Homework IV

Due on April 28, 2023

1. (**Trefethen Bau**) Let A be a square $m \times m$ matrix. Consider the following iterative algorithm: choose a parameter α and define the recurrence (you can initialize x_1 to b):

$$x_{n+1} = x_n + \alpha(b - Ax_n).$$

Show that x_n belongs to the Krylov subspace $\mathcal{K}_n$. We know that any vector in $\mathcal{K}_n$ can be written as $p_{n-1}(A)b$, where $p_{n-1}(x)$ is a degree $n-1$ polynomial. For the above recurrence, what would this polynomial be? Suppose the matrix A is such that all of its eigenvalues are uniformly spread in the circle $|z-2| \leq 1/2$ in the complex plane. What value of α would you recommend for fast convergence?

2. (**Heath**) For the equation

$$f(x) = x^2 - 3x + 2 = 0,$$

each of the following functions yields an equivalent fixed point problem:

- $g_1(x) = (x^2+2)/2$.
- $g_2(x) = \sqrt{3x-2}$.
- $g_3(x) = 3 - 2/x$.
- $g_4(x) = (x^2-2)/(2x-3)$.

Analyze the convergence properties of each of the corresponding fixed point iteration schemes for the root $x = 2$.

3. (**Heath**) Solve the following system of equations by using Newton's method and Broyden's method:

$$\begin{aligned} (x_1+3)(x_2^3-7)+18 &= 0 \\ \sin(x_2 e^{x_1} - 1) &= 0 \end{aligned}$$

Compare the convergence rates of the two methods by computing the error at each iteration, given that the exact solution is $(0,1)$.

4. (**Heath**) Write a program to find a minimum of Rosenbrock's function

$$f(x,y) = 100(y-x^2)^2 + (1-x)^2$$

using each of the following methods : (i) Steepest descent, (ii) Newton, (iii) Damped Newton. You should try each of the methods from each of the starting points $[-1 \;\; 1]^T$, $[0 \;\; 1]^T$, $[2 \;\; 1]^T$. For any line searches and linear system solutions required, you may use MATLAB routines. Plot the path taken in the plane by each of the methods for each of the starting points.

5. **(Boyd, Vandenberghe)** Newton's method with fixed step size ($\eta = 1$) can diverge if the initial point is not close to $x^\star$. In this problem we consider two examples (plot the function values at each of the iterates):

   (a) $f(x) = \log(e^x + e^{-x})$ has a unique minimizer $x^\star = 0$. Run Newton's method with fixed step size $\eta = 1$, starting at $x(0) = 1$ and at $x(0) = 1.1$.

   (b) (b) $f(x) = -\log x + x$ has a unique minimizer at $x^\star = 1$. Run Newton's method with fixed step size $\eta = 1$, starting at $x(0) = 3$.

6. **(Boyd, Vandenberghe)** Consider the optimization problem:

$$\min f(x) = -\sum_{i=1}^{n} x_i \log x_i, \text{ subject to } Ax = b,$$

where $x \in \Re^n$ with all coordinates being positive, and A is $p \times n$ matrix, where $p < n$. Generate a problem instance with $n = 100$ and $p = 30$ by choosing A randomly (checking that it has full rank), choosing $\hat{x}$ as a random positive vector (e.g., with entries uniformly distributed on $[0, 1]$) and then setting $b = A\hat{x}$ (Thus, $\hat{x}$ is feasible). Compute a solution using Newton's method. Plot a graph to show the progress of the method.

7. **(Boyd, Vandenberghe)** Let $\gamma > 1$ and consider the function

$$f(x_1, x_2) = \begin{cases} \sqrt{x_1^2 + \gamma x_2^2} & \text{if } |x_2| < x_1 \\ \frac{x_1 + \gamma |x_2|}{\sqrt{1+\gamma}} & \text{otherwise} \end{cases}$$

Prove that f is convex. Consider the gradient descent algorithm applied to f, with starting point $x(0) = (\gamma, 1)$ and exact line search. Show that the iterates are

$$x_1^k = \gamma \left(\frac{\gamma - 1}{\gamma + 1} \right)^k, x_2^k = \left(-\frac{\gamma - 1}{\gamma + 1} \right)^k.$$

Therefore x^k converges to $(0, 0)$. However, this is not the optimum, since f is unbounded below.