COL 726 Assignment 1

K LAXMAN

TOTAL POINTS

49.5 / 85

QUESTION 1

1 Q1 **5 / 5**

✓ **- 0 pts** *Correct*

- 5 pts Incorrect

- 3 pts Incorrect Condition Number

- 3 pts Incorrect answer for High Condition number - Need to say about x ≈ y

- 2 pts Unsatisfactory Justification - Need to say which part of CN can be eliminated and which is problematic

- 1 pts Not Simplified Condition number - Need to say which part of CN can be eliminated

QUESTION 2

2 Q2 **6 / 10**

- 0 pts Correct

- 10 pts Incorrect

- 3 pts Incorrect a part answer - Need to say about 0

- 3 pts Incorrect b part answer- Need to say $$\frac{2x}{1-x^2}$$

- 1.5 pts Partially Incorrect a answer

✓ **- 4 pts** *Mathematical proof not given / Incorrect Explanation / Condition number not given*

- 1.5 pts Partially Correct b answer

- 2 pts Unsatisfactory explanation/ Condition Number not given

QUESTION 3

3 Q3 **3 / 15**

- 0 pts Correct

✓ **- 15 pts** *Incorrect / No plot*

- 5 pts No Code

- 3 pts No Plot a

- 3 pts Not Marked 1/epsilon-machine

- 5 pts Wrong Part a

- 2.5 pts Accuracy not changing at around 36 in single precision in b

- 2.5 pts Accuracy not changing at around 78 in double precision in b

- 2 pts Unsatisfactory explanation in b - Need to say about precision and storage of bits for higher values

- 2 pts Not written/shown earlier loss of accuracy in c

- 1 pts Not written about rate of loss in accuracy/growth of errors of single precision v/s double in part b and c

- 2.5 pts Unsatisfactory explanation about difference between b and c

- 5 pts Incorrect / No plot in b

- 5 pts Incorrect / No plot in c

+ 3 *Point adjustment*

For explanation

QUESTION 4

4 Q4 **7.5 / 15**

- **- 0 pts** Correct
- **- 3 pts** Part a incorrect - Need to get 2 by mathematical proof
- **- 1.5 pts** Explanation not given in a
- **- 1.5 pts** Incorrect Explanation in b / Explained by example - Need to say about cancellation error using some mathematical proof
- **- 3 pts** Part b is incorrect - Need to say unstable
- **- 4 pts** Part c incorrect - Need to say $$2Sin^2(\frac{x}{2})$$
- ✓ **- 2.5 pts** *Mathematical proof not given / Unsatisfactory explanation in c / Explained by example*
- ✓ **- 5 pts** *Plot not provided / Incorrect*
- **- 2.5 pts** Plot part a incorrect
- **- 2.5 pts** Plot part b incorrect
- **- 15 pts** Incorrect

QUESTION 5

5 Q5 **9 / 10**

- **- 5 pts** Incorrect Plot
- **- 0 pts** Expanded Form $$(x-1)^6$$ has Cancellation Errors
- ✓ **- 0 pts** *Correct Plot*
- ✓ **- 1 pts** *Does not mention about cancellation Errors in the Expanded Form: $$(x-1)^6$$*
- **- 5 pts** No Explanation

QUESTION 6

6 Q6 **9 / 15**

- ✓ **- 0 pts** *Correct Code/Plot*
- ✓ **- 6 pts** *You need to show $$x_k$$ in terms of $$x_1$$ and $$x_2$$, and the problem is ill-conditioned. Missing explanation that the Problem is Ill Conditioned, and the Rounding Errors in $$x_1$$ and $$x_2$$ are enhanced significantly.*
- **- 2 pts** Semilog Plot should have log of the y-axis, not the x-axis.
- **- 10 pts** No Explanation of the Plot
- **- 3 pts** Almost correct Explanation, but you should give $$C_1$$ and $$C_2$$ in terms of $$x_1$$ and $$x_2$$, which makes it clear that the problem is ill-conditioned.
- **- 3 pts** Almost correct Explanation, but you should give $$C_0$$ and $$C_1$$ in terms of $$x_1$$ and $$x_2$$, which makes it clear that the problem is ill-conditioned.
- **- 2 pts** No Code
- **- 3 pts** No Plot
- **- 15 pts** No Submission!
- **- 1 pts** Not Semilog
- **- 5 pts** No Code/Plot

QUESTION 7

7 Q7 **10 / 15**

- ✓ **- 0 pts** *Correct Code and Plot*
- ✓ **- 5 pts** *Missing reasoning that as $$ k \rightarrow \infty, I_k \rightarrow 0 \implies k \times I_{k-1} \rightarrow 1$$, leading to high Cancellation Errors.*
- **- 1 pts** Partially Missing reasoning that as $$ k \rightarrow \infty, I_k \rightarrow 0 \implies k \times I_{k-1} \rightarrow 1$$, leading to high Cancellation Errors.
- **- 0 pts** Correct Reasoning!
- **- 1 pts** Computation in Double Precision not Shown
- **- 5 pts** Incorrect/No Plot

- 10 pts No Explanation of the Errors

- 2 pts No Code

- 3 pts Partially Missing reasoning that as $$ k \rightarrow \infty, I_k \rightarrow 0 \implies k \times I_{k-1} \rightarrow 1$$, leading to high Cancellation Errors.

COL726

Assignment -1

K.Laxman
2018CS50408

(1) $f: \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x - y$

Measure the size of input (x,y) by $|x|+|y|$

The relative condition number of a function measures how sensitive the output of function to small changes here to find relative condition number using formula

$$K(x,y) = \|\nabla f(x,y)\| * \frac{\|x,y\|}{|f(x,y)|} \longrightarrow (3)$$

$$\frac{\partial f}{\partial x} = 1 \longrightarrow (1)$$

$$\frac{\partial f}{\partial y} = -1 \longrightarrow (2)$$

from (1) & (2)

$$\nabla f = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

here vector norm of (x,y) is $|x|+|y|$ and norm of ∇f is 2

From eq (3)

$$\text{relative condition number} = 2 * \frac{|x|+|y|}{|x-y|}$$

here Relative condition number is high when $|x-y|$ is small and $|x|+|y|$ is large

* It can be high when x and y are close in value but have larger magnitudes

In floating point numbers high relative condition number can occur when difference b/w input is less than machine epsilon (ε) leading sensitivity to small changes in input and unexpected results in graphs

$$* \frac{\text{ratio of absolute change in output}}{\text{ratio of small changes in input}}$$

1 Q1 **5 / 5**

✓ **- 0 pts** *Correct*

- 5 pts Incorrect

- 3 pts Incorrect Condition Number

- 3 pts Incorrect answer for High Condition number - Need to say about $x \approx y$

- 2 pts Unsatisfactory Justification - Need to say which part of CN can be eliminated and which is problematic

- 1 pts Not Simplified Condition number - Need to say which part of CN can be eliminated

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② $f(x) = \frac{1}{1-x} - \frac{1}{1+x} \rightarrow ①$

ⓐ The computation will be unstable for values of x close to 1 (or) -1.

If $x \in (-\varepsilon, \varepsilon)$,
Assuming ε is 2^{-24} (single precision)
The value of $(1-x)$ and $(1+x)$ is nearly same. it will lead to truncation error and rounding error.
so, while taking $x \in (-\varepsilon, \varepsilon)$ $f(x)$ is unstable.

ⓑ For the same above expression ①, can be written as

$$\frac{1}{1+x} + \frac{1}{1-x} = \frac{1+x+1-x}{(1-x)(1+x)} = \frac{2}{1-x^2}$$

In this case the function $f(x)$ can be computed accurately in $x \in (-\varepsilon, \varepsilon)$

2 Q2 **6 / 10**

- **- 0 pts** Correct
- **- 10 pts** Incorrect
- **- 3 pts** Incorrect a part answer - Need to say about 0
- **- 3 pts** Incorrect b part answer- Need to say $$\frac{2x}{1-x^2}$$
- **- 1.5 pts** Partially Incorrect a answer

✓ **- 4 pts** *Mathematical proof not given / Incorrect Explanation / Condition number not given*

- **- 1.5 pts** Partially Correct b answer
- **- 2 pts** Unsatisfactory explanation/ Condition Number not given

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(3)

a) plot the numbers f_n, p_n on one log scale plot.

For single precision, ε_{mach} is 2^{-24} and for double precision $\varepsilon_{mach} = 2^{-53}$

$\frac{1}{\varepsilon_{mach}} = 2^{24}$ and $\frac{1}{\varepsilon_{mach}} = 2^{53}$ for double precision

(b) here to recompute f_k for $k = n-2, n-3, \ldots 0$.

$$f_{k+1} = f_k + f_{k-1}$$

$$\Rightarrow f_{k-1} = f_{k+1} - f_k \rightarrow (1)$$

~~and we compute~~

~~when~~. (1) is used to make a plot difference between original $f_0 = 1$ and recomputed f_0 as a function of n,

The value of n values results in low accuracy for recommend

* The value of n ~~values~~ results in low accuracy for recommend f_0 will depend on precision of arithmetic used to compute the recomputed f_0. generally large value of 'n' means less accurate the recomputed f_0. and vice versa.

· In case of single precision:- relative error will be larger than double precision, as the machine epsilon is larger

(c) Striking difference is that loss of precision for the fibonacci numbers is exponential as the ratio b/w consecutive fibonacci numbers increases rapidly. while loss of precision in case of pertubed Fib. number is much slower ~~very~~

3 Q3 **3 / 15**

- **- 0 pts** Correct

✓ **- 15 pts** *Incorrect / No plot*

- **- 5 pts** No Code
- **- 3 pts** No Plot a
- **- 3 pts** Not Marked 1/epsilon-machine
- **- 5 pts** Wrong Part a
- **- 2.5 pts** Accuracy not changing at around 36 in single precision in b
- **- 2.5 pts** Accuracy not changing at around 78 in double precision in b
- **- 2 pts** Unsatisfactory explanation in b - Need to say about precision and storage of bits for higher values
- **- 2 pts** Not written/shown earlier loss of accuracy in c
- **- 1 pts** Not written about rate of loss in accuracy/growth of errors of single precision v/s double in part b and c
- **- 2.5 pts** Unsatisfactory explanation about difference between b and c
- **- 5 pts** Incorrect / No plot in b
- **- 5 pts** Incorrect / No plot in c

+ 3 *Point adjustment*

For explanation

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(4) consider the function $f(x) = 1 - \cos x$

(a) relative condition number at $x = 0$

$$C_f(x) = \frac{x \cdot f'(x)}{f(x)} \quad \because C_f(x) \text{ denotes relative condition number}$$

$$\Rightarrow \frac{x \cdot \sin x}{1 - \cos x}$$

$$\simeq \frac{x \cdot x}{x^2/2} \quad \text{when } x \to 0,$$

$$= 2$$

(b) The numerical evaluation of the formula $1-\cos(x)$ is highly unstable this is likely due to subtraction, of two very similar values (1 and $\cos(x)$) which can result in a cancellation of significant digits.

(c) The stable algorithm for computing $f(x)$ is
The above ~~___~~ $f(x) = 1 - \cos x$ by differentiating $f(x)$

$$1 - \cos\left(\frac{x}{2} + \frac{x}{2}\right) = 1 - \cos\left(\frac{x}{2}\right)^2 + \sin\left(\frac{x}{2}\right)^2$$

$$\Rightarrow 2\sin\left(\frac{x}{2}\right)^2$$

In this case the error is less/small between ~~because~~, computed result and true result because in the case of $(1-\cos x)$ it has ~~to~~ high error rate.
As, the error is small and does not grow excessively as

4 Q4 **7.5 / 15**

- **- 0 pts** Correct
- **- 3 pts** Part a incorrect - Need to get 2 by mathematical proof
- **- 1.5 pts** Explanation not given in a
- **- 1.5 pts** Incorrect Explanation in b / Explained by example - Need to say about cancellation error using some mathematical proof
- **- 3 pts** Part b is incorrect - Need to say unstable
- **- 4 pts** Part c incorrect - Need to say $$2Sin^2(\frac{x}{2})$$
- ✓ **- 2.5 pts** *Mathematical proof not given / Unsatisfactory explanation in c / Explained by example*
- ✓ **- 5 pts** *Plot not provided / Incorrect*
- **- 2.5 pts** Plot part a incorrect
- **- 2.5 pts** Plot part b incorrect
- **- 15 pts** Incorrect

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(5) For the case of polynomial in expanded form, it is more unstable and eratic. here we are calculating power operation once to each and every power(x), which gives different results for all points and adding, multiplication arises the Truncation error.

i.e. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$

but, $(x-1)^6$ is more accurate & stable because we are calculating only one power compared to above.

(6) No, the graph doesn't confirm the expected behavior because Initially we can see the graph is monotonically decreasing till $K=20$, but after that it is increasing as K increases bcz it is ~~generally~~ getting closer ~~and~~. closer to zero. After some time truncating error plays a role and gets accumulated which increases the function value in this case.

(7) There are some errors that may occur due to the program

i) Numerical errors:- as the integral is calculated recursively, floating point errors may occur and accumulate.

ii) Truncation error:- The ~~go~~ program is only calculated the value of integral for $K=0$ to 20, so if we want more precision we need to increase the limit of k.

5 Q5 **9 / 10**

- **- 5 pts** Incorrect Plot
- **- 0 pts** Expanded Form $$(x-1)^6$$ has Cancellation Errors
- ✓ **- 0 pts** *Correct Plot*
- ✓ **- 1 pts** *Does not mention about cancellation Errors in the Expanded Form: $$(x-1)^6$$*
- **- 5 pts** No Explanation

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6 Q6 **9 / 15**

✓ **- 0 pts** *Correct Code/Plot*

✓ **- 6 pts** *You need to show $$x_k$$ in terms of $$x_1$$ and $$x_2$$, and the problem is ill-conditioned. Missing explanation that the Problem is Ill Conditioned, and the Rounding Errors in $$x_1$$ and $$x_2$$ are enhanced significantly.*

- 2 pts Semilog Plot should have log of the y-axis, not the x-axis.

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- 2 pts No Code

- 3 pts No Plot

- 15 pts No Submission!

- 1 pts Not Semilog

- 5 pts No Code/Plot

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7 Q7 **10 / 15**

✓ **- 0 pts** *Correct Code and Plot*

✓ **- 5 pts** *Missing reasoning that as $$ k \rightarrow \infty, I_k \rightarrow 0 \implies k \times I_{k-1} \rightarrow 1$$, leading to high Cancellation Errors.*

- 1 pts Partially Missing reasoning that as $$ k \rightarrow \infty, I_k \rightarrow 0 \implies k \times I_{k-1} \rightarrow 1$$, leading to high Cancellation Errors.

- 0 pts Correct Reasoning!

- 1 pts Computation in Double Precision not Shown

- 5 pts Incorrect/No Plot

- 10 pts No Explanation of the Errors

- 2 pts No Code

- 3 pts Partially Missing reasoning that as $$ k \rightarrow \infty, I_k \rightarrow 0 \implies k \times I_{k-1} \rightarrow 1$$, leading to high Cancellation Errors.

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PLOTS FOR ASSIGNMENT1 COL726

K LAXMAN 2018CS50408

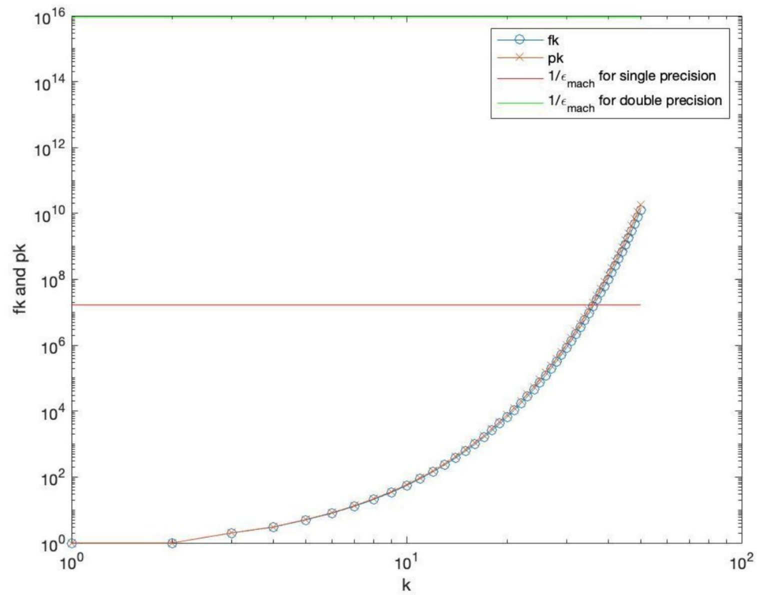


FIG:3(a)

https://drive.google.com/file/d/1orwzpdRux_k1qq2Qqo3H-2BHYDYWPOAl/view?usp=sharing
Google Drive link for MATLAB Code

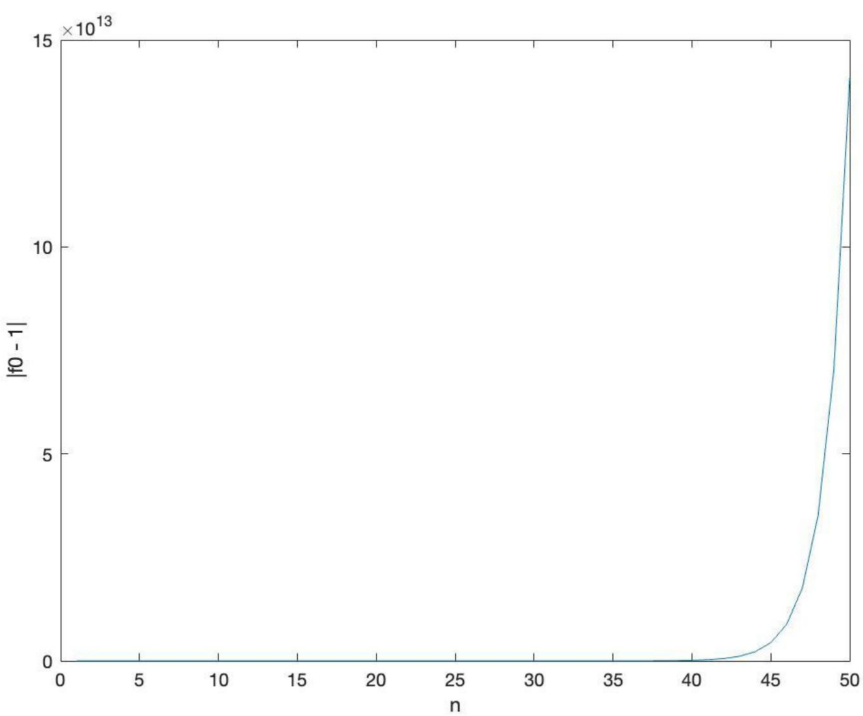
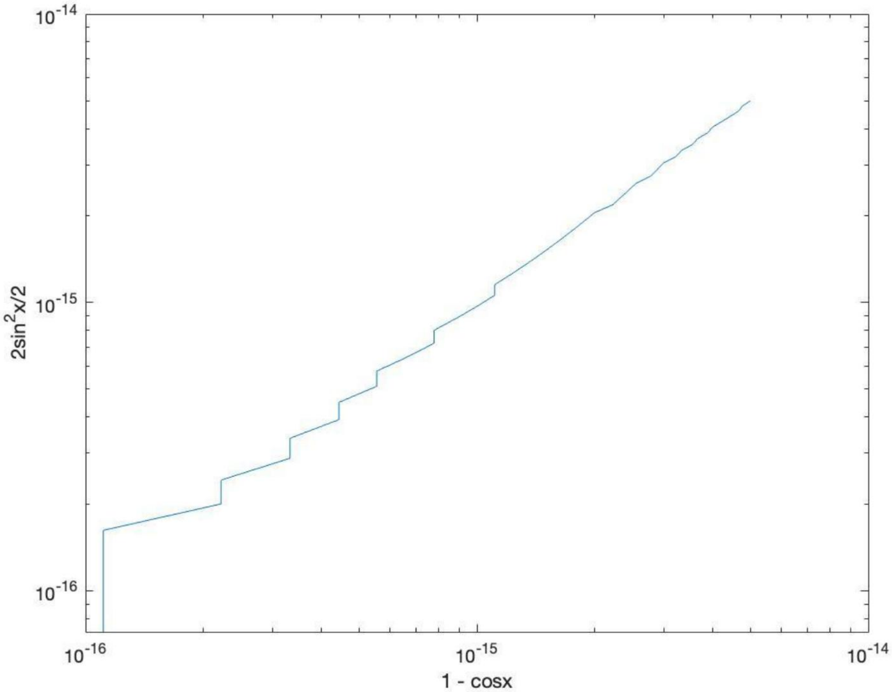
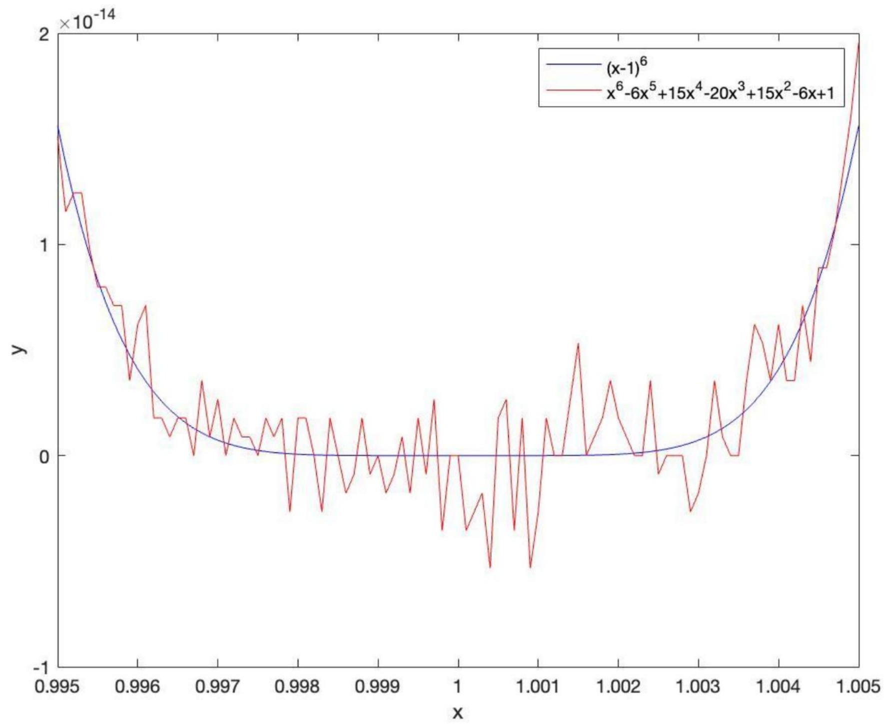


FIG:3(b)

FIG :4(C)



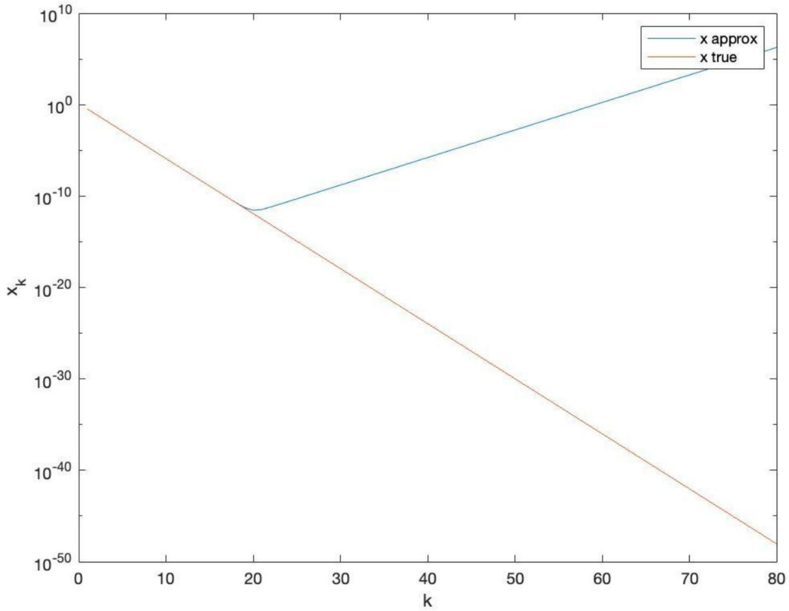
10^{-14}
10^{-15}
10^{-16}
$2\sin^2 x/2$
10^{-16}
10^{-15}
10^{-14}
1 - cosx



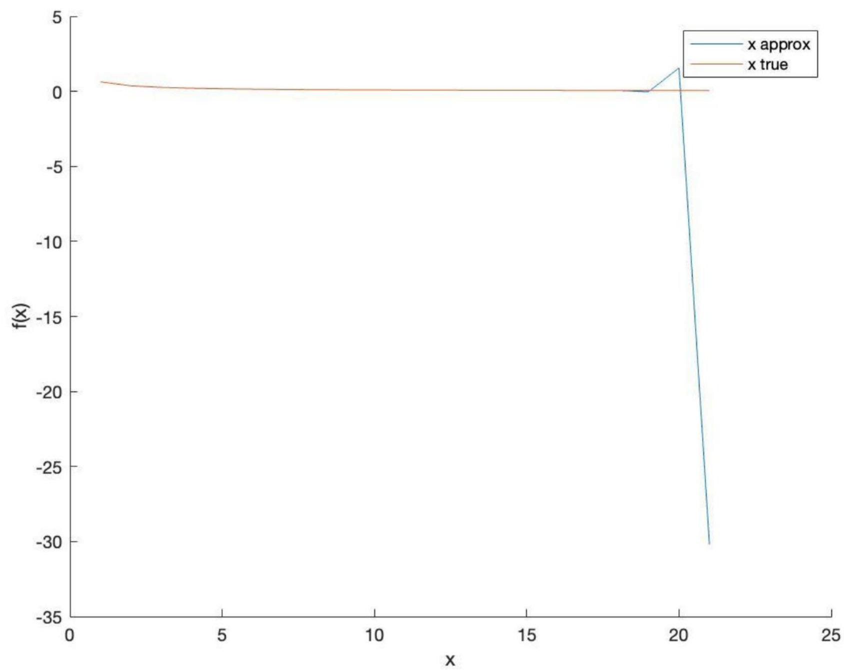
$\times 10^{-14}$
2
1
0
-1
y
0.995
0.996
0.997
0.998
0.999
1
1.001
1.002
1.003
1.004
1.005
x
$(x-1)^6$
$x^6-6x^5+15x^4-20x^3+15x^2-6x+1$

FIG(5)

FIG(6)


x approx
x true
10^{10}
10^{0}
10^{-10}
10^{-20}
10^{-30}
10^{-40}
10^{-50}
x_k
0
10
20
30
40
50
60
70
80
k

FIG(7)


x approx
x true
5
0
-5
-10
-15
-20
-25
-30
-35
f(x)
0
5
10
15
20
25
x

```
%% q3
clc
close all
clear
max_k = 50;

% Initialize the arrays for fk and pk
fk = zeros(1, max_k);
pk = zeros(1, max_k);

% Define the value of c
c = 1 + sqrt(3) / 100;

% Define the initial values of f0 and f1
fk(1) = 1;
fk(2) = 1;

% Define the initial values of p0 and p1
pk(1) = 1;
pk(2) = 1;

% Calculate the values of fk and pk for k > 1
for k = 3:max_k
   fk(k) = fk(k-1) + fk(k-2);
   pk(k) = c*pk(k-1) + pk(k-2);
end

% Plot the values of fk and pk on a log scale plot
figure;
loglog(1:max_k, fk, '-o');
hold on;
loglog(1:max_k, pk, '-x');

% Add the lines for 1/εmach for single and double precision arithmetic
single_prec = 2^24;
double_prec = 2^53;
hold on;
loglog([1 max_k], [single_prec single_prec], 'r');
hold on;
loglog([1 max_k], [double_prec double_prec], 'g');

% Add labels and legend
xlabel('k');
ylabel('fk and pk');
```

```
legend('fk', 'pk', '1/\epsilon_{mach} for single precision', '1/\epsilon_{mach} for double precision');


max_n = 50;

% Initialize the array for f0
f0 = zeros(1, max_n);

% Define the initial values of f0
f0(1) = 1;
f0(2) = 1;

% Initialize the array for f1
f1 = zeros(1, max_n);

% Define the initial values of f1
f1(1) = 1;
f1(2) = 1;

% Calculate the values of f0 and f1 for n > 2
for n = 3:max_n
   f0(n) = f1(n-1);
   f1(n) = f0(n) + f1(n-1);
end

% Calculate the difference between the original f0 and the recomputed f0
difference = abs(f0 - 1);

% Plot the difference as a function of n
figure;
plot(1:max_n, difference);

% Add labels and legend
xlabel('n');
ylabel('|f0 - 1|');




%% q4
clc
clear
close all
x = linspace(-1e-7 , 1e-7 , 101);
```

```
y1 = 1 - cos(x);
y2 = 2*sin(x/2).^2;
loglog(y1,y2)

xlabel('1 - cosx');
ylabel('2sin^2x/2');


%% q5
clc
clear
close all
x = linspace(0.995,1.005,101);
y1 = (x - 1).^6;
y2 = x.^6 - 6*x.^5 + 15*x.^4 - 20*x.^3 + 15*x.^2 - 6*x + 1;
plot(x, y1, 'b', x, y2, 'r');
xlabel('x');
ylabel('y');
legend('(x-1)^6','x^6-6x^5+15x^4-20x^3+15x^2-6x+1')

%% q6

clc;
clear;
close all;
x1 = 1/3;
x2 = 1/12;
x = zeros(1,80);
x_true  = zeros(1,80);
x(1) = x1;
x(2) = x2;
for k = 3:80
   x(k) = 2.25*x(k-1) - 0.5*x(k-2);

end
for k = 1:80
   x_true(k) = 4^(1-k)/3;
end
semilogy(1:80, x);
hold on
semilogy(1:80, x_true);
hold off
xlabel('k');
ylabel('x_k');
```

```
legend('x approx','x true')
%% q7
clc;
clear;
close all;
i0 = 1 - 1/exp(1);
x  = zeros(1,21);
k = 1:20;
x(1) = i0;
q = zeros(1,21);
for i = 2:21
    x(i) = 1 - (i-1)*x(i-1);

end

for i = 1: 21
f = @(x) x.^(i-1) .* exp(x-1);
q(i) = integral(f,0,1);
end
hold on
plot(x)
plot(q)
xlabel('x');
ylabel('f(x)');
legend('x approx','x true')
```