helmholtz

December 28, 2022

1 Numerical solution of the Helmholtz equation

1.1 Description

1.1.1 PDE

We will try to solve the following pde:

$\Delta\psi(x,y) + k^2\psi(x,y) = -f(x,y)$ on $\Omega = [0,1]^2$

where $f(x,y) = 2k^2 e^{jk(x+y)} - k^2 e^{jk(x+y)} = k^2 e^{jk(x+y)}$

1.1.2 Boundary conditions

$\psi(0,y) = e^{jky}, \quad \psi(1,y) = e^{jk(1+y)}, \quad \psi(x,0) = e^{jkx} \quad$ and $ (x,1)=e^{jk(x+1)} $

1.1.3 Loss function

The loss to minimize here is $\mathcal{L} = \frac{1}{N}||\Delta\psi(x,y) + k^2\psi(x,y) + f(x,y)||_2^2$, where N is the size of the training data

1.1.4 Analytical solution

The solution ψ_{th} should be $\psi_{th}(x,y) = e^{jk(x+y)}$

1.1.5 Approximated solution

We want to find a approximated solution $\psi_a(x,y) = A(x,y) + F(x,y)N(x,y)$ s.t:

$F(x,y) = \sin(x)\sin(y)\sin(x-1)\sin(y-1)$

$A(x,y) = (1-x)e^{jky} + xe^{jk(1+y)} + (1-y)\{e^{jkx} - [(1-x) + xe^{jk}]\} + y\{e^{jk(x+1)} - [(1-x)e^{jk} + xe^{j2k}]\}$

2 Libraries

[1]:
```
import jax, optax
import pickle
import functools
import matplotlib.pyplot
import numpy
%matplotlib inline
```

```
# Set and verify device
jax.config.update('jax_platform_name', 'gpu')
jax.config.update("jax_enable_x64", True)
print(jax.lib.xla_bridge.get_backend().platform)
```

gpu

3 Parameters

[2]:
```
# Neural network parameters
parameters = {}
parameters['seed'] = 351
parameters['n_features'] = 2        # Input dimension (x1, x2)
parameters['n_targets'] = 2         # Output dimension. It's a complex number␣
 ↪(y1 + j*y2)
parameters['hidden_layers'] = [50, 50, 50, 50, 50]    # Hidden layers structure
parameters['layers'] = [parameters['n_features']] + parameters['hidden_layers']␣
 ↪+ [parameters['n_targets']]

# Training parameters
parameters['learning_rate'] = optax.linear_schedule(0.005, 0.00001,␣
 ↪transition_steps = 50, transition_begin = 5000)
parameters['optimizer'] = optax.adam(parameters['learning_rate'])
parameters['maximum_num_epochs'] = 50000
parameters['report_steps'] = 1000
parameters['options'] = 1           # 1: we start a new training. 2: We␣
 ↪continue the last training.
                                    # Other cases: We just load the last␣
 ↪training

# Data parameters
parameters['batch_size'] = 100
parameters['domain_bounds'] = jax.numpy.column_stack(([0.0, 0.0], [1.0, 1.0])) ␣
 ↪ # minimal and maximal value of each axis (x, y)
```

4 Neural network

[3]:
```
class MLP:
    """
        Create a multilayer perceptron and initialize the neural network
    Inputs :
        A SEED number and the layers structure
    """
    def __init__(self, key, layers):
        self.key = key
```

```
        self.keys = jax.random.split(self.key,len(layers))
        self.layers = layers
        self.params = []

    def MLP_create(self):
        """
        Initialize the MLP weigths and bias
        Parameters
        ----------
        Returns
        -------
        params : list of parameters[[w1,b1],...,[wn,bn]]
            -- weights and bias
        """
        for layer in range(0, len(self.layers)-1):
            in_size,out_size = self.layers[layer], self.layers[layer+1]
            weights = jax.nn.initializers.glorot_normal()(self.keys[layer],␣
↪(out_size, in_size), jax.numpy.float32)
            bias = jax.nn.initializers.lecun_normal()(self.keys[layer],␣
↪(out_size, 1), jax.numpy.float32).reshape((out_size, ))
            self.params.append((weights, bias))
        return self.params


    @functools.partial(jax.jit, static_argnums=(0,))
    def NN_evaluation(self, params, inputs):
        """
        Evaluate a position (x,y) using the neural network
        Parameters
        ----------
        params : list of parameters[[w1,b1],...,[wn,bn]]
            -- weights and bias
        inputs : jax.numpy.ndarray[[batch_size,batch_size]]
            -- points in the domain
        Returns
        -------
        output : jax.numpy.array[batch_size]
            -- neural network output
        """
        for layer in range(0, len(params)-1):
            weights, bias = params[layer]
            inputs = jax.nn.tanh(jax.numpy.add(jax.numpy.dot(inputs, weights.
↪T), bias))
        weights, bias = params[-1]
        real_and_imaginary_layers = jax.numpy.dot(inputs, weights.T)+bias  #␣
↪The first output of the NN is the real part, the second is the imaginary part
```

```
        output = jax.lax.complex(real_and_imaginary_layers[0],␣
↪real_and_imaginary_layers[1])

        return output
```

5 Operators

[4]:
```
class PDE_operators:
    """
        Class with the operators used to solve the PDE
    Input:
        A function that we want to compute the respective operator
    """
    def __init__(self, function):
        self.function = function

    @functools.partial(jax.jit, static_argnums=(0,))
    def laplacian_2d(self, params, inputs):
        """
        Compute the two dimensional laplacian.
        Parameters
        ----------
        params : list of parameters[[w1,b1],...,[wn,bn]]
            -- weights and bias
        inputs : jax.numpy.ndarray[[batch_size,batch_size]]
            -- coordinates (x, y)
        Returns
        -------
        laplacian : jax.numpy.ndarray[batch_size]
            -- numerical values of the laplacian applied to the inputs
        """
        fun = lambda params,x, y: self.function(params, x, y)
        @functools.partial(jax.jit)
        def action(params,x, y):
            u_xx = jax.jacfwd(jax.jacfwd(fun, 1), 1)(params,x, y)
            u_yy = jax.jacfwd(jax.jacfwd(fun, 2), 2)(params,x, y)
            return u_xx + u_yy
        vec_fun = jax.vmap(action, in_axes = (None, 0, 0))
        laplacian = vec_fun(params, inputs[:,0], inputs[:,1]).flatten()
        return laplacian
```

6 Physics Informed Neural Network

[5]:
```
class PINN:
    """
    Solve a PDE using Physics Informed Neural Networks
    Input:
        The evaluation function of the neural network and the optimizer␣
↪selected to do gradient descent
    """
    def __init__(self, NN_evaluation, optimizer):
        self.NN_evaluation = NN_evaluation
        self.optimizer = optimizer

        self.operators = PDE_operators(self.spatial_solution2d)
        self.laplacian2d = self.operators.laplacian_2d

        self.k_coeff = 0.5  # Wavenumber


    @functools.partial(jax.jit, static_argnums = (0, ))
    def spatial_solution2d(self, params, inputX, inputY):
        """
        Compute the complex solution of the PDE on the points (x, y)
        Parameters
        ----------
        params : list of parameters[[w1,b1],...,[wn,bn]]
            -- weights and bias
        inputX : jax.numpy.array[batch_size]
            -- points on the x-axis of the domain
        inputY : jax.numpy.array[batch_size]
            -- points on the y-axis of the domain
        Returns
        -------
        applied_solution : jax.numpy.array[batch_size]
            -- PINN solution applied to inputs. return F_NN_plus_A, a complex␣
↪array
        """
        inputs = jax.numpy.column_stack((inputX, inputY))
        NN = jax.vmap(functools.partial(jax.jit(self.NN_evaluation),␣
↪params))(inputs)
        F = self.F_function(inputX,inputY)
        A = self.A_function(inputX,inputY)
        F_NN_plus_A = jax.numpy.add(jax.numpy.multiply(F, NN), A).reshape(-1)
        return F_NN_plus_A


    # Definition of the function A(x, y) mentioned above
```

```python
    @functools.partial(jax.jit, static_argnums = (0, ))
    def A_function(self, inputX, inputY):
        """
        Compute A(x, y) on the inputs
        Parameters
        ----------
        inputX : jax.numpy.ndarray[batch_size]
            -- points in the axis x
        inputY : jax.numpy.ndarray[batch_size]
            -- points in the axis y
        Returns
        -------
        A_output : jax.numpy.array[batch_size]
            -- A(x, y) applied to inputs
        """
        j_number = jax.lax.complex(0.0,1.0)
        A1 = (1-inputX)*jax.numpy.exp(j_number*self.k_coeff*inputY) +␣
↪inputX*jax.numpy.exp(j_number*self.k_coeff*(1+inputY))
        A2 = (1-inputY)*(jax.numpy.exp(j_number*self.k_coeff*inputX) -␣
↪((1-inputX) + inputX*jax.numpy.exp(j_number*self.k_coeff)))
        A3 = inputY*(jax.numpy.exp(j_number*self.k_coeff*(inputX + 1)) -␣
↪((1-inputX)*jax.numpy.exp(j_number*self.k_coeff) + inputX*jax.numpy.
↪exp(j_number*2*self.k_coeff)))
        A_output = A1 + A2 + A3
        return A_output


    # Definition of the function F(x, y) mentioned above
    @functools.partial(jax.jit, static_argnums = (0, ))
    def F_function(self, inputX, inputY):
        """
        Compute F(x, y) on the inputs
        Parameters
        ----------
        inputX : jax.numpy.ndarray[batch_size]
            -- points in the axis x
        inputY : jax.numpy.ndarray[batch_size]
            -- points in the axis y
        Returns
        -------
        F_output : jax.numpy.array[batch_size]
            -- F(x, y) applied to inputs
        """
        F1 = jax.numpy.multiply(jax.numpy.sin(inputX), jax.numpy.sin(inputX -␣
↪1))
        F2 = jax.numpy.multiply(jax.numpy.sin(inputY), jax.numpy.sin(inputY -␣
↪1))
```

```
        F_output = jax.numpy.multiply(F1, F2)
        return F_output


    # Definition of the function f(x, y) mentioned above
    @functools.partial(jax.jit, static_argnums = (0, ))
    def exact_function(self, inputs):
        """
        Compute f(x, y) on the inputs
        Parameters
        ----------
        inputs : jax.numpy.ndarray[[batch_size, batch_size]]
            -- (x, y) points from the domain
        Returns
        -------
        exact_output : jax.numpy.array[batch_size]
            -- f(x, y) applied to inputs
        """
        j_number = jax.lax.complex(0.0,1.0)
        exact_output = self.k_coeff**2*jax.numpy.exp(j_number*self.
↪k_coeff*(inputs[:,0] + inputs[:,1]))
        return exact_output


    # Definition of the pde mentioned above
    @functools.partial(jax.jit, static_argnums = (0, ))
    def pde_function(self, params, inputs):
        """
        Compute the pde on the inputs
        Parameters
        ----------
        params : list of parameters[[w1,b1],...,[wn,bn]]
            -- weights and bias
        inputs : jax.numpy.ndarray[[batch_size, batch_size]]
            -- (x, y) points from the domain
        Returns
        -------
        pde_value : jax.numpy.array[batch_size]
            -- pde applied to inputs
        """
        pde_value = self.laplacian2d(params, inputs) + self.k_coeff**2*self.
↪spatial_solution2d(params, inputs[:,0], inputs[:,1])
        return pde_value


    # Definition of the loss function mentioned above
    @functools.partial(jax.jit, static_argnums = (0, ))
```

```python
    def loss_function(self, params, inputs):
        """
        Compute the loss of the pde inside the domain
        Parameters
        ----------
        params : list of parameters[[w1,b1],...,[wn,bn]]
            -- weights and bias
        inputs : jax.numpy.ndarray[[batch_size, batch_size]]
            -- (x, y) points from the domain
        Returns
        -------
        loss_value : a float64
            -- loss function applied to inputs
        """
        residual = self.pde_function(params, inputs) + self.
↪exact_function(inputs)
        loss_value = (jax.numpy.linalg.norm(residual)**2)/inputs.shape[0]
        return loss_value


    # Make one train step
    @functools.partial(jax.jit, static_argnums = (0, ))
    def train_step(self, params, opt_state, inputs):
        """
        Make just one step of the training
        Parameters
        ----------
        params : list of parameters[[w1,b1],...,[wn,bn]]
            -- weights and bias
        opt_state : a tuple given by optax library
            -- state(hystorical) of the gradient descent
        inputs : jax.numpy.ndarray[[batch_size, batch_size]]
            -- (x, y) inputs from the domain
        Returns
        -------
        loss : a float64
            -- loss function applied to inputs
        new_params : list of parameters[[w1,b1],...,[wn,bn]]
            -- weights and bias updated
        opt_state : a tuple given by optax
            -- update the state(hystorical) of the gradient descent
        """
        loss, gradient = jax.value_and_grad(self.loss_function)(params, inputs)
        updates, new_opt_state = self.optimizer.update(gradient, opt_state)
        new_params = optax.apply_updates(params, updates)

        return loss, new_params, new_opt_state
```

7 Analytical solution

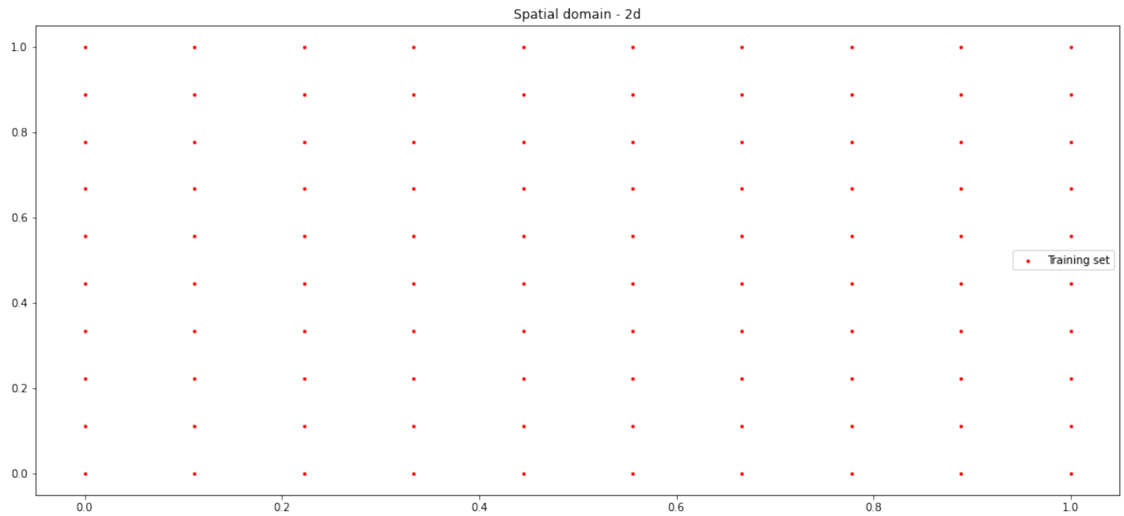
[ ]:
```
@jax.jit
def analytical_solution(x, y, k = 0.5):
    """
    Compute the analytical solution on the domain
    Parameters
    ----------
    inputX : jax.numpy.ndarray[batch_size]
        -- points in the axis x
    inputY : jax.numpy.ndarray[batch_size]
        -- points in the axis y
    Returns
    -------
    sol : jax.numpy.array[batch_size]
        -- analytical solution applied to inputs
    """
    j_number = jax.lax.complex(0.0, 1.0)
    sol = jax.numpy.exp(j_number*k*(x + y))
    return sol
```

8 Dataset creation

[6]:
```
x = jax.numpy.linspace(parameters['domain_bounds'][0,0],␣
 ↪parameters['domain_bounds'][0,1], int(jax.numpy.
 ↪sqrt(parameters['batch_size'])))
y = jax.numpy.linspace(parameters['domain_bounds'][1,0],␣
 ↪parameters['domain_bounds'][1,1], int(jax.numpy.
 ↪sqrt(parameters['batch_size'])))
x, y = jax.numpy.meshgrid(x, y)
x, y = x.flatten(), y.flatten()
XY_train = jax.numpy.column_stack((x, y))
```

9 Dataset plot

[7]:
```
fig, ax = matplotlib.pyplot.subplots()
fig.set_size_inches(18, 8.0)
title = ax.set_title('Spatial domain - 2d')
graph = matplotlib.pyplot.scatter(XY_train[:,0], XY_train[:,1], color = 'red',␣
 ↪s = 5)
__ = ax.legend(['Training set'])

matplotlib.pyplot.savefig('./Images/domain2d.png', facecolor = 'white',␣
 ↪bbox_inches = 'tight')
matplotlib.pyplot.show()
```



10 Model initialization

[8]:
```
key = jax.random.PRNGKey(parameters['seed'])
NN_MLP = MLP(key, parameters['layers'])
params = NN_MLP.MLP_create()            # Create the MLP
NN_eval = NN_MLP.NN_evaluation            # Evaluation function
solver = PINN(NN_eval, parameters['optimizer'])
opt_state = parameters['optimizer'].init(params)
```

11 Training

[9]:
```
loss_history = []
print("Training start")
if parameters['options'] == 1:            # start a new training
    # Main loop to solve the PDE
    for ibatch in range(parameters['maximum_num_epochs']+1):
        loss, params, opt_state = solver.train_step(params, opt_state, XY_train)

        loss_history.append(float(loss))

        if (ibatch%parameters['report_steps']) == parameters['report_steps']-1:
            print("Epoch n°{}: ".format(ibatch+1), loss.item())

        if loss <= numpy.min(loss_history): # save if the current state is the best
            pickle.dump(params, open("./Checkpoints/params_helmholtz", "wb"))
            pickle.dump(opt_state, open("./Checkpoints/opt_state_helmholtz", "wb"))
```

```
            pickle.dump(loss_history, open("./Checkpoints/
↪loss_history_helmholtz", "wb"))

elif parameters['options'] == 2:      # continue the last training
    params = pickle.load(open("./Checkpoints/params_helmholtz", "rb"))
    opt_state = pickle.load(open("./Checkpoints/opt_state_helmholtz", "rb"))
    loss_history = pickle.load(open("./Checkpoints/loss_history_helmholtz",␣
↪"rb"))
    iepoch = len(loss_history)

    # Main loop to solve the PDE
    for ibatch in range(iepoch, parameters['maximum_num_epochs']+1):
        loss, params, opt_state = solver.train_step(params, opt_state, XY_train)

        loss_history.append(float(loss))

        if (ibatch%parameters['report_steps']) == parameters['report_steps']-1:
            print("Epoch n°{}: ".format(ibatch+1), loss.item())

        if loss <= numpy.min(loss_history): # save if the current state is the␣
↪best
            pickle.dump(params, open("./Checkpoints/params_helmholtz", "wb"))
            pickle.dump(opt_state, open("./Checkpoints/opt_state_helmholtz",␣
↪"wb"))
            pickle.dump(loss_history, open("./Checkpoints/
↪loss_history_helmholtz", "wb"))
else:
    params = pickle.load(open("./Checkpoints/params_helmholtz", "rb"))
    opt_state = pickle.load(open("./Checkpoints/opt_state_helmholtz", "rb"))
    loss_history = pickle.load(open("./Checkpoints/loss_history_helmholtz",␣
↪"rb"))
```

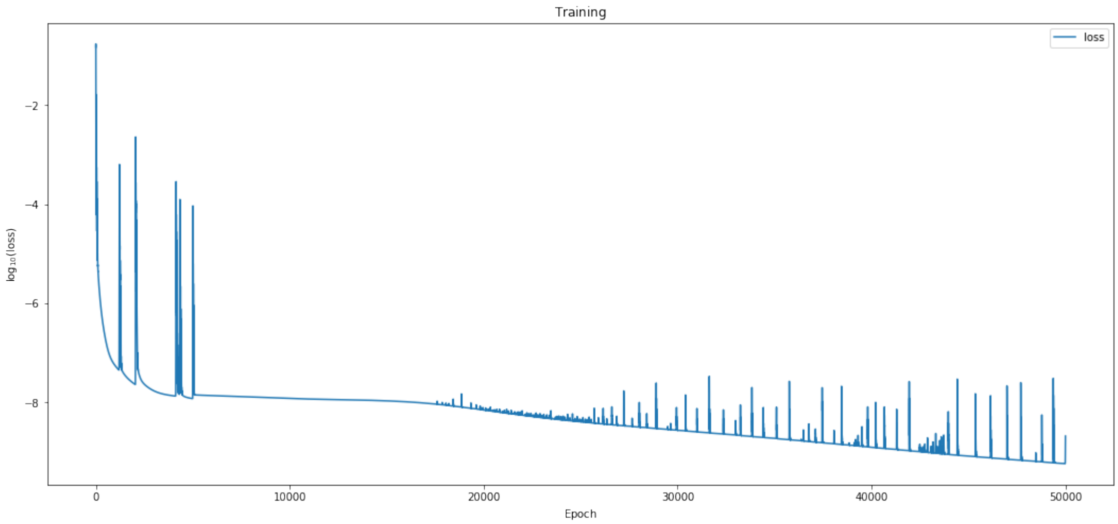
```
Training start
Epoch n°1000:  5.508144004289995e-08
Epoch n°2000:  2.3492064342453148e-08
Epoch n°3000:  1.712214673089992e-08
Epoch n°4000:  1.3584209498974991e-08
Epoch n°5000:  6.782584311577836e-07
Epoch n°6000:  1.3726638647027041e-08
Epoch n°7000:  1.3330959503670555e-08
Epoch n°8000:  1.292204132239227e-08
Epoch n°9000:  1.2481982923987923e-08
Epoch n°10000:  1.2053273138377728e-08
Epoch n°11000:  1.1713291331651193e-08
Epoch n°12000:  1.1489137949481197e-08
Epoch n°13000:  1.1309575373356905e-08
Epoch n°14000:  1.1089247318201054e-08
```

```
Epoch n°15000:  1.077665372165983e-08
Epoch n°16000:  1.0324800746245108e-08
Epoch n°17000:  9.686227017984392e-09
Epoch n°18000:  8.846504053505464e-09
Epoch n°19000:  7.929729709992478e-09
Epoch n°20000:  7.032529485325194e-09
Epoch n°21000:  6.220678566630447e-09
Epoch n°22000:  5.553653758169135e-09
Epoch n°23000:  4.931916761029431e-09
Epoch n°24000:  4.710894492075254e-09
Epoch n°25000:  4.076219021928677e-09
Epoch n°26000:  3.72667566061403e-09
Epoch n°27000:  3.4367460096975306e-09
Epoch n°28000:  3.1837851237343453e-09
Epoch n°29000:  2.953595009631633e-09
Epoch n°30000:  2.741923196143276e-09
Epoch n°31000:  4.641513257320245e-09
Epoch n°32000:  2.362142943674844e-09
Epoch n°33000:  2.947162020402487e-09
Epoch n°34000:  2.0270427864634136e-09
Epoch n°35000:  1.876677980376093e-09
Epoch n°36000:  1.734740948593067e-09
Epoch n°37000:  1.5955786930573046e-09
Epoch n°38000:  1.470499107296139e-09
Epoch n°39000:  1.3568966503466736e-09
Epoch n°40000:  1.24363017358315e-09
Epoch n°41000:  1.1446279565527412e-09
Epoch n°42000:  1.0963765863613892e-09
Epoch n°43000:  9.70507006996731e-10
Epoch n°44000:  9.060656007673109e-10
Epoch n°45000:  8.261451947135163e-10
Epoch n°46000:  7.673061995049718e-10
Epoch n°47000:  1.4628239085135382e-09
Epoch n°48000:  6.666410260233888e-10
Epoch n°49000:  6.207587188713992e-10
Epoch n°50000:  1.8636584057964905e-09
```

12 Loss function plot

[10]:
```
fig, ax = matplotlib.pyplot.subplots(1, 1)
fig.set_size_inches(18, 8.0)
__ = ax.plot(numpy.log10(loss_history))
xlabel = ax.set_xlabel(r'${\rm Epoch}$')
ylabel = ax.set_ylabel(r'$\log_{10}{\rm (loss)}$')
title = ax.set_title(r'${\rm Training}$')
ax.legend(['loss'])
```

```
matplotlib.pyplot.savefig('./Images/loss_function.png', facecolor = 'white',␣
↪bbox_inches = 'tight')
matplotlib.pyplot.show()
```



13 Load best params of the training

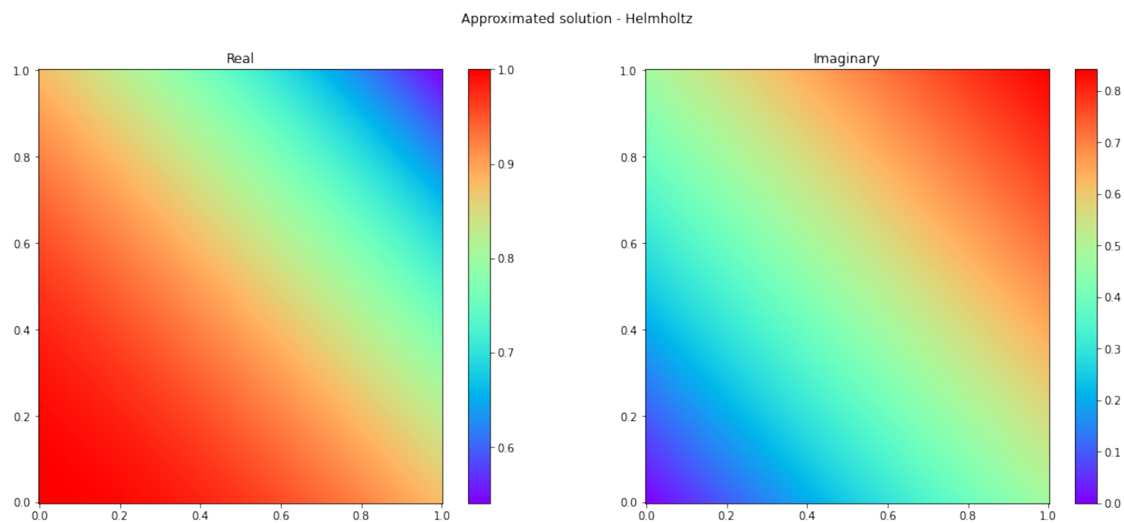
[11]:
```
params = pickle.load(open("./Checkpoints/params_helmholtz", "rb"))
```

14 Approximated solution plot

[12]:
```
npoints = 200
real_values = numpy.zeros((npoints, npoints))
imag_values = numpy.zeros((npoints, npoints))

x, y = numpy.meshgrid(numpy.linspace(parameters['domain_bounds'][0,0],␣
↪parameters['domain_bounds'][0,1], npoints), numpy.
↪linspace(parameters['domain_bounds'][1,0], parameters['domain_bounds'][1,1],␣
↪npoints))

fig, ax = matplotlib.pyplot.subplots(1,2)
fig.set_size_inches(18, 7.2)
title = fig.suptitle('Approximated solution - Helmholtz')

for i in range(npoints):
    print("Plotting: {} out of {}".format(i+1, npoints), end='\r')
    real_values[i,:] = jax.numpy.real(functools.partial(solver.
↪spatial_solution2d, params)(x[i,:], y[i,:]))
```

```
    imag_values[i,:] = jax.numpy.imag(functools.partial(solver.
 ↪spatial_solution2d, params)(x[i,:], y[i,:]))

title = ax[0].set_title('Real')
graph = ax[0].pcolormesh(x, y, real_values, cmap = 'rainbow')
matplotlib.pyplot.colorbar(graph, ax=ax[0])

title = ax[1].set_title('Imaginary')
graph = ax[1].pcolormesh(x, y, imag_values, cmap = 'rainbow')
matplotlib.pyplot.colorbar(graph, ax=ax[1])


matplotlib.pyplot.savefig('./Images/approximated_helmholtz.png', facecolor =␣
 ↪'white', bbox_inches = 'tight')
matplotlib.pyplot.show()
```

```
Plotting: 200 out of 200
```

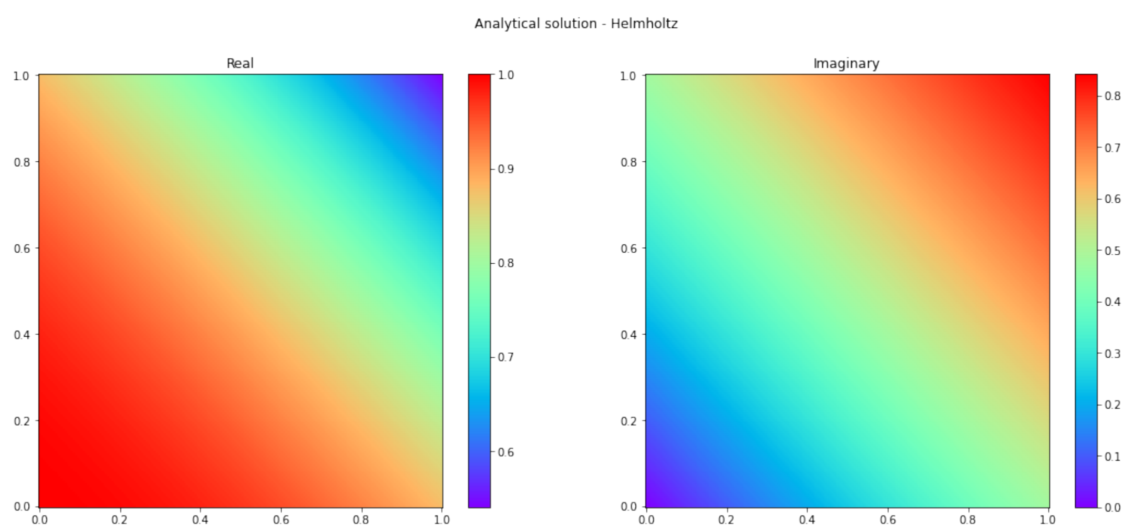


15 Analytical solution plot

[14]:
```
npoints = 200
real_values = numpy.zeros((npoints, npoints))
imag_values = numpy.zeros((npoints, npoints))

x, y = numpy.meshgrid(numpy.linspace(parameters['domain_bounds'][0,0],␣
 ↪parameters['domain_bounds'][0,1], npoints), numpy.
 ↪linspace(parameters['domain_bounds'][1,0], parameters['domain_bounds'][1,1],␣
 ↪npoints))
```

```
fig, ax = matplotlib.pyplot.subplots(1,2)
fig.set_size_inches(18, 7.2)
title = fig.suptitle('Analytical solution - Helmholtz')

for i in range(npoints):
    print("Plotting: {} out of {}".format(i+1, npoints), end='\r')
    real_values[i,:] = jax.numpy.real(functools.
 ↪partial(analytical_solution)(x[i,:], y[i,:]))
    imag_values[i,:] = jax.numpy.imag(functools.
 ↪partial(analytical_solution)(x[i,:], y[i,:]))

title = ax[0].set_title('Real')
graph = ax[0].pcolormesh(x, y, real_values, cmap = 'rainbow')
matplotlib.pyplot.colorbar(graph, ax=ax[0])

title = ax[1].set_title('Imaginary')
graph = ax[1].pcolormesh(x, y, imag_values, cmap = 'rainbow')
matplotlib.pyplot.colorbar(graph, ax=ax[1])


matplotlib.pyplot.savefig('./Images/analytical_helmholtz.png', facecolor =␣
 ↪'white', bbox_inches = 'tight')
matplotlib.pyplot.show()
```

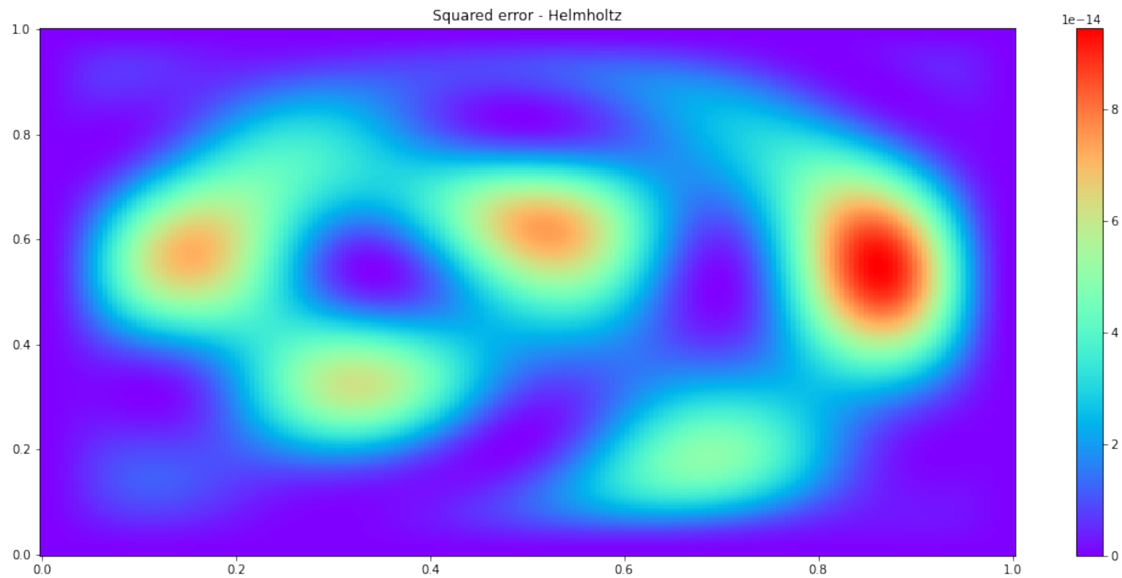
```
Plotting: 200 out of 200
```



16 Squared error plot

[15]:
```
npoints = 200
values = numpy.zeros((npoints, npoints))

x, y = numpy.meshgrid(numpy.linspace(parameters['domain_bounds'][0,0],␣
 ↪parameters['domain_bounds'][0,1], npoints), numpy.
 ↪linspace(parameters['domain_bounds'][1,0], parameters['domain_bounds'][1,1],␣
 ↪npoints))

fig, ax = matplotlib.pyplot.subplots()
fig.set_size_inches(18, 8.0)
title = ax.set_title('Squared error - Helmholtz')

for i in range(npoints):
    print("Plotting: {} out of {}".format(i+1, npoints), end='\r')
    real_squared_error = (jax.numpy.real(functools.partial(solver.
 ↪spatial_solution2d, params)(x[i,:], y[i,:])) - jax.numpy.real(functools.
 ↪partial(analytical_solution)(x[i,:], y[i,:])))**2
    imag_squared_error = (jax.numpy.imag(functools.partial(solver.
 ↪spatial_solution2d, params)(x[i,:], y[i,:])) - jax.numpy.imag(functools.
 ↪partial(analytical_solution)(x[i,:], y[i,:])))**2
    values[i,:] = real_squared_error + imag_squared_error

print("MSE: ", numpy.mean(values.flatten()))
graph = matplotlib.pyplot.pcolormesh(x, y, values, cmap = 'rainbow')
matplotlib.pyplot.colorbar()
matplotlib.pyplot.savefig('./Images/squared_error_helmholtz.png', facecolor =␣
 ↪'white', bbox_inches = 'tight')
matplotlib.pyplot.show()
```

```
MSE:  2.007965818461194e-14
```



Squared error - Helmholtz
1e−14