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CLASSICAL MECHANICS I PROJECT WORK

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Classical Mechanics Project Newtonian Mechanics

Newton constructed a mathematical framework which is powerful enough to explain a broad range of phenomena, the framework needs just a single input: a force. With this in place, it os merely of turning a mathematical handle to reveal what happened next.

Lets start the course by exploring the framework of Newtonian mechanics, understanding the axioms and what they tell us about the way the universe works. We then move on to look at a number of forces that are at play in the world. Nature is kind and the list is surprisingly short. Moreover,Classical mechanics is an ambitious theory. Its purpose is to predict the future and reconstruct the past, to determine the history of every particle in the Universe.

We will cover the basics of classical mechanics as formulated by Galileo and Newton. Starting from a few simple axioms. many of forces that arise have special properties, from which we will see new concepts emerging such as energy and conservation principles. Finally, for each of these forces, we turn the mathematical handle. We turn this handle many many times. In doing so, we will see how classical mechanics is able to explain large swathes of what we see around us.

Despite its wild success, Newtonian mechanics is not the last word in theoretical physics. It struggles in extreme: the realm of the small, the very heavy or the very fast. We finish these lectures with an introduction to special relativity, the theory which replaces Newtonian mechanics when the speed of the particles is comparable to the speed of light. We will see how our common sense ideas of space and time are replaced by something more intricate and more beautiful, with surprising consequences. Time goes slow for those on the move; lengths get smaller; mass is merely another form of energy.

Ultimately, the framework of classical mechanics falls short of its ambitious goal to tell the story of every particle in the universe. Yet its provides the basis for all that follows. Some of the Newtonian ideas do not survive to later, more sophisticated, theories of physics. Even the seemingly primary idea of force will fall by the wayside. Instead other concepts that we will meet along the way, most notably energy, step to the force. But all subsequent theories are built on the Newtonian foundation.

Moreover, developments in the past 300 years have confirmed what is perhaps the most important legacy of Newton: the laws of Nature are written in the language of mathematics. This is one of the great insights of human civilization. It has ushered in scientific, industrial and technological revolutions. It has given us a new way to look at the universe. And, most crucially of all, it means that the power to predict the future lies in hands of mathematicians rather than, say, gypsy astrologer. IN this course, we take the first steps towards grasping this power. **Newton's Motion**

Classical mechanics is all about the motion of particles. We start with a definition.

Definition: A particle is an object of insignificant size. This means that if you want to say what a particle looks like at a given time, the only information you have to specify is its position.

During this course, we will treat electrons, tennis balls, falling cats and planets as particles. In all of these cases, this means that we only care about the position of the object and our analysis will not, for example, be able to say anything about the look on the cat's face as it fall. However, it's not immediately obvious that we can meaningfully assigned a single position to a complicated object such as spinning, mewing cat.

To describe the position of a particle, we need a reference frame. This is a choice of origin, together with a set of axes which, for now, we pick to be Cartesian. With respect to this frame, the position of a particle is specified by a vector **x**, which we denote on time, resulting in a trajectory of the particle described by

In these notes we will also use both the notation $x(t)$ and $r(t)$ to describe the trajectory of a particle.

The velocity of a particle is defined to be

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Throughout these notes, we will often denote differentiation with respect to time by a "dot" above the variable. So we will also write v=x. The acceleration of the particle is defined to be

A Comment on Vector Differentiation

The derivative of a vector is defined by differentiating each of the components. So, if $x = (x_1,x_2,x_3)$ then

Geometrically, the derivative of a path x(t) lies tangent to the path (a fact which you will see in the Vector Calculus course).

In this course, we will be working with vector differential equations. These should be viewed as three, coupled differential equations-one for each component. We will frequently come across situations where we need to differentiate vector dot-products and cross-products. The meaning of these is easy to see if we use the chain rule on each component. For example, Given two vector functions of time, f(t) and g(t), we have

and

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As usual, it doesn't matter what order we write the terms in the dot product, but we have to be more careful with the cross product because, for example, df/dt g $= -g$ df/dt

1.1.1 Newton's laws Newtonian mechanics is a framework which allows us to determine the trajectory x(t) of a particle in any given situation. This framework is usually presented as three axioms known as Newton's laws of motion.They look something like:

Left alone, a particle moves with constant velocity. The acceleration(or,more precisely,the rate of change of momentum) of a particle is proportional to the force acting upon it. Every action has an equal and opposite reaction.

While it is worthy to try to construct axioms on which the laws of physics rest, the trite, minimalistic attempt above falls somewhat short. For example, on first glance, it appears that the first law is nothing more than a special case of the second law. (If the force vanishes, the acceleration vanishes which is the same thing as saying that the velocity is constant). But the truth is somewhat more subtle. In what follows we will take a closer look at what really underlies Newtonian mechanics.

1.2 Inertial Frames and Newton's First Law

Placed in the historical context, it is understandable that Newton wished to stress the first law. It is a rebuttal to the Aristotelian idea that, left alone, an object will naturally come to rest. Instead, as Galileo had previously realized, the natural state of an object is to travel with constant speed. This is the essence of the law of inertia. However, these days we 're not bound to any Aristotelian dogma. Do we really need the first law? The answer is yes, but it has a somewhat different meaning. We've already introduced the idea of a frame of reference: a Cartesian coordinate system in which you measure the position of the particle. But for most reference frames you can think of, Newton's first law is obviously incorrect. For example, suppose the coordinate system that I'm measuring from is rotating. Then, everything will appear to be spinning around me. If I measure a particle's trajectory in my coordinates as x(t), then I certainly won't find that $d^2/dt^2 = 0$, even if I leave the particle alone. In rotating frames, particles do not travel at constant velocity. We see that if we want Newton's first law to fly at all, we must be more careful about the kind of reference frames we're talking about. We define an inertial reference frame to be one in which particles do indeed travel at constant velocity when the force acting on it vanishes. In other words, in an inertial frame

$$0.92\ x = 0 when F = 0$$

The true content of Newton's first law can then be better stated as

N1 Revisited: Inertial frame exist

These inertial frames provide the sitting for all that follows. For example, the second law - which we shall discuss shortly - should be formulated in inertial frames.

One way to ensure that you are in an inertial frame is to insist that you are left alone: fly out into deep space, far from the effects of gravity and other influences, turn off your engine and sit there. This is an inertial frame. Of course, these axes are stationary with respect to the Earth and the Earth is rotating, both about its own axis and about the Sun. This means that the Earth does not quite provide an inertial frame and we will study the consequence of this later.

1.2.1 Galilean Relativity

Inertial frames are not unique. Given one inertial frame, S, in which a particle has coordinates $x^0(t)$ by any combination of the following transformations,

Translation: $x^0 = x + a$, for constant a.

Rotations: $x^0 = Rx$, for 3 3 matrix R obeying $R^T \ R = 1$. (This also allows for reflections if $detR = -1$), although our interest will primarily be on continuous transformations).

Boosts: $x^0 = x + vt$, for constant velocity v.

It is simple to prove that all of these transformations map one inertial frame to another. Suppose that a particle moves with constant velocity with respect to frame S, so that $d^2x/dt^2 = 0$. Then, for each of the transformations above, we also have $d^2x^0/dt^2 = 0$ which tells us that the particle also moves at constant velocity in S^0. Or, in other words, if S is an inertial frame then so too is S^0. The three transformations generate a group known as Galilean group.

The three transformations above are not quite the unique transformation that map between inertial frames. But, for most purposes, they are the only interesting ones! The others are transformations of the form $x^0 = \lambda x$ for some λR. This is just a trivial rescaling of the coordinates. For example, we may choose to measure distances in S in units of meters and distances in s^0 in units of parsecs.

We have already mentioned that Newton's second law is to be formulated in an inertial frame. But, importantly, it doesn't matter which inertial frame. In fact, this is true for all laws of physics: they are the same in any inertial frame. This is known as the *principle of relativity.* The three types of transformation laws that make up thr Galilean group map from one inertial frame to another. Combined with the principle of relativity, each is telling us something about the Universe.

Translations: There is no special point in the Universe.

Rotations: There is no special direction in the Universe.

Boosts: There is no special velocity in the Universe.

The first two fairly unsurprising: position is relative; direction is relative. The third perhaps needs more explanation. Firstly, it is telling us that there is no such thing as "absolutely stationary ". You can only be stationary with respect to something else. Although this is true (and continues to hold in subsequent laws of physics) it is not true that there is no special speed in the Universe. The speed of light is special. We will see how this changes the principle of relativity later.

So position, direction and velocity are relative. But acceleration is not. You do not have to accelerate relative to something else. It makes perfect sense to simply say that you are accelerating or you are not accelerating. In fact, this brings us back to Newton's first law: if you are not accelerating, you are sitting in an inertial frame.

Absolute Time

There is one last issue that we have left implicit in discussion above: the choice of time coordinate t. If observers in two inertial frames, S and S^0, fix the units - seconds, minutes, hours - in which to measure the duration time then the only remaining choice they can make is when to strat the clock. In other words, the time variable in S and S^0 differ only by

$$0.92\ t^0 = t + t_0$$

This is sometimes included among the transformations that make up the Galilean group. The existence of a uniform time, measured equally in all inertial reference frames, is referred to as *absolute time.* It is something that we will have to revisit when we discuss special relativity. As with the other Galilean transformations, the ability to fundamental laws don't care when you start the clock. All evidence suggests that the laws of physics are the same today as they were yesterday. They are time translationally invariant. **Cosmology**

Notably, the Universe itself breaks several of the Galilean transformations. There was a very special time in the universe, around 13.7 billion years ago. This is the time of the Big Bang (which, loosely translated, means "we don't know what happened here").

Similarly, there is one inertial frame in which the background Universe is stationary. The "background" here refers to the sea of photons at a temperature of 2.7 K which fills the universe, known as the Cosmic Microwave Background Radiation. This is the afterglow of the fireball that filled all of space when the Universe was much younger. Different inertial frames are moving relative to this background and measure the radiation differently: the radiation looks more blue in the direction that you're traveling, redder in the direction that you've come from. There is an inertial frame in which this background radiation is uniform, meaning that it is the same color in all directions.

To the best of our knowledge however, the Universe defines neither a special point, nor a special direction. It is, to very good approximation, homogeneous and isotropic.

However, it's worth stressing that this discussion of cosmology in no way invalidates the principle of relativity. All laws of physics are the same regardless of which inertial frame you are in. Overwhelming evidence suggests that the laws of physics are the same in far flung reaches of the Universe.. They were the same in the first few microseconds after the Big Bang as they are now.

Newton's Second law

the second law is the meat of the Newtonian framework. It is the famous "$F = ma$", which tells us how a particle's motion is affected when subjected to a force **F**. The correct form of the second law is

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this is usually referred to as equation of motion. Th quantity in the brackets is called the momentum,

$$0.92\ p \equiv mx$$

Here m is the mass of the particle or, more precisely, the *inertial mass*. It is a measure of the reluctance of the particle to change its motion when subjected to a given force **F**. In most situations, the mass of the particle does not change with time. In this case, we can write the second law in a more familiar form,

$$0.92\ mx = \mathbf{F}(x,x)$$

For much of this course, we will use the form above for the equation of motion. However, we will briefly look at a few cases where the masses are time dependent and we need more general form.

Newton's second law doesn't actually tell us anything until someone else tells us what the force **F** is in any given situation. We will describe several examples in the next section. In general, the force can depend on the position x, and the velocity x of the particle, but does not depend on any higher derivatives. We could also, in principle, consider forces which include an explicit time dependent,$\mathbf{F}(x,x,t\ )$, although we won't do so in these lectures. Finally, if more than one (independent) force is acting on the particle, then we simply take their sum on the right-hand side of $mx = \mathbf{F}(x,x)$

The single most important fact about Newton's equation is that it is a *second order* differential equation. This means that we will have a unique solution only if we specify two initial conditions. These are usually taken to be the position $x(t_0)$ and the velocity $x(t_0)$ at some initial time t_0. However, exactly what boundary conditions you must choose in order to figure out the trajectory depends on the problem you trying to solve. It is not unusual, for example, to have to specify the position at an initial time t_0 and final time t_f to determine the trajectory.

The fact that the equation of motion is second order is a deep statement about the Universe. It carries over, in essence, to all other laws of physics, from quantum mechanics to general relativity to particle physics. Indeed, the fact that all initial conditions must come in pairs - two for each "degree of freedom" in the problem - has important ramifications of both classical and quantum mechanics.

For now, the fact that the equations of motion are second order means the following: if you are given a snapshot of some situation and asked "what happens next?" then there is no way of knowing the answer. It's not enough just to know the positions of the particle at some point of time; you need to know their velocities too. However, once both of these are specified, the future evolution of the system is fully determined for all time.

Looking Forwards: The Validity Of Newtonian Mechanics

Although Newton's laws of motion provide an excellent approximation to many phenomena, when pushed to the extreme situation they are found wanting. Broadly speaking, there are three directions in which Netwonian physics needs replacing with a different framework: they are

When particles travel at speeds close to the speed of light, $c \equiv 3\ 0^{8} ms^{-}1$, the Newtonian concept of absolute time breaks down and Newton's laws need modification. The resulting theory is called special relativity and will be described later. As we will see, although the relationship between space and time is dramatically altered in special relativity, much of it framework of Newtonian mechanics survives unscathed.

On very small scales, much more radical change is needed. Here the whole framework of classical mechanics breaks down so the even the most basic concepts, such as trajectory of particle, becomes ill-defined. The new framework that holds on these small scales is called quantum mechanics. Nonetheless, there are quantities which carry over the classical world to the quantum, in particular energy and momentum.

When we try to describe the force at play between particles, we need to introduce a new concept: the field. This is a function of both space and time. Familiar examples are the electric and magnetic fields of electromagnetism. We won't have to much too say about fields in this course. For now, we mention only that the equations which govern the dynamics of fields are always second order differential equations, similar in spirit to newton's equations. Because of this similarity, field theories are again referred to as "classical".

Eventually, the ideas of special relativity, quantum mechanics and field theories are combined into *quantum field theory.* Here even the concept of particle gets subsumed into the concept of field. This is currently the best framework we have to describe the world around us. But we're getting ahead of ourselves. Let's firstly return to our Newtonian world......

Forces

In this section, we describe a number of different forces that arise in Newtonian mechanics. Throughout, we will restrict attention to the motion of a single particle. (We'll look at what happens when we have more than one particle in section 5). We start by describing the key idea of energy conservation, followed by a description of some common and important forces. **Potentials in One Dimension** let's start by considering a particle moving on a line, so its position is determined bby a single function $x(t)$. For now, suppose that the force on the particle depends only on the position, not the velocity: $F = F(x)$. We define the potential $V\ (x)$ (also called the *potential energy*) by the equation

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The potential is only defined up to an additive constant. We can always invert the equation above by integrating both sides. The integration constant is now determined by the choice of lower limit of the integral,

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Here x^0 is just a dummy variable. (Do not confuse the prime with differential! In this course we will only take derivatives of position x with respect to time and always denote them with a dot over the variable). With this definition, we can write the equation of motion as

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For any force in one-dimension which depends only on the position, there exists a conserved quantity called the energy,

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The fact that this is conserved means that $E = 0$ for any trajectory of the particle which obeys the equation of motion. While $V\ (x)$ is called the potential energy, rId18.png is called the *kinetic energy*. Motion satisfying the equation of motion is called *conservative*. It is not hard to prove that E is conserved. We need only differentiate to get

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where the last equality holds courtesy of the equation of motion.

In any dynamical system, conserved quantities of this kind are very precious. We will spend some time in this course fishing them out of the equations and showing how they help us simplify various problems.

An Example: A Uniform Gravitational Field

In a uniform gravitational field, a particle is subjected to a constant force,

$F = -mg$ where $g \approx 9.8ms^-2$ is the acceleration due to gravity near the Earth. The minus sign arises because the force is downwards while we have chosen to measure position in an upwards direction which we call z. The potential energy is

$$0.92 \;\; V = mgz$$

Notice that we have chosen to have $V = 0$ at $z = 0$. There is nothing that forces us to do this; we could easily add an extra constant to the potential to shift the zero to some other height.

The equation of motion for uniform acceleration is

$$0.92 \;\; z = -g$$

Which can be trivially integrated to give the velocity at time t,

$$0.92 \;\; z = u \; - \; gt$$

where u is the intial velocity at time $t = 0$. (Note that z is measured in the upwards direction, so the particle is moving uo if $z > 0$ and down if $z < 0$). Integrating once more gives the position

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where z_0 is the initial height at time $t = 0$. Many high schools teach that the equations above - the so-called ”suvat” equations - are key equations of mechanics. They are not. They are merely the integration of Newton’s second law for constant acceleration. Do not learn them; learn how to derive them.

Another Simple Example: The Harmonic Oscillator

The harmonic oscillator is, by far, the most important dynamical system in all of theoretical physics. The good news is that it’s very easy. (In fact, the reason that it’s so important is precisely because it’s easy!). The potential energy of the harmonic oscillator is defined to be

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The harmonic oscillator is a good model for, among other things, a particle attached to the end of a spring. The force resulting from the energy V is given by $F = -kx$ which, in context of the spring, is called *Hooke’s law.* The equation of motion is

$$0.92 \;\; mx = kx$$

which has the general solution

$x(t) = A\cos(\omega t) \, + \, B \; \sin(\omega t)$

with

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Here A and B are two integration constants and ω is called the *angular frequency*. We see that all trajectories are qualitatively the same: they just bounce backwards and forwards around the origin. The coefficients A and B determine the amplitude of the oscillations, together with the phase at which you start the cycle. The time taken to complete a full cycle is called the *period*

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The period is independent of the amplitude. (Note that, annoyingly the kinetic energy is also often denoted by T as well. Do not confuse this with the period. It should hopefully be clear from the context).

if we want to determine the integration constants A and B for a given trajectory, we need some initial conditions. For example we're given the position and velocity at time $t = 0$, then it's simple to check that $A = x(0)$ and $B\omega = \omega(0)$.

Moving in a Potential

Let's go back to the general case of a potential $v(X)$ in one dimension. Although the equation of motion is a second order differential equation, the existence of a conserved energy magically allows us to turn this into a first order differential equation,

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This give us our first hint of the importance of conserved quantities in helping solve a problem. Of course, to go from a second order equation to a first order equation, we must have chosen an integration constant. In this case, that is the energy E itself. Given a first order equation, we can always write down a formal solution for the dynamics simple by integrating.

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As before, x^0 is a dummy variable. If we can do the integral, we're solved the problem. If we can't do the integral, you sometimes hear that the problem has been "reduced to quadrature". This rather old-fashioned phrase really means "I can't do the integral". But, it is often the case that having a solution in this form allows some of its properties to become manifest. And, if nothing else, one can always just evaluate the integral numerically (i.e. on your laptop) if need be.

Getting a feel for the Solutions

Given the potential energy $V\ (X)$, it is often very simple to figure out the qualitative nature of any trajectory simple by looking at the form of $V\ (x)$. This allows us to answer some questions with very little work. For example, we may want to know whether the particle is trapped within some region of space or can escape to infinity.

Let's illustrate this with an example. Consider the cubic potential

$V\ (x) = m(x^3 - 3x)$

If we were to substitute this into the general form, we'd get a fearsome looking integral which hasn't been solved since Victorian times.

Even without solving the integral, we can make progress. The potential is plotted in figure 2. Let's start with the particle sitting stationary at some position x_0. This means that the energy is

$$0.92\ E = V\ (x_0)$$

and this must remain constant during the subsequent motion. What happens next depends only on x_0. We can identify the following possibilities.

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Figure 1: The cubic Potential

$x_0 = \pm 1$: These are the local maximum and minimum. If we drop the particle at these points, it stays there for all time.

rId27.png +2): Here the particle is trapped in the dip. It oscillates backwards and forwards between the two points with potential energy $V\ (x_0)$. The particle can't climb to the right because it doesn't have the energy. In principle, it could live off to the left where the potential energy is negative, but to get there it would have to first climb the small bump at $x = -1$ and it doesn't have the energy to do so. (There is an assumption here which is implicit throughout all of classical mechanics: the trajectory of the particle $x(t)$ is a continuous function).

x_0>2: When released, the particle falls into the dip, climbs out the other side, before falling into the void $x \to -\infty$.

$x_0 = \pm 2$: This is a special value, since $E = 2m$ which is the same as the potential energy at the local maximum $x = -1$. The particle falls into the dip and starts to climb up towards $x = -1$. It can never stop before it reaches $x = -1$ for at its stopping point it would have only potential energy V <$2m$. But, similarly, it cannot arrive at $x = -1$ with any excess kinetic energy. The only option is that the particle moves towards $x = -1$ at an ever decreasing speed, only reaching the maximum at time $t \to \infty$. To see that this is indeed the case, we can consider the motion of the particle when it is close to the maximum.

We write rId28.png with rId29.png?. Then, dropping the ^{3} term, the

potential is

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the taken to reach rId31.png is

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The logarithm on the right-hand side gives a divergence as rId33.png 0. This tells us that it indeed takes infinite time to reach the top as promised.

One can easily play a similar game to that above if the starting speed is not zero. In general, one finds that the particle is trapped in the dip

rId34.png ∓ 1 if its energy lies in the interval rId35.png

Equilibrium: Why (Almost) Everything Is a Harmonic Oscillator

A particle at an equilibrium point will stay there for all time. In our last example with a cubic potential, we saw two equilibrium points: $x = \pm 1$. In general, if we want $x = 0$ for all time, then clearly we must have $x = 0$, which, from the form of Newton's equation, tells us that we can identify the equilibrium points with the critical points of the potential,

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What happens to a particle that is close to an equilibrium point, x_0? In this case, we can Taylor expand the potential energy about $x = x_0$. Because, by definition, the first derivative vanishes, we have

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To continue, we need to know about the sign of $V\ ^{00}(x_0)$:

$V\ ^{00}(x_0)$>0: In this case, the equilibrium point is a minimum of the potential and the potential energy is that of a harmonic oscillator. From our discussion of section 2.1.2, we know that the particle oscillator backwards and forwards around x_0 with frequency

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such equilibrium points are called *stable*. This analysis shows that if the amplitude of the oscillations is small enough(so taht we may ignore the $(x-x_0)^3$ terms in the Taylor expansion) then all systems oscillating around a stab;e fixed point look like a harmonic oscillator.

$V\ ^{00}(x_0)$<0: In this case, the equilibrium is a maximum of the potential. The equation of motion again reads

$$0.92\ mx = -V\ ^{00}(x_0)(x - x_0)$$

But with $V\ ^{00}$<0, we have x>0 when $x - x_0$>0. This means that if we displace the system a a little bit away from the equilibrium point, then the acceleration pushes it further away. The general solution is

$0.92\ x - x_0 = Ae^{\alpha t} + Be^{-\alpha t}$ with rId39.png

Any solution with the integration constant A 6= 0 will rapidly move away from the fixed point. Since our whole analysis started from a Taylor expansion, neglecting terms of orde $(x - x_0)^3$ and higher, our approximation will quickly breakdown. We say that such equilibrium points are unstable.

Notice that there are solutions around unstable fixed point with $A = 0$ and B 6= 0 which move back towards the maximum at late times. These finely tuned solutions arise in the kind of solution that we describe for the cubic potential where you drop the particle at a very special point (in the case of the cubic potential, this point was $x = 2$) so that it just reaches the top of a hill in infinite time. Clearly these solutions are not generic: they require very special initial conditions.

Finally, we could have $V\ ^{00}(x_0) = 0$. In this case, there is nothing we can say about the dynamics of the system without Taylor expanding the potential further.

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Yet Another Example: The Pendulum

Consider a particle of mass m attached to the end of a light rod of length

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0.92 Figure 2: fig 3.

l. This counts as a one-dimensional system because we need specify only a single coordinate to say what the system because we need specify only a single coordinate say what the system looks like at a given time. The best coordinate to choose is θ, the angle that the rod makes with the vertical. the equation of motion is

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The energy is

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(Note: Since θ is an angular variable rather than a linear variable, the kinetic energy is a little different. Hopefully this is familiar from earlier courses on mechanics. However, we will redrive this result in Section 4).

There are two qualitatively different motions of the pendulum. If $E>mgl$, then the kinetic energy can never be zero. This means that the pendulum is making complete circles. In contrast, if $E<mgl$, the pendulum completes only part of the circle before it comes to a stop and swings back the other way. If the highest point of the swing is θ_0, then the energy is

$$0.92\ E = -mgl\cos\theta_0$$

We can determine the period T of the pendulum using (2.6). It's actually best to calculate the period by taking 4 times the time the pendulum takes

to go from $\theta = 0$ to $\theta = \theta_0$. We have

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We see that the period is proportional to $\sqrt{l/g}$ multiplied by some dimensionless number given by $(4 times)$ the integral. For what it's worth, this integral turns out to be, once again, an elliptic integral. For small oscillations, we can write cos rId48.png and the pendulum becomes a harmonic oscillator with angular frequency $\omega = \sqrt{g/l}$. If we replace the $\cos\theta$'s in (2.10) by their Taylor expansion, we have

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This agrees with our result (2.5) for the harmonic oscillator.

Potentials in Three Dimensions

Let's now consider a particle moving in three dimensional R^3. Here things are more interesting. Firstly, it is possible to have energy conservation even if the force depends on the velocity.We will see how this can happen in Section 2.4. Conversely, forces which only depend on the position do not necessarily conserve energy: we need an extra condition. For now, we restrict attention to forces of the form $F = F(x)$. We have the following result: **Claim**:There exists a conserved energy if and only if the force can be written in the form

$$F = -5\ v$$

for some potential function $V(x)$. This means that the components of the force must be of the form $f_i = -\delta V/\delta x^i$. The conserved energy is then given by

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Proof:The proof that **E** is conserved if **F** takes the form (2.11) is exactly the same as in the one-dimensional case, together with liberal use of the chain rule. We have

0.92 rId51.png with *usingsummationconvetion*

$= x.\ (mx + 5v) = 0$

where the last equality follows from the equation of motion which is $mx =$

$0.92 - 5\ v.$

To go the other way, we must prove that if there exists a conserved energy E taking the form (2.12) then the force is necessarily given by (2.11). To do this, we need the concept of work. If a force F acts on a particle and succeeds in moving it from $x(t_1)$ to $x(t_2)$ along a trajectory C, then the work done by the force is defined to be

0.92 $$W = \int_c F.dx$$

This is a line integral (of the kind you've met in the Vector Calculus course). The scalar product means that we take the component of the force along the direction of the trajectory at each point. We can make this clearer by writing

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The integrand, which is the rate of doing work, is called the power, $P = F.x$. Using Newton's second law, we can replace $F = mx$ to get

rId53.png

where

rId54.png

is the kinetic energy. (You might think that K is a better name for kinetic energy. I'm inclined to agree. Except in all advanced courses of theoretical physics, kinetic energy is always denoted T which is why I've adopted the same notation here).

So the total work done is proportional to the change in kinetic energy. If we want to have a conserved energy of the form (2.12), then the change in kinetic energy must be equal to the change in potential energy. This means we must be able to write

0.92 $$W = \int_c F.dx = V(x(t_1)) - V(x(t_2))$$

In particular, this result tells us that the work done must be independent of the trajectory C; it can depend only on the end points $x(t_1)$ and $x(t_2)$.

But a simple result (which you will prove in your Vector Calculus course) says that (2.13) holds only for forces of the form

$$F = -\ 5\ V$$

as required

Forces in three dimensions which take the form $F = -\ 5\ V$ are called *conservative*. You will also see in the *Vector Calculus* course that force in R^3 are conservative if and only if 5 F = 0.

Forces of nature

Gravitational force

Electromagnetic force

Strong nuclear force

Weak nuclear force

The two nuclear forces operate on small scales comparable, as the name suggests, to the size of the nucleus $\gamma_0 \approx 10^{-15}$ We can't really give an honest description of these forces without invoking quantum mechanics and for this reason, we won't discuss them in this course. (A very rough, and slightly dishonest, classical description of the strong nuclear force can be given by the potential

rId55.png

In this section we will discuss the force of gravity; in the next, electromagnetism.

Gravity is a conservative force. Consider a particle of mass M fixed of the origin. A particle of mass "m" moving its presence experiences a potential energy

rId56.png

Here G is Newton's constant. It determines the strength of gravitational force and is given by $G \approx 6.67\ 0^{-11} m^3 kg^{-1} s^{-2}$. The force on the particle is given by

rId57.png

or better still

rId58.png

where r is the unit vector in the direction of the particle. This is Newton's famous inverse-square law for gravity. The force points towards the origin. We will devote much of section 4 to studying the motion of a particle under the inverse-square force.

The gravitational field the quantity V in (2.15) is the potential energy of a particle of mass m in the presence of mass M. It is common to define the gravitational field of the mass M to be

rId59.png

ϕ is sometimes called the Newtonian gravitational field to distinguish it from a more sophisticated object later introduced by Einstein.

It is also sometimes called the gravitational potential. It is a property of the mass M alone. The potential energy of the mass m is then given by

0.92 $V_(r) = m\phi_(r)$.

The gravitational force that a moving particle of mass m experiences in this field is

rId60.png

For many body system or the gravitational field due to many particles is simply the sum of the field due to each individual particle. If we fix particles with masses M_i at positions r_i, then the total gravitational field is

rId61.png

The gravitational field of a planets

The fact that contributions to the Newtonian gravitational potential add in a simple linear fashion has an important consequence: the external gravitational field of a spherically symmetric object of mass M such as a star or planet is the same as that of a point mass M positioned at the origin.

....

The proof of this statement is an example of the volume integral that you will learn in the vector calculus course. We include it here only for completeness. We let the planet have density $\rho_(r)$ and radius R. Summing over the contribution from all poins X inside the planet, the gravitational field is given by

rId62.png

It's best to work in spherical polar coordinates ad to choose the polar direction $\theta = 0$, to lie in the direction of r. Then $r \quad x = rx\cos\theta$. We can use this to write an expression for the denominator: $—r - x—^2 = r^2 + x^2 - 2rx\cos\theta$.

The gravitational field then becomes

rId63.png

This means $—r + x— = r + x$ and $—r - x— = r - x$

rId64.png

Escape Velocity

Suppose that you're trapped on the surface of a planet of radius R. (This should be easy). Let's firstly ask what gravitational potential energy you feel. Assuming you can only rise a distance $z << R$ from the planet's surface, we can Taylor expand the potential energy,

rId65.png

rId66.png

If we're only interested in small changes in $Z << R,$ we need focus only on the second term, giving rId67.png

This is the familiar potential energy that gives rise to constant acceleration we usually write rId68.png. For earth gravity $g \approx 9.8ms^{-2}$.

. When an object is projected from the earth surface, the path it takes depend on the speed of projection. At low speeds, it follows a parabolic path back to the earth surface. At very high speeds (about 11 kms-1), the object overcomes the effect of gravitational force and leaves the earth atmosphere into space. The minimum speed required to project an object into space away from the effect of the earth gravitational field is called the escape velocity. Consider a body of mass m projected from the surface of the earth so that if just escapes the earth's force of gravity. The kinetic energy of the body is equal to work done against the earth gravitational field.

rId69.png

where V is the gravitational potential but $V = {}^{GM}{}_{R} = m^{GM}{}_{R}$

q$_{2GM}$

Therefore $V = {}_{R}$

Where V is the escape velocity but rId70.png

Now let's be more ambitious.Suppose we want to escape our parochial, planetbound existence. So we decide to jump. How fast do we have to jump of we wish to truly be free ? This,it turns out, is the same kind of question that we discussed earlier in the context of particles moving in one dimension and can be determined very easily using gravitational energy rId71.png . If you jump directly upwards (i.e radially) with velocity v, your total energy as you leave the surface is

rId72.png

For any energy $E < 0$, you will eventually come to a halt at position $r =$ rId73.png, before falling back. If you want to escape the gravitational attraction of the planet forever, you will need energy $E > 0$. At the minimum value of E = 0, the associated velocity.

rId74.png

$\surd$

This is the escape velocity. from $gR^2 = GM$ $V_{escape} = 2gR$

Earth and Satellite

It can be launched from the earth surface to circle the earth. They are kept in their orbit by the gravitational attraction of the earth. Consider a satellite of mass m, which just circles a mass M of earth, close to a surface m the orbit of radius R.

The centripetal force required to keep the satellite in motion is provided by the gravitational force between the earth and the satellite.

rId75.png

But g = rId76.png where r is the radius of the earth

rId77.png

rId78.png

Let T be the period of eh satellite in orbit, rId79.png

Thus rId80.png

rId81.png

Equating (1) and (2)

rId82.png

Supposing the radius of the orbit R is chosen such that the period of the satellite is equal to the period of rotation of the earth, the satellite will always stay over the same place. When viewed from the earth. The satellite is said to be in a stationary or parpcing orbit. Such a satellite is called a geostationary satellite. From equation (B), if can be seen that for any satellite moving round the earth rId83.png

Thus for two satellites A and B moving around the earth rId84.png where

T_A, R_A and T_B, R_B are the period and orbital radius of the satellites A and B respectively. e.g. Calculate the escape velocity of a rocket whose velocity in orbit is 8.63 km/sec.

solution

0.92 $\surd$

$V_{esc} = 2\ .63 = 4.1545 km/s$

The mass of the earth is 81 times that of the moon and that the distance from the centre of the earth to that of the moon is about $4.0\ 0^5 km$. Calculate the distance from the centre of the earth where the resultant gravitational force becomes zero. When a space craft is launched from the earth to the moon.

rId85.png

At the point where the resultant gravitational force between becomes 0.

rId86.png

since we are only interested in positive distance
$x_1 = 4.0\ 0^4 km$

$$0.92\ S = R - x = 4.0\ 0^5 - 4.0\ 0^4 = 3.6\ 0^5 km.$$

Inertial and Gravitational mass, we have seen two formulae which involve mass, both due to Newton. These are the second law (1.2) and the inverse square law for gravity (2.16). Yet the meaning of mass in these equations is very difficult. The mass appearing in the second law represents the reluctance of a particle to accelerate under any force. In contrast, the mass appearing in the inverse-square law tell us that the strength of a particular force, namely gravity. Since these are very different concepts, we should really distinguish between the two different masses. the second law involves the inertial mass, m_I

$$0.92\ m_I x = F$$

while Newton's law of gravity involves the gravitational mass, m_G

rId87.png

$$0.92\ m_I g = g m_G$$

It is than an experimental fact that $m_I = m_G$ much experimental effort has gone into determining the accuracy of (2.17), most notably by the Hungarian physicist .... at the turn of the (previous) century. We now know that the inertial and gravitational masses are equal to within about one part in 10^{1B}.or (one in part). Currently, the best experiments to study this equivalence, as well as searches for deviations from Newton's law at short distances, are being undertaken by a group at the university of Washington in Seattle who go by the name $Eot - Wash$. A theoretical understanding of the result (2.17) came only with the development of the general theory of relativity.

Electromagnetism

Throughout the universe at each point in space, there exist two vectors, E and B. These are known as electric and magnetic fields. Their role at least for the purposes of this course is to guide any particle that carries electric charge.

The force experienced by a particle with electric charge q is called the Lorentz force,

$F = q(E(x) + x$ B$(x))$

$$0.92\ F = qE(x) + xq\ \mathrm{B}(x)$$

i.e A B $= AB$ sinθ Lorentz force.

$E = -\mathrm{O}\phi$ Electric field.

Here we have used the notation E(x) B(x) to stress that the electric and magnetic fields are functions of space. Both their magnitude and direction can vary from point to point. The electric forces is parallel to the electric field by convention, particles with positive charge q are accelerated in the direction of the electric field; those with negative electric charge are accelerated in the opposite direction. Due to a quick of history, the electron is taken to have a negative charge given by $q_{electron} \approx -1.6\ 0^{-19} coulombs$

As far as fundamental physics is concerned, a much better choice is to simply say that magnitude proportional to the speed of the particle, but with direction perpendicular to that of the particle. We shall see its effect in simple situations shortly.

In particle, both E and B can change in time. However, here we will consider only situations where they are static. In this case, the electric field is always of the form $E = -\mathrm{O}\phi$ where O $= -^{dQ}{}_{dr}$ for some function ϕ_x called the electric potential (or scalar potential or even just the potential as if we didn't already have enough things with that name.) For time independent fields, something special happens: energy is conserved.

Claim. the conserved energy is

rId88.png

rId89.png

$$0.92\ E = mx\ \ x + q\text{O}\phi\ \ x = x(F + q\text{O}\phi)$$

$= qx\ \ (x\ \text{B}) = 0$

where the last equality occurs because x B is necessarily perpendicular to x. Notice that this gives an example of something we promised earlier: a velocity dependent force which conserves energy. The key part of the derivation is that the velocity dependent force is perpendicular to the trajectory of the particle. This ensures that the force does no work.

The electric field of a point charge

Charged objects do not respond to electric fields. They also produce electric fields. A particle of charge Q sitting at the origin will set up an electric field given by

rId90.png

where $r^2 = x\ \ x$. The quantity ϵ_o has grand name Permittivity of free space and is a constant given by $\epsilon_o \approx 8.85\ 0^{-12} m^{-B} kg^{-1} s^2 C^2$

This quantity should be thought of as characterising the strength of the electric interaction.The force between two particles with charges Q and q is given by F = qE with E given by (2.19). In other words,

rId91.png

This is known as the coulomb force. It is a remarkable fact that, mathematically, the force looks identical to the Newtonian gravitational force (2.16): both have the characteristic inverse-square form. We will study motion in this potential in detail in section 4 , with perpendicular forces onthe coulomb force in 4.4.

Although the forces of Newton and coulomb look the same, there is one important difference. Gravity is always attractive because mass $m > 0$. In constrast the electrostatic coulomb force can be attractive or repulsive because charges q come with both signs. Further differences between gravity and electromagnetism come when you ask what happens whe sources (mass or charge) move but that's a story that will be fold in different courses.

2.4.2 Circles in a constant Magnetic Field.

Motion in a constant electric field is simple: the particle undergoes constant acceleration in the direction of E. But what about motion in a constant magnetic field B? The equation of motion is $mx = qx$ B let's pick the magnetic field to lie in the z-direction and write B = (0,0,B) we can now write Lorentz force law (2.18) in components. it reads

$$1.67\ mx = qBy\ my = -qBx$$

$mz = 0$

The last equation is easily solved and the particle just travels at constant velocity in the z direction. The first two equations are more interesting. There are a number of ways to solve them but a particularly elegant way is to construct the complex variable $\epsilon = x + y$ then adding (2.20) to i times

(2.21) gives $m\epsilon = -iqB\epsilon$

which can be integrated to give

$$\epsilon = \propto e^{i\epsilon t+B}$$

where $\propto$ and B are integration constants and ω is given by

rId92.png

if we choose our initial conditions to be that the particle starts life at t = 0 at the origin with velocity-v in the y-direction then $\propto$ and B are fixed to be

rId93.png

Translating this back into x and y coordinates, we have

rId94.png

and rId95.png

The end result is that the particle undergoes circles in the plane with angular frequency ω, known as the cyclotron frequency. The time to undergo a full circle is fixed rId96.png.

In constrast, the size of the circle is rId97.png and arises and an integration constant circles of arbitrary sizes are allowed; the only price that you pay is that you have to go faster.

A comment on solving vector differential equations. The Lorentz force equation (2.18) gives a good example of a vector differential equation. The straight forward way to view these is always in components: they are three, coupled, second order differential equations for x, y and z. This is what we did above when understanding the motion of a particle in magnetic field. However, one can also attack these kinds of equations without reverting to components. Let's see how this would work in the case of Larmor circles. We start with the vector equation

$$0.92\ mx = qx\ \mathrm{B}$$

To begin, we take the dot product with B. Since the right hand side vanishes we are left with
$x - B = 0$
This tells us that the particle travels with constant velocity in the direction of B. This is simply a rewriting of our previous result $z = 0$. For simplicity, let's just assume that the particle doesn't move in B direction remaining at the origin. This tells us that the particles moves in a plane with equation $X \quad B = 0$

However, we're not yet done. We started with (2.22) which was three equations. So there must still be two further equations lurking in (2.22) that we haven't yet taken into account. To find them, the systematic thing to do would be to take the cross product with B. However, in the present case, it turns out that the simplest way forwards is to simply integrate (2.22) once, to get

$$0.92\ mx = qx\ \mathrm{B} + C$$

with C a constant of integratioon. We can now substitute this back into the right-hand side of (2.22) to find

$$0.92\ m^2x = d + q^2(x\ \mathrm{B})\ \mathrm{B}$$

rId98.png

where the integration constant now sits in $d = qc\mathrm{B}$ which, by construction is perpendicular to B. In the last line, we've used the equation. (Note that if we would considered situation in which the particle moving with constant velocity in the B direction, we would have to work a little harder a this point).

The resulting vector equation three harmonic oscillators by the vector d/q^2B^2, oscillating with frequency $\omega = qB/m$.

However, because of constraint (2.23) the motion is necessarily two directions perpendicular to B. The end result is

rId99.png

with $\propto_1, i = 1{,}2$ integration constants satisfying $\propto_1$ B = 0 . This is the same result we found previously.

Admittedly, in this particular example working with components was somewhat easier than manipulating the vector equations directly but this won't always be the case for some problems you will make more progress by playing the kind of games that we've described here.

An aside: Maxwell's equations

In the Lorentz force law, the only hint that the electric and magnetic fields are related is that they both affect a particle in a manner that is proportional to the electric charge.

The connection between them becomes much clearer when things depend on time. A time dependent electric field gives rise to a magnetic field and vice versa. The dynamics of electric and magnetic fields are governed by Maxwell's equations; In the absence of electric charges, these equations are given by

1.21 O $E = 0$,O $B = 0$ O rId100.png

O rId101.png

with C the speed of light. You will learn more about the properties of these equations in the Electromagnetism and Electrodynamics courses.

For now, it's worth making one small comment. When we showed that energy is conserved, we needed both the electric and magnetic field to be time independent. What happens when they change with time? In this case, energy is conserved, but we have to worry about the energy stored in the fields themselves.

Friction

Friction is a messy, dirty business. While energy is always conserved on a fundamental level, it doesn't appear to be conserved in most things that you do every day. If you slide along the floor in your socks you don't keep going forever. At a microspic level, the kinetic energy is transferred to the atoms in the floor where if manifests itself as heat. But if we only want to know how far our socks will slide, the details of all these atomic processes are of little interest. Instead, we try to summarize everything in a single macroscopic force we call friction.

Dry friction

Dry friction occurs when two solid objects are in contact. Think of a heavy box being pushed along the floor, or some idiot sliding in his socks. Experimentally, one finds that the complicated dynamics involved in friction is usually summarized by the force

$$0.92\ F = \mu R$$

where R is the reaction force, normal to they floor, and μ is a constant called the coefficient of friction. Usually $\mu \approx 0.3$, although it depends on the kind of materials that are in contact. Moreover, the coefficient is usually, more or less independent of the velocity. We won't have much to say about dry friction in this course. In fact, we've already said it all.

Fluid Drag

Drag occurs when an object moves although a fluid other liquid or gas. The resistive force is opposite to the direction of the velocity and typically falls into one of two categories.

Linear Drag: $F = -\gamma v$ where the coefficient of friction, γ, is a constant. This form of drag holds for objects moving slowly through very viscous. For a spherical objects of radius L, there is a formula due to stokes which gives $\gamma = 6\pi L$ where is the velocity of the fluid.

Quadratic drag: $F = \gamma$—v—v

Again, γ is called the coefficient of friction. For quadratic friction, γ is usually proportional to the surface area of the object, i.e $\gamma \sim L^2$. (This is in constant to the coefficient for linear friction where stoke's formula gives $\gamma \sim L^2$). Quadratic drag holds for fast moving objects in less viscous fluids. This includes objects falling in air such as, for example, the various farm yard animals dropped by Galileo from the learning tower.

Quadratic drag arises because the object is banging into molecules in the fluid, knocking them out the way. There is an intuitive way to see this. The force is proportional to the charge of momentum that occurs in each collision. That gives one factor of v. But the force is also proportional to the number of collisions. That gives the second factor of v, resulting in a force that scales as v^2

Newtonian Mechanics: A single Particle In the rest of this section, we'll take a flying tour through the basic ideas of classical mechanics handed down to us by Newton. We'll start with a single particle.

A *particle* is defined to be an object of insignificant size. e.g. an electron, a tennis ball or a planet. Obviously, the validity of this statement depends on the context: to first approximation, the earth can be treated as a particle when computing its orbit around the sun. But if you want to understand its spin, it must be treated as an extended object.

The motion of a particle of mass m at the position $\mathbf{r}$ is governed by *Newton's Second Law* $F = m\mathbf{a}$ or, more precisely

$\mathbf{F}(\mathbf{r},\mathbf{r}) = p$ (1)

where $\mathbf{F}$ is the force which in general can depend on both the position $\mathbf{r}$ as well as the velocity $\mathbf{r}$ (for example, friction forces depend on $\mathbf{r}$ and $\mathbf{p} = m\mathbf{r}$ is the momentum. Both $\mathbf{F}$ and $\mathbf{p}$ are 3-vectors which we denote by the bold font. Equation (3) reduces to $\mathbf{F} = m\mathbf{a}$ if $m = 0$. But if $m = m(t)$ (e.g. in rocket science) then the form with $\mathbf{p}$ is correct.

General theorems governing differential equations guarantee that if we are given $\mathbf{r}$ and $\mathbf{r}$ at an initial time $t = t_0$, we can integrate equation (3) to determine $\mathbf{r}(t)$ for all t (as long as $\mathbf{F}$ remains finite). This is the goal of classical dynamics.

Equation(3) is not quite correct as stated: we must add the caveat that it holds only in an *inertial frame.* This is defined to be a frame in which a free particle with $m = 0$ travels in a straight line.

$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$ (2)

Newton's first law is the statement that such frames exist.

An inertial frame is not unique. In fact, there are an infinite number of inertial frames. Let S be an inertial frame. Then there are 10 linearly independent transformations $S \rightarrow S0$ such that $S0$ is also an inertial frame (i.e. if (4)) holds in S, then it also holds in $S0$. These are:

3 Rotations: $\mathbf{r}0 = O\mathbf{r}$ where O is a 3 orthogonal matrix.

3 Translations: $\mathbf{r}0 = \mathbf{r} + \mathbf{c}$ for a constant vector $\mathbf{c}$.

3 Boosts: $\mathbf{r}0 = \mathbf{r} + \mathbf{u}t$ for a constant velocity $\mathbf{u}$.

1 Time Translation: $t0 = t + c$ for a constant real number c.

If the motion is uniform in S, it will also be in $S0$. These transformations make up the *Galilean Group* under which Newton's Laws are invariant. They will be important in Section 2.1 where we will see that these symmetries of shape and time are the underlying reason for conservation laws. As a parenthetical remark, recall from special relativity that Einstein's laws of motion are invariant under Lorentz transformations which together with translation, make up the *Poincare* group. We can recover the Galilean group from the *Poincare* group by taking the speed of light to infinity. **Angular Momentum** We define the *angular momentum* L of a particle and the torque τ acting upon it as

$\mathbf{L} = \mathbf{r}$ p$\tau = \mathbf{r}$ F (3)

Note that, unlike linear momentum $\mathbf{p}$, both $\mathbf{L}$ and τ depend on where we take the origin: we measure angular momentum with respect to a particular point. Let us cross both sides of equation(3) with $\mathbf{r}$. Using the fact that r is parallel to $\mathbf{p}$, we can write rId102.png $\mathbf{p}$. Then we get a version of Newton's second law that holds for angular momentum:

$\tau = \mathbf{L}$(4)

Conservation Laws From (3) and (6), two important conservation laws follow immediately.

If $\mathbf{F} = 0$ then $\mathbf{p}$ is a constant throughout the motion.

If $\tau = 0$ then $\mathbf{L}$ is constant throughout the motion.

Notice that $\tau = 0$ does not require $\mathbf{F} = 0$, but only $\mathbf{r}$ F $= 0$. This means that $\mathbf{F}$ must be parallel to $\mathbf{r}$. This is the definition of a *central force*. An example is given by the gravitational force between the earth and the sun: the earth's angular momentum about the sun is constant. As written above in terms of forces and torques, these conservation laws appear trivial. In Section 2.4 we'll see how they rise as a property of the symmetry of space as encoded in the Galilean group.

Energy Let's now recall the definitions of energy. We firstly define the *kinetic energy* T as

rId103.png $\mathbf{r}$ (5)

Suppose from now on that the mass is constant. We can compute the change of kinetic energy with time: rId104.png $\mathbf{r}$. If the particle travels from the position $\mathbf{r}_1$ at time t_1 to position $\mathbf{r}_2$ at time t_2 then this change in kinetic energy is given by rId105.png **r** (6)

where the final expression involving the integral of the force over the path is called the *work done* by the force. So we see that the work done is equal to the change in kinetic energy. From now on we will mostly focus on a very special type of force known as a *conservative* force. Such a force depends only on position **r** rather than the velocity **r** and is such that the work done is independent of the path taken. In particular, for a closed path, the work done vanishes.

I

$F \quad d\mathbf{r} = 0 \Leftrightarrow \nabla$ F = 0 (7)

It is a deep property of flat space $\mathbf{R}^3$ that this property implies we may write the force as

$\mathbf{F} = -\nabla V\ (\mathbf{r})$ (8)

for some *potential* V(**r**). Systems which admit a potential of this form include gravitational, electrostatic and interatomic forces. When we have a conservative force, we necessarily have a conservation law for energy. To see this, return to equation(8) which now reads

rId106.png) (9)

So $E = T + V$ is also a constant of motion. It is energy. When the energy is considered to be a function of position **r** and momentum **p** it is referred to as the *Hamiltonian H*. In section 4, we will be seeing much more of the Hamiltonian.

0.0.1 Examples

Example 1: The Simple Harmonic Oscillator

This is a one-dimensional system with a force proportional to the distance x to the origin: $F(x) = -kx$. This force arises from a potential rId107.png. Since F 6= 0, momentum is not conserved (the

object oscillates backwards and forwards) and, since the system lives in only one dimension, angular momentum is not defined. But energy rId108.png is conserved.

Example 2: The Damped Simple Harmonic Oscillation

We now include a friction term so that $F(x,x = -kx - \gamma x)$. Since F is not conservative, energy is conserved. This system loses energy until it comes to rest.

Example 3: Particle Moving Under Gravity

Consider a particle of mass m moving in 3 dimensions under the gravitational pull of much larger particle of mass M. The force is

$\mathbf{F} = -(GMm/r^2)\mathbf{r}$ which arises from the potential $V = -GMm/r$. Again, the linear momentum $\mathbf{p}$ of the smaller particle is not conserved, but the force is both central and conservative, ensuring the particle's total energy E and the angular momentum L are conserved.

Newtonian Mechanics: Many Particles

It's easy to generalize the above discussion to many particles: we simply add an index to everything in sight! Let particle i have mass m_i and position $\mathbf{r}_i$ where $i = 1,......N$ is the number of particles. Newton's law now reads

$\mathbf{F}_i = \mathbf{p}_i$ (11)

where $\mathbf{F}_i$ is the force on the i^{th}. The subtlely is that forces can now be working between particles. In general, we can decompose the force in the following way

$\mathbf{F}$ rId109.png (12)

$j6=i$

1.11 where $\mathbf{F}_{ij}$ is the force acting on the i^{th} particle due to the j^{th} particle, while $\mathbf{F}^{ext}{}_i$ is the external force on the i^{th} particle. We now sum over all N particles.

X X X ext

$\mathbf{F}_i = + \mathbf{F}_i$

$i\ i,jwithj6=i\ i$

X X X ext

$\mathbf{F}_i = (\mathbf{F}_{ij} + \mathbf{F}_{ji}) + \mathbf{F}_i$ (13)

$i\ i<j\ i$

where in the second line, we've re-written the sum to be over all pairs $i < j$. At this stage we make use of *Newton's third law of motion*: every action has an equal and opposite reaction. Or, in other words, $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$. We see that the first term vanishes and we are left simply with

X *ext*

$\mathbf{F}_i = \mathbf{F}$ (14)

i

rId110.png where we've defined the total external force to be $\mathbf{F}^{ext} = \sum_i \mathbf{F}^{ext}{}_i$. We now define the total mass of the system $M = \sum_i m_i$, as well as the *center of mass* $\mathbf{R}$.

$\mathbf{R}$(15)

Then using (13), and summing over all particles, we arrive at the simple formula.

$\mathbf{F}^{ext} = M\mathbf{R}$ (16)

which is identical to that of a single particle. This is an important formula. It tells that the center of mass of a system of particles acts just as if all the mass were concentrated there. In other words, it doesn't matter if you throw a tennis ball or a very lively cat: the center of mass of each traces the same path.

Momentum Revisited

The total *momentum* is defined to be $\mathbf{P} = \sum_i \mathbf{p}_i$ and, from the formulae above, it is simple to derive $\mathbf{P} = \mathbf{F}^{ext}$. So we find the conservation law of total linear momentum for a system of many particles: $\mathbf{P}$ is a constant if $\mathbf{F}^{ext}$ vanishes.

Similarly, we define *total angular momentum* to be $\mathbf{L} = \sum_i \mathbf{L}_i$. Now let's see what happens when we compute the time derivative.

$\mathbf{L} = \sum \mathbf{r}_i \text{ p } _i$

i

rId111.png !

$\mathbf{L}$(17)

The last term in this expression is the definition of *total external torque*: $\tau^{ext} = \sum_i \mathbf{r}_i \text{ F}^{ext}{}_i$. But what we are going to do with the first term on the right hand side? Ideally we would like it to vanish! Let's look at the circumstances under which this will happen. We can again rewrite it as a sum over pairs $i < j$ to get

X

$(\mathbf{r}_i - \mathbf{r}_j)$ F$_{ij}$ (19)

$i<j$

which will vanish if and only if the force $\mathbf{F}_{ij}$ is parallel to the line joining to two particles $(\mathbf{r}_i - \mathbf{r}_j)$.This is the strong form of Newton's third law. If this is true, then we have a statement about the conservation of momentum, namely $\mathbf{L}$ is constant if $\tau^{ext} = 0$.

Most forces do indeed obey both forms of Newton's third law:$\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, and $\mathbf{F}_{ji}$ is parallel to $(\mathbf{r}_i - \mathbf{r}_j)$. For example, gravitational and electrostatic forces have this property. And the total momentum and angular momentum are both conserved in these systems. But some forces don't have these properties! The most famous example is the Lorentz force on two moving particles with electric charge Q. This is given by

$\mathbf{F}_{ij} = Q\mathbf{v}_i$ B$_j$ (20)

where $\mathbf{v}_i$ is the velocity of the i^{th} particle and $\mathbf{B}_j$ is the magnetic field generated by the j^{th} particle. Consider two particles crossing each other in a "T" as shown in the diagram. The forces on the particle 1 from particle 2 vanishes. Meanwhile, the force on particle 2 from particle 1 is non-zero and in the direction

$\mathbf{F}_{21} \sim\uparrow \otimes \sim\leftarrow$ (21)

Does this mean that conservation of total linear and angular momentum is violated? Thankfully, no! We need to realize that the electromagnetic field itself carries angular momentum which restores the conservation law. Once we realize this, it becomes a rather: cheap counterexample to Newton's third law, little different from an underwater swimmer who can appear to violate Newton's third law if we don't take into account the momentum of the water. **Energy Revisited**

The total kinetic energy of a system of many particles is `rId112.png` .

Let us decompose the position vector $\mathbf{r}_i$, as

$\mathbf{r}_i = \mathbf{R} + r_i$ (22)

where r, is the distance from the center of mass to the particle i. Then we can write the total kinetic energy as

`rId113.png` (23)

Which shows us that the kinetic energy splits up into the kinetic energy of the center of mass, together with an *internal energy* describing how the system is moving around its center of mass. As for a single particle, we may calculate the change in total kinetic energy,

rId114.png (24)

Like before, we need to consider conservative forces to get energy conservation. But now we need both

Conservative external forces: $\mathbf{F}^{ext}{}_i = -\nabla_i V_i(\mathbf{r}_1.....\mathbf{r}_N)$

Conservative internal forces: $\mathbf{F}_{ij} = -\nabla_i V_{ij}(\mathbf{r}_1.....\mathbf{r}_N)$

where $\nabla_i \equiv \partial/\partial \mathbf{r}_i$. To get Newton's third law $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, together with the requirement that this is parallel to $(r_i - r_j)$, we should take the internal potentials to satisfy $V_{ij} = V_{ji}$ with

$V_{ij}(\mathbf{r}_1.....\mathbf{r}_N) = V_{ij}(|\mathbf{r}_i.....\mathbf{r}_j|)$ (25)

so that V_{ij} depends only on the distance between the i^{th} and j^{th} particles. We also insist on a restriction for the external forces. $V_i(\mathbf{r}_1.....\mathbf{r}_N) = V_i(\mathbf{r}_i)$ so that the force on particle i does not depend on the positions of the particles. Then, following the steps we took in the single particle case, we can define the *total potential energy* $V = \sum_i V_i + \sum_{i<j} V_{ij}$ and we can show that $H = T + V$ is conserved.

Example

Let us return to the case of gravitational attraction between two bodies but unlike in Section 1.1.1 now including both particles. We have $T =$

rId115.png . The potential is $V = -Gm_1m_2/|\mathbf{r}_1 - \mathbf{r}_2|$. This system has

total linear momentum and total angular momentum conserved, as a well as the total energy $H = T + V$.

Potentials in One Dimension

Lets start by considering a particle moving on a line, so its position is determined by a single function x(t). For now, suppose that the force on the particle depends only on the position, not the velocity: F = F(x). We define the potential V (x) (also called the potential energy) by the equation

rId116.png (26)

The potential is only defined up to an additive constant. We can always invert (1) by integrating both sides. The integration constant is now determined by the choice of lower limit of the integral,

rId117.png) (27)

Here x' is just a dummy variable. (Do not confuse the prime with differentiation! In this course we will only take derivatives of position x with respect to time and always denote them with a dot over the variable). With this definition, we can write the equation of motion as

rId118.png (28)

For any force in one-dimension which depends only on the position, there exists a conserved quantity called the energy,

rId119.png

rId120.png is called the kinetic energy, While V(x) is called the potential

energy

The fact that this is conserved means that $E = 0$

rId121.png

rId122.png

rId123.png

rId124.png from equation 3 above.

rId125.png

rId126.png

An Example: A Uniform Gravitational Field

In a uniform gravitational field, a particle is subjected to a constant force, $F = mg$ where g ’ $9.8ms^{-2}$is the acceleration due to gravity near the surface of the Earth. The minus sign arises because the force is downwards while we have chosen to measure position in an upwards direction which we call z. The potential energy is V = mgz Notice that we have chosen to have V = 0 at z = 0. There is nothing that forces us to do this; we could easily add an extra constant to the potential to shift the zero to some other height. The equation of motion for uniform acceleration is

1.83 $mz = mg$ $z = g$

rId127.png

rId128.png

where z^o is the initial height at time t = 0

0.1 Another Simple Example: The Harmonic Oscillator

The harmonic oscillator is, by far, the most important dynamical system in all of theoretical physics. The good news is that its very easy. (In fact, the reason that its so important is precisely because its easy!). The potential energy of the harmonic oscillator is defined to be

rId129.png (29)

1.12 The harmonic oscillator is a good model for, among other things, a particle attached to the end of a spring. The force resulting from the energy V is given by $F = kx$ which, in the context of the spring, is called Hookes law. The equation of motion is $mx = kx$

rId130.png

0.92 $x = a(acceleration) = \omega^2 x$

rId131.png

rId132.png

1.85 taking $x(t) = A\cos\omega t + B\ \sin\omega t\ x = -A\omega\ \sin\omega t + B\omega\ \cos\omega t\ x = -A\omega^2\ \cos\omega t - B\omega^2\ \sin\omega t\ \omega = \omega t - A\sin\omega t + B\ \cos\omega t$

1.91 squaring x, ω and x $x^2 = A^2\cos^2\omega t + B^2\sin^2\omega t\ v^2 = -(A\omega)^2\sin^2\omega t + (B\omega)^2\cos\omega t\ \omega^2 = (A\omega)^2 - A^2\sin^2\omega t + B^2\cos^2\omega t\ \omega^2 x^2 = A^2\omega^2\cos^2\omega t + B^2\omega^2\sin^2\omega t\ \omega 2x2 + v2 = A2\omega 2 + B2\omega 2$

$v2 = A2\omega 2 + B2\omega 2 - \omega 2x2$

rId133.png

NB at the maximum velocity $X = 0$

0.92 $\surd$

0.92 $V_{max} = \omega\ A2 + B2$

Here A and B are two integration constants and ω is called the angular frequency. We see that all trajectories are qualitatively the same: they just bounce backwards and forwards around the origin. The coefficients A and B determine the amplitude of the oscillations, together with the phase at which you start the cycle. The time taken to complete a full cycle is called the period rId134.png

The period is independent of the amplitude. (Note that, annoyingly, the kinetic energy is also often denoted by T as well. Do not confuse this with the period. It should hopefully be clear from the context). If we want to determine the integration constants A and B for a given trajectory, we need some initial conditions. For example, if were given the position and velocity at time t = 0, then its simple to check that $A = x(0)$ and $B\omega = x(0)$.

Moving in a Potential

Lets go back to the general case of a potential V (x) in one dimension. Although the equation of motion is a second order differential equation, the existence of a conserved energy magically allows us to turn this into a first order differential equation

rId135.png

rId136.png

rId137.png

rId138.png

rId139.png

As before, x' is a dummy variable. If we can do the integral, weve solved the problem. If we cant do the integral, you sometimes hear that the problem has been reduced to quadrature. This rather old-fashioned phrase really means I cant do the integral. But, it is often the case that having a solution in this form allows some of its properties to become manifest. And, if nothing else, one can always just evaluate the integral numerically (i.e. on your laptop) if need be

Getting a Feel for the Solutions

Given the potential energy V(x), it is often very simple to figure out the qualitative nature of any trajectory simply by looking at the form of V(x). This allows us to answer some questions with very little work. For example, we may want to know whether the particle is trapped within some region of space or can escape to infinity. Lets illustrate this with an example. Consider the cubic potential $V\ (x) = m(x^3 - 3x)$ If we were to substitute this into the general form, wed get a fearsome looking integral which hasnt been solved since Victorian times1. Even without solving the integral, we can make progress. The potential is plotted Lets start with the particle sitting stationary at some position x_o. This means that the energy is $E = V\ (x_o)$ and this must remain constant during the subsequent motion. What happens next depends only on x_o. We can identify the following possibilities

$x_o = \pm 1$: These are the local maximum and minimum. If we drop the particle at these points, it stays there for all time.

$x_o \in$ (1,+2) : Here the particle is trapped in the dip. It oscillates backwards and forwards between the two points with potential energy V (x_o). The particle cant climb to the right because it doesnt have the energy. In principle, it could live off to the left where the potential energy is negative, but to get there it would have to first climb the small bump at $x = 1$ and it doesnt have the energy to do so. (There is an assumption here which is implicit throughout all of classical mechanics: the trajectory of the particle x(t) is a continuous function).

$x_o > 2$: When released, the particle falls into the dip, climbs out the other side, before falling into the void $x \longrightarrow \infty$.

$x_o < 1$: The particle just falls off to the left.

$x_o = +2$: This is a special value, since $E = 2m$ which is the same as the potential energy at the local maximum x = 1. The particle falls into the dip and starts to climb up towards $x = 1$. It can never stop before it reaches $x = 1$ for at its stopping point it would have only potential energy $V < 2m$. But, similarly, it cannot arrive at $x = 1$ with any excess kinetic energy. The only option is that the particle moves towards $x = 1$ at an ever decreasing speed, only reaching the maximum at time $t \longrightarrow \infty$. To see that this is indeed the case, we can consider the motion of the particle when it is close to the maximum. We write $x \approx 1 + \in$ with $\in 1$. Then, dropping the ^{3} term, the

potential is rId140.png

Taking the formular below;

rId141.png

using rId142.png

rId143.png

rId144.png

rId145.png

if rId146.png

0.92 $t - t_o \longrightarrow \infty$

Equilibrium: Why (Almost) Everything is a Harmonic Oscillator A particle placed at an equilibrium point will stay there for all time. In our last example with a cubic potential, we saw two equilibrium points: $x = \pm 1$. In general, if we want $x = 0$ for all time, then clearly we must have $x = 0$, which, from the form of Newtons equation, tells us that we can identify the equilibrium points with the critical points of the potential, we continue solving this equation by using tailor's series if $f(x) = v(x)$

rId147.png

rId148.png

$v^0(x_o)(x - x_o) = 0$

rId149.png

$V^{00}(x_o) > 0$: In this case, the equilibrium point is a minimum of the potential and the potential energy is that of a harmonic oscillator. we know that the particle oscillates backwards and forwards around x_o with frequency.

rId150.png

0.92 $v^0(x) = kx$

$v^{00}(x) = k$

0.92 $k = v^{00}(x) > 0$

rId151.png

$V^{00}(x_o) < 0$: In this case, the equilibrium point is a maximum of the potential. The equation of motion again reads $mx = V^{00}(x_o)(x - x_o)$ But with $V^{00} < 0$, we have $x > 0$ when $x - x_o > 0$. This means that if we displace the system a little bit away from the equilibrium point, then the acceleration pushes it further away. The general solution is

4. Central Forces

In this section we will study the three-dimensional motion of a particle in a central force potential. Such a system obeys the equation of motion

$$0.92\ mx = -\nabla V\ (r)$$

where the potential depends only on $r = —x—$

Our first line of attack in solving(4.1) is to use angular momentum. Recall that this is defined as

$$0.92\ L = mx \text{ x}$$

We already saw in Section 2.2.2 that angular momentum is conserved in a central potential. The proof is straightforward:

rId152.png

where the final equality follows because ∇V is parallel to x. The conservation of angular momentum has an important consequence: all motion takes place in a plane. This follows because L is a fixed, unchanging vector which, by construction, obeys

$L\ \ x = 0$

So the position of the particle always lies in a plane perpendicular to L. By the same argument,$L\ \ x = 0$ so the velocity of the particle also lies in the same plane. In this way the three-dimensional dynamics is reduced to dynamics on a plane.

4.1 Polar Coordinates in the Plane

To start, we rotate our coordinate system so that the angular momentum points in the z-direction and all motion takes place in the (x; y) plane. We then define the usual polar coordinates

$$0.92\ x = r\ \cos\pi, y = rsin\pi$$

Our goal is to express both the velocity and acceleration in polar coordinates.We introduce two unit vectors,r and θ in the direction of increasing r and θ respectively as shown in the diagram. Written in Cartesian form, these vectors are

rId153.png

These vectors form an orthornormal basis at every point on the plane. But the basis itself depends on θ which angle we sit at. Moving in the radial direction doesn't change the basis, but moving in the angular direction we have

rId154.png

This means that if the particle moves in a way such that θ changes with time, then the basis vectors themselves will also change with time. Let's see what this means for the velocity expressed in these polar coordinates. The position of a particle is written as the simple, if somewhat ugly, equation

$$0.92\ x = rr$$

From this we can compute the velocity, remembering that both r and the basis vector r can change with time. We get

rId155.png

$$0.92 = rr + r\theta\theta$$

The second term in the above expression arises because the basis vectors change with time and is proportional to the angular velocity,θ. (Strictly speaking, this is the angular speed. In the next section, we will introduce a vector quantity which is the angular velocity).

Differentiating once more gives us the expression for acceleration in polar coordinates,

rId156.png

$$= (r - r\theta^2)r + (r\theta + 2r\theta)\theta$$

0.92 The two expressions (4.2) and (4.3) will be important in what follows.

4.2 Back to Central Forces

We've already seen that the three-dimensional motion in a central force potential actually takes place in a plane. Let's write the equation of motion (4.1) using the plane polar coordinates that we've just introduced. Since V

= V (r), the force itself can be written using

rId157.png

and, from (4.3) the equation of motion becomes

rId158.png

The θ component of this is particularly simple. It is

rId159.png

It looks as if we've found a new conserved quantity since we've learnt that

$$0.92\ l = r^2\theta$$

does not change with time. However, we shouldn't get too excited. This is something that we already know. To see this, let's look again at the angular momentum L. We already used the fact that the direction of L is conserved when restricting motion to the plane. But what about the magnitude of L?

Using (4.2), we write

$$0.92\ L = mx \text{ x} = mrrrr{+}\ r\theta\theta) = mr^2\theta(r\theta)$$

Since r and θ are orthogonal unit vectors, $r\ \theta$ is also a unit vector. The magnitude of the angular momentum vector is therefore

$$0.92\ \text{—}L\text{—} = ml$$

and l, given in (4.5), is identified as the angular momentum per unit mass, although we will often be lazy and refer to l simply as the angular momentum. Let's now look at the r component of the equation of motion

(4.4). It is

rId160.png

Using the fact that $l = r^2\theta$ is conserved, we can write this as

rId161.png

It's worth pausing to reflect on what's happened here. We started in (4.1) with a complicated, three dimensional problem. We used the direction of the angular momentum to reduce it to a two dimensional problem, and the magnitude of the angular momentum to reduce it to a one dimensional problem. This was all possible because angular momentum is conserved. This should give you some idea of how important conserved quantities are when it comes to solving anything. Roughly speaking, this is also why it's not usually possible to solve the N-body problem with $N \geq 3$. In Section 5.1.5, we'll see that for the N = 2 mutually interacting particles, we can use the symmetry of translational invariance to solve the problem. But for$N \geq 3$, we don't have any more conserved quantities to come to our rescue. Returning to our main storyline, we can write (4.6) in the suggestive form

rId162.png

where $V_{eff}(r)$ is called the effective potential and is given by

rId163.png

The extra term, $ml^2/2r^2$ is called the angular momentum barrier (also known as the centrifugal barrier). It stops the particle getting too close to the origin, since there is must pay a heavy price in effective energy".

4.2.1 The Effective Potential: Getting a Feel for Orbits

Let's just check that the effective potential can indeed be thought of as part of the energy of the full system. Using (4.2), we can write the energy of the full three dimensional problem as

rId164.png

This tells us that the energy E of the three dimensional system does indeed coincide with the energy of the effective one dimensional system that we've reduced to. The effective potential energy is the real potential energy, together with a contribution from the angular kinetic energy. We already saw in Section 2.1.1 how we can understand qualitative aspects of one dimensional motion simply by plotting the potential energy. Let's play the same game here. We start with the most useful example of a central potential: $V\ (r) = -k/r$, corresponding to an attractive inverse square law for $k > 0$. The effective potential is

rId165.png

and is drawn in the figure.

The minimum of the effective potential occurs at $r = ml^2/k$ and takes the value $V_{eff}(r) = -k^2/2ml^2$. The possible forms of the motion can be characterised by their energy E.

$E = Emin = -k^2/2ml^2$: Here the particle sits at the bottom of the well r and stays there for all time. However, remember that the particle also has angular velocity, given by $\theta = l/r^2$ So although the particle has fixed radial position, it is moving in the angular direction. In other words, the trajectory of the particle is a circular orbit about the origin.

Notice that the radial position of the minimum depends on the angular momentum l. The higher the angular momentum, the further away the minimum. If there is no angular momentum, and l = 0, then $V_{eff} = V$ and the potential has no minimum. This is telling us the obvious fact that there is no way that r can be constant unless the particle is moving in the θ direction. In a similar vein, notice that there is a relationship between the angular velocity θ and the size of the orbit, r, which we get by eliminating l: it is $\theta^2 = k/mr^3$. We'll come back to this relationship in Section 4.3.2 when we discuss Kepler's laws of planetary motion.

$Emin < E < 0$: Here the 1d system sits in the dip, oscillating backwards and forwards between two points. Of course, since l 6= 0, the particle also has angular velocity in the plane. This describes an orbit in which the radial distance r depends on time. Although it is not yet obvious, we will soon show that for $V = -k/r$, this orbit is an ellipse.

The smallest value of r that the particle reaches is called the periapsis. The furthest distance is called the apoapsis. Together, these two points are referred to as the apsides. In the case of motion around the Sun, the periapsis is called the perihelion and the apoapsis the aphelion.

$E > 0$. Now the particle can sit above the horizontal axis. It comes in from infinity, reaches some minimum distance r, then rolls back out to in infinity. We will see later that, for the $V = -k/r$ potential, this trajectory is hyperbola.

Lagragian and Hamiltonian Formulation

What are the pros/cons of each approach? What questions are more naturally solved in each? For example, I believe Fermat's Principle of Least Time is something that's very naturally explained in Lagrangian mechanics ("minimize the time it takes to get between these two points"), but more difficult to explain in Newtonian mechanics since it requires knowing your endpoint.

What's the overall difference in layman's terms? From what I've read so far, it sounds like Newtonian mechanics takes a more local "cause-andeffect"/"apply a force, get a reaction" view, while Lagrangian mechanics takes a more global "minimize this quantity" view. Or, to put it more axiomatically, Newtonian mechanics starts with Newton's three laws of motion, while Lagrangian mechanics starts with the Principle of Least Action. How do the approaches differ mathematically/when you're trying to solve a problem? Kind of similar to above, I'm guessing that Newtonian solutions start with drawing a bunch of force vectors, while Lagrangian solutions start with defining some function (calculating the Lagrangian...?) you want to minimize, but I really have no idea.

In Newtonian mechanics you have to use mainly rectangular co ordinate system and consider all the constraint forces. Lagrange's scheme avoids the considerations of the constraint forces deftly and you can use any set of "generalized coordinates" like angle, radial distance etc. consistent with the constraint relations. The number of those generalized coordinates are the same with the number of degrees of freedom of the system.

In all dynamical systems we arbitrarily choose some generalized co ordinates consistent with the constraints of the system. In Newtonian mechanics, the difference between the kinetic and potential energy of the system gives you the so called Lagrangian. Then we have n number of differential equations.

Advantages of Lagragian Mechanics

The main advantage of Lagrangian mechanics is that we don't have to consider the forces of constraints and given the total kinetic and potential energies of the system we can choose some generalized coordinates and blindly calculate the equation of motions totally analytically unlike Newtonian case where one has to consider the constraints and the geometrical nature of the system.

Calculus of variation

he calculus of variations involves finding an extremum (maximum or minimum) of a quantity that is expressible as an integral. Lets look at a couple of examples:

The shortest path between two points

Fermats principle (light follows a path that is an extremum)

What is the shortest path between two points in a plane? You certainly know the answera straight linebut you probably have not seen a proof of thisthe calculus of variations provides such a proof.

Consider two points in the x-y plane, as shown in the figure.

An arbitrary path joining the points follows the general curve $y = y(x)$, and an element of length along the path is

$ds = \text{p} \, dx^2 + dy^2$ (30)

We can rewrite this (pay attention, because you will be doing this a lot in the next few chapters) as

which is valid because rId166.png

Thus, the length is

rId167.png (32)

Shortest Path Between 2 Points

Note that we have converted the problem from an integral along a path, to an integral over x:

rId168.png (33)

We have thus succeeded in writing the problem down, but we need some additional mathematical machinery to find the path for which L is an extremum (a minimum in this case). **Fermats Principle:**

A similar but somewhat more interesting problem is to find the path light will take through a medium that has some index of refraction $n \neq 1$. You may recall that light travels more slowly through such a medium, and we define the index of refraction as $n = c/v$. where c is the speed of light in vacuum, and v is the speed of light in the medium. The total travel time is then

rId169.png (34)

Here we are allowing the index of refraction to vary arbitrarily vs. x and y.

Variational Principles

Obviously, both problems on the previous slide are similar, and such cases arise in many other situations.

In our usual minimizing or maximizing of a function f(x), we would take the derivative and find its zeroes (i.e. the values of x for which the slope of the function is zero). These points of zero slope may be minima, maxima, or points of inflection, but in each case we can say that the function is stationary at those points, meaning for values of x near such a point, the value of the function does not change (due to the zero slope).

In analogy with this familiar approach, we want to be able to find solutions to these integrals that are stationary for infinitesimal variations in the path. This is called calculus of variations.

The methods we will develop are called variational methods, and a principle like Fermats Principle are called variational principles.

These principles are common, and of great importance, in many areas of physics (such as quantum mechanics and general relativity).

Euler-Lagrange Equations

Euler-Lagrange Equation-1

We are now going to discuss a variational method due to Euler and Lagrange, which seeks to find an extremum (to be definite, lets consider this a minimum) for an as yet unknown curve joining two points x 1 and x 2 ,

satisfying the integral relation

rId170.png (35)

The function f is a function of three variables, but because the path of integration is $y = y(x)$, the integrand can be reduced to a function of just one variable, x.

To start, lets consider two curves joining points 1 and 2, the right curve $y(x)$, and a wrong curve $Y\ (x)$ that is a small displacement from the right curve, as shown in the figure.

We will write the difference between these curves as some function $\eta(x)$

$Y\ (x) = y(x) + \eta(x);\ \eta(x_1) = \eta(x_1) = 0$ (36)

Euler-Lagrange Equation-2

There are infinitely many functions $h(x)$, that can be wrong, but we require that they each be longer that the right path. To quantify how close the wrong path can be to the right one, lets write $Y = y + a\eta$, so that

$$S(a) = \int_{x_1}^{x_2} f[Y, Y^{0}(x), x]\,dx \qquad (37)$$

$$= \int_{x_1}^{x_2} f[y + a\eta, y^0 + a\eta^0, x]\,dx \qquad (38)$$

This is going to allow us to characterize the shortest path as the one for which the derivative $dS/da = 0$ when $a = 0$. To differentiate the above equation with respect to a , we need to evaluate the partial derivative dS/da via the chain rule

rId171.png (39)

so $dS/da = 0$ gives

rId172.png $= 0$ (40)

Euler-Lagrange Equation-3

We can handle the second term in the previous equation by integration by parts:

rId173.png (41)

but the first term of this relation (the end-point term) is zero because h (x) is zero at the endpoints.

Our modified equation is then

1.23 rId174.png (42) This leads us to the Euler-Lagrange equation

rId175.png $= 0$ (43)

We come to this conclusion because the modified equation has to be zero for any $\eta(x)$. See the text for more discussion of this point.

Geodesics: Shortest Path Between Two Points

We earlier showed that the problem of the shortest path between two points can be expressed as

rId176.png (44)

The integrand contains our function

$f(y,y^0,x) = \mathrm{p}1 + y^0(x)^2$ (45)

The two partial derivatives in the Euler-Lagrange equation area

rId177.png rId178.png
= 0 and (46)

Thus, the Euler-Lagrange equation gives us

rId179.png = 0 (47)

This says that

rId180.png or $y^{o2} = C^2(1 + y^{o2})$ (48)

A little rearrangement gives the final result: rId181.png (call it m^2), so $y(x) = mx + b$. In other words, a straight line is the shortest path.

The Brachistochrone

0.92 This is a famous problem in the history of the calculus of variations.

Statement of the problem:

Given two points 1 and 2, with 1 higher above the ground, in what shape could we build a track for a frictionless roller-coaster so that a car released from point 1 would reach point 2 in the shortest possible time? See the figure, which takes point 1 as the origin, with y positive downward.

Solution:

rId182.png $\surd$

The time to travel from point 1 to 2 iswhere $v = 2gy$ from kinetic energy considerations.

Since this depends on y, we will take y as the independent variable, hence

rId183.png (50)

From the Euler-Lagrange equation:

rId184.png (51)

Since

rId185.png (52)

Evaluating this derivative and squaring it for convenience, we have

rId186.png (53)

where the constant is renamed $1/2a$ for future convenience.
Solving for x^0we have

rId187.png (54)

Finally, to get x we integrate

rId188.png (55)

It is not obvious, but this can be solved by the substitution $y = a(1-cos\theta)$, which gives

Z

$x = a\ (1\ -\ cos\theta)d\theta = a(\theta\ -\ sin\theta)\ +\ const.$ (56)

The two equations that give the path are then:

0.92 $x = a(\theta-sin\theta)$ $y = a(1-cos\theta)$ *in terms of* θ (57)

This curve is called a cycloid, and is a very special curve indeed. As you will show in the homework, it is the curve traced out by a wheel rolling (upside down) along the x axis.

Another remarkable thing is that the time it takes for a cart to travel this path from $2 \rightarrow 3$ is the same, no matter where 2 is placed, from 1 to 3!

Thus, oscillations of the cart along that path are exactly isochronous (period perfectly independent of amplitude).

Generalized Coordinates and constraints

Generalized Coordinates

In analytical mechanics, specifically the study of the rigid body dynamics of multibody systems, the term generalized coordinates refers to the parameters that describe the configuration of the system relative to some reference configuration. These parameters must uniquely define the configuration of the system relative to the reference configuration. This is done assuming that this can be done with a single chart. The generalized velocities are the time derivatives of the generalized coordinates of the system.

An example of a generalized coordinate is the angle that locates a point moving on a circle. The adjective "generalized" distinguishes these parameters from the traditional use of the term coordinate to refer to Cartesian coordinates:

for example, describing the location of the point on the circle using x and y coordinates.

Although there may be many choices for generalized coordinates for a physical system, parameters which are convenient are usually selected for the specification of the configuration of the system and which make the solution of its equations of motion easier. If these parameters are independent of one another, the number of independent generalized coordinates is defined by the number of degrees of freedom of the system

Constraints and degrees of freedom

Generalized coordinates are usually selected to provide the minimum number of independent coordinates that define the configuration of a system, which simplifies the formulation of Lagrange's equations of motion. However, it can also occur that a useful set of generalized coordinates may be dependent, which means that they are related by one or more constraint equations.

Holonomic Constraints

For a system of N particles in 3D real coordinate space, the position vector of each particle can be written as a 3-tuple in Cartesian coordinates:

$$0.92\ r_1 = (x_1,y_1,z_1),r_2 = (x_2,y_2,z_2),,r_N = (x_N,y_N,z_N)$$

$\mathbf{r}_1 = (x_1,y_1,z_1),$

$$\mathbf{r}_2 = (x_2,y_2,z_2),\ldots\ ,\mathbf{r}_N = (x_N,y_N,z_N).$$

$\mathbf{r}_1 = (x_1,y_1,z_1),$

$$\mathbf{r}_2 = (x_2,y_2,z_2),\ldots\ ,\mathbf{r}_N = (x_N,y_N,z_N).$$

particle k

$f(rk,t) = 0f(\mathbf{r}_k,t) = 0f(\mathbf{r}_k,t) = 0$

which connects all the 3 spatial coordinates of that particle together, so they are not independent. The constraint may change with time, so time t will appear explicitly in the constraint equations. At any instant of time, when t is a constant, any one coordinate will be determined from the other coordinates, e.g. if xk and zk are given, then so is yk. One constraint equation counts as one constraint. If there are C constraints, each has an equation, so there will be C constraint equations. There is not necessarily one constraint equation for each particle, and if there are no constraints on the system then there are no constraint equations. Any of the position vectors can be denoted rk where $k =$ 1,2,...,N labels the particles. A holonomic constraint is a constraint equation of the form for particle

So far, the configuration of the system is defined by $3N$ quantities, but C coordinates can be eliminated, one coordinate from each constraint equation. The number of independent coordinates is $n = 3N - C$. (In D dimensions, the original configuration would need ND coordinates, and the reduction by constraints means $n = ND - C$). It is ideal to use the minimum number of coordinates needed to define the configuration of the entire system, while taking advantage of the constraints on the system. These quantities are known as generalized coordinates in this context, denoted $qj(t)$. It is convenient to collect them into an n-tuple $q(t) = (q1(t),q2(t),,qn(t))\mathbf{q}(t) = (q_1(t),q_2(t),...,q_n(t))\mathbf{q}(t) = (q_1(t),q_2(t),...,q_n(t))$

which is a point in the configuration space of the system. They are all independent of one other, and each is a function of time. Geometrically they can be lengths along straight lines, or arc lengths along curves, or angles; not necessarily Cartesian coordinates or other standard orthogonal coordinates. There is one for each degree of freedom, so the number of generalized coordinates equals the number of degrees of freedom, n. A degree of freedom corresponds to one quantity that changes the configuration of the system, for example the angle of a pendulum, or the arc length traversed by a bead along a wire. If it is possible to find from the constraints as many independent variables as there are degrees of freedom, these can be used as generalized coordinates. The position vector rk of particle k is a function of all the n generalized coordinates and time and time $rk = rk(q(t),t),\mathbf{r}_k = \mathbf{r}_k(\mathbf{q}(t),t),\mathbf{r}_k = \mathbf{r}_k(\mathbf{q}(t),t),$

and the generalized coordinates can be thought of as parameters associated with the constraint. The corresponding time derivatives of q are the generalized velocities,

0.92 $d\mathbf{q}$ $d\mathbf{q}$

$q = \mathbf{q} = = (q_1(t),q_2(t),...,q_n(t))\mathbf{q} = = (q_1(t),q_2(t),...,q_n(t))$ dt dt

(each dot over a quantity indicates one time derivative). The velocity vector vk is the total derivative of rk with respect to time

n n

0.92

rId189.png

$d\mathbf{r}$ $\partial\mathbf{r}$ $\partial\mathbf{r}$ $d\mathbf{r}$ $\partial\mathbf{r}$ $\partial\mathbf{r}$

.

j j $j{=}1$ $j{=}1$

and so generally depends on the generalized velocities and coordinates. Since we are free to specify the initial values of the generalized coordinates and velocities separately, the generalized coordinates and velocities can be treated as independent variables. The generalized coordinates qj and velocities dqj/dt are treated as independent variables.

Non Holonomic Constraints

A mechanical system can involve constraints on both the generalized coordinates and their derivatives. Constraints of this type are known as nonholonomic. First-order non-holonomic constraints have the form

$g(g,q.t) = 0,g(\mathbf{q},\mathbf{q},t) = 0,g(\mathbf{q},\mathbf{q},t) = 0,$

An example of such a constraint is a rolling wheel or knife-edge that constrains the direction of the velocity vector. Non-holonomic constraints can also involve next-order derivatives such as generalized accelerations. **Generalized Momentum** The generalized momentum "canonically conjugate to" the coordinate qi is defined by

rId190.png

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If the Lagrangian L does not depend on some coordinate qi, then it follows from the EulerLagrange equations that the corresponding generalized momentum will be a conserved quantity, because the time derivative is zero implying the momentum is a constant of the motion;

rId191.png .

The Hamiltonian Formalism

We'll now move onto the next level in the formalism of classical mechanics, due initially to Hamilton around 1830. While we won't use Hamilton's approach to solve any further complicated problems, we will use it to reveal much more of the structure underlying classical dynamics. If you like, it will help us understands what questions we should ask.

Hamiltonian's Equations

Recall that in the Lagrangian formulation, we have the function $L(q_i;\ q_i;t)$ where qi (i = 1,...n) are n generalized coordinates. The equations of motion are

rId192.png

1.07 These are $n2^{nd}$ order differential equations which require 2n initial conditions, say $q_i(t = 0)$ and $q_i(t = 0)$. The basic idea of Hamilton's approach is to try and place qi and q_i on a more symmetric footing. More precisely, we'll work with the n generalised momenta that we introduced in section 2.3.3,

rId193.png

so $pi = pi(q_j;\ q_j;t)$. This coincides with what we usually call momentum only if we work in Cartesian coordinates (so the kinetic term is rId194.png). If we rewrite Lagrange's equations (4.1) using the definition of the momentum (4.2), they become

rId195.png

The plan will be to eliminate q_i in favor of the momenta p_i, and then to place q_i and p_i on equal footing.

An Example: The Pendulum

Consider a simple pendulum. The configuration space is clearly a circle, S^1, parameterized by an angle rId196.png). The phase space of the pendulum is a cylinder R S^1, with the R factor corresponding to the momentum. We draw this by flattening out the cylinder. The two different types of motion are clearly visible in the phase space flows. For small θ and small

rId197.jpg

Figure 3: Flows in the phase space of a pendulum

momentum, the pendulum oscillates back and forth, motion which appears as an ellipse in phase space. But for large momentum, the pendulum swings all the way around, which appears as lines wrapping around the S^1 of phase space. Separating these two different motions is the special case where the pendulum starts upright, falls, and just makes it back to the upright position.

This curve in phase space is called the separatix.

The Legendre Transform

We want to find a function on phase space that will determine the unique evolution of q_i and p_i. This means it should be a function of qi and pi (and not of q_i) but must contain the same information as the Lagrangian $L(q_i;\, q_i,t)$. There is a mathematical trick to do this, known as the Legendre transform. To describe this, consider an arbitrary function f(x, y) so that the total derivative is

Now define a function rId198.png) which depends on three

variables, x,y and also u. If we look at the total derivative of g, we have

rId199.png

At this point u is an independent variable. But suppose we choose it to be a specific function of x and y, defined by

rId200.png

Then the term proportional to dx in (4.5) vanishes and we have

rId201.png

Or, in other words, g is to be thought of as a function of u and y : $g = g(u,y)$. If we want an explicit expression for g(u,y), we must first invert (4.6) to get $x = x(u;y)$ and then insert this into the definition of g so that

$$0.92\ g(u,y) = ux(u,y) - f(x(u,y),y)$$

This is the Legendre transform. It takes us from one function $f(x,y)$ to a different function g(u; y) where $u = \delta f/\delta x$. The key point is that we haven't lost any information. Indeed, we can always recover $f(x,y)$ from $g(u,y)$ by noting that

rId202.png) and rId203.png

which assures us that the inverse Legendre transform $f = (\delta g/\delta u)u - g$ takes us back to the original function. The geometrical meaning of the Legendre transform is captured in the diagram. For fixed y, we draw the two curves f(x, y) and ux. For each slope u, the value of $g(u)$ is the maximal distance between the two curves. To see this, note that extremising this distance means

rId204.png

Hamiltonian's Equations

The Lagrangian $L(qi,q_i,t)$ is a function of the coordinates q_i, their time derivatives q_i and (possibly) time. We de ne the Hamiltonian to be the Legendre transform of the Lagrangian with

respect to the q_i variables,

rId205.png

where q_i is eliminated from the right hand side in favour of p_i by using

rId206.png

0.92 and inverting to get $q_i = q_i(q_j,p_j,t)$. Now look at the variation of H:

rId207.png

rId208.png

but we know that this can be rewritten as

rId209.png

So we can equate terms. So far this is repeating the steps of the Legendre transform. The new ingredient that we now add is Lagrange's equation which reads $p_i = \delta L/\delta q_i$. We find

rId210.png

These are Hamilton's equations. We have replaced n $2^n d$ order differential equations by 2n $1^s t$ order differential equations for q_i and p_i. In practice, for solving problems, this isn't particularly helpful. But, as we shall see, conceptually it's very useful!

Example A Particle in a Potential

Let's start with a simple example: a particle moving in a potential in

3-dimensional space. The Lagrangian is simply

rId211.png

We calculate the momentum by taking the derivative with respect to r

rId212.png

which, in this case, coincides with what we usually call momentum. The Hamiltonian is then given by

rId213.png

where, in the end, we've eliminated r in favor of $\mathbf{p}$ and written the Hamiltonian as a function of $\mathbf{p}$ and $\mathbf{r}$. Hamilton's equations are simply

rId214.png

rId215.png

which are familiar: the first is the definition of momentum in terms of velocity; the second is Newton's equation for this system.

A Particle in an Electromagnetic Field

We saw in section 2.5.7 that the Lagrangian for a charged particle moving in an electromagnetic field is

rId216.png

From this we compute the momentum conjugate to the position

rId217.png

which now differs from what we usually call momentum by the addition of the vector potential A. Inverting, we have

rId218.png

So we calculate the Hamiltonian to be

$$0.92\ H(p,r) = p.r - L$$

rId219.png

Now Hamilton's equations read

rId220.png

while the $p_a = -\delta H/\delta r$ equation is best expressed in terms of components

rId221.png

An Example of the Example

Let's illustrate the dynamics of a particle moving in a magnetic field by looking at a particular case. Imagine a uniform magnetic field pointing in the z-direction: $B = (0{,}0{,}B)$. We can get this from a vector potential $B = \mathrm{O}A$ with

$A = (-By{,}0{,}0)$

This vector potential isn't unique: we could choose others related by a gauge transform as described in section 2.5.7. But this one will do for our purposes. Consider a particle moving in the $(x;y)$-plane. Then the Hamiltonian for this system is

rId222.png

From which we have four, first order differential equations which are Hamilton's equations

rId223.png

rId224.png

If we add these together in the right way, we find that

rId225.png

and rId226.png

which is easy to solve: we have

rId227.png

with a, b,R and t_0 integration constants. So we see that the particle makes circles in the (x,y)-plane with frequency

rId228.png

This is known as the cyclotron frequency.

Some Conservation Laws In Section 2, we saw the importance of conservation laws in solving a given problem. The conservation laws are often simple to see in the Hamiltonian formalism. For example,

Claim: If $\delta H/\delta t = 0$ (i.e. H does not depend on time explicitly) then H itself is a constant of motion. **Proof**:

rId229.png

Claim: If an ignorable coordinate q doesn't appear in the Lagrangian then, by construction, it also doesn't appear in the Hamiltonian. The conjugate momentum p_q is then conserved.

Proof

rId230.png

The Principle of Least Action

Recall that in section 2.1 we saw the principle of least action from the Lagrangian perspective. This followed from defining the action

rId231.png

Then we could derive Lagrange's equations by insisting that $\delta S = 0$ for all paths with fixed end points so that $\delta q_i(t_1) = \delta q_i(t_2) = 0$. How does this work in the Hamiltonian formalism? It's quite simple! We define the action

rId232.png

where, of course, $q_i = q_i(q_i,p_i)$. Now we consider varying qi and pi independently. Notice that this is different from the Lagrangian set-up, where a variation of qi automatically leads to a variation of q_i. But remember that the whole point of the Hamiltonian formalism is that we treat q_i and p_i on equal footing. So we vary both. We have

rId233.png

rId234.png

Except there's a very slight subtlety with the boundary conditions. We need the last term in (4.38) to vanish, and so require only that

$\delta q_i(t_i) = \delta q_i(t_2) = 0$

while δp_i can be free at the end points $t = t_1$ and $t = t_2$. So, despite our best efforts, q_i and p_i are not quite symmetric in this formalism.

Note that we could simply impose $\delta p_i(t_1) = \delta p_i(t_2) = 0$ if we really wanted to and the above derivation still holds. It would mean we were being more restrictive on the types of paths we considered. But it does have the advantage that it keeps qi and pi on a symmetric footing. It also means that we have the freedom to add a function to consider actions of the form

rId235.png

so that what sits in the integrand differs from the Lagrangian. For some situations this may be useful.

Conic Sections

Numerous functions are encountered in the day-to-day life of a scientist. These functions include polynomials,trigonometry, hyperbolic functions amongst others. However throughout history of science one group of functions, the *conics*, arise time and time again not only in the development of mathematical theory but also in practical application. The conics were first studied by the Greek mathematician Apollonius more than 200 years BC. However, it was not until the 17th century that Sir Isaac Newton found their application to physics.

Conic sections have been known since the ancient Greek era. In mathematics, a *conic section* is a curve obtained as the intersection of the surface of a cone with a plane. The three types of conic section are the hyperbola, the parabola, and the ellipse.The circle is a special case of the ellipse, and is sufficient interest in its own right that it was sometimes called the fourth type of conic section.

The Ellipse

An ellipse informally is an oval or a squished circle. an ellipse is obtained by the plane which is not perpendicular to the core-axis, but cutting the cone in a closed curve. Various ellipses are obtained as the plane continues to rotate.

The sum of the distances to any point on the ellipse (x,y) from the two foci (c,0) and (-c,0) is a constant That will be 2a. If we let d_1 and d_2 bet the distances from the foci to the point then $d_1 + d_2 = 2$a. You can use that definition to derive the equation of an ellipse, but I will give you the short form below.

The ellipse is a stretched circle. Begin with the unit circle (circle with a radius of 1) centered at the origin. Stretch the vertex from x = 1 to x = a and the point y=1 to y = b. What you have done is multiplied every x by a and multiplied every y by b.

rId236.jpg

0.92 Figure 4: The Conics.

In translation form, you represent that by X divided by a and y divided by b. So, the equation of the circle changes from rId237.png = 1 and the standard equation for an ellipse centered at the origin.

The center is the starting point at (h,k).

The major axis contains the foci and the vertices.

Major axis length = 2a. This is also the constant the sum of the distances added to be.

Minor axis length = 2b.

Distance between foci = 2c.

The foci are within the curve.

Since the vertices are the farthest away from the center, a is the largest of the three lengths, and the Pythagorean relationship is $a^2 = b^2 + c^2$.

rId238.jpg

0.92 Figure 5: An Ellipse.

The Circle

A circle is "the set of all the points in a plane equidistant from a fixed point(circle)".

The standard form for a circle with center at the origin is $x^2 + y^2$ - r^2 , where r is the radius of the circle. Here the center of the circle is located at the origin (0,0) and the radius of the circle the center of the circle at a point (α,β) then we use the form:

$$0.92\ (x - \alpha)^2 + (y - \beta)^2 = r^2.$$

The Parabola

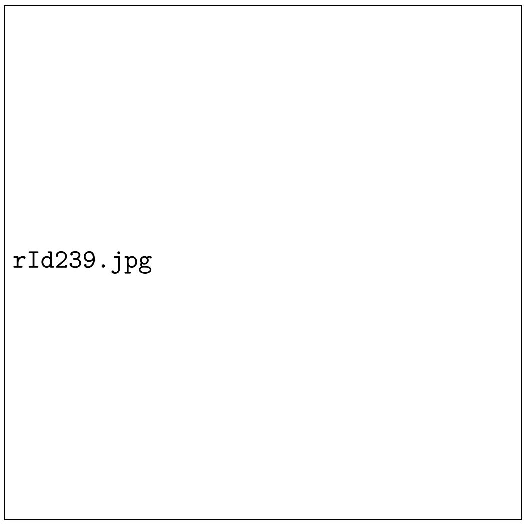
A parabola is "the set of all points in a plane equidistant from a fixed point (focus) and a fixed line (directrix)". The distances to any point (x,y) on the parabola from the focus (0,p) and the directrix y = -p, are equal to each other. This can be used to develop the equation of a parabola.

If you take the definition of a parabola and work out the algebra, you can develop the equation of a parabola. This is a short but standard version on how to develop that standard form of x^2 = 4py.

The standing point is the vertex at (h,k)

There is an axis of symmetry that contains the focus and thr vertex and is perpendicular to the directrix.

Move p units along the axis of symmetry from the vertex to the focus.



0.92 Figure 6: A Circle.

Move -p units along the axis of symmetry from the vertex to the directrix (which is a line)

The focus is within the curve.

The parabola has the property that any signal (light, sound, etc) entering the parabola parallel to the axis of symmetry will be reflected through the focus (this is why satellite dishes and those parabolic antennas that the detectives use to eavesdrop on conversations work). Also, any signal originating at the focus will be reflected out parallel to the axis of symmetry (this is why flashlights work).

The Hyperbola

A hyperbola is "the set of all points in a plane such that the difference of the distances from two fixed points (foci) is constant ". The difference of the distances to any point on the hyperbola(x,y) from the two foci (c,0) and (-c,0) is a constant. That constant will be 2a. If we let d_1 and d_2 be the distances from the foci to the point, them —d_1 - d_2 — = 2a. The absolute value is around the difference so that it is always positive. You can use the definition to derive the equation of a hyperbola, but below is a short but standard way of deriving it. The only difference in the definition of a hyperbola and that of an ellipse is that the hyperbola is the difference of the distances from the rId240.jpg

0.92 Figure 7: A Parabola.

foci that is constant and the ellipse is the sum of the distances from the foci that is constant.

Instead of the equation being (rId241.png = 1, the equation is (rId242.png

= 1

The graphs, however , are very different.

The center is starting point at (h,k).

The Transverse axis contains the foci and the vertices.

Transverse axis length = 2a. This is also the constant that the difference od the distances must be.

Conjugate axis length = 2b

Distance between foci = 2c.

the foci within the curve.

0.92 Since the foci are the farthest away from the center, c is the largest of the three lengths, and the Pythagorean relationship is $a^2 + b^2 = c^2$.

General Conics

The conics we have considered above-the ellipse, the parabola and the hyperbol-have all been presented in standard form; their axes are parallel

rId243.jpg

0.92 Figure 8: A Hyperbola.

to either the x or y-axes. However, conics m ay be rotated to any angle with respect to the axes; they clearl remain conics, but what equations do they have?

It can be shown that the equation of any conic, can be described by the quadratic expression

Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0

where A,B,C,D,E,F, are constants. If not all of A,B,C are zero the graph of this equation is

an ellipse if B^2 ¡ 4AC (circle if A = C)

a parabola if B^2=4AC

a hyperbola if B^2 ¿ 4AC.

Newton and the Conics

These amongst a few are some of the things that Newton accomplished. He mathematically showed that the orbit of any planet around the Sun should be a conic section with one of its foci located at the Sun, that is if the force of gravity acting between then is inversely proportional to the square of their distances apart. This also fit into Kepler's first law which shows that the orbit of any planet is an ellipse with the sun at one foci. With more observations and experiment ranging from the falling of apples to the moon orbiting around the earth, he asserted that every massive object attracts another with the amount of force being inversely proportional to their distance squared.

Another of these real life conics is that the the trajectory of a planet is perfectly elliptical when only the gravitation due to the sun is considered. This however, is not very true since the trajectory deviates slightly away from a perfect ellipse due to the presence of gravitational attraction of other planets; the line connecting the two foci of a planet rotates very slowly rather than being fixed. In the case of Mercury, it rotates by 0.160 degrees per century. However, if you calculate the gravitational effect of other planets, it should rotate by only 0.148 degrees. In 1915, Einstein successfully accounted for this difference by discovering theory of general relativity., In other words, he discovered that the gravitational force is not exactly, but only approximately, inversely proportional to the square of the distance.

When a ball is bouncing off, the maths of conics can be used to map its trajectory. A satellite is also designed to be a parabola. If they are designed otherwise, they wont function properly. A signal sent to it will be lost of an odea of where the focus would be is not considered. The solar cook is also not left out. It is designed in a form of parabola in other to receive maximum solar energy for efficiency. The touch light is designed the same way in other for the emerging beam to shine on a wider area. Telescopes also apply the idea of conics.