2 Q) we are given that N_t is a poisson process with parameter λ. hence the PMF is given by

$$p(N_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

Now, $\operatorname{cov}(N_t, N_s) = E[N_t N_s] - E[N_s]E[N_t]$

assuming $t > s$, we have

$$\operatorname{Cov}(N_t, N_s) = E[((N_t - N_s) + N_s)N_s] - E[N_s]E[N_t]$$

$$= E[(N_t - N_s)N_s] + E[N_s^2] - E[N_s]E[N_t]$$

since N_t is a stochastic process, $N_t - N_s$ and $N_s - N_0 = N_s$ are independent due to property of independent increments

hence $E[(N_t - N_s)N_s] = E[(N_t - N_s)] = E[(N_t - N_s)]E[N_s]$

putting above we get

$$\operatorname{Cov}(N_t, N_s) = E[N_t - N_s]E[N_s] + E[N_s^2] - E[N_s]E[N_t]$$

$$= E[N_s^2] + E[N_s](E[N_t - N_s] - E[N_t])$$

$$= E[N_s^2] - (E[N_s])^2$$

$$= \operatorname{var}(N_s) = \lambda s$$ (poisson process have mean and variance $= \lambda t$)

similarly if $t < s$, $\operatorname{cov}(N_t, N_s) = \lambda t$

hence $\operatorname{Cov}(N_t, N_s) = \underline{\underline{\lambda \min(t, s)}}$

hence proved

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(3) (i)

Given

Let (N_t^1) and (N_t^2) be two independent poisson process with parameters λ_1 and λ_2 respectively then the sum of two poisson process is also a poisson process with $\lambda_1 + \lambda_2$

PMF (probability mass function) is given by

$$P(N_t = k) = \frac{e^{-\lambda t} (\lambda t)^k}{k!}$$

as we know that sum of two poisson process is also a poisson process,

we have $(N_t^1 + N_t^2)$ is a poisson distribution with $\lambda_1 + \lambda_2$

So, $$P(N_t^1 = k \mid N_t^1 + N_t^2 = n) = \frac{P(N_t^1 = k,\ N_t^2 + N_t^1 = n)}{P(N_t^1 + N_t^2 = n)}$$

using the independence of two poisson processes, we can write it as

$$P(N_t^1 = k, N_t^1 + N_t^2 = n) = P(N_t^1 = k)\, P(N_t^2 = n-k)$$

$$= \frac{e^{-\lambda_1 t} (\lambda_1 t)^k}{k!} \times \frac{e^{-\lambda_2 t} (\lambda_2 t)^{n-k}}{(n-k)!}$$

$$\therefore P(N_t^1 = k \mid N_t^1 + N_t^2 = n) = \frac{\frac{e^{-\lambda_1 t} (\lambda_1 t)^k}{k!} \times \frac{\cancel{e^{-\lambda_2 t}} (\lambda_2 t)^{n-k}}{(n-k)!}}{\frac{e^{-(\lambda_1+\lambda_2)t} (\lambda_1 + \lambda_2)^n t^n}{n!}}$$

$$= \frac{n!}{k!(n-k)!} \frac{(\lambda_1)^k (\lambda_2)^{n-k}}{(\lambda_1 + \lambda_2)^n}$$

putting $\frac{\lambda_1}{\lambda_1+\lambda_2} = p$ we get

$$P(N_t^1 = k / N_t^1 + N_t^2 = n) = {}^nC_k \, p^k (1-p)^{n-k}$$

hence proved

(Q4) Let N_t^1 be a poisson process that represents the number of cyclones in t time

Then $\lambda_1 = 3/12$

similarly N_t^2 be the poisson process that represent no. of earth quakes in t time

Then $\lambda_2 = \frac{5}{12}$

So, we need to find

$$P(N_t^1 = 8 / N_t^1 + N_t^2 = 20)$$

we can use the formula given in the above question (3)

with $p = \frac{3/12}{3/12 + 5/12} = 3/8$

$$P(N_t^1 = 8 / N_t^1 + N_t^2 = 20) = {}^{20}C_8 \left(3/8\right)^8 \left(5/8\right)^{12}$$

Q5) Let's denote the no of subscriptions as $N(t)$ for a period t.

" no. of 1-year subscription as $X(t)$

" no of 2-year subscription as $Y(t)$

Given that subscription follows poisson process with mean rate of 6 per day.

we know that no. of subs in period t, $N(t)$ follows poisson distribution with mean of $6t$

expected value of $N(t) = E[N(t)] = 6t$. we can calculate expected no of 1-year subscriptions and 2-year subs.

Given - subs may subscribe for 1 & 2 years independently with probability $2/3$ & $1/3$. we can express in terms of $N(t)$

$$E[X(t)] = \frac{2}{3} E[N(t)] = \left(\frac{2}{3}\right) 6t = 4t$$

$$E[Y(t)] = \frac{1}{3} E[N(t)] = \left(\frac{1}{3}\right) 6t = 2t$$

Let 'A' be commission earned for 1 year & B for 2 year

expected commission are calculated as

$$E[\text{commission}1(t)] = A * E[X(t)] = A * 4t \rightarrow ①$$

$$E[\text{commission}2(t)] = B * E[Y(t)] = B * 2t \rightarrow ②$$

at last, expected total commission in period t. is sum of both years from ① & ②

$$A * 4t + B * 2t$$

$$= 4At + 2Bt$$

Q6) The arrival follows the poisson process with $\lambda = 20$ per hour $= 20/60$ per min and the customer buys with probability $= 0.4$

hence the no. of sales follows a poisson process with

$\lambda = 20 \times 0.4$

$= 8$ per hour $\Rightarrow 8/60$ per min

a) $$P(N_{10} = 0) = \frac{e^{-8/60 \times 10} \times \left(\frac{80}{60}\right)^0}{0!} = e^{-8/6}$$

b) Expected No. of sales in $t = 8$ hours. $= \mu(Nt)$

$= \lambda t$ $\quad (\lambda = 8$ per hour$)$

$= 8 \times 8 \Rightarrow 64$

c) Since N_t is a poisson process, we know from independent increment property that no. of sales in first 1.5 hours (N_t^1) is independent of no. of sales in the next 1.5 hours (N_t^2)

So, we need to find

$$P(N_t^1 = 25 \mid N_t^1 + N_t^2 = 40)$$

we can now use the formula in q3 here with

$$P = \frac{\lambda}{\lambda + \lambda} = 1/2$$

hence, $$P(N_t^1 = 25 / N_t^1 + N_t^2 = 40) = {}^{40}C_{25} \left(\tfrac{1}{2}\right)^{25} \left(\tfrac{1}{2}\right)^{15}$$

$$= {}^{40}C_{25} \left(\tfrac{1}{2}\right)^{40}$$

d) we need to find

$$P(N_{2hour} \geq 20) = 1 - \sum_{k=0}^{19} P(N_{2hour} = k)$$

$$= \sum_{k=0}^{19} \frac{e^{-16} (16)^k}{k!}$$

Q7) The No. of accidents follows poisson process with mean 2 per day

$P(Y_i = k) = 1/2^k, \quad k \geq 1$

Let N_t be the no. of accidents in time t. Then N_t is a poisson process with intensity $\lambda = 2$ per day.

Let Y_1, Y_2 be independent random variable with distribution

$P(Y_i = k) = 1/2^k, \; k \geq 1.$

Then total no of people involved in accidents in time t is

$S_t = Y_1 + Y_2 + \cdots + Y_{(N_t)}$

The distribution of S_t is called compound poisson distribution.

The mean and variance can be calculated as

$E[S_t] = E[N_t]\,E[Y_1] = \lambda E[Y_i]$

$Var[S_t] = E[N_t]\,Var[Y_1] + Var[N_t]\,(E[Y_1])^2$

$= \lambda Var[Y_1] + \lambda E[Y_1]^2$

since Y_i has distribution $P(Y_i = k) = 1/2^k \quad k \geq 1$

we have

$E[Y_i] = \sum_{k=1}^{\infty} k \cdot (1/2^k) = 2$

$var[Y_i] = \sum_{k=1}^{\infty} k^2 \cdot \left(\frac{1}{2}\right)^k - 4 = 8 - 4 = \underline{4}$

$\therefore \; E[S_t] = \lambda E[Y_1] = 4t \Rightarrow 28 \quad (\text{since } t = 7)$

$Var[S_t] = \lambda Var[Y_1] + \lambda E[Y_1]^2 = \underline{\underline{8t}} = 56$

Q 8) Let N_t be no. of delayed flights in time t, then N_t is a poisson process with intensity $\lambda = 5$ per month

Let $Y_1, Y_2 \dots$ be independent random variables with mean $\mu = 50$ and

Variance $\sigma^2 = 100$.

Then total amount of compensation paid in time t is

$S_t = 500(Y_1 + Y_2 + \dots Y_{N_t})$

The distribution of S_t is called compound poisson process distribution.

The mean and variance of S_t can be calculated as

$$E[S_t] = E[N_t]\,E[Y_1] = \lambda\mu\, 500t$$

$$Var[S_t] = E[N_t]\,Var[Y_1] + Var[N_t]\,(E[Y_1])^2$$
$$= (\lambda\sigma^2 + \lambda\mu^2) \times 250000t$$

Therefore, putting $t = 12$,

$$E[S_{12}] = \lambda\mu 500t = 300000 \times 5 = 1500000$$

$$Var[S_{12}] = 250000 \times (5 \times 12) \times (50^2 + 100)$$
$$= 250000 \times (60) \times (2600)$$
$$= 39000000000$$

hence, expected total amount compensation for delayed flight is Rs. 1500000 and standard deviation of flight

is Rs. 1,97,484.17

Q9)

Let $\{X(t) : t \geq 0\}$ be a pure death process $\mu_i = i\mu$ for $i = 1, 2.$

Where $X(0) = N$

i) To find $p(X(t) = j)$ for $j = 0, 1, 2, \ldots$ we can use formula

$$p(X(t) = j) = {}^{N}C_{j} (1 - e^{-\mu t})^{N-j} e^{-j\mu t}$$

where N is initial population size & C_j is binomial coefficient.

The binomial coeff ${}^{N}C_{j}$ represents no. of ways to choose j individuals from population N.

The term $(1 - e^{-\mu t})^{N-j}$ represent the probability that $(N-j)$ individual will survive untill time t.

The $e^{-j\mu t}$ represents the probability that each of j individual will die by t time

ii) $p(X(t) = j)$ represents the binomial distribution with $p = e^{-\mu t}$

hence we know that the expected values of binomial distribution $= np = N e^{-\mu t}$

Variance $= Np(1-p) = N(e^{-\mu t})(1 - e^{-\mu t})$

Q 10) We are given

$$X(t) = X(0)(-1)^{N(t)}, \quad t > 0$$

where $N(t)$ is a poisson process with $\lambda = 6$.

a) The kolmogorov forward eq's for this CTMC are given by

$$p'(t) = p(t)\,Q$$

where $p(t)$ is PDV(probability distribution vector) at time t and Q is infinitesimal generator matrix. So, we have

$$Q = \begin{bmatrix} -6 & 6 \\ 6 & -6 \end{bmatrix}$$

$\therefore$

$$p'(t) = \begin{bmatrix} -6 & 6 \\ 6 & -6 \end{bmatrix} * p(t)$$

So, from the above, kolmogrov equations can be written as

$$p'_{00}(t) = -6P_{00}(t) + 6p_{01}(t)$$

$$p'_{01}(t) = +6P_{00}(t) - 6p_{01}(t)$$

$$p'_{10}(t) = -6P_{10}(t) + 6P_{10}(t)$$

$$p'_{11}(t) = 6P_{10}(t) + 6P_{11}(t)$$

b) $p(X(t)=1) = p(X(0)=1)\, p(N(t)=\text{even}) + p(X(0)=-1)\, p(N(t)=\text{odd})$

$$p(N(t)=\text{even}) = \sum_{k\in\text{even}} \frac{e^{-\lambda t}(\lambda t)^k}{k!} = e^{-\lambda t}\sum_{k\in\text{even}} \frac{(\lambda t)^k}{k!}$$

$$= e^{-\lambda t}\left(\frac{e^{\lambda t}+e^{-\lambda t}}{2}\right)$$

parallely

$$p(N(t)=\text{odd}) = e^{-\lambda t}\left(\frac{e^{-\lambda t} - e^{-\lambda t}}{2}\right)$$

by putting above with $p(X(0)=1) = 0.5$

$p(X(0)=-1) = 0.5$

we get

$$p(x(t)=1) = \frac{1}{2} \times e^{-\lambda t}\left(\frac{e^{\lambda t}+e^{-\lambda t}}{2}\right) + \frac{1}{2} e^{-\lambda t}\left(\frac{e^{\lambda t}-e^{-\lambda t}}{2}\right)$$

$$= \frac{1}{2} e^{-\cancel{\lambda t}} \left(\cancel{e^{\lambda t}}\right) = \frac{1}{2}$$

Parallely for $p(x(t) = -1) = \frac{1}{2}$

Q 11) Let T be the time taken to reach 20 individuals, and T_i is the time required to increase one individual from i to $i+1$. Then

$$T = T_1 + T_2 + \cdots T_{20}$$

Since all the events are independents hence T_i's are independent. The distribution of T_i is given by

$$T_i \sim Exp(\lambda_i) = Exp(i\lambda)$$

$\therefore$

$$E[T] = \sum_{i=1}^{20} E[T_i] = \sum_{i=1}^{20} \frac{1}{i\lambda}$$ (Since expected value of exponential distribution is $1/\lambda$)

also,

$$Var[T] = \sum_{i=1}^{20} Var(T_i)$$ (All T_i are independent)

$$= \sum_{i=1}^{20} \frac{1}{i^2\lambda^2}$$ (Var of exp distribution is $1/\lambda^2$)

Q12) we are given matrix

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{bmatrix}.$$

we know that $p' = PQ$ and at stationary distribution we have $p' = 0$, hence we need to find solution of

$PQ = [0, 0, 0]$

Let $P = [P_1, P_2, P_3]$,

$$-2P_1 + P_2 + 2P_3 = 0$$
$$P_1 - P_2 + P_3 = 0$$
$$P_1 \quad -3P_3 = 0$$

by solving the above equations we get

$$P_1 = 3P_3, \quad P_2 = 4P_3$$

Now, $P_1 + P_2 + P_3 = 1$

hence $3P_3 + 4P_3 + P_3 = 1$

$8P_3 = 1 \quad \Rightarrow P_3 = 1/8$

$\therefore \quad P_2 = 4/8$

$P_1 = 3/8$

hence we get $P = [3/8, 4/8, 1/8]$

(Q14) we know that $\pi'(t) = \pi(t) Q$

for steady state $\pi'(t) = 0$ & $\pi(t) = \pi$

Therefore $0 = \pi . Q$ $\quad \sum_{i \in s} \pi_i = 1$

$$0 = -\lambda_0 \pi_0 + \mu_1 \pi_1$$

$$0 = -(\lambda_i \mu_i) \pi_i + \lambda_{i-1} \pi_{i-1} + \mu_{i+1} \pi_{i+1}, \quad i \geq 1$$

on solving we get

$$\pi_n = \frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i} \pi_0$$

$$\pi_n = \frac{1 \times 3 \times \cdots (2n-1)}{3^n n!} \pi_0$$

so, $\pi_n = \frac{(2n)!}{n! n!} \frac{1}{6^n} = \left({}^{2n}C_n\right)\left(\frac{1}{6^n}\right) \quad \left(\because 1 \times 3 \times \cdots (2n-1) = \frac{(2n)!}{2^n n!}\right)$

We know that from ratio test.

$\pi_n = \left({}^{2n}C_n\right) \frac{1}{6^n}$ is convergent $\left(\frac{\pi_n}{\pi_{n+1}} = \frac{1}{3} < 1\right)$

Now using $\sum \pi_i = 1$, we have

$$\pi_0 \left(\sum_{n=0}^{\infty} {}^{2n}C_n \left(\frac{1}{6^n}\right)\right) = 1$$

$$\therefore \quad \pi_0 = \frac{1}{\left(\sum_{n=0}^{\infty} {}^{2n}C_n \frac{1}{(6)^n}\right)}$$

$$= \frac{1}{\frac{1}{\sqrt{1-4/6}}} \qquad \left(\sum_{n=0}^{\infty} {}^{2n}C_n\, x^n = \frac{1}{\sqrt{1-4x}} \right)$$

$$= \frac{1}{\sqrt{3}}$$

$$\therefore \pi_n = 1/\sqrt{3}\ {}^{2n}C_n \cdot 1/(6)^n$$

$\pi = [\pi_0, \pi_1, \dots]$ is the required stationary distribution

Assignment - 7

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1 a) We are given that

$$X(t) := N(t+L) - N(t), \ t \geq 0$$

$\hookrightarrow$ poisson process

i) To show that $X(t)$ is second order process.

Since, $X(t) = N(t+L) - N(t)$ and $N(t)$ is poisson hence $X(t)$ is a stochastic process and in order to show that it is a second order process. we need to show that

$$E((X(t))^2) < \infty$$

$$E(X(t)^2) = E\left[(N(t+L) - N(t))(N(t+L) - N(t))\right]$$

$$= E\left[(N(t+L))(N(t+L) - N(t))\right] - E\left[N(t)(N(t+L) - N(t))\right]$$

$$= E[N(t+L)]\,E[N(t+L) - N(t)] - E[N(t)]\,E[(N(t+L)) - N(t)]$$

(using the independent increment property)

$$= \lambda(t+L)\lambda(L) - \lambda(t)\lambda L$$

$$= \lambda^2 L^2 < \infty$$

hence, it is a second order process

Let's Assume $s > t$

ii) $\mathrm{Cov}(X(s), X(t)) = \mathrm{cov}(N(s+L) - N(s), N(t+L) - N(t))$

Let's assume $t+L > s$ otherwise $\mathrm{Conv}(X(s), X(t)) = 0$ due to independent increment property

hence

$$\mathrm{Cov}(X(s), X(t)) = \mathrm{cov}(N(s+L) - N(t+L) + N[t+L) - N(s), \quad N(t+L) - N(s) + N(s) - N(t))$$

$$= \mathrm{Cov}(N(s+L)-N(t+L), N(t+L)-N(s)) + \mathrm{Cov}(N(s+L)-N(t+L), N(s)-N(t))$$

$$+ \mathrm{Cov}(N(t+L)-N(s), N(t+L)-N(s)) + \mathrm{Cov}(N(t+L)-N(s), N(s)-N(t))$$

$$\Rightarrow \mathrm{Var}(N(t+L)-N(s)) = \lambda(t+L-s)$$

Therefore $\mathrm{Cov}(X(t), X(s))$ only depends on the time difference $|t-s|$ and not on the absolute time points t and s.

This shows that $X(t)$ is covariance stationary

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b) using backward Kolmogrov equation, we get

$$P'_{1i}(t) = \sum_{j=1}^{3} P_{ji}(t)\, q_{1j}(t) = -2P_{1i}(t) + P_{2i}(t) + P_{3i}(t)$$

$$= P'_{3i}(t) = \sum_{j=1}^{3} P_{ji}(t)\, q_{3j}(t) = P_{1i}(t) + 2P_{2i}(t) - 3P_{3i}(t)$$

c) Inorder to calculate $\lim_{t\to\infty} P_{i2}(t)\ \forall\, i \in S$, we need to find the limiting distribution hence

$$P'_{ij}(t) = 0 = \Pi Q$$

Let $\Pi = [\Pi_1\ \Pi_2\ \Pi_3] \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$

hence

$0 = -2\Pi_1 + \Pi_2 + \Pi_3$

$0 = \Pi_1 - 2\Pi_2 + 2\Pi_3$

$0 = \Pi_1 + \Pi_2 - 3\Pi_3$

on solving 3 equations we get

$-3\Pi_1 + 4\Pi_3 = 0$ $\qquad \Pi_1 = 4/3\, \Pi_3$

$10/3\, \Pi_3 - 2\Pi_2 = 0$ $\qquad \Pi_2 = 5/3\, \Pi_3$

also $\Pi_1 + \Pi_2 + \Pi_3 = 1$

$\Pi_3 (1 + 4/3 + 5/3) = 1$

$\Pi_3 = 1/4,\ \Pi_2 = 5/12,\ \Pi_1 = 4/12$

hence $P_{i2} = 5/12$