Honours Algebra: Eigenpictures

Junchen Hu, Lawrence He,
Loana Gericke & Sebastian Müksch

University of Edinburgh

April 16, 2019

What are Eigenpictures?

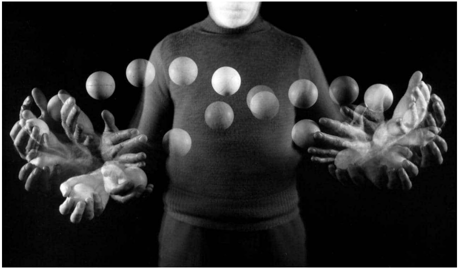


Figure 1: Stroboscopic Photography by Gjon Mili [4]

What are Eigenpictures?

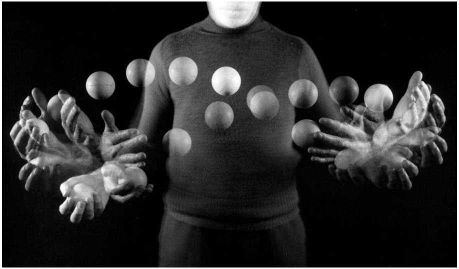


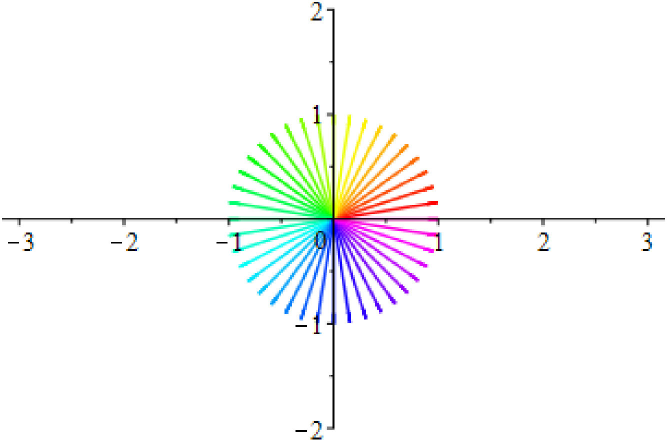
Figure 1: Stroboscopic Photography by Gjon Mili [4]

We will construct an Eigenpicture step-by-step for the matrix

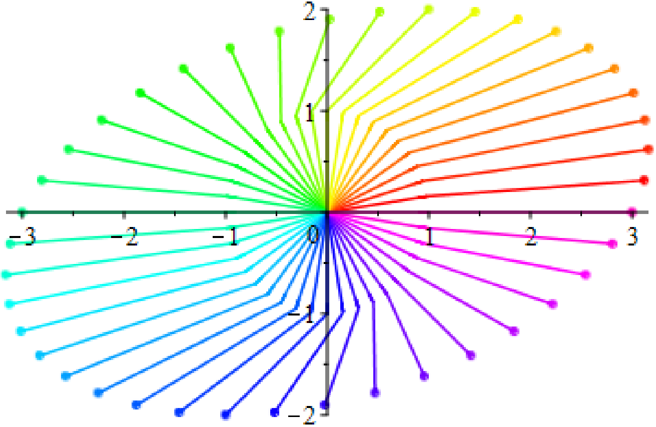
$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

Constructing an Eigenpicture

STEP 1: Draw unit vectors around the origin.

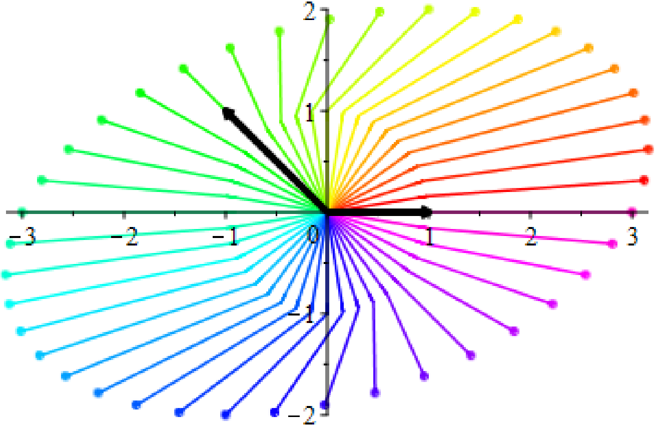


STEP 2: Draw each transformed unit vector from the head of the unit vector.



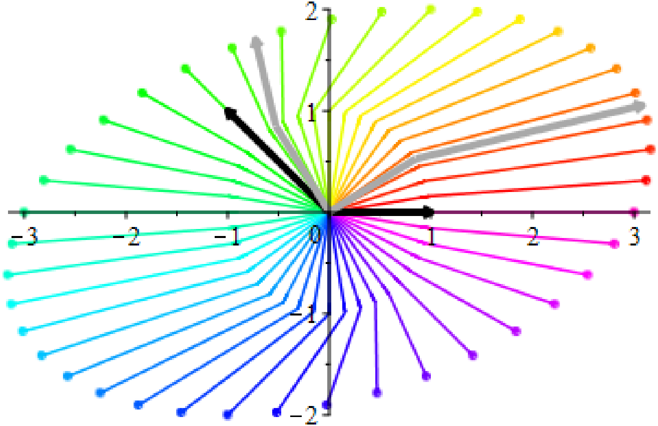
Constructing an Eigenpicture

STEP 3: Draw each eigenvector.

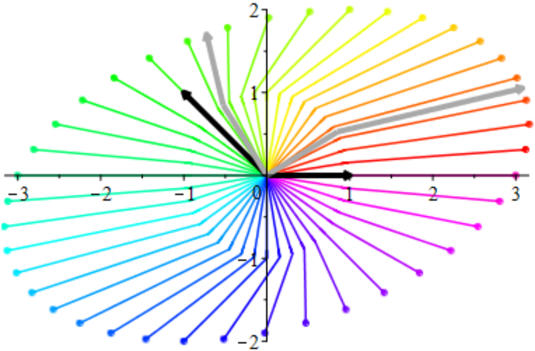


Constructing an Eigenpicture

EXTRA: Draw in vectors with most and least expansion by A.

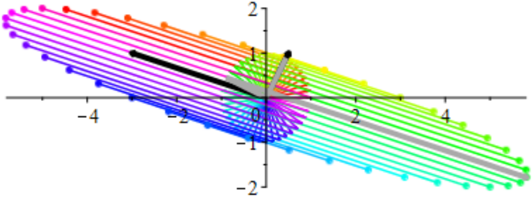


Finished Eigenpicture - Distinct Eigenvalues



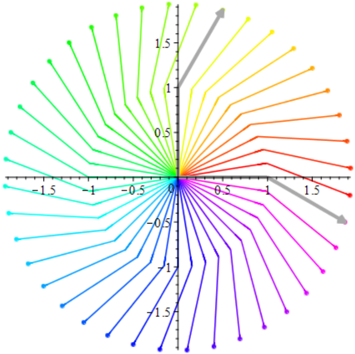
Matrix	Eigenvalues	Eigenvectors
$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$	$1, 2$	$\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

One Zero Eigenvalue



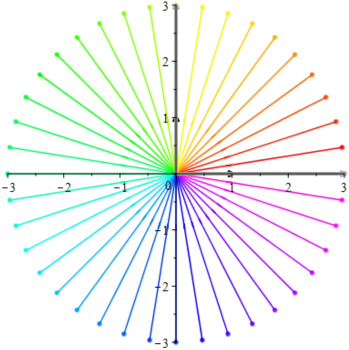
Matrix	Eigenvalues	Eigenvectors
$\begin{pmatrix} -6 & 3 \\ 2 & -1 \end{pmatrix}$	$-7, 0$	$\begin{pmatrix} -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Complex Eigenvalues and Eigenvectors



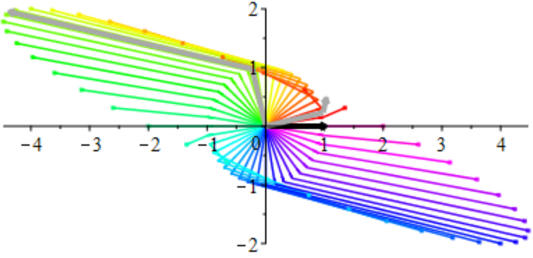
Matrix	Eigenvalues	Eigenvectors
$\begin{pmatrix} \cos(\frac{\pi}{6}) & \sin(\frac{\pi}{6}) \\ -\sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \end{pmatrix}$	$\frac{1}{2}(\sqrt{3} \pm i)$	$\begin{pmatrix} \pm i \\ 1 \end{pmatrix}$

Repeated Eigenvalue - Independent Eigenvectors



Matrix	**Eigenvalue**	**Eigenvectors**
$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	2	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Repeated Eigenvalue - Dependent Eigenvectors



Matrix	Eigenvalue	Eigenvector
$\begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$	1	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Open Questions

How did we find vectors of most and least expansion?

Will there always be an ellipse?

Singular Value Decomposition

By **Singular Value Decomposition (SVD)** (derived from both [2, section 7.4] and [3]), A can be rewritten as:

$$A = U\Sigma V^T$$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

- The columns of U are eigenvectors of AA^T.
- The rows of V^T are eigenvectors of $A^T A$.
- The diagonal of $\Sigma^T\Sigma$ stores the corresponding eigenvalues.

Most and Least Expansion

Proposition

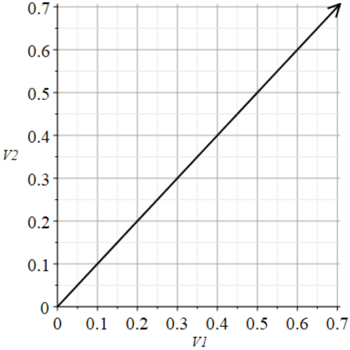
Suppose $A \in \mathrm{Mat}(2; \mathbb{R})$, let $\vec{v_1}$ and $\vec{v_2}$ be the normalised eigenvectors of $A^T A$, then they are the unit vectors subject to the **most and least expansion** by A.

Most and Least Expansion

Proposition

Suppose $A \in \mathrm{Mat}(2;\mathbb{R})$, let $\vec{v_1}$ and $\vec{v_2}$ be the normalised eigenvectors of $A^T A$, then they are the unit vectors subject to the **most and least expansion** by A.

- $\{\vec{v_1}, \vec{v_2}\}$ an orthogonal basis.
- $\vec{x} = \lambda_1 \vec{v_1} + \lambda_2 \vec{v_2}$.
- $\lambda_1^2 + \lambda_2^2 = 1$.



Most and Least Expansion

Proof.

With **standard inner product** we have:

$$
\begin{aligned}
||A\vec{x}||^2 &= (A\vec{x})^T(A\vec{x}) \\
&= \vec{x}^T(A^TA)\vec{x} \\
&= (\lambda_1\vec{v}_1^T + \lambda_2\vec{v}_2^T)(A^TA)(\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2) \\
&= (\lambda_1\vec{v}_1^T + \lambda_2\vec{v}_2^T)(\lambda_1(A^TA)\vec{v}_1 + \lambda_2(A^TA)\vec{v}_2) \\
&= (\lambda_1\vec{v}_1^T + \lambda_2\vec{v}_2^T)(\lambda_1\sigma_1^2\vec{v}_1 + \lambda_2\sigma_2^2\vec{v}_2) \\
&= \lambda_1^2\sigma_1^2 + \lambda_2^2\sigma_2^2
\end{aligned}
$$

Most and Least Expansion

Proof Continued.

WLOG, suppose $\sigma_1^2 \geq \sigma_2^2$:

$$||A\vec{x}||^2 = \lambda_1^2\sigma_1^2 + \lambda_2^2\sigma_2^2 = (1-\lambda_2^2)\sigma_1^2 + \lambda_2^2\sigma_2^2 = \sigma_1^2 + \lambda_2^2(\sigma_2^2 - \sigma_1^2) \leq \sigma_1^2$$

- **Case 1**: $\sigma_1^2 = \sigma_2^2$, then every unit vector is **expanded equally**;
- **Case 2**: $\sigma_1^2 \neq \sigma_2^2$, then $\lambda_2 = 0$. Hence, $||A\vec{x}|| = |\sigma_1|$ and $\vec{x} = \vec{v}_1$.

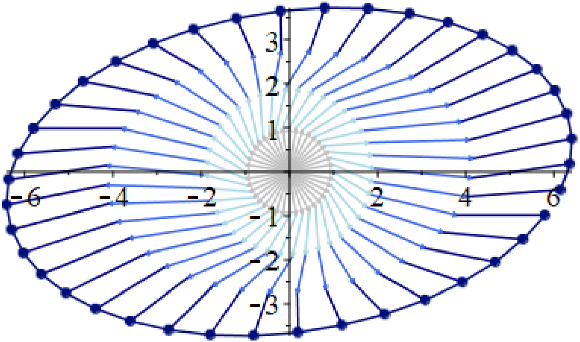
Most and Least Expansion

Recall the Raleigh Quotient $R : V \backslash \{\vec{0}\} \to \mathbb{R}$:

$$\vec{v} \to \mathrm{R}(\vec{v}) = \frac{((A^T A)\vec{v}, \vec{v})}{(\vec{v}, \vec{v})}$$

It retains an extreme value at the eigenvector of $A^T A$ (subject to $||\vec{v}|| = 1$).

Always an Ellipse?

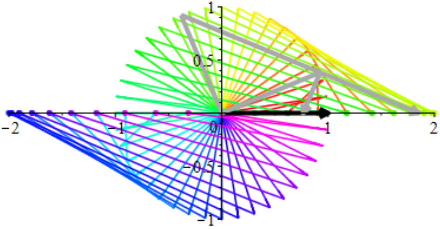


$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

Always an Ellipse?

NO!

Always an Ellipse?



Matrix	Eigenvalues	Eigenvectors
$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$	-1	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

References

[1] David Poole (2014)
'Linear Algebra: A Modern Introduction'.
US: Cengage Learning Publishers.

[2] MIT OpenCourseWare (2016, May 6)
'Singular Value Decomposition (the SVD)' [online].
Available at: https://www.youtube.com/watch?v=mBcLRGuAFUk.

[3] Steven Schonefeld (1995, September)
Eigenpictures: Picturing the Eigenvector Problem
The College Mathematics Journal , Vol. 26, No. 4

[4] Stroboscopic Photography by Gjon Mili
Available at: http://albania.al/article/42/gjon_mili_/

[5] The University of Edinburgh (2019, March 20)
'Honours Algebra Skills Project 1: Eigenpictures' [online].
Availatble at: Learn, Myed.