

# NN\_Jax\_PDE8

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## 1 Solving PDEs with Jax - Problem 8

### 1.1 Description

#### 1.1.1 Average time of execution

Between 2 and 3 minutes on GPU

#### 1.1.2 PDE

We will try to solve the problem 8 of the article <https://ieeexplore.ieee.org/document/712178>:

$$\Delta\psi(x, y) + \psi(x, y) \cdot \frac{\partial\psi(x, y)}{\partial y} = f(x, y) \text{ on } \Omega = [0, 1]^2$$

where  $f(x, y) = \sin(\pi x)(2 - \pi^2 y^2 + 2y^3 \sin(\pi x))$

#### 1.1.3 Boundary conditions

$$\psi(0, y) = \psi(1, y) = \psi(x, 0) = 0 \text{ and } \frac{\partial\psi}{\partial y}(x, 1) = 2 \sin(\pi x)$$

#### 1.1.4 Loss function

The loss to minimize here is  $\mathcal{L} = \|\Delta\psi(x, y) + \psi(x, y) \cdot \frac{\partial\psi(x, y)}{\partial y} - f(x, y)\|_2$

#### 1.1.5 Analytical solution

The true function  $\psi$  should be  $\psi(x, y) = y^2 \sin(\pi x)$

This solution is the same of the problem 7

#### 1.1.6 Approximated solution

We want find a solution  $\psi(x, y) = A(x, y) + F(x, y)N(x, y)$  s.t:

$$F(x, y) = \sin(x - 1) \sin(y - 1) \sin(x) \sin(y) \quad A(x, y) = y \sin(\pi x)$$

## 2 Importing libraries

```
[14]: # Jax libraries
from jax import value_and_grad, vmap, jit, jacfwd
from functools import partial
from jax import random as jran
from jax.example_libraries import optimizers as jax_opt
```

```

from jax.nn import tanh
from jax.lib import xla_bridge
import jax.numpy as jnp

# Others libraries
from time import time
import matplotlib.pyplot as plt
import numpy as np
import os
import pickle
print(xla_bridge.get_backend().platform)

```

### 3 Multilayer Perceptron

```

[15]: class MLP:
        """
        Create a multilayer perceptron and initialize the neural network
        Inputs :
        A SEED number and the layers structure
        """

        # Class initialization
        def __init__(self, SEED, layers):
            self.key = jran.PRNGKey(SEED)
            self.keys = jran.split(self.key, len(layers))
            self.layers = layers
            self.params = []

        # Initialize the MLP weights and bias
        def MLP_create(self):
            for layer in range(0, len(self.layers)-1):
                in_size, out_size = self.layers[layer], self.layers[layer+1]
                std_dev = jnp.sqrt(2/(in_size + out_size))
                weights = jran.truncated_normal(self.keys[layer], -2, 2,
                ↪ shape=(out_size, in_size), dtype=np.float32)*std_dev
                bias = jran.truncated_normal(self.keys[layer], -1, 1, shape=(out_size,
                ↪ 1), dtype=np.float32).reshape((out_size,))
                self.params.append((weights, bias))
            return self.params

        # Evaluate a position XY using the neural network
        @partial(jit, static_argnums=(0,))
        def NN_evaluation(self, new_params, inputs):
            for layer in range(0, len(new_params)-1):
                weights, bias = new_params[layer]
                inputs = tanh(jnp.add(jnp.dot(inputs, weights.T), bias))

```

```

weights, bias = new_params[-1]
output = jnp.dot(inputs, weights.T)+bias
return output

# Get the key associated with the neural network
def get_key(self):
    return self.key

```

## 4 PDE operators

```

[16]: class PDE_operators:
    """
        Class with the most common operators used to solve PDEs
        Input:
            A function that we want to compute the respective operator
    """

    # Class initialization
    def __init__(self,function):
        self.function=function

    # Compute the two dimensional laplacian
    def laplacian_2d(self,params,inputs):
        fun = lambda params,x,y: self.function(params, x,y)
        @partial(jit)
        def action(params,x,y):
            u_xx = jacfwd(jacfwd(fun, 1), 1)(params,x,y)
            u_yy = jacfwd(jacfwd(fun, 2), 2)(params,x,y)
            return u_xx + u_yy
        vec_fun = vmap(action, in_axes = (None, 0, 0))
        laplacian = vec_fun(params, inputs[:,0], inputs[:,1])
        return laplacian

    # Compute the derivative in x
    @partial(jit, static_argnums=(0,))
    def du_dx(self,params,inputs):
        fun = lambda params,x,y: self.function(params, x,y)
        @partial(jit)
        def action(params,x,y):
            u_x = jacfwd(fun, 1)(params,x,y)
            return u_x
        vec_fun = vmap(action, in_axes = (None, 0, 0))
        return vec_fun(params, inputs[:,0], inputs[:,1])

    # Compute the derivative in y
    @partial(jit, static_argnums=(0,))

```

```

def du_dy(self,params,inputs):
    fun = lambda params,x,y: self.function(params, x,y)
    @partial(jit)
    def action(params,x,y):
        u_y = jacfwd(fun, 2)(params,x,y)
        return u_y
    vec_fun = vmap(action, in_axes = (None, 0, 0))
    return vec_fun(params, inputs[:,0], inputs[:,1])

```

## 5 Physics Informed Neural Networks

```

[17]: class PINN:
    """
    Solve a PDE using Physics Informed Neural Networks
    Input:
        The evaluation function of the neural network
    """

    # Class initialization
    def __init__(self,NN_evaluation):
        self.operators=PDE_operators(self.solution)
        self.laplacian=self.operators.laplacian_2d
        self.NN_evaluation=NN_evaluation
        self.dsol_dy=self.operators.du_dy

    # Definition of the function A(x,y) mentioned above
    @partial(jit, static_argnums=(0,))
    def A_function(self,inputX,inputY):
        return jnp.multiply(inputY,jnp.sin(jnp.pi*inputX)).reshape(-1,1)

    # Definition of the function F(x,y) mentioned above
    @partial(jit, static_argnums=(0,))
    def F_function(self,inputX,inputY):
        F1=jnp.multiply(jnp.sin(inputX),jnp.sin(inputX-jnp.ones_like(inputX)))
        F2=jnp.multiply(jnp.sin(inputY),jnp.sin(inputY-jnp.ones_like(inputY)))
        return jnp.multiply(F1,F2).reshape((-1,1))

    # Definition of the function f(x,y) mentioned above
    @partial(jit, static_argnums=(0,))
    def target_function(self,inputs):
        return jnp.multiply(jnp.sin(jnp.pi*inputs[:,0]),2-jnp.pi**2*inputs[:,
→,1]**2+2*inputs[:,1]**3*jnp.sin(jnp.pi*inputs[:,0])).reshape(-1,1)

    # Compute the solution of the PDE on the points (x,y)
    @partial(jit, static_argnums=(0,))
    def solution(self,params,inputX,inputY):

```

```

        inputs=jnp.column_stack((inputX,inputY))
        NN = vmap(partial(jit(self.NN_evaluation), params))(inputs)
        F=self.F_function(inputX,inputY)
        A=self.A_function(inputX,inputY)
        return jnp.add(jnp.multiply(F,NN),A).reshape(-1,1)

    # Compute the loss function
    @partial(jit, static_argnums=(0,))
    def loss_function(self,params,batch,targets):
        targets=self.target_function(batch)
        laplacian=self.laplacian(params,batch).reshape(-1,1)
        dsol_dy_values=self.dsol_dy(params,batch)[: ,0].reshape((-1,1))
        preds=laplacian+jnp.multiply(self.solution(params,batch[: ,0],batch[:
→,1]),dsol_dy_values).reshape(-1,1)
        return jnp.linalg.norm(preds-targets)

    # Train step
    @partial(jit, static_argnums=(0,))
    def train_step(self,i, opt_state, inputs, pred_outputs):
        params = get_params(opt_state)
        loss, gradient = value_and_grad(self.loss_function)(params,inputs,
→pred_outputs)
        return loss, opt_update(i, gradient, opt_state)

```

## 6 Initialize neural network

```

[18]: # Neural network parameters
SEED = 351
n_features, n_targets = 2, 1          # Input and output dimension
layers = [n_features,30,30,n_targets] # Layers structure

# Initialization
NN_MLP=MLP(SEED,layers)
params = NN_MLP.MLP_create()          # Create the MLP
NN_eval=NN_MLP.NN_evaluation          # Evaluate function
solver=PINN(NN_eval)
key=NN_MLP.get_key()

```

## 7 Train parameters

```

[19]: batch_size = 10000
num_batches = 5000
report_steps=100
loss_history = []

```

## 8 Adam optimizer

It's possible to continue the last training if we use options=1

```
[20]: opt_init, opt_update, get_params = jax_opt.adam(0.001)

options=0
if options==0: # Start a new training
    opt_state=opt_init(params)

else:          # Continue the last training
    # Load trained parameters for a NN with the layers [2,30,30,1]
    best_params = pickle.load(open("./NN_saves/NN_jax_params.pkl", "rb"))
    opt_state = jax_opt.pack_optimizer_state(best_params)
    params=get_params(opt_state)
```

## 9 Solving PDE

```
[21]: # Main loop to solve the PDE
for ibatch in range(0,num_batches):
    ran_key, batch_key = jran.split(key)
    XY_train = jran.uniform(batch_key, shape=(batch_size, n_features), minval=0,
    ↪maxval=1)

    targets = solver.target_function(XY_train)
    loss, opt_state = solver.train_step(ibatch,opt_state, XY_train,targets)
    loss_history.append(float(loss))

    if ibatch%report_steps==report_steps-1:
        print("Epoch n°{}: ".format(ibatch+1), loss.item())
    if ibatch%5000==0:
        trained_params = jax_opt.unpack_optimizer_state(opt_state)
        pickle.dump(trained_params, open("./NN_saves/NN_jax_checkpoint.pkl",
    ↪"wb"))
```

```
Epoch n°100: 51.43021774291992
Epoch n°200: 46.608707427978516
Epoch n°300: 42.7242546081543
Epoch n°400: 33.08360290527344
Epoch n°500: 26.27277183532715
Epoch n°600: 25.79522705078125
Epoch n°700: 25.579057693481445
Epoch n°800: 25.45096778869629
Epoch n°900: 25.371774673461914
Epoch n°1000: 25.321613311767578
Epoch n°1100: 25.28913116455078
Epoch n°1200: 25.267547607421875
```

```

Epoch n°1300: 25.252656936645508
Epoch n°1400: 25.241756439208984
Epoch n°1500: 25.232988357543945
Epoch n°1600: 25.224929809570312
Epoch n°1700: 25.21621322631836
Epoch n°1800: 25.205018997192383
Epoch n°1900: 25.188005447387695
Epoch n°2000: 25.1571102142334
Epoch n°2100: 25.086633682250977
Epoch n°2200: 24.853960037231445
Epoch n°2300: 23.273897171020508
Epoch n°2400: 4.551088333129883
Epoch n°2500: 2.347841739654541
Epoch n°2600: 2.106163740158081
Epoch n°2700: 1.8725926876068115
Epoch n°2800: 1.6912328004837036
Epoch n°2900: 1.5889019966125488
Epoch n°3000: 1.5024032592773438
Epoch n°3100: 1.4266735315322876
Epoch n°3200: 1.3593132495880127
Epoch n°3300: 1.301129937171936
Epoch n°3400: 1.250550627708435
Epoch n°3500: 1.2060681581497192
Epoch n°3600: 1.1661580801010132
Epoch n°3700: 1.1299446821212769
Epoch n°3800: 1.0965420007705688
Epoch n°3900: 1.0653166770935059
Epoch n°4000: 1.0356481075286865
Epoch n°4100: 1.007233738899231
Epoch n°4200: 0.9798058271408081
Epoch n°4300: 0.9529896378517151
Epoch n°4400: 0.9269103407859802
Epoch n°4500: 0.901094913482666
Epoch n°4600: 0.875564694404602
Epoch n°4700: 0.8502098321914673
Epoch n°4800: 0.8249480724334717
Epoch n°4900: 0.7997944951057434
Epoch n°5000: 0.774591863155365

```

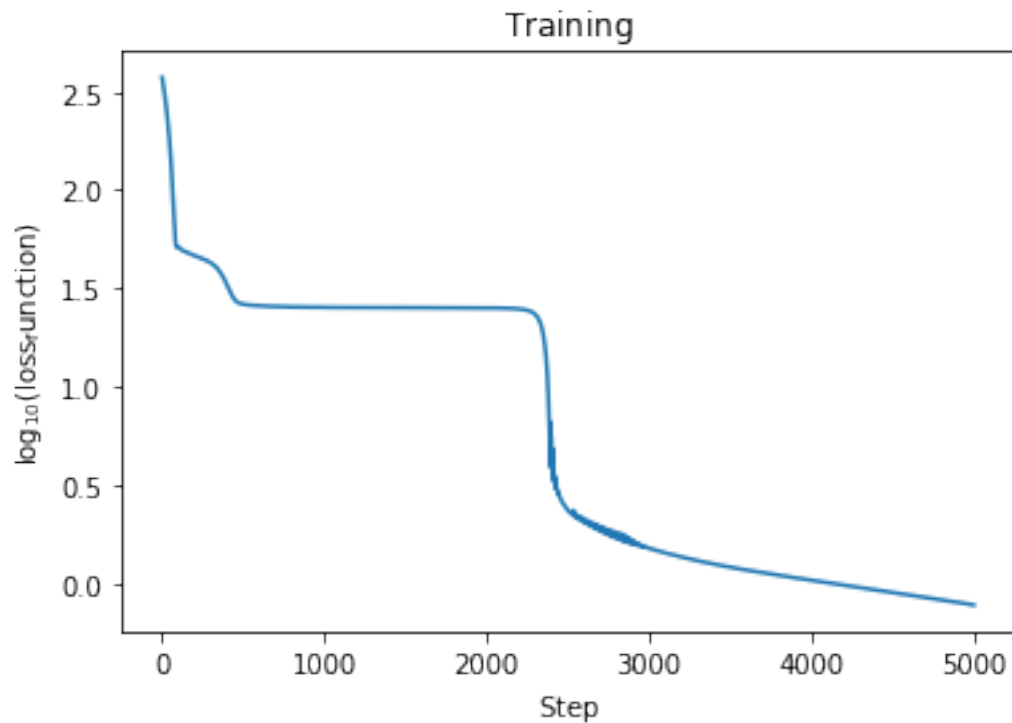
## 10 Plot loss function

```

[22]: fig, ax = plt.subplots(1, 1)
      __=ax.plot(np.log10(loss_history))
      xlabel = ax.set_xlabel(r'$\{\rm Step\}$')
      ylabel = ax.set_ylabel(r'$\log_{10}\{\rm (loss\_function)\}$')
      title = ax.set_title(r'$\{\rm Training\}$')
      plt.show

```

```
[22]: <function matplotlib.pyplot.show(close=None, block=None)>
```



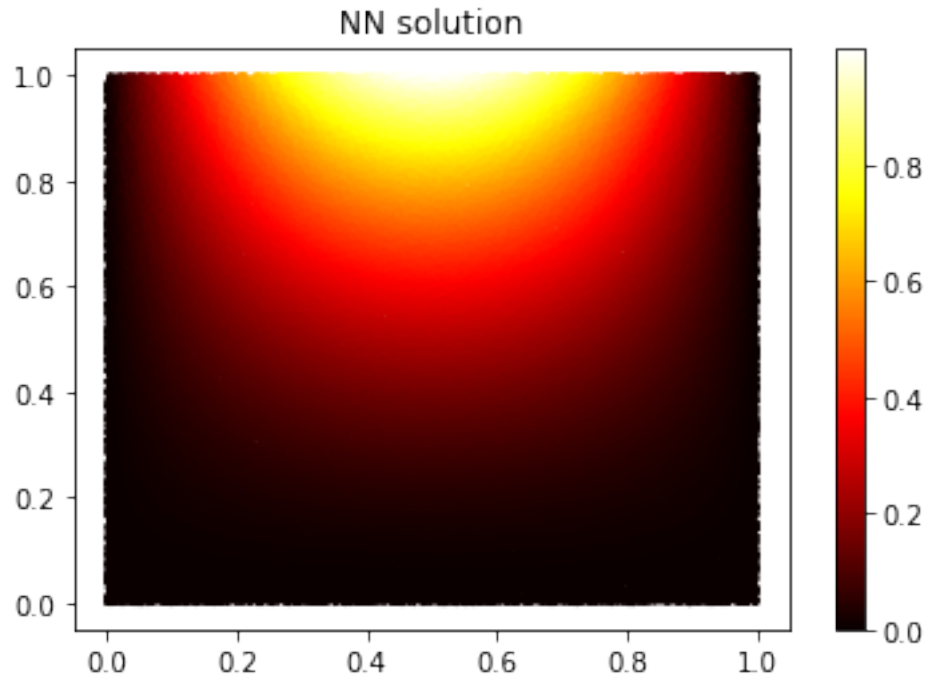
## 11 Approximated solution

We plot the solution obtained with our NN

```
[23]: plt.figure()
      params=get_params(opt_state)
      n_points=100000
      ran_key, batch_key = jran.split(key)
      XY_test = jran.uniform(batch_key, shape=(n_points, n_features), minval=0,
      ↪maxval=1)

      predictions = solver.solution(params,XY_test[:,0],XY_test[:,1])
      plt.scatter(XY_test[:,0],XY_test[:,1], c=predictions, cmap="hot",s=2)
      plt.clim(vmin=jnp.min(predictions),vmax=jnp.max(predictions))
      plt.colorbar()
      plt.title("NN solution")
      plt.show()
```





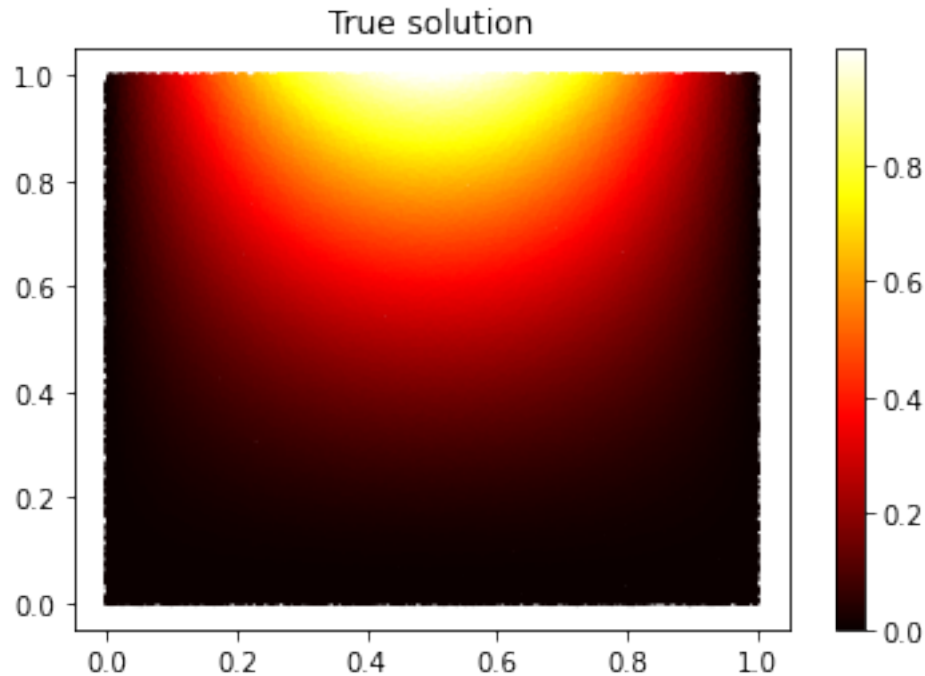
## 12 True solution

We plot the true solution, its form was mentioned above

```
[27]: def true_solution(inputs):
        return jnp.multiply(inputs[:,1]**2,jnp.sin(jnp.pi*inputs[:,0]))

plt.figure()
n_points=100000
ran_key, batch_key = jran.split(key)
XY_train = jran.uniform(batch_key, shape=(n_points, n_features), minval=0,
    ↪maxval=1)

true_sol = true_solution(XY_test)
plt.scatter(XY_test[:,0],XY_test[:,1], c=true_sol, cmap="hot",s=2)
plt.colorbar()
plt.title("True solution")
plt.show()
```



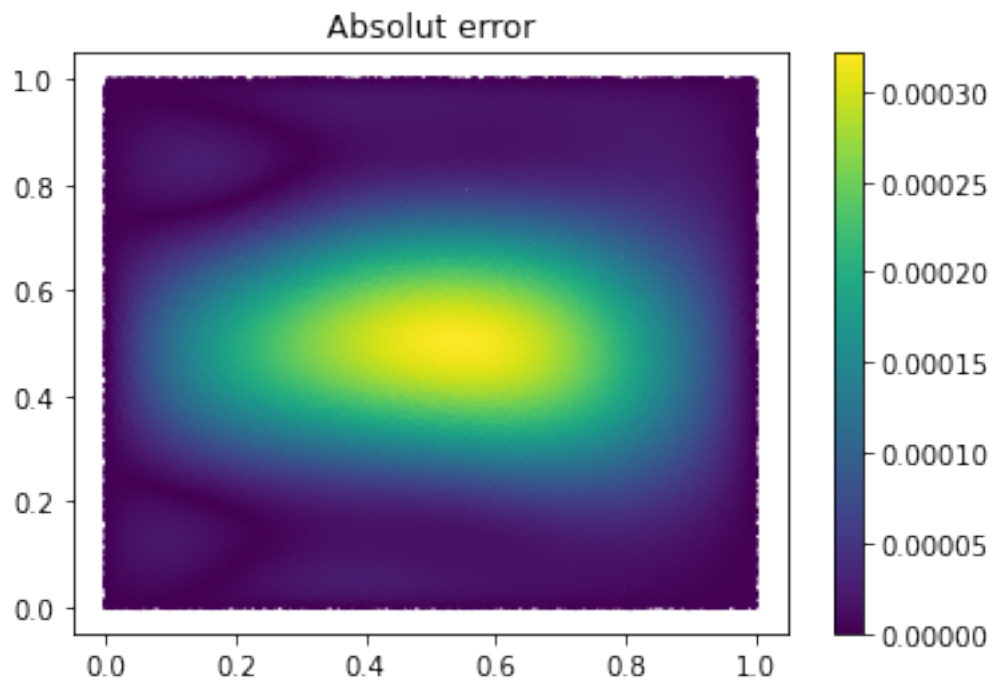
### 13 Absolut error

We plot the absolut error, it's  $|\text{true solution} - \text{neural network output}|$

```
[25]: plt.figure()
      params=get_params(opt_state)
      n_points=100000
      ran_key, batch_key = jran.split(key)
      XY_test = jran.uniform(batch_key, shape=(n_points, n_features), minval=0,
      ↪maxval=1)

      predictions = solver.solution(params,XY_test[:,0],XY_test[:,1])[:,0]
      true_sol = true_solution(XY_test)
      error=abs(predictions-true_sol)

      plt.scatter(XY_test[:,0],XY_test[:,1], c=error, cmap="viridis",s=2)
      plt.colorbar()
      plt.title("Absolut error")
      plt.show()
```



## 14 Save NN parameters

```
[26]: trained_params = jax_opt.unpack_optimizer_state(opt_state)
      pickle.dump(trained_params, open("./NN_saves/NN_jax_params.pkl", "wb"))
```