Department of Mathematics, IIT Delhi

MTL106: Exam-2.

Time: 1 hour 30 minutes **Date:** 15-05-2021 **Total Marks:** 35

Q.1) Let $\{Y_n, n = 1, 2, \ldots\}$ be a sequence of non-negative **i.i.d** random variables, defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with mean 1 and variance 2. Let $X_n = \dfrac{Y_n}{n^2}, n = 1, 2, \ldots$. Examine whether X_n converges almost surely or not. Show that

$$\lim_{n\to\infty} \mathbb{E}\big[e^{-X_n}\big] = 1.$$

3+3 marks

Q.2) a) Let $\{X_n, n = 0, 1, \ldots\}$ be a discrete-time Markov chain (DTMC) with state space $\mathcal{S} = \{1, 2, 3\}$, transition probability matrix $P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$, and initial distribution $\pi_0 = (0.2, 0.7, 0.1)$. Find

$$\mathbb{P}(X_2 = 3), \quad \mathbb{P}(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2).$$

b) Let $\{X_n : n \geq 0\}$ be a DTMC with state space $\mathcal{S} = \{1, 2, \ldots, K\}$ and transition probability matrix $P = (p_{ij})$ where

$$p_{ij} = \begin{cases} \frac{1}{3}, & j = i+1 \\ \frac{2}{3}, & j = i-1 \end{cases} \quad 1 < i < K; \quad p_{ii} = 0,\ i \neq 1, K; \quad p_{12} = \frac{1}{3} = p_{KK}; \quad p_{11} = \frac{2}{3} = p_{K,K-1}\,.$$

Examine whether there exists a limiting distribution of the given DTMC or not. Calculate $p_{r2}^{(n)}$ for $n \to \infty$ and for $r = 1, 2, \cdots, K$.

(2+1)+(2+5) marks

Q.3) a) Let $\{N(t) : t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$.

i) Show that, for any $a > 0$, there holds

$$\frac{a^2}{\lambda t + a^2} \leq \mathbb{P}(N(t) - \lambda t < a) \leq 1.$$

ii) Find $\mathbb{E}[N(t)N(t+s)]$ for $t > 0$ and $s > 0$.

b) In a particular city of India, three types of natural disasters are occurred: **A**, **B** and **C**. The occurrence of each disasters are independent and follows a Poisson process with rate 3, 1 and 5 per year respectively.

i) Calculate the probability that 3 or more disasters occur in a six-month period.

ii) Find the expected number of disasters in a two years time.

ii) What is the probability that there is 6 times type **A** disaster given that total no of disasters is 24 in a 36 month time period.

(2+3)+(2+1+2) marks

Q.4) Consider a CTMC $\{X(t) : t \geq 0\}$ with state space $S = \{1, 2, 3\}$, rate matrix $Q = \begin{pmatrix} -2 & 1 & 1 \\ 4 & -5 & 1 \\ 1 & 2 & -3 \end{pmatrix}$, and the initial distribution $(1/3, 1/3, 1/3)$.

a) Let τ be the spent time in state 2 before moving to another state. Find $\mathbb{P}(\tau \leq t)$.

b) Explain whether the stationary distribution for the given CTMC exists or not.

c) Let $P(t) = \big(p_{ij}(t)\big)_{i,j \in S}$ be the transition probability matrix of the given CTMC. Show that, for all $j \in S$

$$p'_{2j}(t) = 4p_{1j}(t) - 5p_{2j}(t) + p_{3j}(t).$$

d) Calculate $\lim_{t \to \infty} p_{i3}(t)$ for all $i \in S$.

1+ 2 +1+ 5 marks

Best of Luck!!!