**COL 726**

# Homework I

*Due on Jan 23, 2023*

Use single precision (24 bits) unless specified otherwise. Whenever you are asked to explain some results, you should explain the observation in a quantitative manner. For example, if a curve has a bend at say $n = 1000$, you need to explain why you would expect the bend to happen roughly around this value of $n$.

1. Consider the function $f : \Re^2 \to \Re$ defined by $f(x, y) = x - y$. Measure the size of the input $(x, y)$ by $|x| + |y|$. What is the condition number of this function? When is the condition number very high ? Justify your answer.

2. Consider the function
$$f(x) = \frac{1}{1-x} - \frac{1}{1+x},$$
assuming $x \neq \pm 1$.
   (a) Suppose we compute this function using the expression above. For what range of values of $x$ is the computation unstable?
   (b) How will you compute accurately the value of this expression in the range of $x$ lying in part (a) ?

3. The *fibonacci* numbers $f_k$ are defined by $f_0 = 1, f_1 = 1$, and
$$f_{k+1} = f_k + f_{k-1} \tag{1}$$
for any integer $k > 1$. A small perturbation of them, the *pib* numbers, $p_k$, are defined by $p_0 = 1, p_1 = 1$ and
$$p_{k+1} = c \cdot p_k + p_{k-1} \tag{2}$$
for any integer $k > 1$ where $c = 1 + \frac{\sqrt{3}}{100}$.
   (a) Plot the numbers $f_n$ and $p_n$ together in one log scale plot. On the plot, mark $1/\epsilon_{mach}$ for single and double precision arithmetic.
   (b) Rewrite (1) to express $f_{k-1}$ in terms of $f_k$ and $f_{k+1}$. Use the computed $f_n$ and $f_{n-1}$ to recompute $f_k$ for $k = n-2, n-3, \ldots, 0$. Make a plot of the difference between the original $f_0 = 1$ and the recomputed $f_0$ as a function of $n$. What $n$ values result in low accuracy for the recomputed $f_0$ ? How do the results in single and double precision differ ?
   (c) Repeat part (b) for the pib numbers. Comment on the striking difference in the way precision is lost in the two cases. Explain the results.

4. Consider the function $f(x) = 1 - \cos x$.

   (a) What is the relative condition number at $x = 0$ ?

   (b) Suppose that we compute the function $f(x)$ as in the expression above (assume that we have a stable way of computing trigonometric functions). Is this algorithm stable ? Why ?

   (c) Suggest a stable algorithm for computing $f(x)$. Plot the results of both the algorithms for small values of $x$ in a log-log plot.

5. The polynomial $(x-1)^6$ has the value zero at $x = 1$ and is positive elsewhere. The expanded form of the polynomial, $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$, is mathematically equivalent but may not give the same results numerically. Compute and plot the values of this polynomial, using each of the two forms, for 101 equally spaced points in the interval $[0.995, 1.005]$, i.e., with a spacing of 0.0001. Can you explain the difference ?

6. Write a program to generate the first 80 terms in the sequence given by the difference equation $x_{k+1} = 2.25x_k - 0.5x_{k-1}$, with starting values $x_1 = \frac{1}{3}$ and $x_2 = \frac{1}{12}$. Make a semilog plot of the values you obtain as a function of $k$. The exact solution to the equation above is $x_k = \frac{4^{1-k}}{3}$, which decrease monotonically as $k$ increases. Does your graph confirm this expected behaviour ? Can you explain your results ?

7. Consider the problem of determining the value of the integral

$$\int_0^1 x^{20} e^{x-1} dx.$$

   If we let

$$I_k = \int_0^1 x^k e^{x-1} dx,$$

   then integration by part gives

$$\begin{aligned} I_k &= 1 - kI_{k-1} \\ I_0 &= \int_0^1 e^{x-1} dx = 1 - 1/e \end{aligned}$$

   Thus, we can compute $I_{20}$ by successively computing $I_1, I_2, \ldots$. Write a program to compute this, and plot $(k, I_k)$ values. What are the errors due to ?