MTL-106

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① Given that

$\{Y_n, n=1,2,\dots\}$ be a sequence of iid random variables defined with mean 1 and Variance 2

To show $\lim_{n\to\infty} E[e^{-X_n}] = 1$ for sequence $X_n = \frac{Y_n}{n^2}$, $n=1,2$

Solution :-

The sequence $X_n = \frac{Y_n}{n^2}$, $n=1,2$ converges almost surely to zero.

This is because of strong law of large numbers (SLLN) which states that if $\{X_n\}$ is the sequence of independent and identically distributed random variables with finite mean μ.

then the avg. of first 'n' terms approaches μ as $n \to \infty$ (approaches) with the probability 1.

$\therefore$ since Y_n has mean $1 \ \forall n$. and $X_n = \frac{Y_n}{n^2}$

we have that X_n has mean $\frac{1}{n^2}$

Thus by (SSLN) X_n converges almost surely to zero.

To show $\lim_{n\to\infty} E[e^{-X_n}] = 1$

In this case we can use the dominated convergence theorem (DCT)

Since Y_n is non-negative $\forall n$ and $E[Y_n] = 1 \ \forall n$

we have that $E[e^{(-Y_n)}] \le e^{-1}$ for all n by markov inequality

Thus by DCT we have that

$$\lim_{n\to\infty} E[e^{-(X_n)}] = \lim_{n\to\infty} E(e^{(-Y_n/n)_2})] : E[\lim_{n\to\infty} e^{(-Y_n/n^2)}] =$$

$E, 1 = 1$

Q3) Given

$E(N(t)N(t+s)]$ for $t>0, s>0$

a)

ii) ~~E(N(t)N(t+s))~~ ~~t≥0~~

we know that $N(t)$ is a poisson process in which

$E(N(t)] = \lambda t$, $var(N(t)) = \lambda t$, $E[N^2 t] = \lambda^2 t^2 + \lambda t$

Now, from the above

$E[N(t)N(t+s)] = E[(N(t+s) - N(t) + N(t))N(t)]$

$= E[N(t+s) - N(t))N(t))] + E[(N^2 t)]$

$= E[N(t+s) - N(t)) E[N(t)] + \lambda^2 t^2 + \lambda t$

from the independent increment property

~~$= \lambda(t)\lambda t$~~

$= \lambda(s)\lambda t + \lambda^2 t^2 + \lambda t$

$= \lambda^2 t(t+s) + \lambda t$

3Q) b) we know that A follows poisson process with rate 3, B with 1 and C with 5 per year.

hence $A_t \sim P(3t), B_t \sim P(1t), C_t \sim P(5t)$

we know that $X_t = A_t + B_t + C_t \sim (3t + 1t + 5t)$

$\sim P(9t)$

i) $P(X_{0.5} \geq 3) = 1 - P(X_{0.5} < 3)$

$= 1 - P(X_{0.5} = 0) - P(X_{0.5} = 1) - P(X_{0.5} = 2)$

$= 1 - e^{-4.5}\left(1 + 4.5 + \frac{(4.5)^2}{2}\right)$

$= 1 - e^{-4.5}(15.625)$

Probability of 3 & more disasters in 6 months $= 0.826$

ii) $\underbrace{E[X_{2t}]}_{\text{Poisson process}} = \lambda t = 9 \times 2 = 18$

iii) $P(A_t = 6 \mid X_t = 24) = \dfrac{P(A_t = 6, B_t + C_t = 18)}{P(X_t = 24)}$

$= \dfrac{e^{-3t}(3t)^6 e^{-6t}(6t)^{18}\, 24!}{6!\; 18!\; e^{-9t}(9t)^{24}}$

$= {}^{24}C_6 \dfrac{(3t)^6 (6t)^{18}}{(9t)^{(24)}} \Rightarrow {}^{24}C_6 \dfrac{3^6 6^{18}}{9^{24}}$

$= {}^{24}C_6 \dfrac{2^{18}}{3^{24}}$

Q 4) we are required to calculate

d) $\lim_{t \to \infty} P_{i3}(t)$ for all $i \in S$.

hence we will find the limiting probability π_i by

setting
$$P'(t) = P(t)Q = 0$$

Let $\pi = [\pi_1, \pi_2, \pi_2]$ we will equate
$$0 = \pi Q$$

$$0 = [\pi_1, \pi_2, \pi_3] \begin{bmatrix} -2 & 1 & 1 \\ 4 & -5 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

hence

$$0 = -2\pi_1 + 4\pi_2 + \pi_3 \rightarrow ①$$

$$0 = \pi_1 - 5\pi_2 + 2\pi_3 \rightarrow ②$$

$$0 = \pi_1 + \pi_2 - 3\pi_3 \rightarrow ③$$

on solving the above ③ equations

$$6\pi_2 - 5\pi_3 = 0 \qquad \pi_3 = 6/_3 \pi_2$$

$$\pi_1 - 13/_5 \pi_2 = 0 \qquad \pi_1 = 13/_5 \pi_2$$

we know that $\pi_1 + \pi_2 + \pi_3 = 1$ we get

$$\pi_2 \left(13/_5 + 1 + 6/_5 \right) = 1 \qquad \therefore \pi_2 = 5/_{24}$$

$$\pi_1 = 13/_{24}$$

$$\pi_3 = 6/_{24}$$

Hence $P_{i3}(t) = 6/_{24} = 1/_4 \ \forall \ i$

4 c) $p(t) = (P_{ij}(t))_{i,j \in S}$

$$P'_{2j}(t) = 4P_{1j}(t) - 5P_{2j}(t) + P_{3j}(t)$$

In this case,

using Backward kolmogrov equation we can write it as

$$P'_{ij}(t) = \sum_{i=1}^{3} P_{ij}(t) q_{3i}(t)$$

$$= 4P_{1j}(t) - 5P_{2j}(t) + P_{3j}(t)$$

2) a) $p(X_2=3)$, $p(X_3=2, X_2=3, X_1=3, X_0=2)$

$$TPM = P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

$\cancel{p(X_3=2/X_2=3)\, p(X_2=3/X_1=3)\, p(X_1=3/X_0=2)\, p(X_0=2)}$

$\cancel{= 0.4 \times 0.3 \times 0.2 \times 0.7}$

$X_1 = X_0 P$

a) $$[0.2,\ 0.7,\ 0.1] \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$= [0.47,\ 0.28,\ 0.25]$$

$X_2 = X_1 P$

$$[0.47,\ 0.28,\ 0.25] \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$ multiplication

$$= 0.319$$

similarly $p(X_2) =$ 0.0618 using multiplica