Question 1

1.We have

$$X(t) = N(t+L) - N(t) \sim N(L)$$

Hence

$$\mathbb{E}[X^2(t)] = \lambda L + \lambda^2 L^2$$

2. $\mathbb{E}[X(t)] = \lambda L$ Finally wlog $s < t$, we find $Cov(X(t), X(s))$ Two cases arise:

- $s < s+L < t < t+L$. In this case, we have $N(t+L) - N(t)$ and $N(s+L) - N(s)$ are independent and hence $Cov(X(t), X(s)) = 0$
- $s < t < s+L < t+L$, In this case, we have $Cov(X(t), X(s)) = Cov(N(t+L) - N(s+L) + N(s+L) - N(t), N(s+L) - N(t) + N(t) - N(s))$ which by linearity of covariance and independent increments is given by $Cov(N(s+L) - N(t), N(s+L) - N(t)) = \lambda(s+L-t)$

Hence in both cases covariance is a function of $t - s$

Question 2

WLOG $s < t$, we have $Cov(N_t, N_s) = Cov(N_t - N_s + N_s, N_s) = Cov(N_s, N_s) = \lambda s = \lambda min(s, t)$

Quesstion 3

$$\mathbb{P}(N_t^1 = k | N_t^1 + N_t^2 = n) = \mathbb{P}(N_t^1 = k, N_t^2 = n-k)/\mathbb{P}(N_t^1 + N_t^2 = n)$$

Using the fact that sum of independt poisson process is a poisson process with rate parameter as sum of the rate parameters of individual poissons, we have

$$\mathbb{P}(N_t^1 + N_t^2 = n) = e^{-\lambda_1 - \lambda_2} \frac{(\lambda_1 + \lambda_2)^n}{n!}$$

Also, due to independence

$$\mathbb{P}(N_t^1 = k, N_t^2 = n-k) = \mathbb{P}(N_t^1 = k)\mathbb{P}(N_t^2 = n-k) = e^{-\lambda_1 - \lambda_2} \frac{\lambda_1^k}{k!} \frac{\lambda_2^{n-k}}{(n-k)!}$$

Hence,

$$\mathbb{P}(N_t^1 = k | N_t^1 + N_t^2 = n) = \binom{n}{r} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-k}$$

Question 4

Using 3 find the answer

Question 5

Total comission is given by $S(t) = N(t, \lambda p) * A + N(t, \lambda(1-o)) * B$
Hence, we have $\mathbb{E}(S(t)) = Ap\lambda + B(1-p)\lambda$

Question 6

1. The total number of sales is a poisson process with intensity $20 \times 0.4 = 8/hr$
   Hence Probability that no sales are made in first 10 minutes is given by $\mathbb{P}(N_{1/6} = 0) = e^{-8/6}$
2. Expected number of sales during eight hours is given by $\mathbb{E}(N_8) = 8 * 8 = 64$
3. $\mathbb{P}(N_{1.5} = 25 | N_3 = 40)$
   Write $N_3 = N_3 - N_{1.5} + N_{1.5}$, use independent increments, homogenity of poisson process and question 3 to find the value
4. Easy

Question 7

Compound poisson process, mean is given by $\lambda t \mathbb{E}[Y_i]$ and variance is given by $\lambda_t \mathbb{E}(Y_i^2)$

Question 8

Compensation amount is given by $S(t) = 500\sum_{i=1}^{N}(t) Y_i$ where Y_i are iid and mean of Y_i is 50 and standard deviation is 100
$\mathbb{E}(S(t)) = 500\lambda t \mathbb{E}[Y_i]$
$Var(S(t)) = 2500\lambda t \mathbb{E}[Y_i^2]$

Question 9

The generator matrix is given by $q_{ii} = -i\mu$ and $q_{i,i-1} = i\mu$
Writing forward kolmogrov equations , we have

$$p'_{N,N}(t) = \sum_{i=0}^{N} p_{N,i}(t) q_{i,N} = -N\mu p_{N,N}(t)$$

Solving, we get

$$p_{N,N}(t) = Ae^{-N\mu t}$$

Since $p_{N,N}(0) = 1$, we have $A = 1$
Hence $p_{N,N}(t) = e^{-N\mu t}$
Writing forward kolmogrov equation for $j < N$, we have

$$p'_{N,j}(t) = \sum_{i=0}^{N} p_{N,i}(t) q_{i,j} = -j\mu p_{N,j}(t) + (j+1)\mu p_{N,j+1}(t)$$

$$\Rightarrow p'_{N,j}(t) + j\mu p_{N,j}(t) = (j+1)\mu p_{N,j+1}(t)$$

Multiplying both sides by $e^{j\mu t}$, we get

$$(e^{j\mu t} p_{N,j}(t))' = (j+1)\mu p_{N,j+1}(t) e^{j\mu t}$$

Let $e^{j\mu t} p_{N,j}(t) = R_j(t)$ Claim $R_j(t) = \binom{N}{j}(1 - e^{-\mu t})^{N-j}$
Clearly the claim is true for $j = N$
So let it be true for all $j \geq k+1$
For $j = k$, we have

$$R'_k(t) = (k+1)\mu \binom{N}{k+1} e^{-\mu t}(1 - e^{-\mu t})^{N-k+1}$$

Integrating both sides, we get

$$R_k(t) = \frac{k+1}{N-k} \binom{N}{k+1} (1 - e^{-\mu t})^{N-k} + C$$

Using $R_j(0) = 0, j < N$ and $\frac{k+1}{N-k}\binom{N}{k+1} = \binom{N}{k+} e^{-j\mu t}$ we get

$$R_k(t) = \binom{N}{k}(1 - e^{-\mu t})^{N-k}$$

Hence

$$\mathbb{P}(X(t) = j) = \binom{N}{j}(1 - e^{-\mu t})^{N-j} e^{-j\mu t}$$

which is binomial distribution with $p = e^{-\mu t}, n = N$
$\mathbb{E}(X(t)) = Ne^{-\mu t}$ and $Var(X(t)) = Ne^{-\mu t}(1 - e^{-\mu t})$

Question 10

Q matrix is given by

$$Q = \begin{bmatrix} -6 & 6 \\ 6 & -6 \end{bmatrix}$$

Hence forward kolmogrov equations are

$$p'_{00}(t) - -6p_0 0(t) + 6p_{01(t)}$$

$$p'_{01}(t) = 6p_{00}(t) - 6p_{01}(t)$$

$$p'_{10}(t) - 6p_{10}(t) + 6p_{11}(t)$$

$$p'_{11}(t) = 6p_{10}(t) - 6p_{11}(t)$$

$$\mathbb{P}(X(t) = -1) = \mathbb{P}(X(0) = -1)\mathbb{P}(N(t) = even) + \mathbb{P}(X(0) = 1)\mathbb{P}(N(t) = odd) = 0.5\mathbb{P}(N(t) = even) + 0.5\mathbb{P}(N(t) = odd)$$

$$= 0.5e^{-\lambda t}\left(\frac{e^{-\lambda t} + e^{\lambda t}}{2} + \frac{e^{\lambda t} - e^{-\lambda t}}{2}\right) = 0.5$$

Similar for other one

Question 11

Time taken to reach 20 individuals is given by $T = T_1 + T_2 + \ldots + T_n$, where T_i is the time to increase population from i to $i+1$
We know that $T_i's$ are independent and $T_i \sim Exp(i\lambda)$ Hence, we have

$$\mathbb{E}[T] = \sum_{i=1}^{20} \mathbb{E}[T_i] = \sum_{i=1}^{20} \frac{1}{i\lambda}$$

Similarly variance is given by

$$\sum_{i=1}^{20} Var[T_i] = \sum_{i=1}^{20} \frac{1}{i^2\lambda^2}$$

Question 12

Given by $\Pi Q = 0$ Solving, we get $\Pi = (3/8, 4/8, 1/8)$

Question 13

- Transtion probability matrix of embedded markov chain is given by

$$\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/3 & 2/3 & 0 \end{bmatrix}$$

- From backward kolmogrov equations, we have

$$p'_{1i}(t) = \sum_{j=1}^{3} p_{ji}(t) q_{1j}(t) = -2p_{1i}(t) + p_{2i}(t) + p_{3i}(t)$$

$$p'_{3i}(t) = \sum_{j=1}^{3} p_{ji}(t) q_{3j}(t) = p_{1i}(t) + 2p_{2i}(t) - 3p_{3i}(t)$$

Embedded markov chain is ergodic and $\tilde{\lambda} = \frac{1}{4}$ hence a limiting distribution exists
Solving $\Pi Q = 0, \Pi_1 + \Pi_2 + \Pi_3 = 1$, we get
$\Pi_1 = 4/12, /Pi_2 = 5/12, \Pi_3 = 3/12$

$$\lim_{t \to \infty} p_{i2}(t) = \Pi_2 = 5/12$$

Question 14

For stationary distribution $\Pi Q = 0$

$$\Pi_n = \frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i} \Pi_0$$

$$\Pi_n = \frac{1 \times 3 \times \ldots \times (2n-1)}{3^n n!} \Pi_0$$

$$1 \times 3 \times \ldots \times (2n-1) = \frac{(2n)!}{2^n n!}$$

Hence, we have

$$\Pi_n = \frac{(2n)!}{n!n!} \frac{1}{6^n} = \binom{2n}{n} \frac{1}{6^n}$$

We know that

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$$

Also

$$\lim_{n \longrightarrow \infty} (\Pi_n)^{1/n} < 1$$

so the series $\sum_{n=0}^{\infty} \Pi_n$ converges
Hence, we have

$$\sum_{n=0}^{\infty} \Pi_n = \Pi_0 \frac{1}{\sqrt{1-2/3}} = \sqrt{3}\Pi_0$$

Hence, $\Pi_0 = 1/\sqrt{3}$ and $\Pi_n = \binom{2n}{n} \frac{1}{6^n} \frac{1}{\sqrt{3}}$ which is the required stationary distribution