COL 726 Homework III

K LAXMAN

TOTAL POINTS

35 / 95

QUESTION 1

1 **0 / 10**

- **- 0 pts** Correct Algorithm and runs in $$O(n^2)$$ time.
- **- 5 pts** No Details of Conversion to Hessenberg Matrix using Householder Transformations
- **- 2 pts** Missing Details in the Algorithm (Conversion to Hessenberg Matrix using Householder Transformations)
- ✓ **- 10 pts** *No Attempt*

QUESTION 2

2 **0 / 15**

- **- 0 pts** (a) Correct
- **- 2 pts** (a) Does not take errors due to Floating Point Operations into Account
- **- 3 pts** (a) Details Missing
- **- 5 pts** (a) No Attempt
- **- 0 pts** (b) Correct
- **- 1 pts** (b) Minor Details Missing
- **- 2 pts** (b) Ignores the Errors due to Floating Point Computation of $$\tilde{b}$$
- **- 2 pts** (b) Incomplete Proof
- **- 5 pts** (b) No Attempt
- **- 0 pts** (c) Correct
- **- 2 pts** (c) Incomplete Proof
- **- 5 pts** (c) No Attempt
- ✓ **- 15 pts** *Incorrect/No Attempt*

QUESTION 3

3 **6 / 15**

- ✓ **- 0 pts** *Correct*
- **- 15 pts** Incorrect
- **- 9 pts** Implementation
- ✓ **- 6 pts** *Reasoning*
- **- 3 pts** generate data function
- **- 4.5 pts** Solving using Cholesky function
- **- 4.5 pts** Solving using QR decomposition
- **- 3 pts** Function to analyze error for reasoning
- **- 3 pts** Reasoning after analyzation is incorrect / not given

- 3 *Point adjustment*

you should give QR code here also..................+3 after confirmation of this

QUESTION 4

4 **9 / 15**

- **- 0 pts** Correct
- **- 15 pts** Incorrect
- **- 5 pts** has not done converse proof
- **- 5 pts** has not done forward proof
- **- 5 pts** part b not done (similar to a)
- **- 7 pts** Proper mathematical proof not given
- **- 8 pts** not said about schur / any other relevant method/not said about some decomposition; not said about diagonality; ambiguous partially correct answer /

- 7 pts Assumption not justified / Reasons not mentioned / Partially correct Answer

- 6 *Point adjustment*

missing details..........b converse missing

QUESTION 5

5 **0 / 10**

- 0 pts Correct Proof (Shows that $$\frac{w}{||w||} \sim \frac{\tilde{w}}{||\tilde{w}||} $$)

✓ **- 10 pts** *No Attempt*

- 7 pts Incomplete Proof

- 3 pts Incorrect/Incomplete Proof: Doesn't take into account that the condition number of A is large (as the smallest eigenvalue is much smaller than other eigenvalues).

- 2 pts Details Missing

QUESTION 6

6 **10 / 10**

✓ **- 0 pts** *Correct Proof (Shows that all eigenvalues are distinct by showing that $$GM(\lambda) = 1$$)*

- 10 pts No Attempt

- 7 pts Incomplete Proof

QUESTION 7

7 **10 / 10**

✓ **- 0 pts** *Correct*

- 5 pts Incomplete/Incorrect "if" Proof

- 5 pts Incomplete/Incorrect "only if" Proof

QUESTION 8

8 **0 / 10**

- 0 pts Correct

✓ **- 10 pts** *Incorrect*

- 5 pts original solution and answer analysis not given

- 5 pts random noise not added for modification

- 8 pts Implementation not given but idea is correct

- 4 pts missing reasons / explaination of code / answers

for efforts

1 **0 / 10**

- **- 0 pts** Correct Algorithm and runs in $O(n^2)$ time.
- **- 5 pts** No Details of Conversion to Hessenberg Matrix using Householder Transformations
- **- 2 pts** Missing Details in the Algorithm (Conversion to Hessenberg Matrix using Householder Transformations)
- ✓ **- 10 pts** *No Attempt*

2 **0 / 15**

- **- 0 pts** (a) Correct
- **- 2 pts** (a) Does not take errors due to Floating Point Operations into Account
- **- 3 pts** (a) Details Missing
- **- 5 pts** (a) No Attempt
- **- 0 pts** (b) Correct
- **- 1 pts** (b) Minor Details Missing
- **- 2 pts** (b) Ignores the Errors due to Floating Point Computation of $$\tilde{b}$$
- **- 2 pts** (b) Incomplete Proof
- **- 5 pts** (b) No Attempt
- **- 0 pts** (c) Correct
- **- 2 pts** (c) Incomplete Proof
- **- 5 pts** (c) No Attempt

✓ **- 15 pts** *Incorrect/No Attempt*

https://drive.google.com/drive/folders/1zoF2tepBrWqpgzPa63BAPLAFWvFoWpa8?usp=sharing

```
m = 21;
n = 12;
epsilon = 10^-10;

matA = ones(m,n);
for i = 1:m
   ti = (i-1)/(m-1);
   temp = 1;
   for j = 1:n
      matA(i,j) = temp;
      temp = temp*ti;
   end
end

matX = ones(n,1);
matB = matA*matX;

for i = 1:m
   u = rand;
   matB(i) = matB(i) + (2*u - 1)*epsilon;
end

% Cholesky Factorisation

Ac = matA'*matA;
L = chol(Ac);

y = L'\(matA'*matB);
matX1 = L\y;
disp(matX1);

error1 = norm(matX1-matX)/norm(matX);
disp(['Error in Cholesky factorisation: ', num2str(error1)]);

% QR Factorisation

[Q,R] = qr(matA);
matX2 = R\(Q'*matB);
disp(matX2);

error2 = norm(matX2-matX)/norm(matX);
```

3 **6 / 15**

✓ **- 0 pts** *Correct*

- 15 pts Incorrect

- 9 pts Implementation

✓ **- 6 pts** *Reasoning*

- 3 pts generate data function

- 4.5 pts Solving using Cholesky function

- 4.5 pts Solving using QR decomposition

- 3 pts Function to analyze error for reasoning

- 3 pts Reasoning after analyzation is incorrect / not given

- 3 *Point adjustment*

you should give QR code here also..................+3 after confirmation of this

(4)

(a) $\lim_{n\to\infty} \|A^n\| = 0$ iff $\rho(A) < 1$

Suppose that $\rho(A) < 1$ then $\exists$ a constant c such that $|\lambda| \leq \rho(A) < 1$
for all eigenvalues of A

$\therefore \|A^n v\| = \|\lambda^n v\| = |\lambda|^n \|v\| \leq \rho(A)^n \|v\|$

for all $n \geq 0$ since $\|v\|$ is nonzero and fixed we have

$\lim_{n\to\infty} \|A^n v\| = 0$ By the triangle inequality $\|A^n\| \leq \|A\|^n$

$\forall\ n \geq 0$, $\therefore \lim_{n\to\infty} \|A^n\| = 0$

Conversely suppose that $\lim_{n\to\infty} \|A^n\| = 0$, then for any $\varepsilon > 0$, $\exists$ an integer
N such that $\|A^N\| < \varepsilon$ $\forall\ n \geq N$ and let λ be eigen value of A with
maximum real part. Then we have

$$|\lambda^n| = \frac{|(A^n)v|}{\|v\|} \leq \frac{\|A^n\|}{\|v\|} \leq \frac{\|A^N\|}{\|v\|} < \varepsilon/\|v\|$$

by taking limits $n \to \infty$, we get $|\lambda| \leq \frac{\varepsilon}{\|v\|}$ since $\varepsilon > 0$,
since v is arbitrary $|\lambda| = 0 \Rightarrow \lambda = 0$

$\therefore$ eigenvalue of A have $\rho(A) < 1$

(b) suppose that $\alpha(A) < 0$, then $\exists$ a complex eigenvalue λ
with positive eigenvalue λ with positive real part. Let v be a
corresponding eigen vector then

$\|e^{t(tA)} v\| = \|e^{(t\lambda)} v\| = e^{(t\,Re(\lambda))} \|v\|$

since $Re(\lambda) > 0$ the right hand side $\to \infty$, as $t \to \infty$

by defn of norm

$\therefore$

$\lim_{t\to\infty} \|e^{tA}\| = \infty$.

4 **9 / 15**

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- **- 7 pts** Assumption not justified / Reasons not mentioned / Partially correct Answer

- 6 *Point adjustment*

missing details..........b converse missing

5 **0 / 10**

- 0 pts Correct Proof (Shows that $$\frac{w}{||w||} \sim \frac{\tilde{w}}{||\tilde{w}||} $$)

✓ **- 10 pts** *No Attempt*

- 7 pts Incomplete Proof

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- 2 pts Details Missing

K. Laxman 2018CS50408

Q6

(b) Show that :- To prove that eigen values are distinct

Initially we fix $\forall \lambda \in \mathbb{C}$. Let us consider $A-\lambda I$ and here notice that $B = (A-\lambda I)_{2:m, 2:m}$ i.e. by eliminating 1st row & last column is full rank as it is an upper triangular with non zero diagonals which correspond to A's sub diagonal.

$$\begin{bmatrix} \times & \times & & \vdots \\ \times & \times & \times & \vdots \\ \times & \times & \times & \times \\ & & \times & \times \end{bmatrix} \qquad : B_{ii} = (A-\lambda I)_{i+1,i} = A_{i+1,i} \ (1 \le i \le m-1)$$

and $B_{ij} = (A-\lambda I)_{i+1,j} = A_{i+1,j} = 0$ for $\forall i > j$, by defn of tridiagonal.

So, if rank $(A-\lambda I) \le m-2$, there should be linearly dependent pairs of rows in B, which ~~coordinate~~ contradicts to full rank of B.

$\Rightarrow$ ran $(A-\lambda I) \ge m-1$

Now, from the theorem in book (24.7) it diagnolizable by $A = X^* \Lambda X$. and by thm 24.3, Λ & A have the exactly same eigenvalues which appear on Λ.

$A - \lambda I = X^* \Lambda X - \lambda I = X^* (\Lambda - \lambda I) X$.

as $A - \lambda I$ & $\Lambda - \lambda I$ are similar have same rank i.e rank $(\Lambda - \lambda I) \ge m-1$

i.e $\lambda_i = (\Lambda)_{ii}$ $(1 \le i \le m)$ and each λ_i is the eigen value of A.

Suppose the eigenvalues of A are not distinct then

i.e. $\lambda_i = \lambda_j$ for some $i \ne j$. Fix $\lambda = \lambda_i = \lambda_j$

Since $\Lambda - \lambda I$ is diagonal, rank $(\Lambda - \lambda I) =$ No of non zero diagonals

6 **10 / 10**

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- 10 pts No Attempt

- 7 pts Incomplete Proof

But as $(\Lambda - \lambda I)_{jj} = (\Pi)_{jj} = 0$,

$\text{rank}(\Lambda - \lambda I) = \text{rank}(A - \lambda I) \leq m-2$ which is contradiction

So, from this we can say that eigenvalues are distinct.

(7) To prove that a complex number z is Rayleigh quotient of A iff it is a diagonal entry of $Q^* A Q$ for some unitary matrix Q.

$(\rightarrow)$

Let us say that $r(x) = \dfrac{x^* A x}{x^* x} = z$ for some $x \in \mathbb{C}^m$

extend the equation $\left\{ \dfrac{x}{\|x\|_2} \right\}$ to the orthonormal basis for $\mathbb{C}^m$, $\left\{ \dfrac{x}{\|x\|_2}, f_2, \dots, f_m \right)$

we have to note here that $\|x\|_2 = \sqrt{x^* x}$ and set

$Q = \left[\dfrac{x}{\|x\|_2} \middle| f_2 \middle| \dots \middle| f_m \right]$ then

$$(Q^* A Q)_{11} = \left(\frac{x}{\|x\|_2} \right)^* (AQ)_1 = \underbrace{\left(\frac{x}{\|x\|_2} \right)^*}_{\text{first row of } Q^*} A \underbrace{\frac{x}{\|x\|}}_{\text{1st column of } Q} = \frac{x^* A x}{x^* x} = z$$

$(\leftarrow)$

Let $z = (Q^* A Q)_{ii}$ for some $1 \leq i \leq m$

choose f_i (ith column of Q)

As we can see then $z = f_i^* A f_i = \dfrac{f_i^* A f_i}{f_i^* f_i} = r(f_i)$

7 **10 / 10**

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for efforts

```
disp(['Error in QR factorisation: ', num2str(error2)]);
```