K. Laxman 2018CS50408

Q6

(b) Show that :- To prove that eigen values are distinct

Initially we fix $\forall \lambda \in \mathbb{C}$. Let us consider $A-\lambda I$ and here notice that $B = (A-\lambda I)_{2:m, 1:m}$ i.e. by eliminating 1st row & last column is full rank as it is an upper triangular with non zero diagonals which correspond to A's subdiagonal.

$$\begin{bmatrix} \times & \times & & \vdots \\ \times & \times & \times & \vdots \\ & \times & \times & \times \\ & & \times & \times \end{bmatrix}$$

$: B_{ii} = (A-\lambda I)_{i+1,i} = A_{i+1,i} \; (1 \leq j \leq m-1)$

and $B_{ij} = (A-\lambda I)_{i+1,j} = A_{i+1,j} = 0$ for $\forall i > j$, by defn of tridiagonal.

So, if $\text{rank}(A-\lambda I) \leq m-2$, there should be linearly dependent pairs of rows in B, which ~~coordinate~~ contradicts to full rank of B.

$\Rightarrow \text{ran}(A-\lambda I) \geq m-1$

Now, from the theorem in book (24.7) it diagnolizable by $A = X^* \Lambda X$. and by thm 24.3, Λ & A have the exactly same eigenvalues which appears on Λ.

$A - \lambda I = X^* \Lambda X - \lambda I = X^* (\Lambda - \lambda I) X$.

as $A - \lambda I$ & $\Lambda - \lambda I$ are similar have same rank i.e. $\text{rank}(\Lambda - \lambda I) \geq m-1$

i.e. $\lambda_i = (\Lambda)_{ij} \; (1 \leq i \leq m)$ and each λ_i is the eigen value of A.

Suppose the eigenvalues of A are not distinct then i.e. $\lambda_i = \lambda_j$ for some $i \neq j$. Fix $\lambda = \lambda_i = \lambda_j$

Since $\Lambda - \lambda I$ is diagonal, $\text{rank}(\Lambda - \lambda I) =$ No. of non zero diagonals

But as $(\Lambda - \lambda I)_{jj} = (\,)_{jj} = 0$,

$\text{rank}(\Lambda - \lambda I) = \text{rank}(A - \lambda I) \le m-2$ which is contradiction

So, from this we can say that eigenvalues are distinct.

(7) To prove that a complex number z is Rayleigh quotient of A iff it is a diagonal entry of $Q^* A Q$ for some unitary matrix Q.

$(\rightarrow)$

Let us say that $r(x) = \dfrac{x^* A x}{x^* x} = z$ for some $x \in \mathbb{C}^m$

extend the equation $\left\{\dfrac{x}{\|x\|_2}\right\}$ to the orthonormal basis for $\mathbb{C}^m$, $\left\{\dfrac{x}{\|x\|_2}, f_2, \ldots, f_m\right)$

we have to note here that $\|x\|_2 = \sqrt{x^* x}$ and set

$Q = \left[\dfrac{x}{\|x\|_2} \middle| f_2 | \ldots | f_m\right]$ then

$$(Q^* A Q)_{11} = \left(\frac{x}{\|x\|_2}\right)^* (AQ)_1 = \underbrace{\left(\frac{x}{\|x\|_2}\right)^*}_{\text{first row of } Q^*} A \underbrace{\frac{x}{\|x\|_2}}_{1\text{st column of } Q} = \frac{x^* A x}{x^* x} = z$$

$(\leftarrow)$

Let $z = (Q^* A Q)_{ii}$ for some $1 \le i \le m$

choose f_i (ith column of Q)

As we can see then $z = f_i^* A f_i = \dfrac{f_i^* A f_i}{f_i^* f_i} = r(f_i)$

(4)

(a) $\lim_{n\to\infty} \|A^n\| = 0$ iff $\rho(A) < 1$

Suppose that $\rho(A) < 1$ then $\exists$ a constant c such that $|\lambda| \le \rho(A) < 1$ for all eigenvalues of A

$\therefore \|A^n v\| = \|\lambda^n v\| = |\lambda|^n \|v\| \le \rho(A)^n \|v\|$

for all $n \ge 0$ since $\|v\|$ is nonzero and fixed we have

$\lim_{n\to\infty} \|A^n v\| = 0$ By the triangle inequality $\|A^n\| \le \|A\|^n$

$\forall\ n \ge 0$, $\therefore \lim_{n\to\infty} \|A_n\| = 0$

Conversely suppose that $\lim_{n\to\infty} \|A_n\| = 0$, then for any $\varepsilon > 0$, $\exists$ an integer N such that $\|A^N\| < \varepsilon$ $\forall\ n \ge N$ and let λ be eigen value of A with maximum real part. Then we have

$$|\lambda n| = \frac{|(A^n)v|}{\|v\|} \le \frac{\|A^n\|}{\|v\|} \le \frac{\|A^N\|}{\|v\|} < \varepsilon/\|v\|$$

by taking limits $n \to \infty$, we get $|\lambda| \le \frac{\varepsilon}{\|v\|}$ since $\varepsilon > 0$.

since v is arbitrary $|\lambda| = 0 \Rightarrow \lambda = 0$

$\therefore$ eigenvalue of A have $\rho(A) < 1$

(b) suppose that $\alpha(A) < 0$, then $\exists$ a complex eigenvalue λ with positive eigenvalue λ with positive real part. Let v be a corresponding eigen vector then

$\|e^t(tA)v\| = \|e^{(t\lambda)} v\| = e^{(t\,Re(\lambda))}\|v\|$

since $Re(\lambda) > 0$ the right hand side $\to \infty$, as $t \to \infty$

by defn of norm

$\therefore$

$\lim_{t\to\infty} \|e^{tA}\| = \infty$.

https://drive.google.com/drive/folders/1zoF2tepBrWqpgzPa63BAPLAFWvFoWpa8?usp=sharing

```
m = 21;
n = 12;
epsilon = 10^-10;

matA = ones(m,n);
for i = 1:m
    ti = (i-1)/(m-1);
    temp = 1;
    for j = 1:n
        matA(i,j) = temp;
        temp = temp*ti;
    end
end

matX = ones(n,1);
matB = matA*matX;

for i = 1:m
    u = rand;
    matB(i) = matB(i) + (2*u - 1)*epsilon;
end

% Cholesky Factorisation

Ac = matA'*matA;
L = chol(Ac);

y = L'\(matA'*matB);
matX1 = L\y;
disp(matX1);

error1 = norm(matX1-matX)/norm(matX);
disp(['Error in Cholesky factorisation: ', num2str(error1)]);

% QR Factorisation

[Q,R] = qr(matA);
matX2 = R\(Q'*matB);
disp(matX2);

error2 = norm(matX2-matX)/norm(matX);
```

```
disp(['Error in QR factorisation: ', num2str(error2)]);
```