Question 3.

a) show that a good estimate for the condition number of A^TA is $k(A^TA) \simeq k(A)^2$.

Answer.

$$M = \|A\| = \max \frac{\|Ax\|}{\|x\|}$$

$$m = \min \frac{\|Ax\|}{\|x\|} = \|A^{-1}\|$$

$$k(A) = \|A\| \cdot \|A^{-1}\|$$

$$k(A^TA) = \|A^TA\| \, \|(A^TA)^{-1}\|.$$

$$= \|A\|^2 \cdot \|A^{-1}\|^2 = k(A)^2.$$

b) Given two orthonormal matrices U and Q, show that product of these matrices, UQ, is also orthonormal.

$$Q^T = Q.$$

$$U^T = U^{-1}$$

$$(UQ)^T = Q^TU^T$$

$$= Q^{-1}U^{-1}$$

$$= (UQ)^{-1}$$

UQ is orthonormal.

c) if A is an invertible matrix and Q is an orthonormal matrix, show that $k(QA) = k(A)$.

$$k(QA) = \|(QA)^{-1}\| \, \|(QA)\|$$

$$= \|(QA)^{-1}(QA)\|$$

$$= \|A^{-1}Q^{-1}QA\|$$

$$= \|A^{-1}A\|$$

$$k(A) = \|A^{-1}\| \, \|A\|$$

$$= \|A^{-1}A\|$$

$$k(QA) = k(A).$$