KTH ROYAL INSTITUTE
OF TECHNOLOGY

DT2212: Music Acoustics

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Lecture 5: Strings (ideal and real)
Reading: Ch 7 Hartmann: Ch 2 Fletcher and Rossing

Outline of course

~~Lecture 1: Introduction; basic acoustics~~

~~Lecture 2: Modes of vibration~~

~~Lecture 3: The ear; tuning and temperament~~

~~Lecture 4: Tuning and temperament and audio analysis~~

Lecture 5: Vibrating strings

Lecture 6: Guitar and piano

LAB 1: Guitar

Lecture 7: Synthesis methods

Lecture 8: Singing voice

LAB 2: Singing voice

Lecture 9: Bowed strings

Lecture 10: Wind instruments

Lecture 11: Unconventional instruments + Project workshop

Lecture 12: Final project presentations

What is a sound wave?

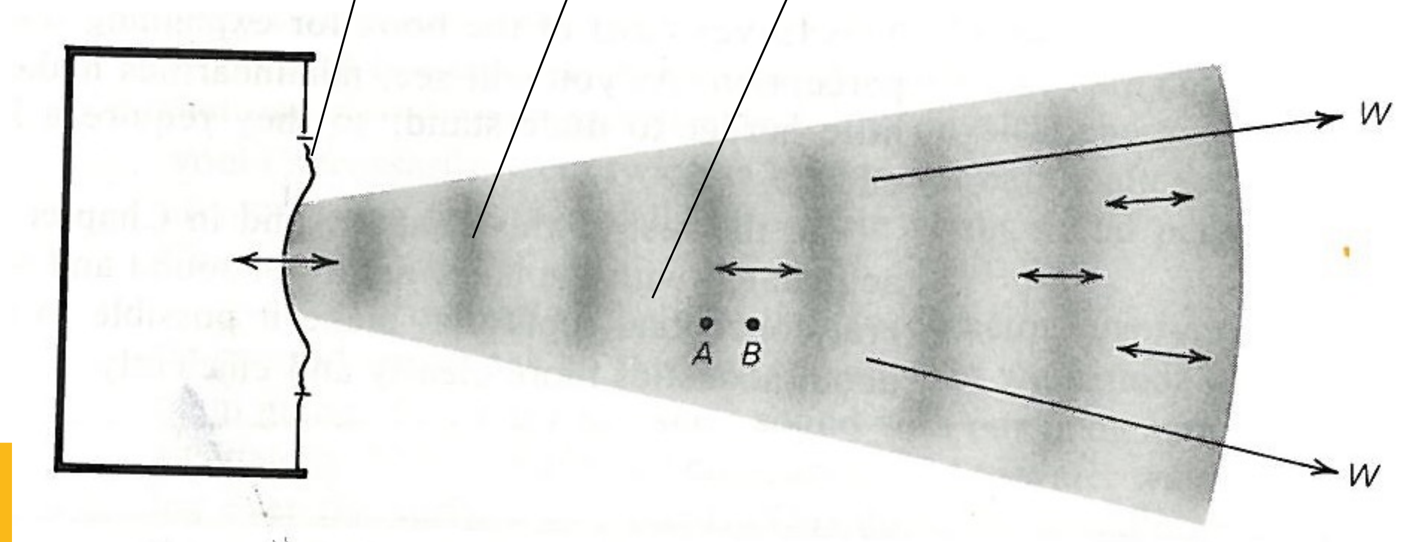
A “longitudinal” propagation of energy.

Vibrating surface

loudspeaker membrane
violin body, trumpet bell
(pipe end)

condensation

rarefaction
(decompression)

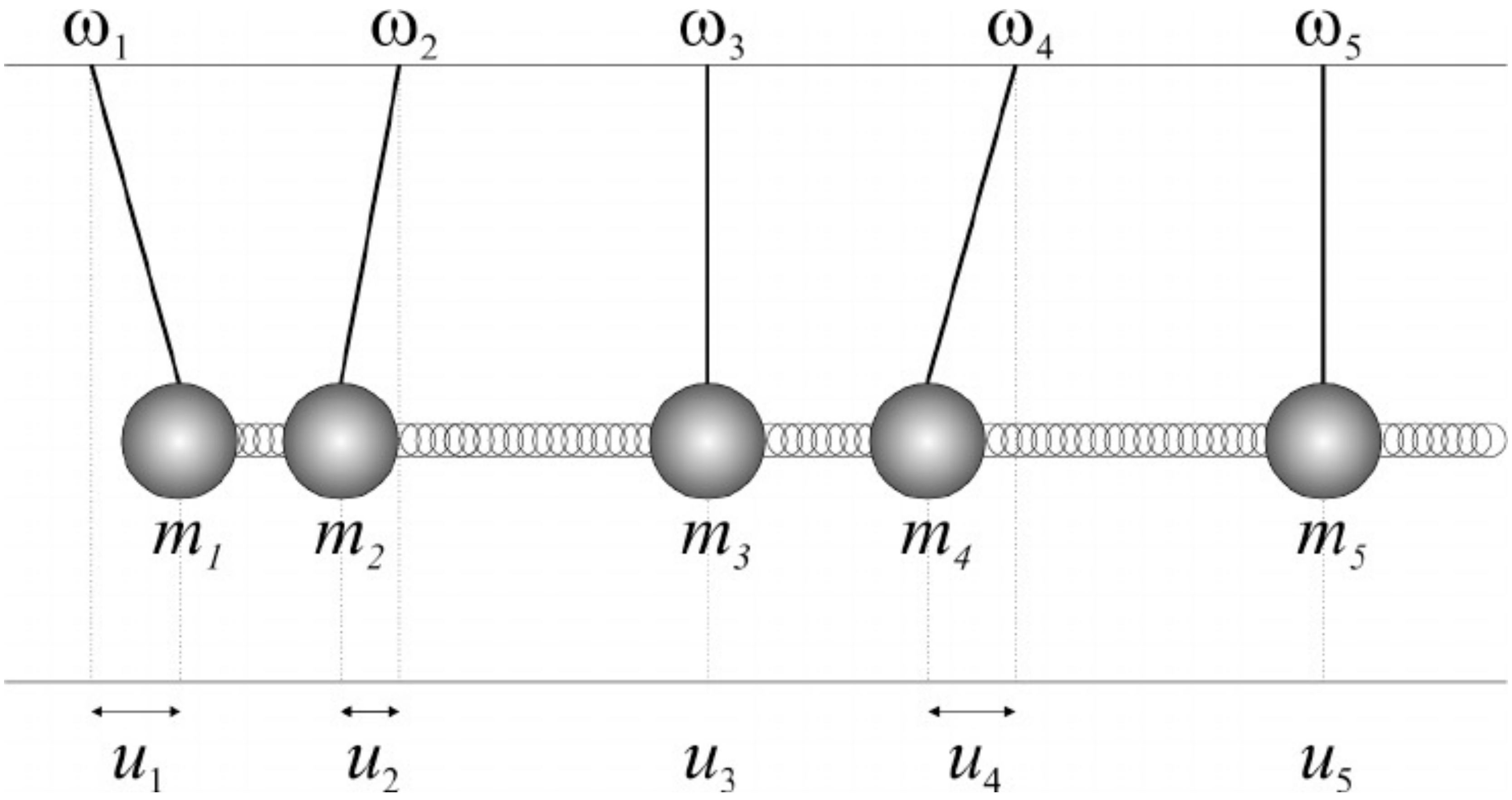


Acoustical sizes

Is an object large or small from an acoustical perspective?
Depends on the wavelength!

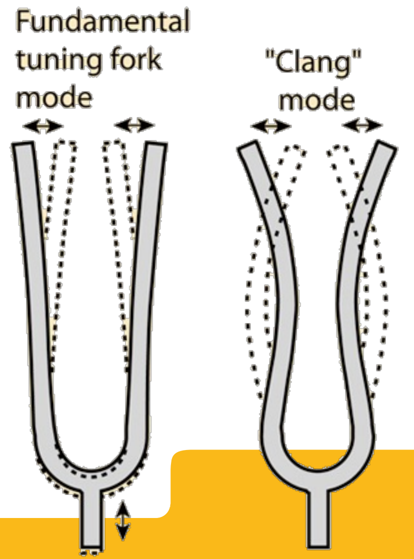
Sound Frequency *f* (Hz)	Wavelength λ (m)	Wavelength comparable to	Acoustical limit large – small object $He = 2\pi r/\lambda = 1$
20	17	Semi Truck	r = 3 m
41	8	Small Yacht	1 m
110	3	Sofa	50 cm
440	0.8	Cello	10 cm
660	0.5	Chair	8 cm
1000	0.3	Violin	5 cm
10 000	0.03	Outer ear	5 mm

Coupled Oscillators



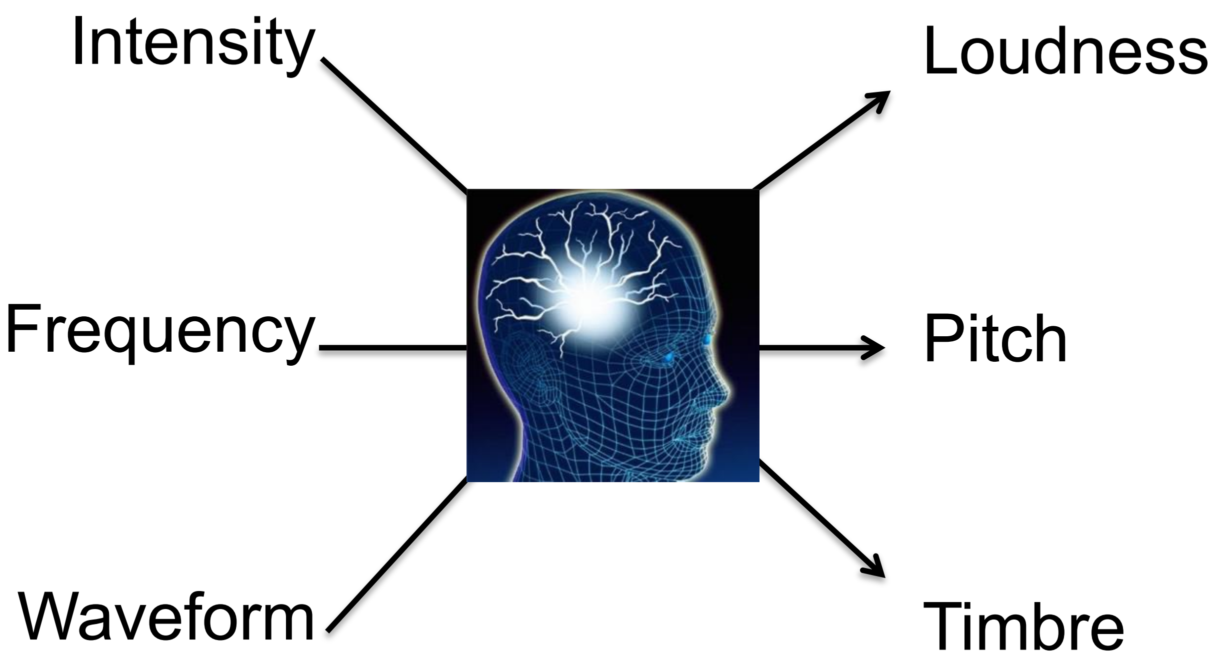
Complex Vibrating Objects

- Any material object has “natural” modes (all components undergo simple harmonic motion)
- Each natural mode has a different frequency, shape and damping
- Any vibration of an object is a linear combination (superposition) of its *natural modes*

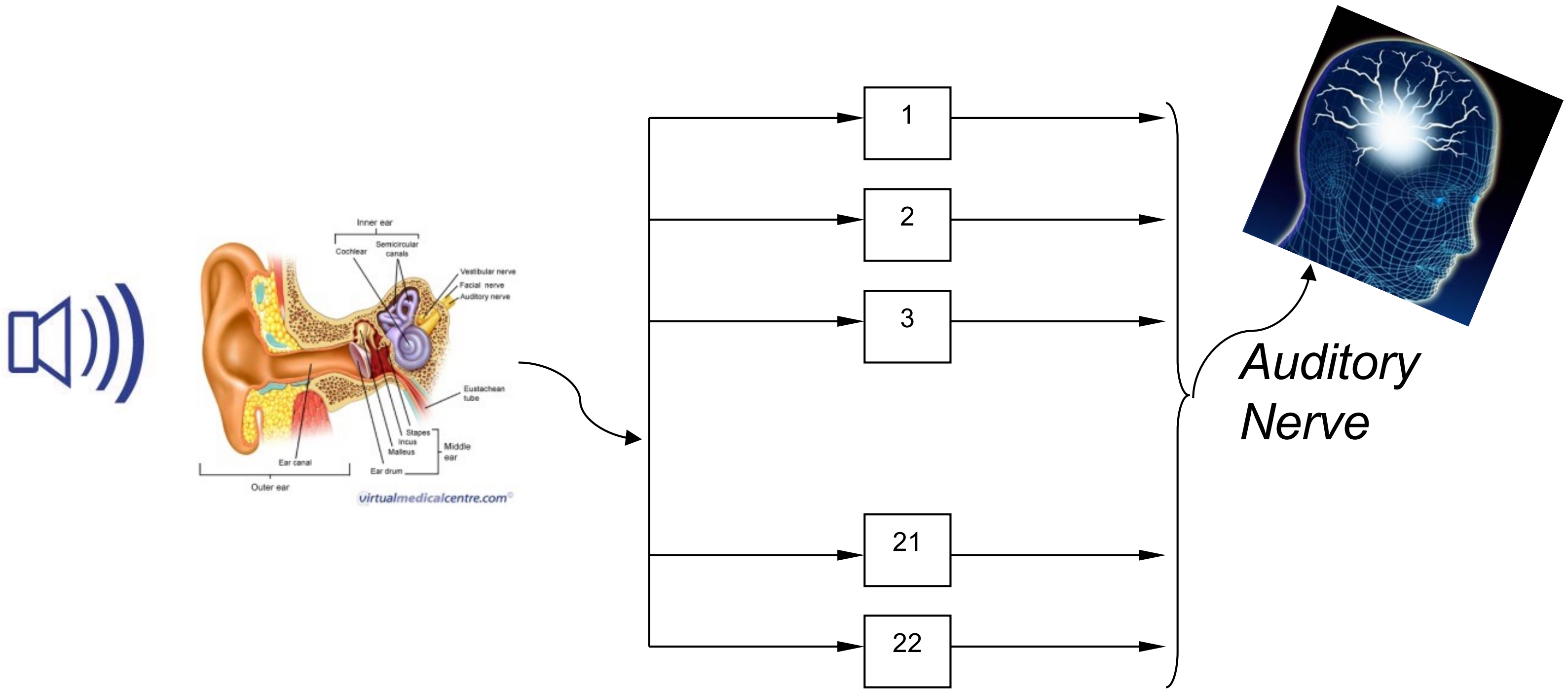


Attributes of sound

Physical *Perceptual*

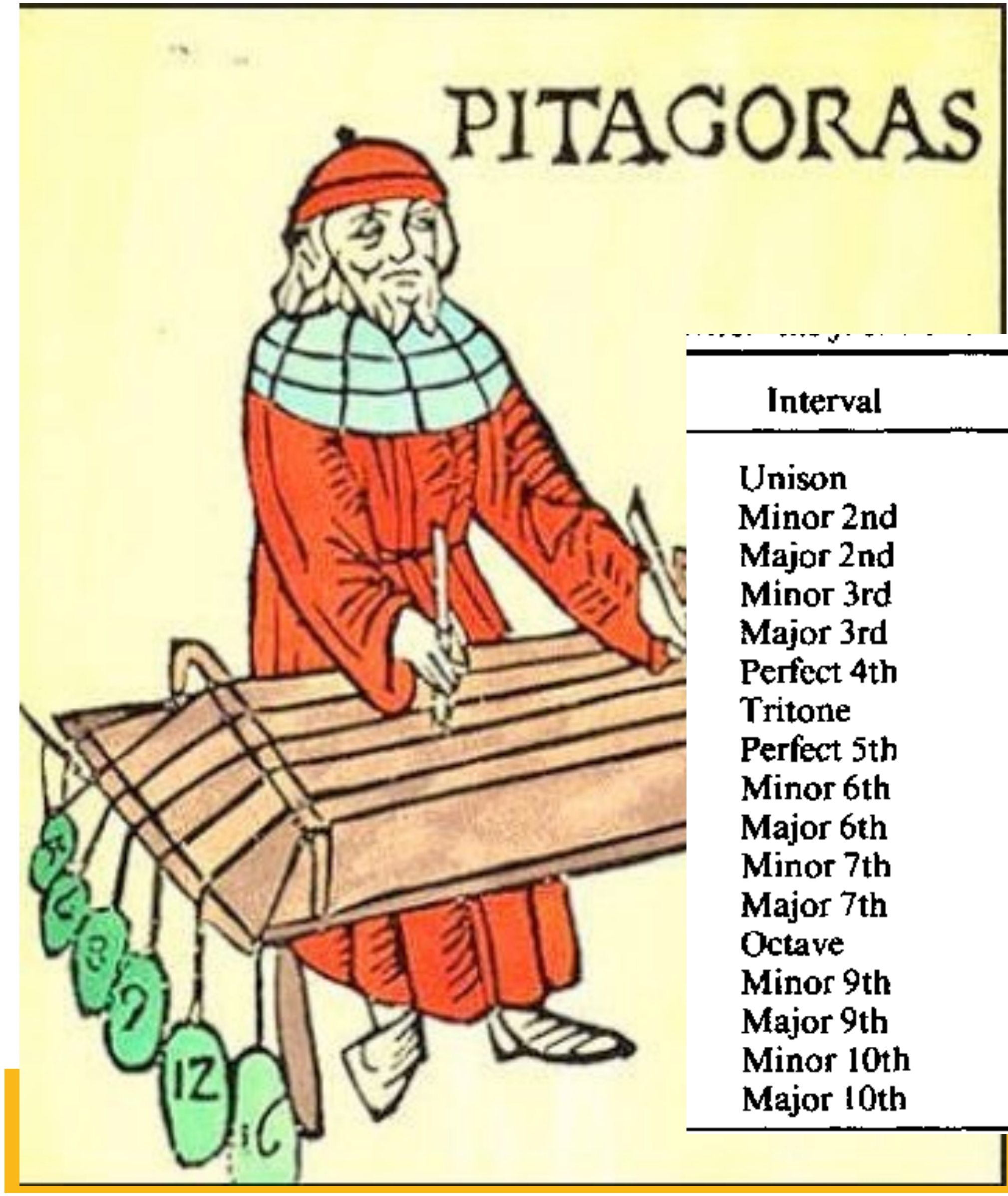


Simple model of auditory system

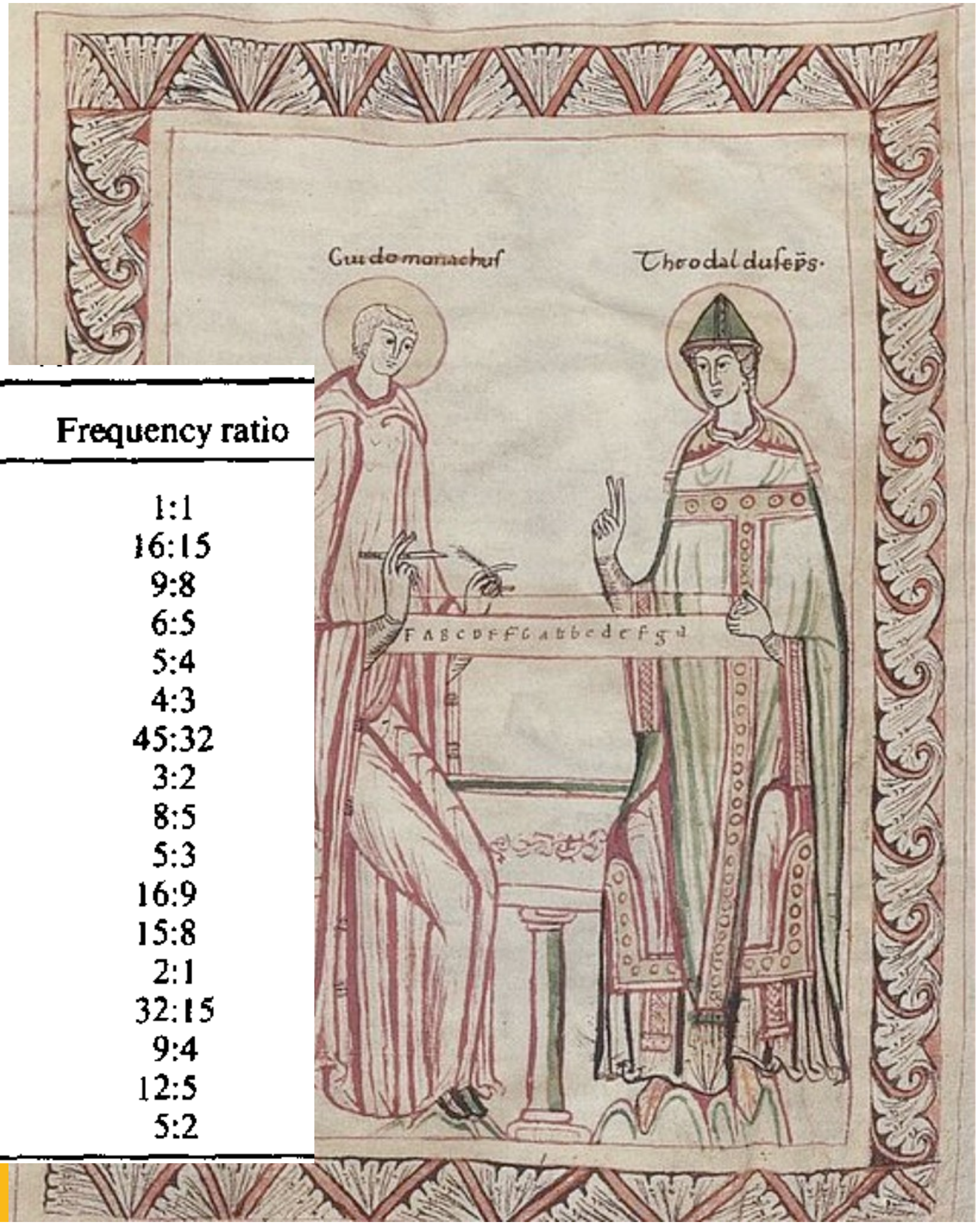


Filterbank
"frequency bands"

Consider the monochord



Interval	Frequency ratio
Unison	1:1
Minor 2nd	16:15
Major 2nd	9:8
Minor 3rd	6:5
Major 3rd	5:4
Perfect 4th	4:3
Tritone	45:32
Perfect 5th	3:2
Minor 6th	8:5
Major 6th	5:3
Minor 7th	16:9
Major 7th	15:8
Octave	2:1
Minor 9th	32:15
Major 9th	9:4
Minor 10th	12:5
Major 10th	5:2

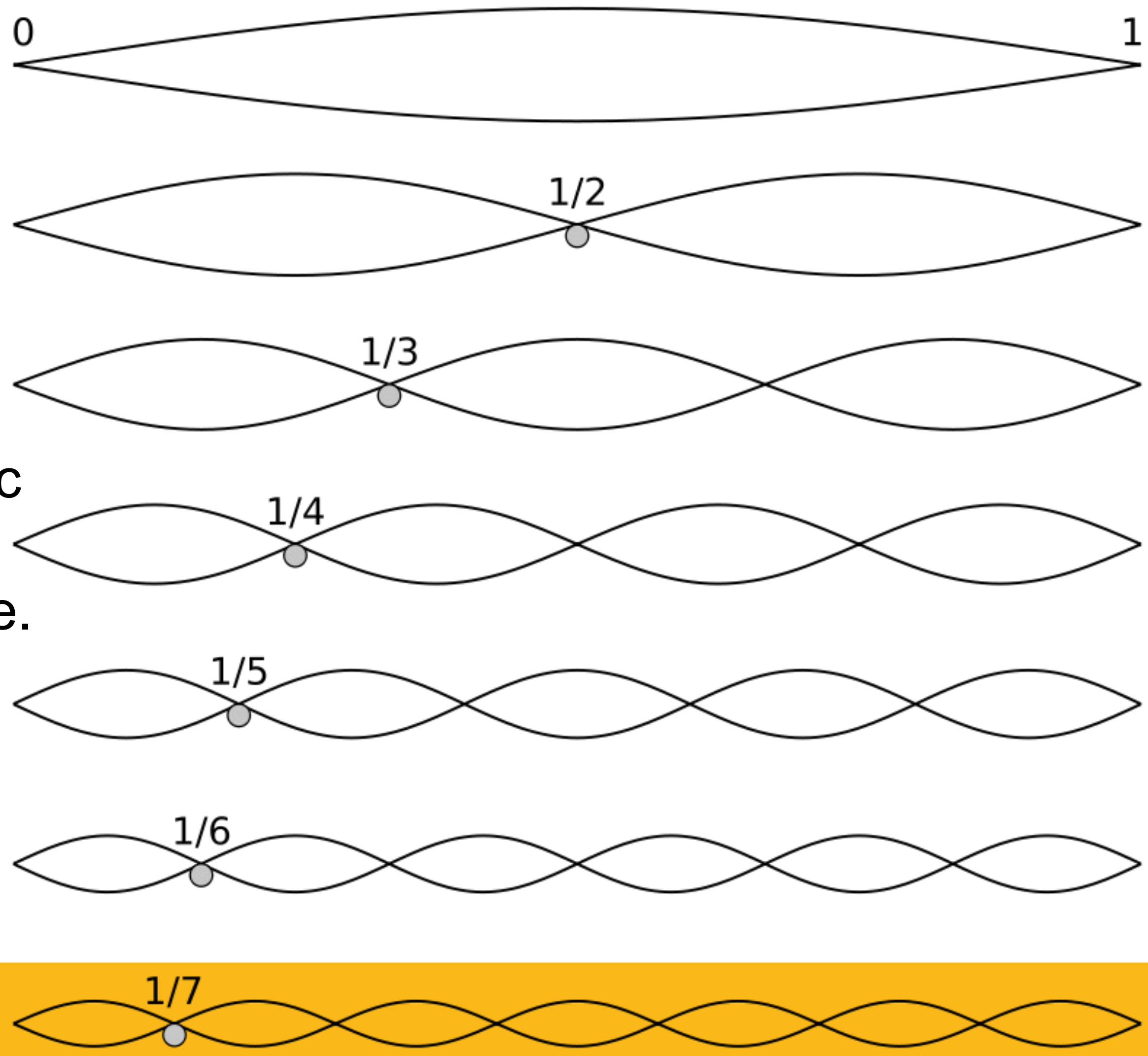


Harmonic series

We know that the string motion at the nuts must be zero

So, only certain wavelengths are permitted!

Frequency of nth harmonic is nf_0, where f_0 is the frequency of the first mode.



Harmonic series

Ideal string of length L
(no stiffness, homogenous)

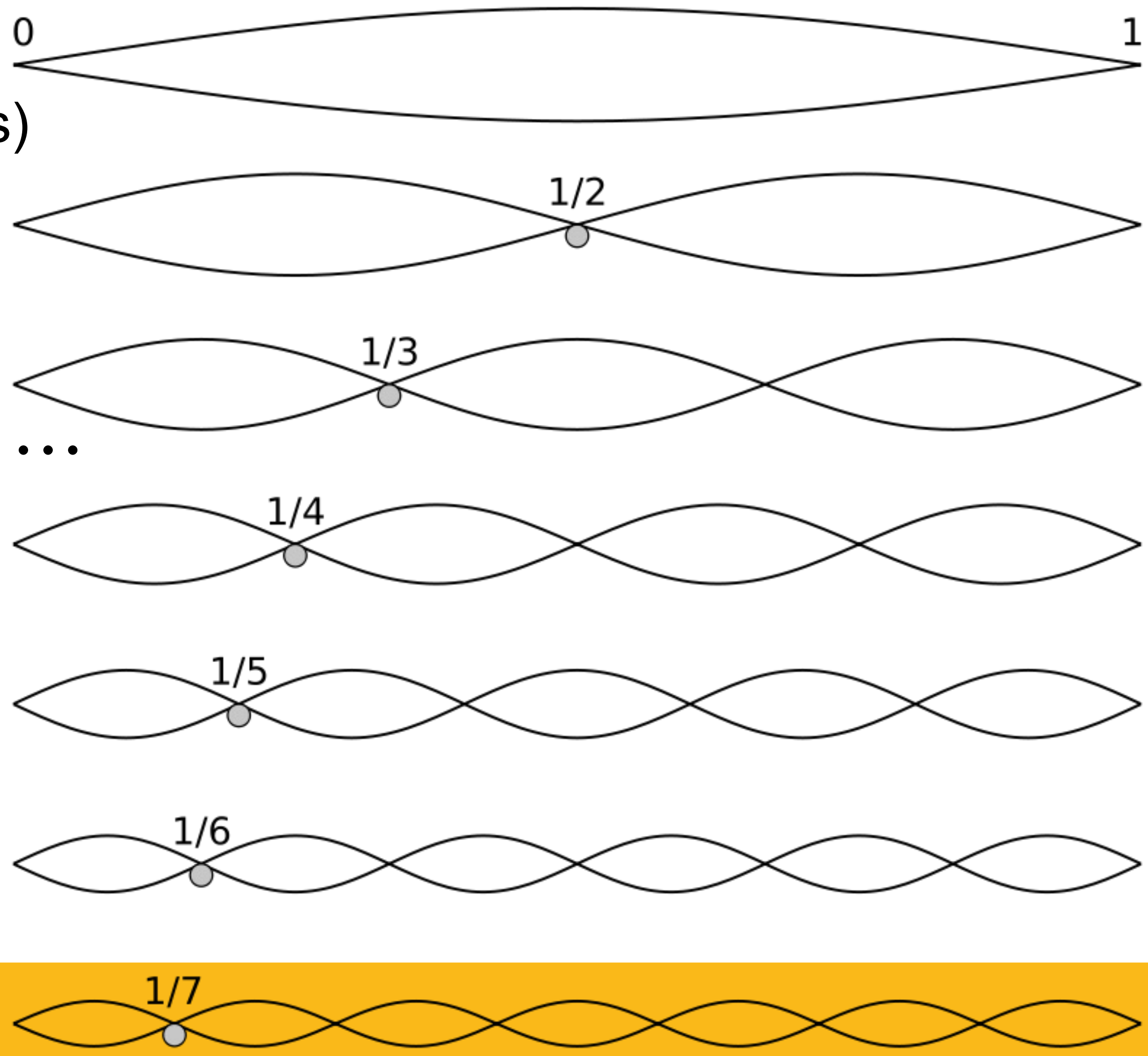
Only certain wavelengths can occur:

$$\lambda_n = \frac{2L}{n}, n = 1, 2, \ldots$$

Equivalently, only certain frequencies can occur:

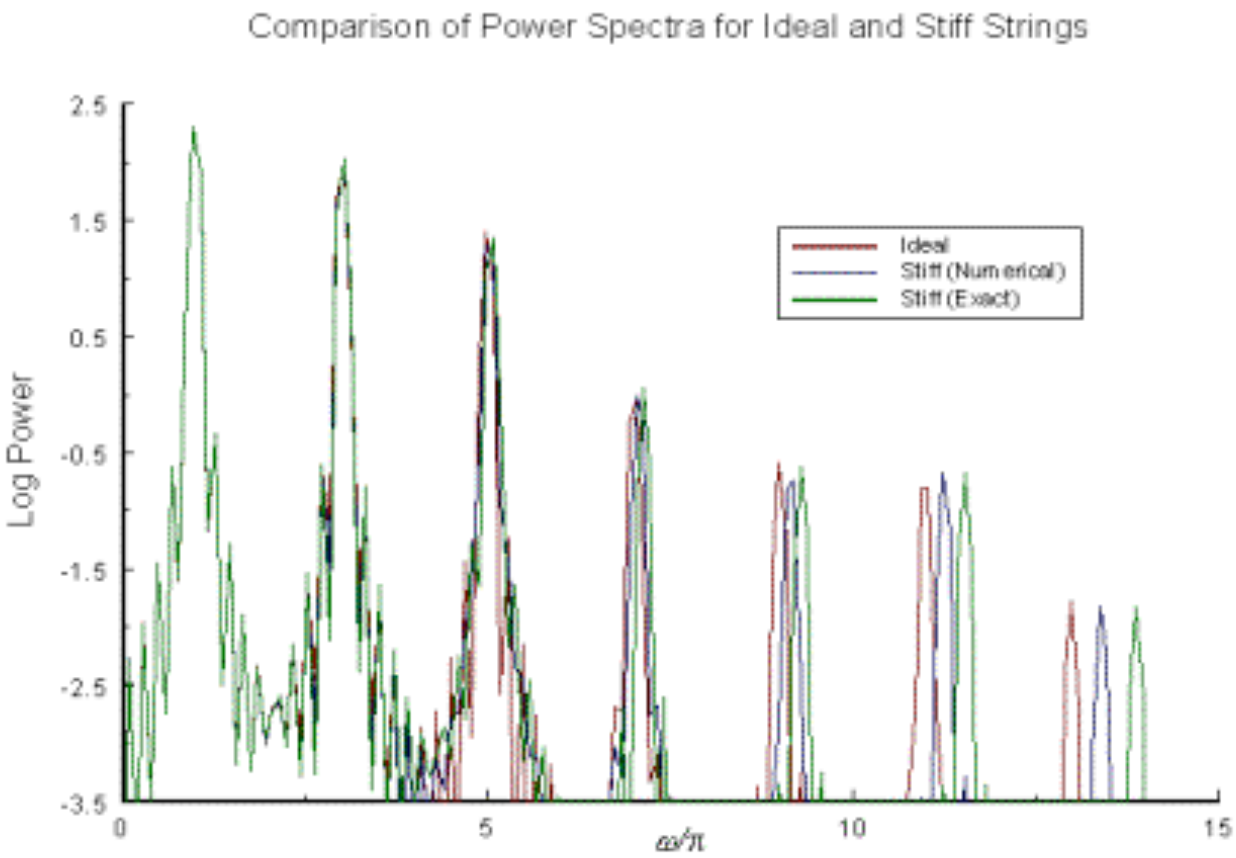
$$f_n = v/\lambda_n$$

Velocity of transverse string vibration

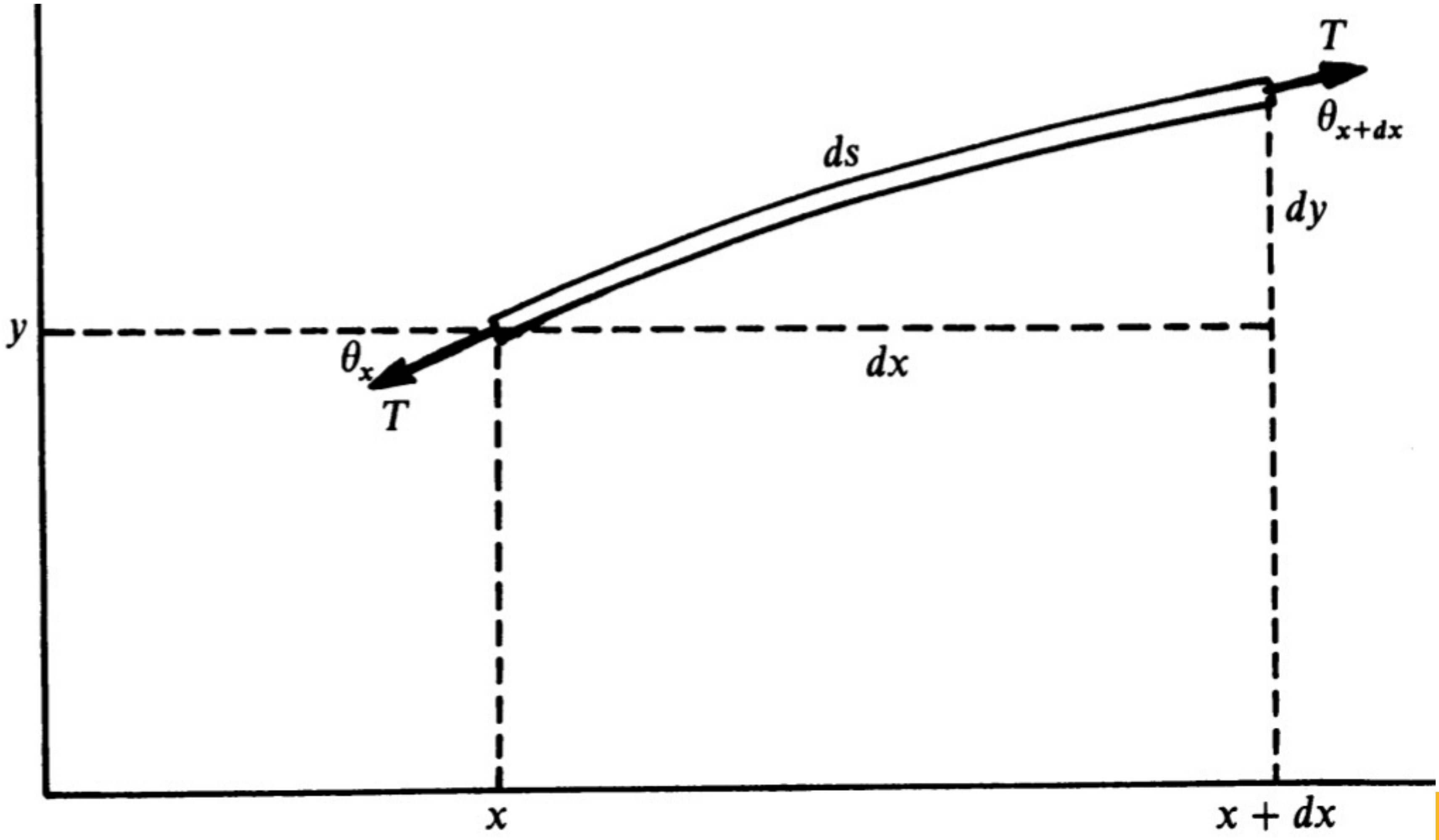


Question:

Why do real vibrating strings exhibit inharmonicities?



Ideal string from a physical perspective



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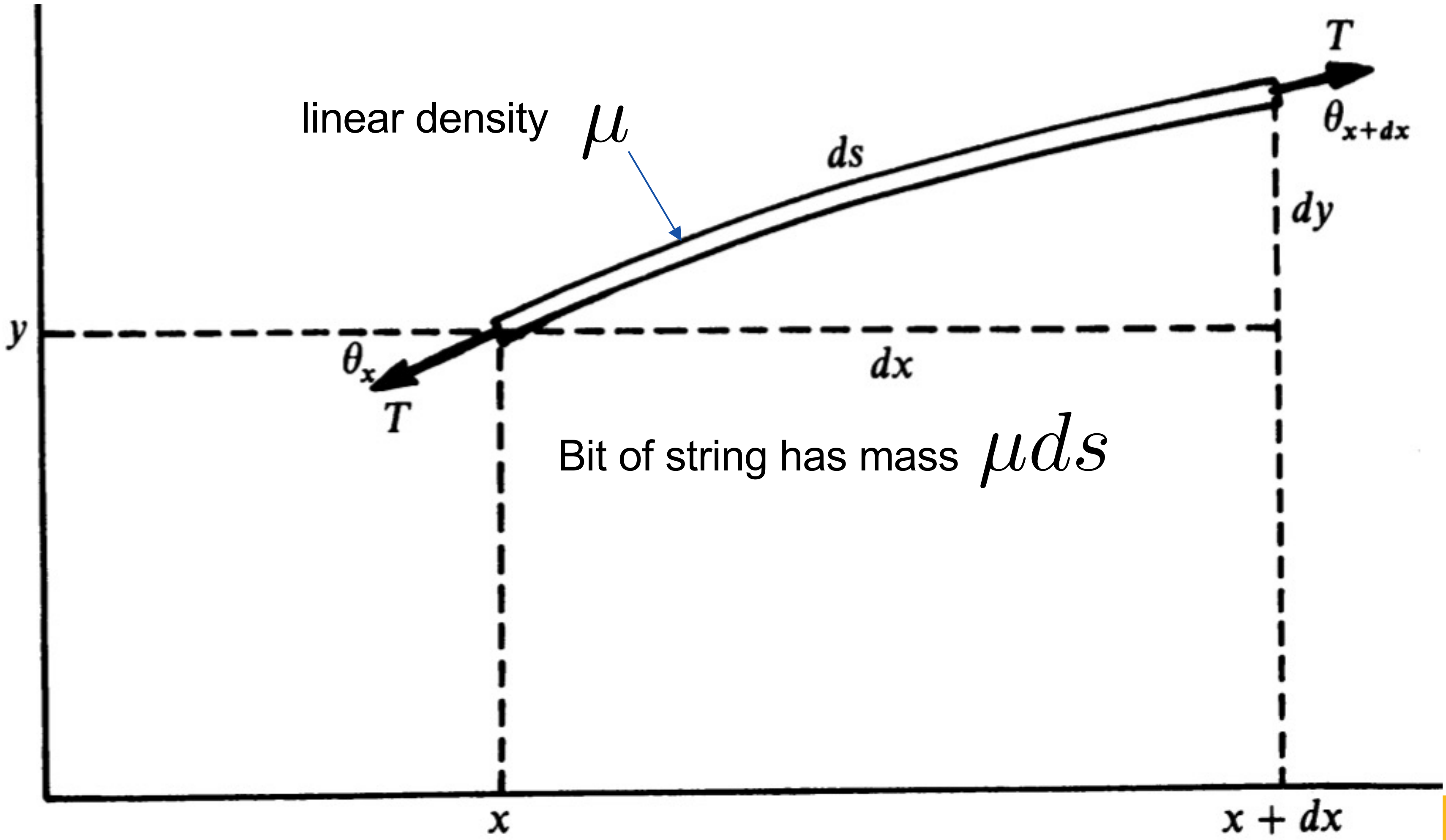
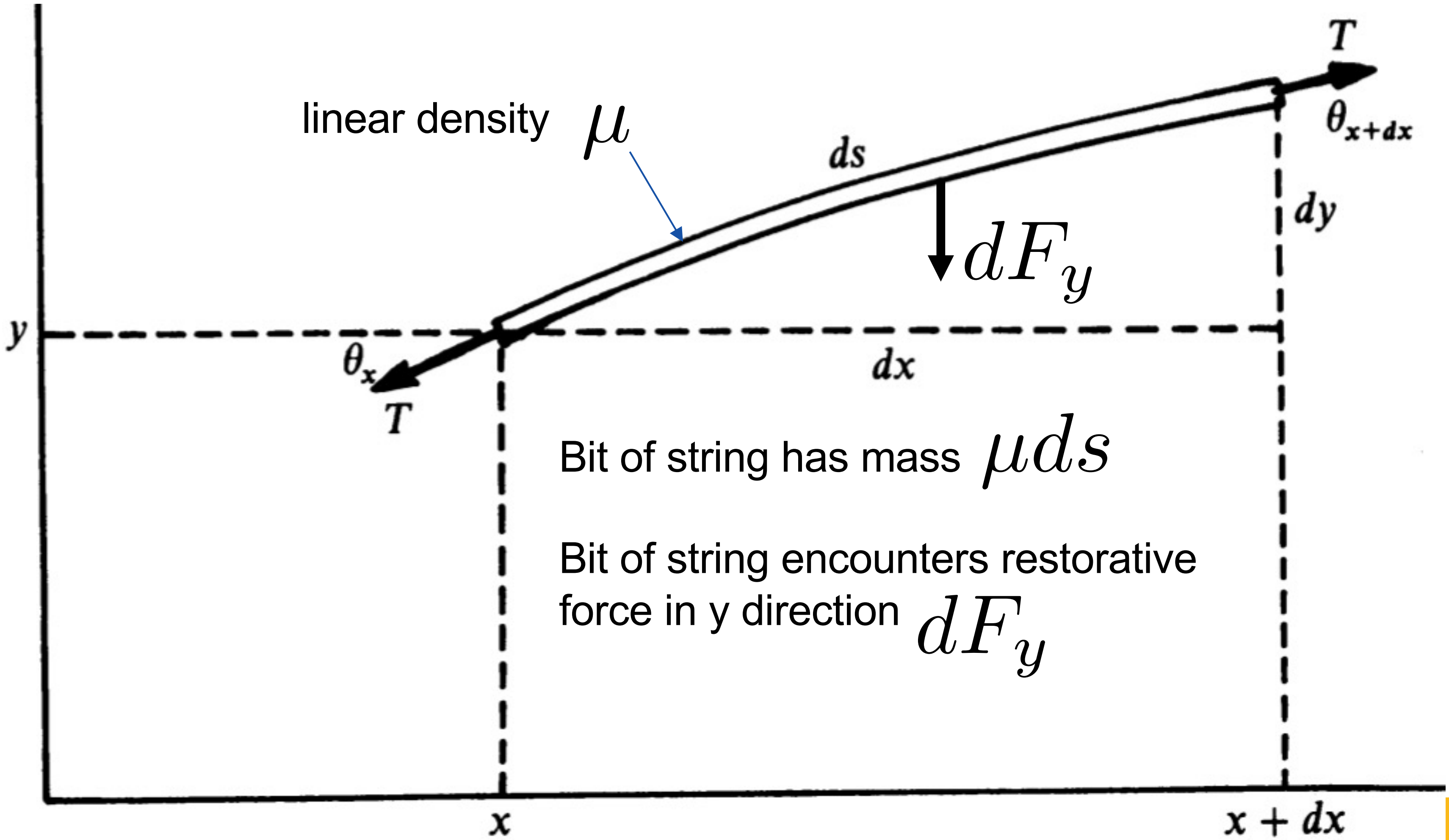
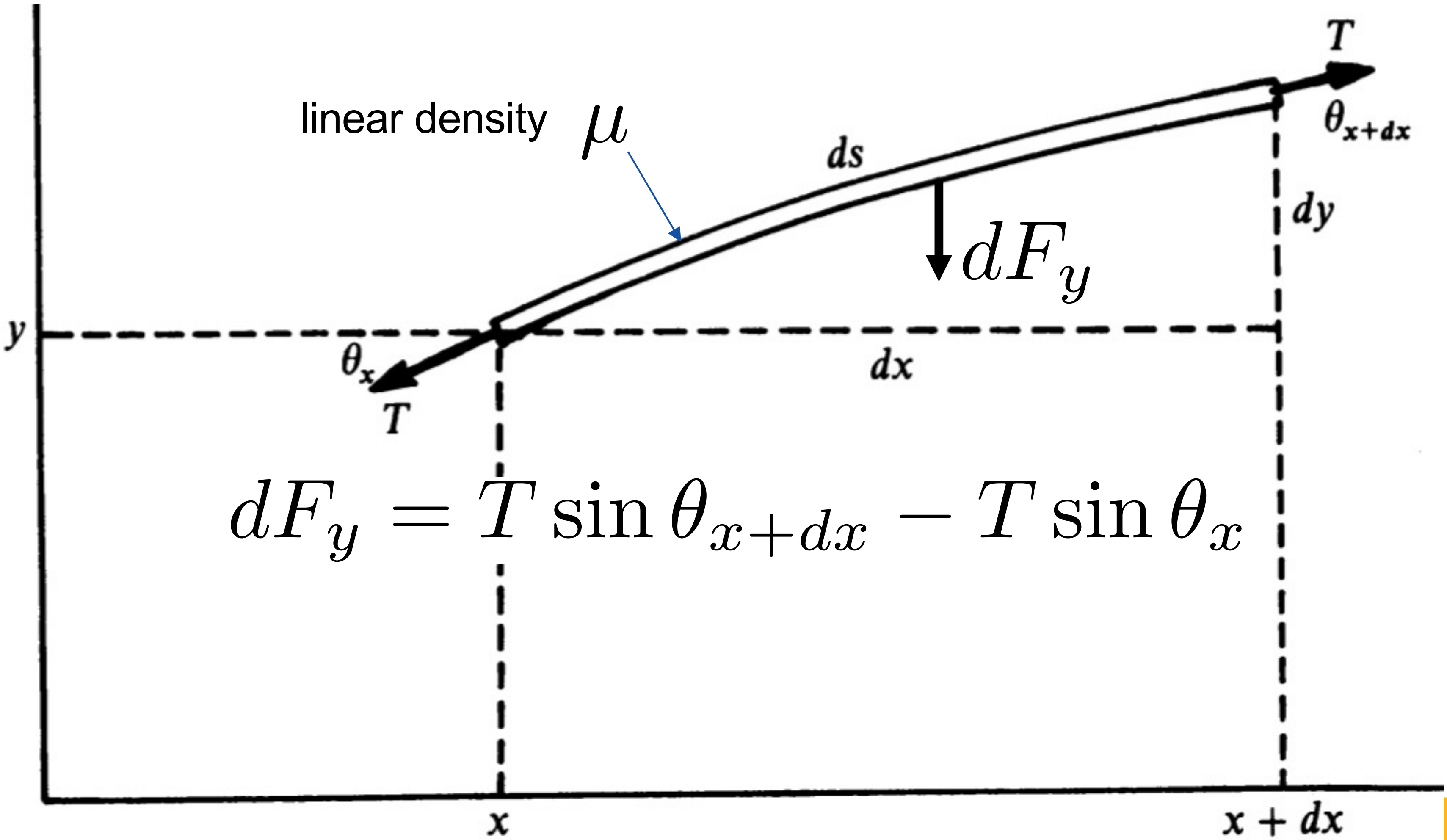


Ideal string from a physical perspective



There is force in x direction too (see Fletcher and Rossing 2.14)

Ideal string from a physical perspective

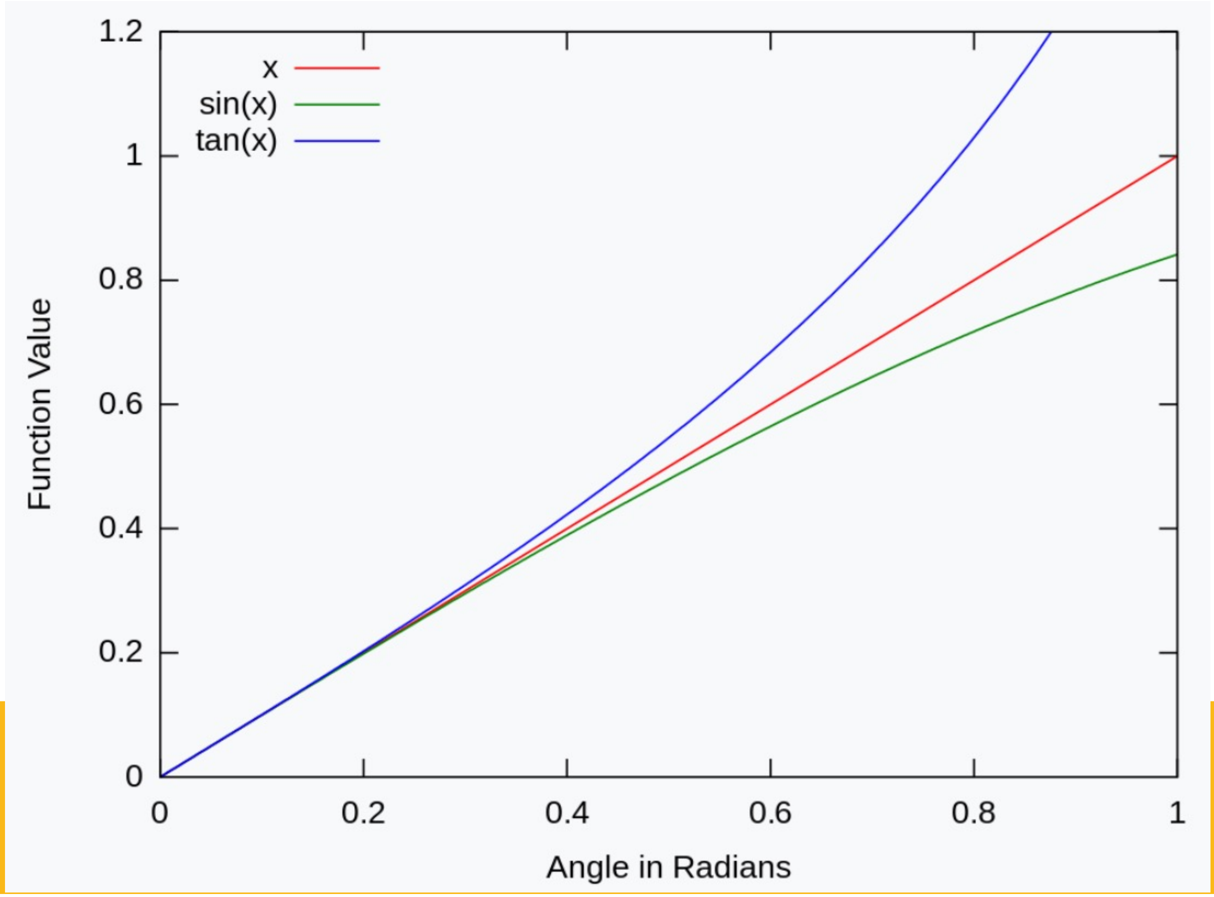


$$dF_y = T \sin \theta_{x+dx} - T \sin \theta_x$$

Ideal string from a physical perspective

$$dF_y = T\sin\theta_{x+dx} - T\sin\theta_x$$

If an angle is small, $\tan\theta = \sin\theta / \cos\theta \approx \sin\theta$



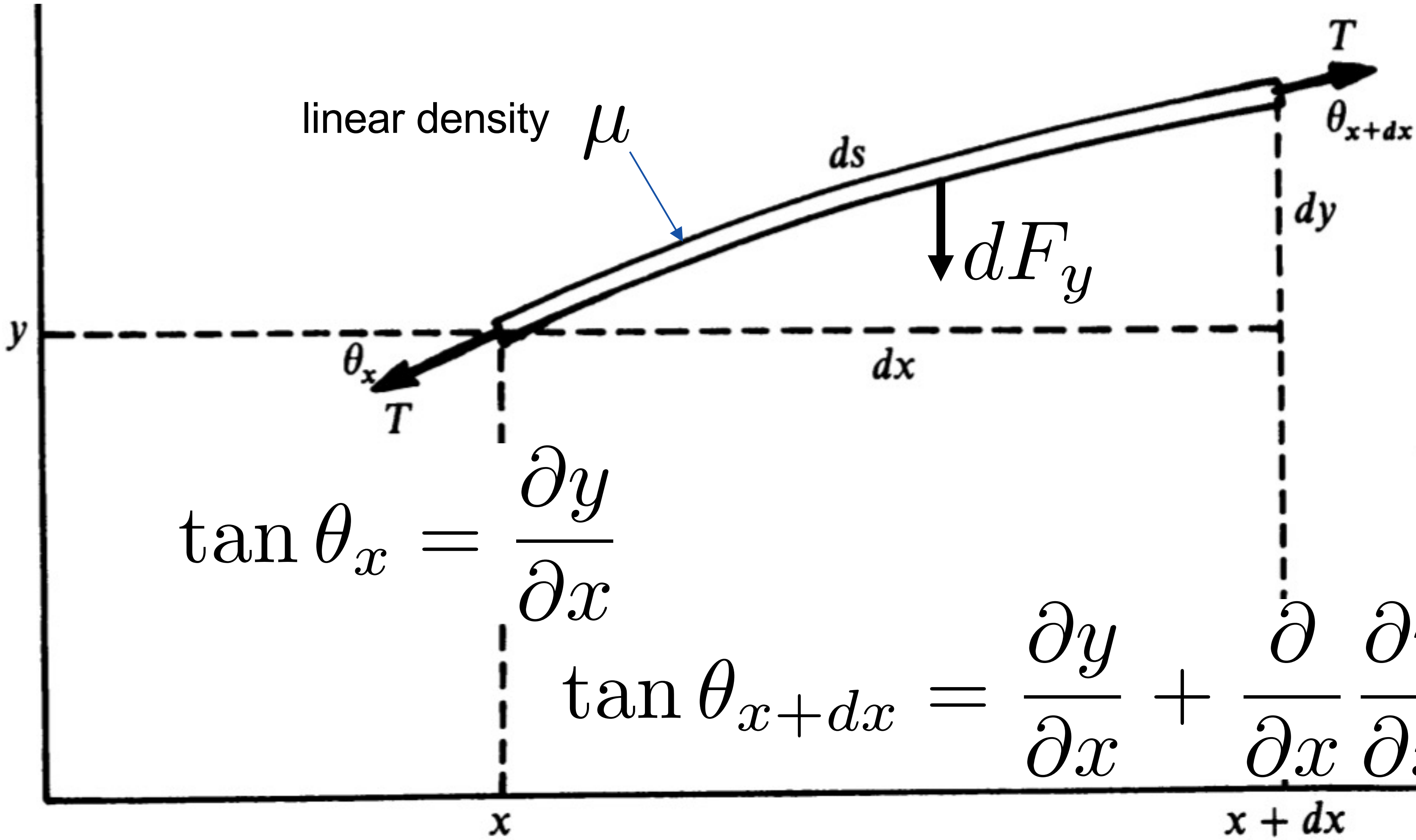
Ideal string from a physical perspective

$$dF_y = T \sin \theta_{x+dx} - T \sin \theta_x$$

If an angle is small, $\tan \theta = \sin \theta / \cos \theta \approx \sin \theta$

$$dF_y = T \tan \theta_{x+dx} - T \tan \theta_x$$

Ideal string from a physical perspective



$$\tan \theta_x = \frac{\partial y}{\partial x}$$

$$\tan \theta_{x+dx} = \frac{\partial y}{\partial x} + \frac{\partial}{\partial x}\frac{\partial y}{\partial x} dx$$

Ideal string from a physical perspective

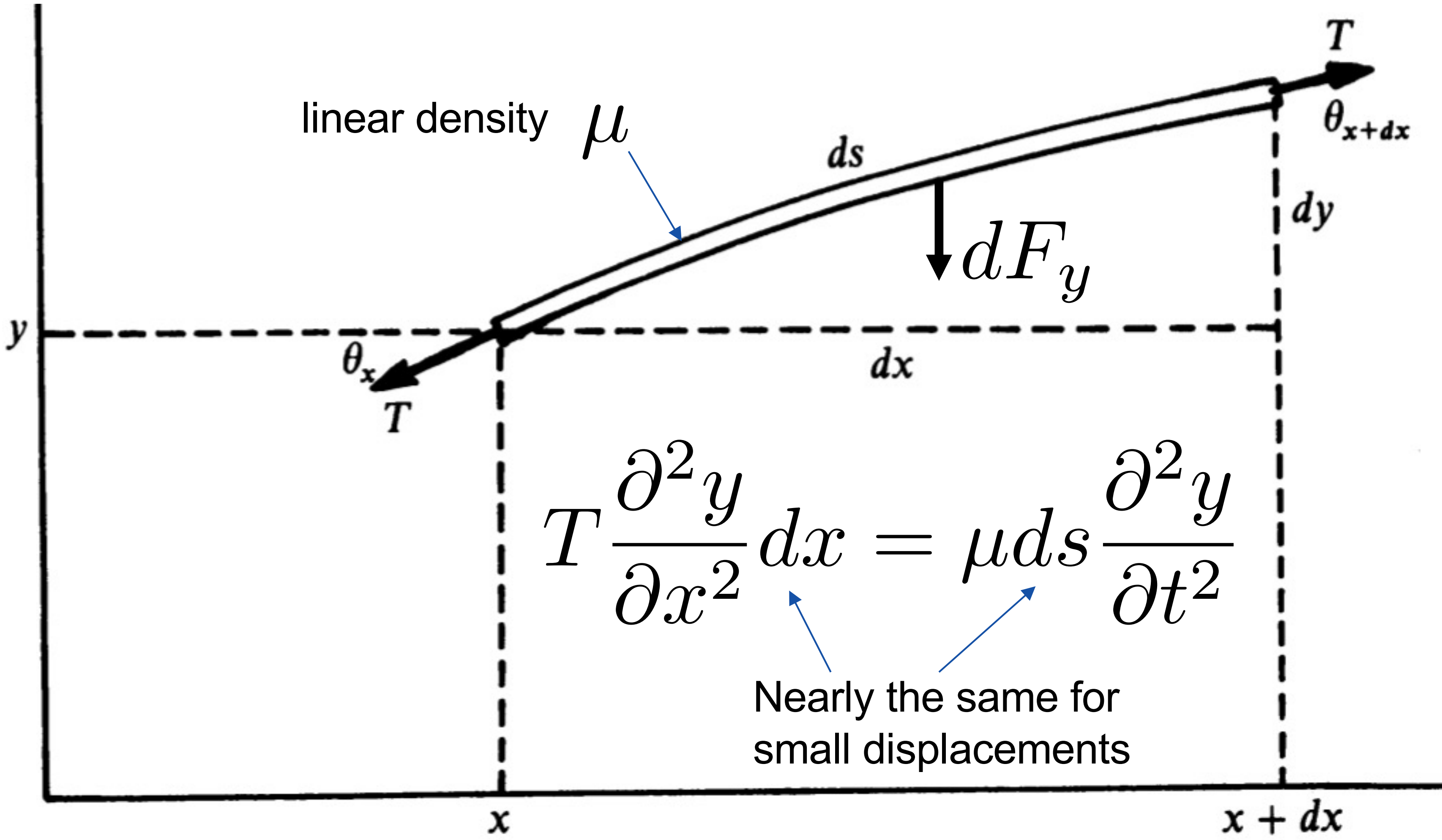
$$dF_y = T\sin\theta_{x+dx} - T\sin\theta_x$$

$$dF_y = T\tan\theta_{x+dx} - T\tan\theta_x$$

$$dF_y = T\left(\frac{\partial y}{\partial x} + \frac{\partial}{\partial x}\frac{\partial y}{\partial x}dx\right) - T\frac{\partial y}{\partial x}$$

$$dF_y = T\frac{\partial^2 y}{\partial x^2}dx$$

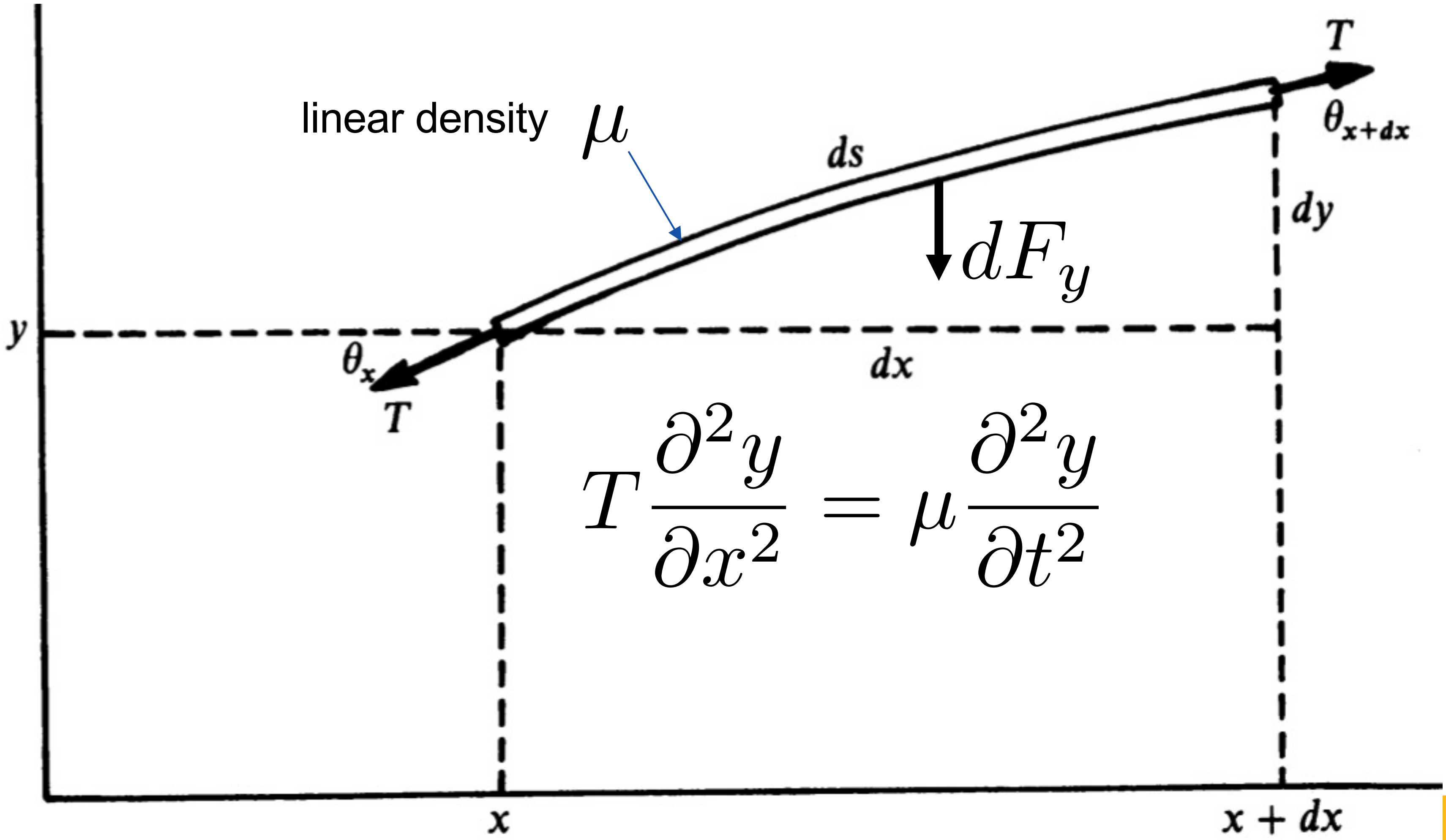
Ideal string from a physical perspective



$$T\frac{\partial^2 y}{\partial x^2}dx = \mu ds \frac{\partial^2 y}{\partial t^2}$$

Nearly the same for small displacements

Ideal string from a physical perspective



Wave equation for ideal string

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{\mu}{T} \frac{\partial^2 y(x,t)}{\partial t^2}$$

$$\frac{1}{c^2}$$

Transverse propagation velocity
(independent of frequency)

The “ideal” string assumes:

- Deformations are small
- Cross sectional area is constant
- Density is uniform
- Energy is not lost

Wave equation for ideal string: *Interpretation*

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

At any position and time on the string, the curvature (slope of the slope) of the ideal string is proportional to the transverse acceleration.

Now, how do we solve it?

Wave equation for ideal string: *Solution*

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

In essence, this equation says the curvature at any position and time looks like the acceleration at any position and time.

What about the below?

$$y(x,t) = A\cos(\omega t + kx)$$

Wave equation for ideal string: *Solution*

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

$$y(x,t) = A\cos(\omega t + kx)$$

Derivatives w.r.t. x

$$\frac{\partial}{\partial x} y(x,t) = \frac{\partial}{\partial x} A\cos(\omega t + kx) = Ak\sin(\omega t + kx)$$

$$\frac{\partial^2}{\partial x^2} y(x,t) = \frac{\partial}{\partial x} Ak\sin(\omega t + kx) = -Ak^2\cos(\omega t + kx)$$

Wave equation for ideal string: *Solution*

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

$$y(x,t) = A\cos(\omega t + kx)$$

Derivatives w.r.t. t

$$\frac{\partial}{\partial t} y(x,t) = \frac{\partial}{\partial t} A\cos(\omega t + kx) = A\omega\sin(\omega t + kx)$$

$$\frac{\partial^2}{\partial t^2} y(x,t) = \frac{\partial}{\partial x} A\omega\sin(\omega t + kx) = -A\omega^2\cos(\omega t + kx)$$

Wave equation for ideal string: *Solution*

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

$$y(x,t) = A\cos(\omega t + kx)$$

$$-Ak^2\cos(\omega t + kx) = -\frac{1}{c^2}A\omega^2\cos(\omega t + kx) =$$

$$k^2 = \frac{1}{c^2}\omega^2 \rightarrow k = \pm\frac{1}{c}\omega$$ ”wave number”

Wave equation for ideal string: *Solution*

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

Substitute:

$$k = \pm\frac{1}{c}\omega \qquad y(x,t) = A\cos(\omega t + kx)$$

$$y(x,t) = A\cos(\omega t + \omega x/c) + A\cos(\omega t - \omega x/c)$$

Note too:

$$y(x,t) = A\sin(\omega t + \omega x/c) + A\sin(\omega t - \omega x/c)$$

Wave equation for ideal string: *Solution*

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

One more possible solution:

$$\begin{aligned} y(x,t) = {} & A\cos(\omega t + \omega x/c) + B\cos(\omega t - \omega x/c) \\ & + C\sin(\omega t + \omega x/c) + D\sin(\omega t - \omega x/c) \end{aligned}$$

Wave equation for ideal string: *Solution*

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

One more possible solution:

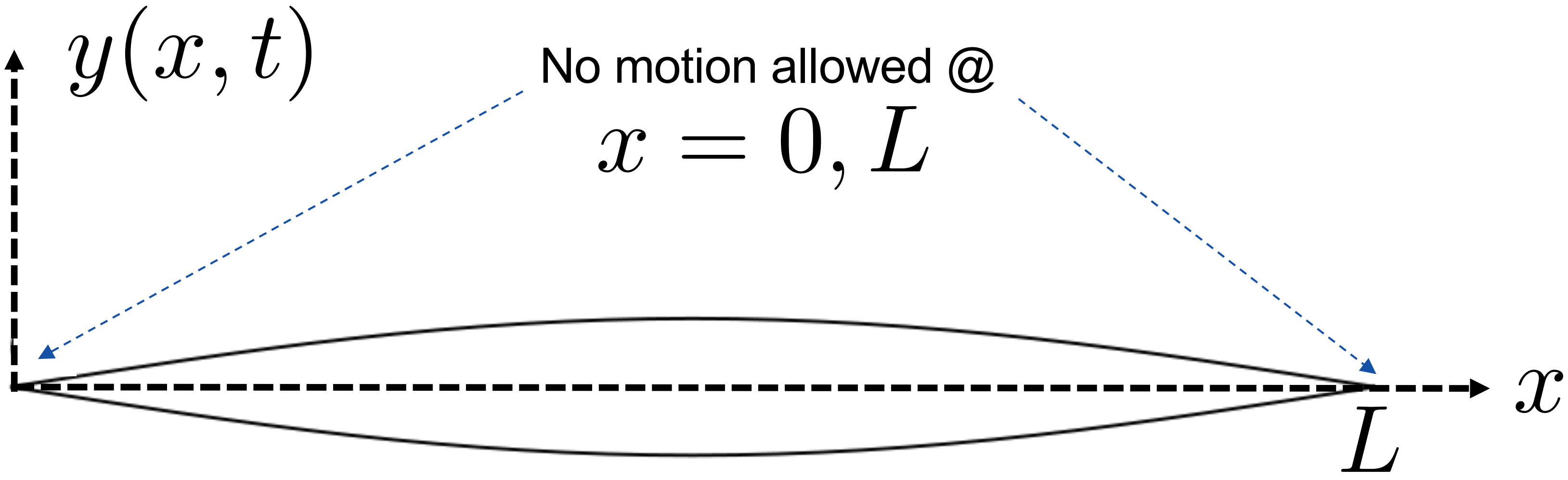
$$\begin{aligned} y(x,t) = A\cos(\omega t + \omega x/c) + B\cos(\omega t - \omega x/c) \\ + C\sin(\omega t + \omega x/c) + D\sin(\omega t - \omega x/c) \end{aligned}$$

What are: $A, B, C, D?$

Wave equation for ideal string: *Solution*

$$y(x,t) = A\cos(\omega t + \omega x/c) + B\cos(\omega t - \omega x/c)$$
$$+ C\sin(\omega t + \omega x/c) + D\sin(\omega t - \omega x/c)$$

What are $A, B, C, D?$



Wave equation for ideal string: *Solution*

$$y(x,t) = A\cos(\omega t + \omega x/c) + B\cos(\omega t - \omega x/c) \\ + C\sin(\omega t + \omega x/c) + D\sin(\omega t - \omega x/c)$$

$$y(0,t) = A\cos(\omega t) + B\cos(\omega t) \\ + C\sin(\omega t) + D\sin(\omega t) = 0$$

$$A = -B$$

$$C = -D$$

Wave equation for ideal string: *Solution*

$$y(x,t) = A[\cos(\omega t + \omega x/c) - \cos(\omega t - \omega x/c)]$$
$$+ C[\sin(\omega t + \omega x/c) - \sin(\omega t - \omega x/c)]$$

Recall the following identities:

$$\cos(w \pm z) = \cos w \cos z \mp \sin w \sin z$$

$$\sin(w \pm z) = \sin w \cos z \pm \cos w \sin z$$

Wave equation for ideal string: *Solution*

$$y(x,t) = A[\cos(\omega t + \omega x/c) - \cos(\omega t - \omega x/c)]$$
$$+ C[\sin(\omega t + \omega x/c) - \sin(\omega t - \omega x/c)]$$

$$\cos(\omega t \pm \omega x/c) = \cos(\omega t)\cos(\omega x/c) \mp \sin(\omega t)\sin(\omega x/c)$$

$$\sin(\omega t \pm \omega x/c) = \sin(\omega t)\cos(\omega x/c) \pm \cos(\omega t)\sin(\omega x/c)$$

$$\cos(\omega t + \omega x/c) - \cos(\omega t - \omega x/c) =$$
$$\cos(\omega t)\cos(\omega x/c) - \sin(\omega t)\sin(\omega x/c)$$
$$- [\cos(\omega t)\cos(\omega x/c) + \sin(\omega t)\sin(\omega x/c)]$$
$$= -2\sin(\omega t)\sin(\omega x/c)$$

Wave equation for ideal string: *Solution*

$$y(x,t) = A[\cos(\omega t + \omega x/c) - \cos(\omega t - \omega x/c)]$$
$$+ C[\sin(\omega t + \omega x/c) - \sin(\omega t - \omega x/c)]$$

$$\cos(\omega t \pm \omega x/c) = \cos(\omega t)\cos(\omega x/c) \mp \sin(\omega t)\sin(\omega x/c)$$

$$\sin(\omega t \pm \omega x/c) = \sin(\omega t)\cos(\omega x/c) \pm \cos(\omega t)\sin(\omega x/c)$$

$$\sin(\omega t + \omega x/c) - \sin(\omega t - \omega x/c) =$$
$$\sin(\omega t)\cos(\omega x/c) + \cos(\omega t)\sin(\omega x/c)$$
$$-\left[\sin(\omega t)\cos(\omega x/c) - \cos(\omega t)\sin(\omega x/c)\right]$$
$$= 2\cos(\omega t)\sin(\omega x/c)$$

Wave equation for ideal string: *Solution*

$$
\begin{aligned}
y(x,t) = {} & A[\cos(\omega t + \omega x/c) - \cos(\omega t - \omega x/c)] \\
& + C[\sin(\omega t + \omega x/c) - \sin(\omega t - \omega x/c)]
\end{aligned}
$$

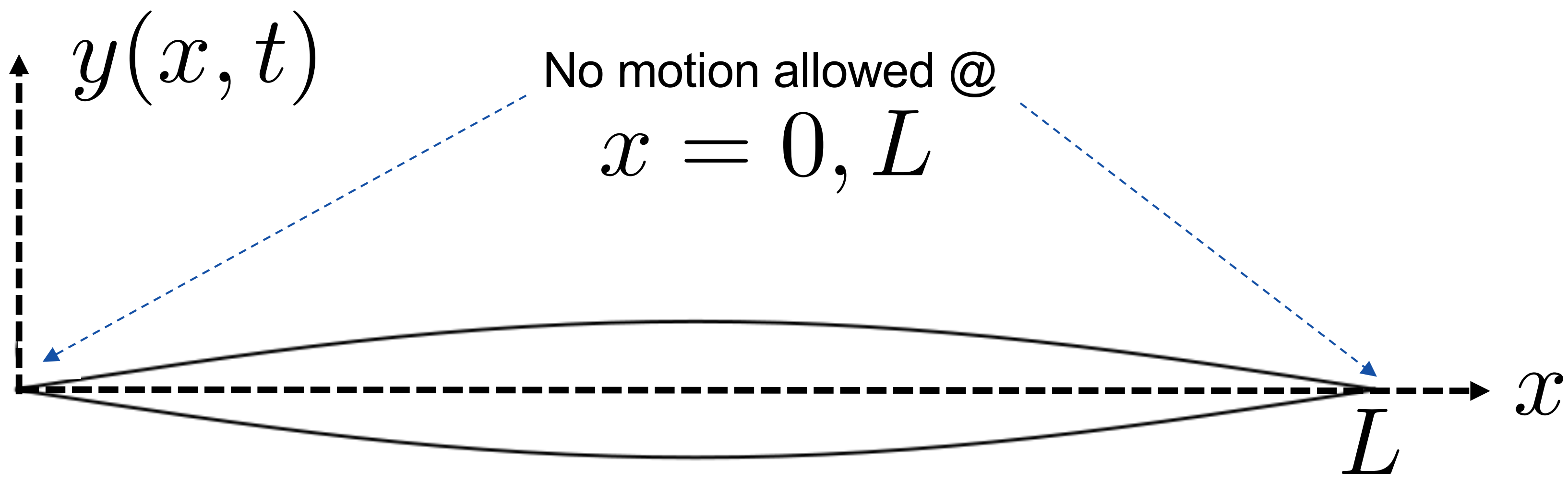
$$
\begin{aligned}
&= -2A\sin(\omega t)\sin(\omega x/c) \\
&\quad + 2C\cos(\omega t)\sin(\omega x/c) \\
&= 2[C\cos(\omega t) - A\sin(\omega t)]\sin(\omega x/c)
\end{aligned}
$$

Only depends on time!

Only depends on position!

Wave equation for ideal string: *Solution*

$$y(x,t) = 2[C\cos(\omega t) - A\sin(\omega t)]\sin(\omega x/c)$$



Wave equation for ideal string: *Solution*

$$y(x,t) = 2[C\cos(\omega t) - A\sin(\omega t)]\sin(\omega x/c)$$

$$\sin(\omega L/c) = 0 \rightarrow \omega L/c = n\pi$$

Define $\omega_n = n\pi c/L$ $\quad f_n = \dfrac{\omega_n}{2\pi} = \dfrac{nc}{2L}$ *Natural harmonics!*

$$y(x,t) = 2[C\cos(\omega_n t) - A\sin(\omega_n t)]\sin(n\pi x/L)$$

Shape of "standing wave" (natural mode)

A, C determined from initial conditions

Wave equation for ideal string: *Solution*

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

Bernoulli's solution

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}ct + \phi_n\right)$$

Wave equation for ideal string: *Solution*

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

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$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}ct + \phi_n\right)$$

$$\omega_1 = \frac{\pi c}{L} = \frac{\pi}{L}\sqrt{T/\mu}$$ Fundamental (angular) frequency

$$f_1 = \omega_1/2\pi = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

Wave equation for ideal string: *Solution*

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

A general solution (d'Alembert):

$$y(x,t) = f_1(x - ct) + f_2(x + ct)$$

Wave equation for ideal string: *Solution*

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

A general solution (d'Alembert):

$$y(x,t) = f_1(x-ct) + f_2(x+ct)$$

$$\frac{\partial^2}{\partial x^2} f_1(x-ct) = f_1''(x-ct)$$

$$\frac{\partial^2}{\partial t^2} f_1(x-ct) = c^2 f_1''(x-ct)$$

Wave equation for ideal string: *Solution*

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

A general solution (d'Alembert):

$$y(x,t) = f_1(x - ct) + f_2(x + ct)$$

Waveform travelling to the right

Waveform travelling to the left

Wave equation for ideal string: *Solution*

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

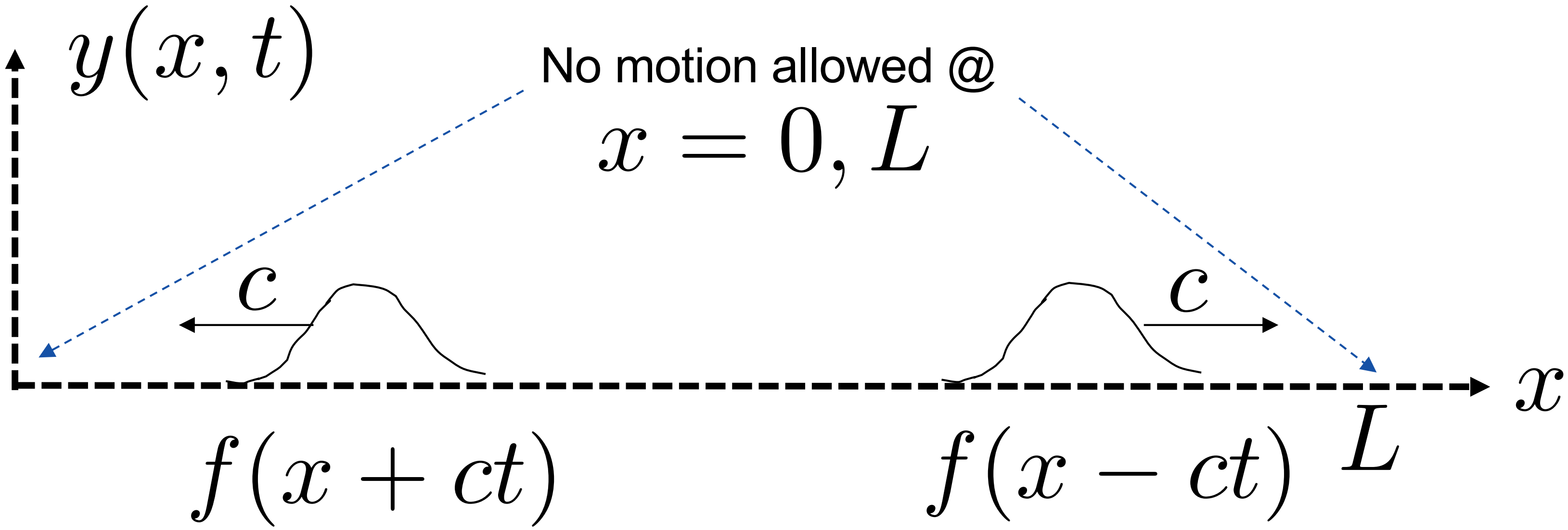
With the boundary conditions: $y(0,t) = y(L,t) = 0$

$$y(x,t) = f(x-ct) + f(x+ct)$$

Travelling waveforms have the same shape but move in opposite directions.

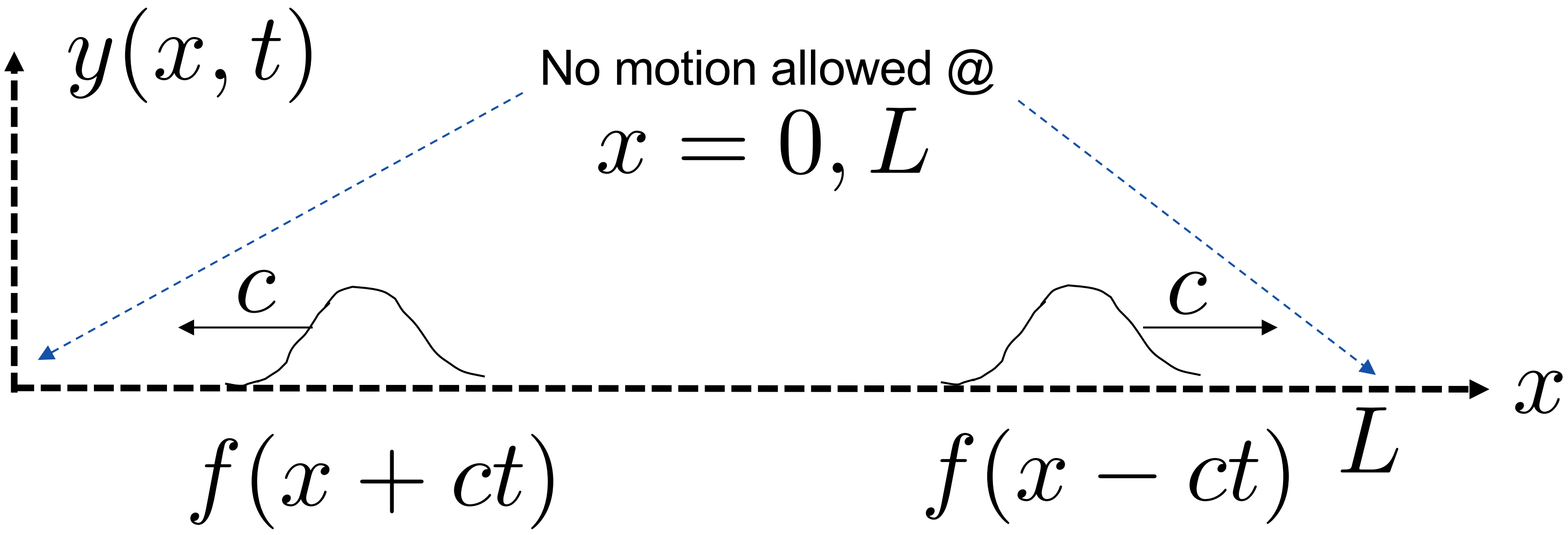
Wave equation for ideal string: *Solution*

$$y(x,t) = f(x-ct) + f(x+ct)$$



What happens at the ends?

Wave equation for ideal string: *Solution*

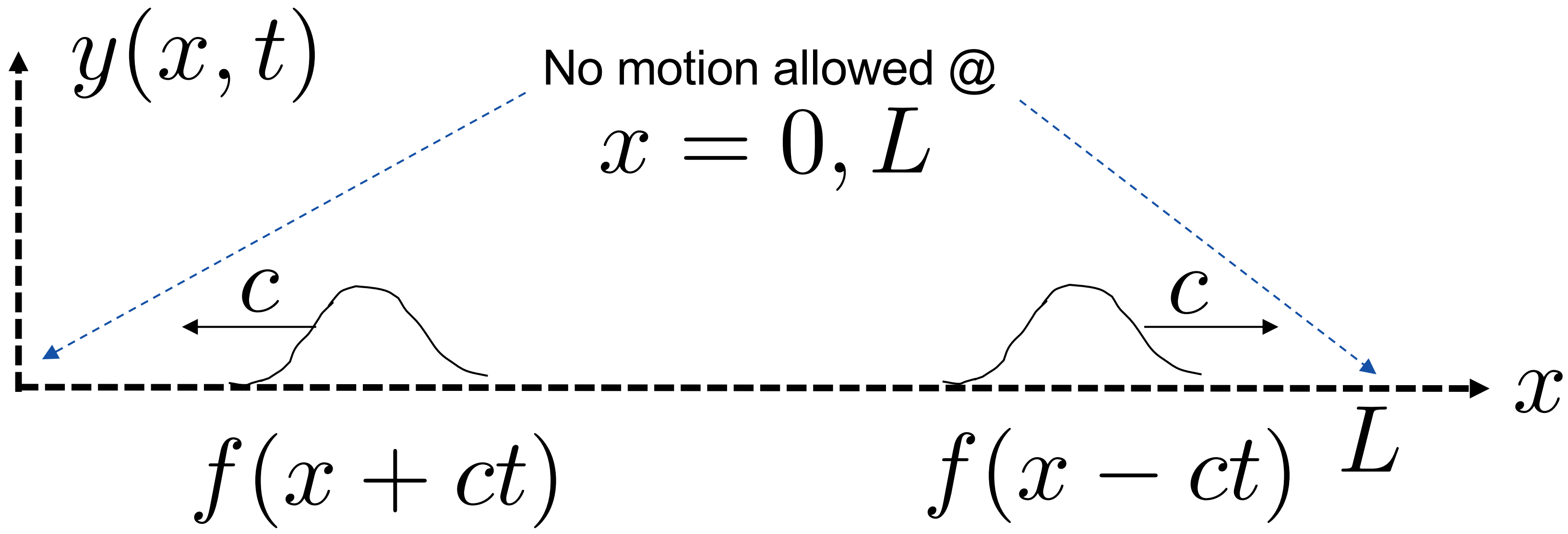


No motion allowed means:

$$f(x-ct)+f(x+ct)=0$$

@ $x = 0, L$

Wave equation for ideal string: *Solution*

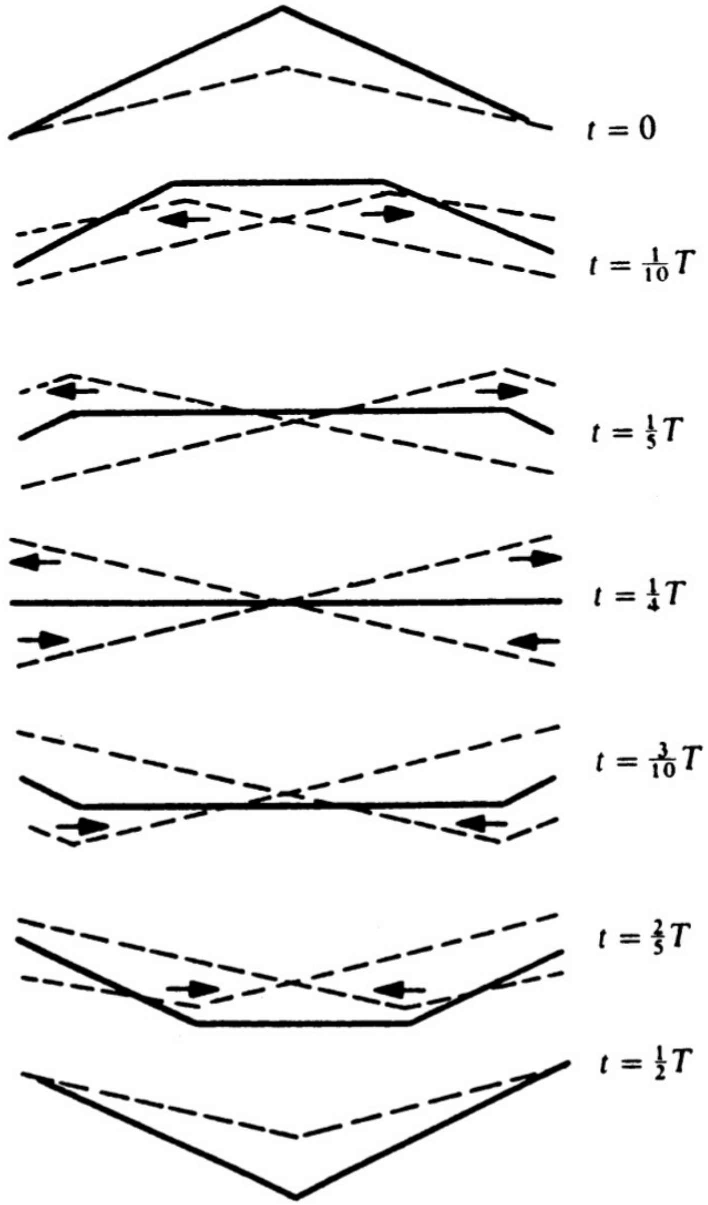


No motion allowed means: $f(-ct) + f(ct) = 0$

$$\rightarrow f(-ct) = -f(ct)$$

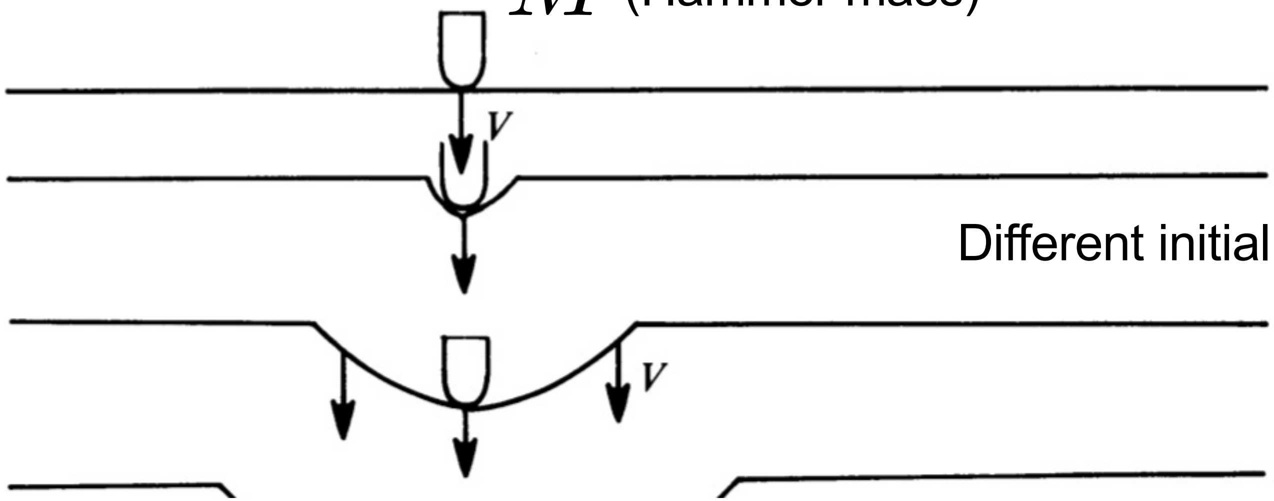
At $x = 0$ the left-travelling wave meets a right-travelling wave with a flipped polarity: reflection + flip

Plucked Ideal String



Struck Ideal String

M (Hammer mass)



Different initial conditions!

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}ct + \phi_n\right)$$

The coefficients will be different
between the plucked and struck string.

Strings in the real world

1. **Not perfectly flexible (resistance to bending)**

$$T\frac{\partial^2 y(x,t)}{\partial x^2} - ESK^2\frac{\partial^4 y(x,t)}{\partial x^4} = \mu\frac{\partial^2 y(x,t)}{\partial t^2}$$

Tension

Young's modulus

Cross-sec. area

Radius of gyration

density

Strings in the real world

1. **Resistance to bending (not perfectly flexible)**
   - The frequency components (modes) of a transverse wave on such a string have velocities depending on frequency

$$f_n = v_{t,n}/\lambda_n$$

   - The overtones are stretched (their frequencies increase, equivalently, the transverse velocity increases)

Strings in the real world

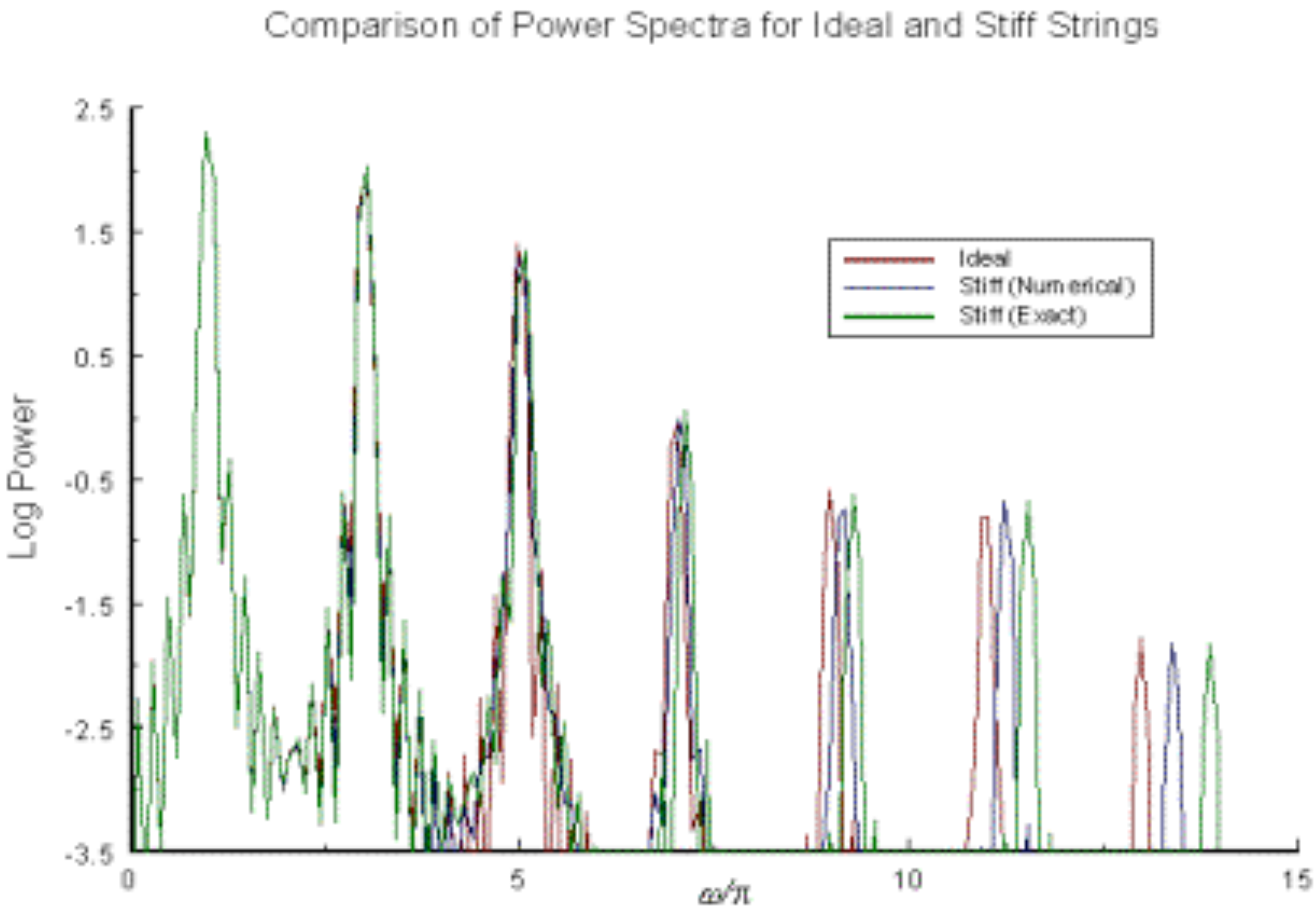
1. **Resistance to bending (not perfectly flexible)**
   - The frequency components (modes) of a transverse wave on such a string have velocities depending on frequency

$$f_n = v_{t,n}/\lambda_n$$

   - The overtones are stretched (their frequencies increase, equivalently, the transverse velocity increases)
   - Approximation:

$$f_n \approx n f_1 \sqrt{1 + n^2 \frac{\pi S^2 E}{4TL^2}}$$

Strings in the real world



Strings in the real world

1. **Resistance to bending (not perfectly flexible)**
   - The frequency components (modes) of a transverse wave on such a string have velocities depending on frequency

$$f_n = v_{t,n}/\lambda_n$$

   - The overtones are stretched (their frequencies increase)
   - Approximation:

$$f_n \approx n f_1 \sqrt{1 + n^2 \frac{\pi S^2 E}{4TL^2}}$$

   - What pitches does this inharmonicity affect the most?
   - How can we control this inharmonicity?

Strings in the real world

1. Resistance to bending (not perfectly flexible)

- The frequency components (modes) of a transverse wave on such a string have velocities depending on frequency

2. Not perfectly terminated

- String imparts a force on the termination.

$$F_{tr} = -T \frac{\partial y(x,t)}{\partial x}\bigg|_{x=0}$$

Assuming displacement is small

Strings in the real world

1. Resistance to bending (not perfectly flexible)

- The frequency components (modes) of a transverse wave on such a string have velocities depending on frequency

2. Not perfectly terminated

- Vibrating string is imparting a force on the termination.
- If the termination can undergo some displacement, the vibrating string will do work on it (work = force x distance)

Mechanical Impedance

- We need a way to describe how a string not perfectly terminated transmits its energy through the termination.

$$\underbrace{F(\omega)}_{\text{Force}} = \underbrace{Z(\omega)}_{\text{Impedance}}\underbrace{v(\omega)}_{\text{Velocity}}$$

- A transverse wave of some frequency moving on a string at some velocity applied to a termination on a system with some mechanical impedance results in a force!

Strings in the real world

1. Resistance to bending (not perfectly flexible)

- The frequency components (modes) of a transverse wave on such a string have velocities depending on frequency

2. Not perfectly terminated

- Vibrating string is imparting a force on the termination.
- If the termination can undergo some displacement, the vibrating string will do work on it (work = force x distance)
- Characteristic impedance of the ideal string.

$$Z_0 = \frac{T}{c} = 2L\mu f_1$$

- Due to stiffness, the impedance will not be constant over modal frequencies!

Strings in the real world

1. **Not perfectly flexible**
2. **Not perfectly terminated**
3. **Damping is occurring**

- Friction due to air, longitudinal waves, loss of energy at terminations

https://www.falstad.com/loadedstring

Before next lecture

1. Start lab 1 preparatory exercises
2. Read Chapter 25.1 in Hartmann
3. Assignment 1 due Feb 3