COL 726

Homework II

Due on Feb 16, 2023

1. Prove that any $m \times n$ matrix is the limit of a sequence of matrices of full rank. You can use 2-norm for measuring distance between two matrices.

2. Consider the following algorithm for computing dot product of two vectors x, y in $\Re^n$:

```
S = 0;
for i=1 to n
   S = S + x_i y_i
 Output S.
```

   Prove that this algorithm is backward stable. Now consider the following algorithm for computing matrix-vector product (you should treat both A and x as input): to compute Ax, we compute dot product of each row of A with x as above. Prove that this algorithm is also backward stable.

3. Let A and B be two square $n \times n$ matrices. Let $C = AB$. Let $c_1 \geq c_2 \geq \ldots c_n$ be the singular values of C (define $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ similarly for A and B respectively). Prove that $c_i \geq b_i \cdot a_n$.

4. Let A, B, C be three matrices of appropriate dimension so that the product ABC is defined. Prove that $||ABC||_F \leq ||A||_2 ||B||_F ||C||_2$.

5. Consider the 4×3 matrix A given below, with $\mu = O(\epsilon_m)$ (where ϵ_m denotes the machine precision):

$$A = \begin{bmatrix} 1 & 1 & 1 \\ \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

   Suppose its QR factorization is computed in floating-point arithmetic with (a)the classical Gram-Schmidt method, and (b)the modified Gram-Schmidt method. For both cases, work out on paper the columns $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3$ and $\tilde{r}_1, \tilde{r}_2, \tilde{r}_3$ of the computed factorization, along with bounds on the errors of all the entries. For example,

$$\tilde{q}_1 = \begin{bmatrix} 1 + O(\epsilon_m) \\ \mu + O(\epsilon_m) \\ 0 \\ 0 \end{bmatrix}.$$

   Assume that addition of 0 and multiplication by 0 or 1 do not incur any rounding error. What can you conclude about the orthogonality of the computed Q in both cases?

6. Consider the following matrix:
$$\begin{bmatrix} I & F \\ 0 & I \end{bmatrix},$$
where I is the square $n \times n$ matrix, and F is an $n \times n$ matrix. Prove that the 2-norm of this matrix is equal to $\sqrt{\frac{2+||F||_2^2+||F||_2\sqrt{4+||F||_2^2}}{2}}$

7. Write a MATLAB program `[W, R] = house(A)` that computes an implicit representation of a full QR factorization $A = QR$ of an $m \times n$ matrix A with $m \geq n$ using Householder reflections. The output variables are a lower traingular matrix W (which is $m \times n$) whose column vectors define the successive Householder reflections, and a triangular $m \times n$ matrix R. Write a MATLAB program `Q = formQ(W)` that takes the matrix W above and generates the corresponding $m \times m$ orthogonal matrix Q.

8. Write a MATLAB program `x = leastSquare(A,b)` that solves the least squares problem $Ax = b$. You should use the the program for QR factorization described above. You will use this implementation of least squares in the problem below.

9. (**Heath**) A planet follows an elliptic orbit, which can be represented in a Cartesian (x, y) coordinate system by the equation
$$ay^2 + bxy + cx + dy + e = x^2.$$
   (a) Determine, using least squares, the parameters a, b, c, d, e given the following observations of the planet's positions :

x	1.02	0.95	0.87	0.77	0.67	0.56	0.44	0.30	0.16	0.01
y	0.39	0.32	0.27	0.22	0.18	0.15	0.13	0.12	0.13	0.15

   In addition to printing the values for the orbital paramters, plot the resulting orbit and the given data points (you may want to use the matlab function `ezplot`).

   (b) This least squares problem is nearly rank deficient. To see what effect this has on the solution, perturb the input data slightly by adding to each coordinate of each data point a random number uniformly distributed on the interval [-0.005, 0.005] and solve the least squares problem with this perturbed data. Compare the new values for the parameters with the previously obtained ones. What effect does this difference have on the plot of the orbit ? Can you explain the behaviour ?