1 ⓐ To compute the LU decomposition of a banded matrix A without pivoting in O(lum) flops using the doolittle algorithm

set L to be the m×m identify matrix

For k = 1 to m do:

For i = k to min (m, k+u) do:

compute (L(i, k) = A(i,k) / A(k, k)

For j = k+1 to min (m, k+l) do:

For i = max(k, j−l) to k−1 do:

compute A(i,j) = A(i,j) − L(i,k) * A(k,j).

Compute A(k,j) = A(k,j) − L(k, k+1 : k+l) * A(k+1 : k+l, j)

The resulting matrices L and U are the lower and upper triangular matrices of LU decomposition of A.

1 ⓑ we can use following matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

LU decomposition with partial pivoting will get this

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

② Given Let A be a $m \times m$ SPD matrix with cholesky factorisation $A = LL^T$. We want to compute cholesky factorisation of $M = \begin{bmatrix} A & b \\ b^T & c \end{bmatrix}$ where, A is $m \times m$ SPD matrix

Let L be the $m \times m$ lower triangular matrix such that $A = LL^T$. We want to compute C.F of M

the algorithm is as follows

$y^T = L^{-1}b$

$z = \sqrt{(c - y^T y)}$

then $L' = \begin{bmatrix} L & 0 \\ y & z \end{bmatrix}$

The resulting matrix (L') is lower triangular matrix of cholesky factorisation (C.F.) of M.

to justify this algo, need to show L' satisfies the following

L' is lower triangular

$L'L' = M$

$$\Rightarrow \begin{bmatrix} L & 0 \\ y & z \end{bmatrix} \begin{bmatrix} L^T & y^T \\ 0 & z^T \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} LL^T & Ly^T \\ L^T y & y^T y + zz^T \end{bmatrix}$$

put $Ly^T = b$

& $zz^T = c - y^T$

we get

$$\Rightarrow \begin{bmatrix} A & b \\ b^T & c \end{bmatrix}$$

$\therefore$ L' is indeed the lower triangular matrix of cholesky factorisation of M.

K. Laxman
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COL726

Assignment - 2

(3) To prove that all the eigenvalues of A lies in the interval $[0.4, 1.6]$, we will use the fact that A is Symmetric positive definite matrix.

* given condition $\|A-I\|_2 = 0.6$

Let λ be an eigen value of A and x be its associated eigen vector. we have

$$Ax = \lambda x$$

$$(A-I)x = (\lambda-1)x \quad \text{(by rearranging eq)}$$

taking the 2 norm on both sides

$$\|(A-I)x\|_2 \leq \|A-I\|_2 \|x\|_2$$

by substituting

$$|\lambda-1| \, \|x\|_2 \leq 0.6 \|x\|_2$$

dividing both sides by $\|x\|_2$ (assuming $x \neq 0$)

$$|\lambda-1| \leq 0.6$$

hence, from this (inequality) we can say that absolute difference b/w any eigenvalue λ of A and 1 (eigenvalue of I) is less than & equal to one

$\therefore$ eigenvalue of A lie in the interval $[1-0.6, 1+0.6]$

$$\Rightarrow [0.4, 1.6]$$

The upper bound for error e_n in A-norm after 'n' steps

$$\frac{\|e_n\|_A}{\|e_0\|_n} \leq 2\left(\frac{\sqrt{K}-1}{\sqrt{K}+1}\right)^n$$

here K is the condition number of A.

which is given by $\quad K = \|A\|_2 \|A^{-1}\|_2 = \dfrac{\lambda_{max}}{\lambda_{min}}$

from above we know $\lambda_{max} \leq 1.6$ and $\lambda_{min} \leq \dfrac{1}{0.4}$

hence $\quad K \leq \dfrac{1.6}{0.4} \leq 4$

hence $\frac{\|e_n\|_A}{\|e_0\|_A} \le 2\left(\frac{\sqrt{4}-1}{\sqrt{4}+1}\right)^n \le \boxed{2\left(\frac{1}{3}\right)^n}$

(b) we are given symmetric positive definite matrix A with eigenvalue $\lambda_1 = 1$ with associated eigen vector v_1 remaining eigen vectors are $\lambda_2, \lambda_3, \lambda_4 \cdots \lambda_m$

Now we are given, $B = A + ww^T$

$B = A + 49vv^T$

$Bx = Ax + 49vv^Tx$

$Bx = \lambda x + 49(v^Tx)v$

if $x = v$, we get

$Bv = \lambda v + 49(v^Tv)v$ and $v^Tv = 1$

so, $Bv = \lambda v + 49v$

$= 50v$

we get $\boxed{\lambda_1 = 50}$. Now, if $x \neq v$, $v^Tx = 0$

$Bx = Ax + 49(0)v$

$Bx = Ax = \lambda x$

Hence, we get eigen values of B as $(50, \lambda_2, \lambda_3 \cdots \lambda_m)$ and eigen vectors as $(v_1, v_2, \ldots v_m)$

Now,

$$\frac{\|e_n\|_B}{\|e_0\|_B} = 2\left(\frac{\sqrt{K_B}-1}{\sqrt{K_B}+1}\right)^n \text{ and } K_B = \frac{(\lambda_{max})_B}{(\lambda_{min})_B}$$

$$\frac{1}{(\lambda_{min})_B} \leq \frac{1}{0.4} \leq 2.5$$ (since all λ_i are same except λ_1)

also

$$(\lambda_{max})_B \leq \max(1.6, 50) \leq 50$$

hence $K_B \leq 50 \times 2.5 \leq 125$

$$\therefore \frac{\|e_n\|_B}{\|e_0\|_B} \leq 2\left(\frac{\sqrt{125}-1}{\sqrt{125}+1}\right)^n$$

$$\leq 2\left(\frac{10.18}{12.18}\right)^n$$

4(a) since 'A' is nonsingular symmetric matrix. The eigenvalues will be greater than zero and the highest eigenvalue of A^{-1} will be equal to smallest eigen value of A, with same eigen vectors for both.

hence in order to find eigenvector of A, corresponding to smallest eigenvalue in absolute terms

we will apply the power iteration method on A^{-1}

The algorithm starts with initial vector x_0, repeatedly applies transformation $x_{k+1} = A^{-1} x_k$ until convergence

The eigenvector corresponding to the largest eigenvalue is obtained by normalizing the final vector x_n

(b) Since A is symmetric real matrix hence it is hermitian and hence can be represented as tridiagonal.

In order to find k^{th} smallest eigen vector we will apply Lanczős iteration, that will generate tridiagonal matrix T, which is similar to A.

The eigenvectors of T are related to eigenvector of A, & can be used to compute the eigenvector of A.

So, here the Lanczős algorithm can be modified to compute the k th smallest eigenvalue and its corresponding eigenvectors by using using a variant called implicity restarted Lanczős method (IRLM)

The IRLM uses a shift & invert strategy to shift the eigenvalue of T, so that

k th smallest eigenvalue becomes the largest eigenvalue.

* IRLM then applies power iteration method to T^{-1} to obtain eigenvector corresponding to kth smallest eigenvalue.

5Q) We are Given that

$$Q^T A Q = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$ where Q is an orthogonal matrix

$$\therefore A = Q \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} Q^T$$

Consider that 'V' is an eigen vector of A_{11} with eigen value λ. Consider v' to be a R^m vector such that

$$v' = \begin{bmatrix} v \\ 0_{(m-n)} \end{bmatrix}$$

we claim the Qv' is an eigenvector of A with eigen value λ

Proof: $$A(Q.V_1) = Q \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} Q^T (QV')$$

$$Q \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} (Q^T Q) V'$$

$$= Q \begin{bmatrix} A_{11} & A_{22} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} V \\ 0 \end{bmatrix}$$

$$= Q \begin{bmatrix} A_{11}V + A_{12}0 \\ 0V + A_{22}0 \end{bmatrix} = Q \begin{bmatrix} A_{11}V \\ 0 \end{bmatrix}$$

$$= Q \begin{bmatrix} \lambda V \\ 0 \end{bmatrix} = \lambda Q \begin{bmatrix} V \\ 0 \end{bmatrix}$$

$$= \lambda Q V'$$

Since V was arbitrary hence all eigenvectors of A_{11} will generate eigenvectors of A as done above hence A will have all the eigenvalues of A_{11}.

Now consider an eigenvector w of A_{22}^T with eigenvalue N

Consider $w' = \begin{bmatrix} 0_n \\ w \end{bmatrix}$
R^m vector

we have

$$A = Q \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} Q^T$$

$$A^T = Q \begin{bmatrix} A_{11}^T & 0 \\ A_{12}^T & A_{22}^T \end{bmatrix} Q^T$$

we claim that Qw' is an eigenvector of A^T with eigenvalue μ

$$A^T(Qw') = Q \begin{bmatrix} A_{11}^T & 0 \\ A_{12}^T & A_{22} \end{bmatrix} (Q^T Q) w'$$

$$= Q \begin{bmatrix} A_{11}^T & 0 \\ A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} 0 \\ w \end{bmatrix} = Q \begin{bmatrix} A_{11}^T 0 + 0w \\ A_{12}^T 0 + A_{22}^T w \end{bmatrix}$$

$$= Q \begin{bmatrix} 0 \\ A_{22}^T w \end{bmatrix} = Q \begin{bmatrix} 0 \\ \mu w \end{bmatrix} = \mu Q w'$$

Since ω was arbitrary, every eigenvector of A_{22}^T will provide eigenvector of A^T with eigenvalue μ.

So, A^T will have all the eigenvalues of A_{22}^T. we also know that A and A^T have Some characteristic equation and same eigenvalues. So A will have all the eigenvalues of A_{22}

For eigenvectors, we have produced eigenvectors corresponding to eigenvalues of A_{11}. for producing eigenvectors ~~corr~~ corres- ponding to A_{22}, we have to solve equation

$$AX = \mu X$$

6@ Table attached screenshot

6(b) table " "

(6) c) The set $\{y : \tilde{f}(y)=0\}$ is the tanget line to the curve $f(x)=0$ at point $(x, f(x))$.

if x lies on the curve ~~then~~ $f(x)=0$, then this set is simply the tangent line to the curve at that point

* we can verify quadratic convergence by computing the ratio of error at each iteration to the squared error at previous iteration. so, if this ratio approaches const value as iteration increases then we have quadratic convergence

$$\lim_{K\to\infty} \frac{|x^{K+1} - x^*|}{(x^K - x^*)^2} = C$$

* The relationship of x^{k+1} is given by

$$x^{K+1} = x^K - [df(x^k)]^{-1} \cdot f(x^k)$$

Where $df(x)$ is Jacobian matrix of partial derivative of f_1 and f_2.