus not only due to bad temperament of intervals and beating string trichords, but also to increased inharmonicity.

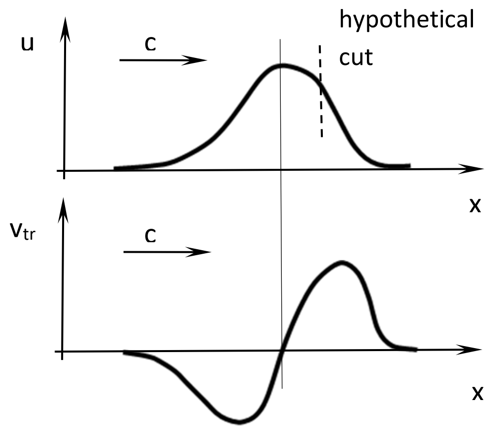
Characteristic impedance

A vibrating string exerts a transversal force F_{tr} on the terminations. This effect is the sound-generating source of the stringed instruments. The excitation force is transmitted through the string termination, the bridge, to the instrument body. It turns out that the excitation force is related to the transversal velocity of the string trough a parameter which describes the mechanical properties of the taut string, the *characteristic impedance* Z_o.

It is the slope of the string in combination with the string tension which gives the transversal force a wave exerts on the string termination. The slope has a simple relation to the transversal velocity of the wave.

Consider a wave on a string which propagates to the right with tranversal displacement $u(x, t) = f(ct-x)$. The constant c is the propagation velocity.

The corresponding transversal velocity v_{tr} of the wave is by definition



$$v_{tr} = \frac{\partial u}{\partial t} = c\, f'(ct - x) \qquad (28)$$

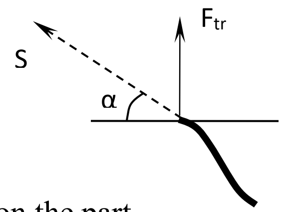
The slope of the string at an arbitrary point x is

$$\frac{\partial u}{\partial x} = -f'(ct - x) \qquad (29)$$

(the minus sign originates from the inner derivative)

Eqs, (28) and (29) give

$$v_{tr} = -c \frac{\partial u}{\partial x}$$



Now, imagine a cut through the string at position x (vertical dashed line in the figure above)

The transversal force F_{tr} that the string part to the left of the cut exerts on the part to the right is

$$F_{tr} = S \sin\alpha \approx S \tan\alpha = -S \frac{\partial u}{\partial x} \qquad (30)$$

The minus sign is due to the definition of positive force in the same direction as displacement u (upwards). The string displacement u is always small in reality (strongly exaggerated in the figures) which makes the angle α small as well and justifies the approximation sin α ≈ tan α.

The relation between transversal force and velocity of the wave is called the *characteristic impedance* Z_o of the string

$$Z_0 = \frac{F_{tr}}{v_{tr}} = \frac{-S \frac{\partial u}{\partial x}}{-c \frac{\partial u}{\partial x}} = \frac{S}{c}$$

The important result is that excitation force on the string support F_{tr} is proportional to the transversal velocity v_{tr} of the wave on the string multiplied by a string parameter Z_o

$$F_{tr} = Z_o v_{tr} \qquad (32)$$

Using the expression for the fundamental frequency f_0 of the string

$$f_1^o = \frac{c}{2L} = \frac{1}{2L}\sqrt{\frac{S}{\rho_\ell}} = \frac{1}{2L}\sqrt{\frac{\sigma}{\rho}} \qquad (33)$$

useful alternatives of the expression for Z_o are obtained

$$Z_0 = \frac{S}{c} = \sqrt{S\,\rho_\ell} = 2L\,\rho_\ell\, f_0 = 2m_s f_1^o \qquad (34)$$

where σ is tensile stress and ρ density of the string material, ρ_ℓ the linear density and m_s the total mass of the speaking length L of the string.

The characteristic impedance Z_o is not solely determined by the geometry and material properties. It also depends on tension. The basic rule is evident from Eq. 34. For a given fundamental frequency (pitch), Z_o is proportional to total string mass m_s. The influence of tension is inherent as a heavy string needs to be taut to higher tension than a light string of the same pitch.

A note on instrument design

It is the transversal force F_{tr} that makes the instrument body vibrate. The obvious way to make a string instrument louder is to increase Z_o. This will require heavier strings under higher tensions. A basic problem is then to make the instrument body strong enough to withstand the higher stresses. A second issue is to be able to excite the heavier strings under higher tensions to at least the same transversal velocity as the lighter strings. This requires higher plucking forces, higher bow forces and heavier hammers and harder touch for, plucked, bowed and struck string instruments, respectively. Such a process towards louder instruments characterizes the development of all string instruments during the last centuries.