ISyE/CS 719: Stochastic Programming
Fall 2016
Assignment #2

Example Solutions

Reminder – Midterm exam is Wed., October 26, 7:15-9:15 PM

1. In this problem, we will let $f(x) : \mathbb{R} \to \mathbb{R}$ be

$$f(x) \stackrel{\text{def}}{=} \mathbb{E}_\xi Q(x, \xi),$$

with

$$Q(x, \xi) = \begin{cases} 2(\xi - x) & \text{if } x \leq \xi \\ 3(x - \xi) & \text{if } x \geq \xi \end{cases}$$

Further, let ξ be a random variable taking value 1 with probability 1/4, 4 with probability 3/8 and 8 with probability 3/8. Consider the 1-D optimization problem

$$\min_{0 \leq x \leq 10} f(x). \tag{1}$$

In the problems below, you may stop after 3 iterations (in case it is not solved by then).

(a) Solve (1) with the single-cut L-shaped method. Show all your iterations (by hand), and start at the initial guess $x_0 = 0$.

Answer:

With $x_0 = 0$ we first evaluate $Q(x_0, \xi)$ for each of the three ξ values, and collect the corresponding subgradient lower bound.

	$\xi = 1$	$\xi = 4$	$\xi = 8$
Subgrad lb	$2(1 - x)$	$2(4 - x)$	$2(8 - x)$
$Q(x_0, \xi)$	2	8	16
Probability	1/4	3/8	3/8

We thus obtain that $f(x_0) = 2(1/4) + 8(3/8) + 16(3/8) = 9.5$ and the initial master problem Benders cut is:

$$\Theta \geq [2(1/4) + 8(3/8) + 16(3/8)] - [2(1/4) + 2(3/8) + 2(3/8)]x = 9.5 - 2x.$$

Thus, the master problem to be solved at iteration 1 is:

$$\begin{aligned} \min\ & \Theta \\ \text{s.t. } & \Theta \geq 9.5 - 2x \\ & \Theta \geq 0 \\ & 0 \leq x \leq 10 \end{aligned}$$

[Note: I added the constraint $\Theta \geq 0$ because this can be seen to be valid by analyzing the value function. It is also fine if this constraint is not included in the master problem.]

The optimal solution to the master problem at iteration 1 is $x_1 = 10$ and $\Theta_1 = 0$. Thus, the lower bound at this iteration is 0.

We next evaluate $Q(x_1, \xi)$ for each of the three ξ values and collect the corresponding subgradient lower bound.

	$\xi = 1$	$\xi = 4$	$\xi = 8$
Subgrad lb	$3(x-1)$	$3(x-4)$	$3(x-8)$
$Q(x_1, \xi)$	27	18	6
Probability	1/4	3/8	3/8

We thus obtain $f(x_1) = 27(1/4) + 18(3/8) + 6(3/8) = 15.75$, and the master problem Benders cut is:

$$\Theta \geq [-3(1/4) + (-12)(3/8) + (-24)(3/8)] + 3x$$

The master problem to be solved at iteration 2 is:

$$\begin{aligned} \min\ & \Theta \\ \text{s.t. } & \Theta \geq 9.5 - 2x \\ & \Theta \geq 3x - 14.25 \\ & \Theta \geq 0 \\ & 0 \leq x \leq 10 \end{aligned}$$

The optimal solution to this master problem is $x_2 = 4.75$ and $\Theta_2 = 0$ (Θ_2 is also the current lower bound).

We next evaluate $Q(x_2, \xi)$ for each of the three ξ values and collect the corresponding subgradient lower bound.

	$\xi = 1$	$\xi = 4$	$\xi = 8$
Subgrad lb	$3(x-1)$	$3(x-4)$	$2(8-x)$
$Q(x_2, \xi)$	11.25	2.25	6.5
Probability	1/4	3/8	3/8

We thus obtain $f(x_2) = 11.25(1/4) + 2.25(3/8) + 6.5(3/8) = 6.09375$, and the master problem Benders cut is:

$$\Theta \geq [-3(1/4) + (-12)(3/8) + 16(3/8)] + [3(1/4) + 3(3/8) - 2(3/8)]x = 0.75 + 1.125x$$

The master problem at iteration 3 is:

$$\begin{aligned} \min\ & \Theta \\ \text{s.t. } & \Theta \geq 9.5 - 2x \\ & \Theta \geq 3x - 14.25 \\ & \Theta \geq 0.75 + 1.125x \\ & \Theta \geq 0 \\ & 0 \leq x \leq 10 \end{aligned}$$

The optimal solution to this master problem is $x_3 = 2.8$ and $\Theta_3 = 3.9$.

We next evaluate $Q(x_3, \xi)$ for each of the three ξ values and collect the corresponding subgradient lower bound.

	$\xi = 1$	$\xi = 4$	$\xi = 8$
Subgrad lb	$3(x-1)$	$2(4-x)$	$2(8-x)$
$Q(x_3, \xi)$	5.4	2.4	10.5
Probability	1/4	3/8	3/8

We thus obtain $f(x_3) = 5.4(1/4) + 2.4(3/8) + 10.4(3/8) = 6.15$, and the master problem Benders cut is:

$$\Theta \geq [-3(1/4) + 8(3/8) + 16(3/8)] + [3(1/4) - 2(3/8) - 2(3/8)]x = 8.25 - 0.75x$$

The master problem at iteration 4 is:

$$\begin{aligned} \min\ & \Theta \\ \text{s.t. } & \Theta \geq 9.5 - 2x \\ & \Theta \geq 3x - 14.25 \\ & \Theta \geq 0.75 + 1.125x \\ & \Theta \geq 8.25 - 0.75x \\ & \Theta \geq 0 \\ & 0 \leq x \leq 10 \end{aligned}$$

The optimal solution to this master problem is $x_4 = 4$ and $\Theta_4 = 5.25$. We next evaluate $Q(x_4, \xi)$ for each of the three ξ values and collect the corresponding subgradient lower bound.

	$\xi = 1$	$\xi = 4$	$\xi = 8$
Subgrad lb	$3(x-1)$	$2(4-x)$	$2(8-x)$
$Q(x_4, \xi)$	9	0	8
Probability	1/4	3/8	3/8

We thus obtain $f(x_4) = 9(1/4) + 8(3/8) = 5.25$. As this upper bound matches the lower bound, we concluce that $x_4 = 4$ is optimal, and with optimal objective value 5.24.

$\Diamond$

(b) Solve (1) with the level method. Show all your iterations (by hand), and start at the initial guess $x_0 = 0$. Use the "optimal" λ of 0.2929.

Answer:

[For consistency with the statement of Level method in the lectures, we start with $x_1 = 0$, rather than $x_0 = 0$.] As in the initialization step for the Benders algorithm, we first obtain for $x_1 = 0$ $f(x_1) = 9.5$ and the initial master problem Benders cut is:

$$\Theta \geq 9.5 - 2x.$$

Also the initial master problem is identical to the initial master problem in Benders, and hence yields lower bound value $\underline{z}_1 = \Theta_1 = 0$. Since $\bar{z}_1 = 9.5$, the level at iteration 1 is $\ell_t = 0 + (0.2929)9.5 = 2.78255$. We then solve the following probelm to find the next point x_2:

$$\begin{aligned} \min\ & (x-0)^2 \\ \text{s.t. } & \Theta \geq 9.5 - 2x \\ & \Theta \geq 0 \\ & \Theta \leq 2.78255 \\ & 0 \leq x \leq 10 \end{aligned}$$

The optimal solution is $x_2 = 3.359$. We next evaluate $Q(x_2, \xi)$ for each of the three ξ values and collect the corresponding subgradient lower bound.

	$\xi = 1$	$\xi = 4$	$\xi = 8$
Subgrad lb	$3(x-1)$	$2(4-x)$	$2(8-x)$
$Q(x_2, \xi)$	7.077	1.282	9.282
Probability	1/4	3/8	3/8

We thus obtain $\bar{z}_2 = f(x_2) = 7.077(1/4) + 1.282(3/8) + 9.282(3/8) = 5.73075$, and the master problem Benders cut is:

$$\Theta \geq [-3(1/4) + 8(3/8) + 16(3/8)] + [3(1/4) - 2(3/8) - 2(3/8)]x = 8.25 - 0.75x$$

Then, the next lower bounding problem is:

$$\begin{aligned} \min\ & \Theta \\ \text{s.t. } & \Theta \geq 9.5 - 2x \\ & \Theta \geq 8.25 - 0.75x \\ & \Theta \geq 0 \\ & 0 \leq x \leq 10 \end{aligned}$$

The optimal value of this problem is $\underline{z}_2 = \Theta_2 = 0.75$. Thus, the level in this iteration is $\ell_2 = 0.75 + 0.2929 * (5.7305 - 0.75) = 2.2088$.
We therefore solve the following problem to choose x_3:

$$\begin{aligned} \min\ & (x - 3.359)^2 \\ \text{s.t. } & \Theta \geq 9.5 - 2x \\ & \Theta \geq 8.25 - 0.75x \\ & \Theta \geq 0 \\ & \Theta \leq 2.2088 \\ & 0 \leq x \leq 10 \end{aligned}$$

The optimal solution is $x_3 = 8.055$. We next evaluate $Q(x_3, \xi)$ for each of the three ξ values and collect the corresponding subgradient lower bound.

	$\xi = 1$	$\xi = 4$	$\xi = 8$
Subgrad lb	$3(x-1)$	$3(x-4)$	$3(x-8)$
$Q(x_3, \xi)$	21.165	12.165	0.165
Probability	1/4	3/8	3/8

The value is $f(x_3) = 9.915$. As this is larger than the previous upper bound, we maintain $\bar{z}_3 = 5.7305$. The Benders cut is $\Theta \geq 3x - 14.25$, and therefore the next lower bounding problem is:

$$\begin{aligned} \min\ & \Theta \\ \text{s.t. } & \Theta \geq 9.5 - 2x \\ & \Theta \geq 8.25 - 0.75x \\ & \Theta \geq 3x - 14.25 \\ & \Theta \geq 0 \\ & 0 \leq x \leq 10 \end{aligned}$$

The optimal value is $\Theta_3 = 3.75$. The level in this iteration is $\ell_3 = 3.75 + 0.2929 * (5.7305 - 3.75) = 4.33$. We therefore solve the following problem to choose x_4:

$$\begin{aligned} \min\ & (x - 8.055)^2 \\ \text{s.t. } & \Theta \geq 9.5 - 2x \\ & \Theta \geq 8.25 - 0.75x \\ & \Theta \geq 3x - 14.25 \\ & \Theta \geq 0 \\ & \Theta \leq 4.33 \\ & 0 \leq x \leq 10 \end{aligned}$$

The optimal solution is $x_4 = 6.1933$. We next evaluate $Q(x_4, \xi)$ for each of the three ξ values and collect the corresponding subgradient lower bound.

	$\xi = 1$	$\xi = 4$	$\xi = 8$
Subgrad lb	$3(x - 1)$	$3(x - 4)$	$2(8 - x)$
$Q(x_3, \xi)$	15.58	6.58	3.6134
Probability	1/4	3/8	3/8

The value is $f(x_4) = 7.7175$. As this is larger than the previous upper bound, we maintain $\bar{z}_4 = 5.7305$. The Benders cut is $\Theta \geq 1.125x + 0.75$, and therefore the next lower bounding problem is:

$$\begin{aligned} \min\ & \Theta \\ \text{s.t. } & \Theta \geq 9.5 - 2x \\ & \Theta \geq 8.25 - 0.75x \\ & \Theta \geq 3x - 14.25 \\ & \Theta \geq 0.75 + 1.125x \\ & \Theta \geq 0 \\ & 0 \leq x \leq 10 \end{aligned}$$

The optimal value is $\Theta_4 = 5.25$. The level in this iteration is $\ell_4 = 5.25 + 0.2929 * (5.7305 - 5.25) = 5.3907$. We therefore solve the following problem to choose x_5:

$$\begin{aligned} \min\ & (x - 6.1933)^2 \\ \text{s.t. } & \Theta \geq 9.5 - 2x \\ & \Theta \geq 8.25 - 0.75x \\ & \Theta \geq 3x - 14.25 \\ & \Theta \geq 0.75 + 1.125x \\ & \Theta \geq 0 \\ & \Theta \leq 5.3907 \\ & 0 \leq x \leq 10 \end{aligned}$$

The optimal solution is $x_5 = 4.214$. We next evaluate $Q(x_5, \xi)$ for each of the three ξ values and collect the corresponding subgradient lower bound.

	$\xi = 1$	$\xi = 4$	$\xi = 8$
Subgrad lb	$3(x-1)$	$3(x-4)$	$2(8-x)$
$Q(x_3, \xi)$	9.642	0.642	7.572
Probability	1/4	3/8	3/8

The value is $f(x_5) = 5.49075$, and thus we update $\bar{z}_5 = 5.49075$. The Benders cut obtained at this solution has previously been added to the lower bound problem, and so we don't add it again, and we obtain $\Theta_5 = \Theta_4 = 5.25$. The level in this iteration is $\ell_5 = 5.25 + 0.2929 * (5.49075 - 5.25) = 5.32$. We therefore solve the following problem to choose x_6.

$$\begin{aligned} \min\ & (x - 4.214)^2 \\ \text{s.t.}\ & \Theta \geq 9.5 - 2x \\ & \Theta \geq 8.25 - 0.75x \\ & \Theta \geq 3x - 14.25 \\ & \Theta \geq 0.75 + 1.125x \\ & \Theta \geq 0 \\ & \Theta \leq 5.32 \\ & 0 \leq x \leq 10 \end{aligned}$$

The optimal solution is $x_6 = 4.062$. We next evaluate $f(x_6) = 5.31975$ (I'm skipping the calculations) and therefore update $\bar{z}_6 = 5.31975$. The lower bound problem again does not change, and so we obtain $\Theta_6 = 5.25$. The level in this iteration is thus $\ell_6 = 5.25 + 0.2929 * (5.31975 - 5.25) = 5.27$. We therefore solve the following problem to choose x_7.

$$\begin{aligned} \min\ & (x - 4.062)^2 \\ \text{s.t.}\ & \Theta \geq 9.5 - 2x \\ & \Theta \geq 8.25 - 0.75x \\ & \Theta \geq 3x - 14.25 \\ & \Theta \geq 0.75 + 1.125x \\ & \Theta \geq 0 \\ & \Theta \leq 5.27 \\ & 0 \leq x \leq 10 \end{aligned}$$

The optimal solution is $x_7 = 4.018$. We next evaluate $f(x_7) = 5.27$, and therefore $\bar{z}_7 = 5.27$. Again, $\Theta_7 = 5.25$. Thsu, $\ell_7 = 5.25 + 0.2929 * (5.27 - 5.25) = 5.256$. The following problem is solved to choose x_8:

$$\begin{aligned}
\min\ & (x - 4.018)^2 \\
\text{s.t. } & \Theta \geq 9.5 - 2x \\
& \Theta \geq 8.25 - 0.75x \\
& \Theta \geq 3x - 14.25 \\
& \Theta \geq 0.75 + 1.125x \\
& \Theta \geq 0 \\
& \Theta \leq 5.256 \\
& 0 \leq x \leq 10
\end{aligned}$$

The optimal solution is $x_8 = 4.005$. We next evaluate $f(x_8) = 5.256$. We terminate now as this is "close enough" to the lower bound of 5.25. ◊

2. The State of Texas needs your help positioning its wildfire response resources. There are a set of fire bases B, and each base $j \in B$ currently has r_j units of fire-fighting resources. Prior to the wildfire season, the state has the option to reposition the resources, in order to try to assure sufficient resources are "close enough" to a wildfire to perform a successful "initial response". A unit of resource can be moved from base $i \in B$ to base $j \in B$ at a cost of c_{ij}. The state can also purchase new resources at a location $i \in B$ at a cost of h_i per unit. (In reality these resources are discrete things like bulldozers, but for the purpose of this problem we'll assume they are continuous and so can be fractionally moved around and purchased.) The forests in Texas are divided into a set of fire districts F. For each district $f \in F$, the set $S_f \subseteq B$ is the set of bases that are "close enough" to the district to be able to respond quickly enough to perform the initial response. The distribution of (simultaneously igniting) wildfires is represented by a set of equally likely scenarios $(d^k, p^k), k = 1, \ldots, K$, where d^k_f represents the amount of resources that would be required to perform an initial attack on a fire in district f under scenario k. (It is rare for more than one fire to ignite nearly simultaneously, so in most scenarios only one or two of these values is positive.) Once a wildfire scenario is observed, the resources at fire bases that are "close enough" to a fire district f can be dispatched to the district, with the goal to meet the target response quantity d^k_f. If the target is not met, a cost is incurred at the rate of p^k_f per unit below the target (this cost rate is also part of the scenario, reflecting the uncertainty of the cost of inadequate initial response). The state has a budget C for performing repositioning and purchasing additional resources at the beginning of the season. Formulate a two-stage stochastic program that minimizes the expected cost incurred due to inadequate responses.

Answer:
First-stage decision variables:

- x_{ij}: Amount of resource to relocate from base $i \in B$ to position $j \in B$. For convenience, we define this variable even for $i = j$, but for that case the cost is set to $c_{ii} = 0$.
- y_i: Amount of resource to purchase at base $i \in B$.
- z_i: Amount of resource available at base $i \in B$ after relocations and purchases. (These variables aren't strictly needed, but simplify derivation of Benders cuts, as this is all that is needed in second-stage.)

Second-stage decision variables (all depend on scenario):

- w_{if}: Amount of resource at base $i \in S_f$ used to meet a fire at fire district $f \in F$.
- u_f: Shortage of resource at fire district $f \in F$.

The objective is to minimize expected second-stage costs (this model has no first-stage costs):

$$\frac{1}{K}\sum_{k=1}^{K} Q_k(z).$$

The constraints on the first-stage variables are:

$$\sum_{j \in B} x_{ij} = r_i, \quad i \in B \tag{2}$$

$$\sum_{i \in B} x_{ij} + y_j = z_j, \quad j \in B \tag{3}$$

$$\sum_{i \in B}\sum_{j \in B} c_{ij}x_{ij} + \sum_{i \in B} h_i y_i \le C \tag{4}$$

$$x_{ij} \ge 0, y_j \ge 0, z_j \ge 0 \quad i, j \in B$$

The constraints (2) ensure that only available resources are repositioned. (Recall that x_{ii} represents resources kept at base i.) Constraints (3) define z_j to be the total available resource at base j (the z variables are passed to the second-stage). Finally, constraint (4) enforces the budget.

The second-stage problem for scenario k is defined as:

$$Q_k(z) = \min \sum_{f \in F} p_f^k u_f \tag{5}$$

$$\text{s.t.} \sum_{i \in S_f} w_{if} + u_f = d_f^k, \quad f \in F \tag{6}$$

$$\sum_{f \in F_i} w_{if} \le z_i, \quad i \in B \tag{7}$$

$$u_f \ge 0, w_{if} \ge 0 \quad f \in F, i \in B$$

The objective (5) minimizes the penalty for shortage below the targets. The constraint (6) determines the shortfall based on what resources are supplied from the bases, and the constraints (7) limit what is supplied from a base to near enough fires to what is available at the base. In those constraints, the notation $F_i = \{f \in F : i \in S_f\}$ is the set of fire districts that are close enough to the base i. $\diamondsuit$

3. For the instance 'forest4-10-31.pdat' posted on the course web site, solve the problem formulated in the previous question using two methods:

   (a) The single-cut Benders decomposition algorithm.

   (b) The level method, using paramater $\lambda = 0.2929$.

For each method, have your code output the upper and lower bound obtained after each iteration. A python file 'hw2-q3.py' is provided for reading the data of the instance into python (it also has the budget C hard-coded in the file). (If you wish to use something other than python, just write out the data from the python file into a format that is convenient for you to read.)In class turn in a printout of the output, and a printout of the code. Also submit the code on the Learn@UW web site Dropbox.

Answer:

Note that data set had 30 scenarios with just one fire, and one scenario with many fires. (This was a mistake – Instead of one with many fires, I meant to have several with two fires. But, the model and algorithms should still work on this "weird" data.)

I first derive the form of the Benders cut, which is used in both algorithms. Let π^d_{kf} be the dual variables associated with (6) and π^z_{ki} be the dual variables associated with (7) in the second-stage subproblem for scenario k. Then, the Benders cut has the form:

$$\Theta \geq \frac{1}{K} \sum_{k=1}^{K} \sum_{f \in F} \pi^d_{kf} d^k_f + \sum_{i \in B} \left(\frac{1}{K} \sum_{k=1}^{K} \pi^z_{ki} \right) z_i.$$

See the implementation files on the course web site for example solution. The Benders algorithm solved in 6 iterations, while the level algorithm solved in 14. (This is a pretty easy instance for the Benders algorithm, so the level method did not have much chance to improve on it.) They converged to the same solution having objective value 1.99967. (If you investigate the scenarios individually, you would find that there is no unmet demand in any scenario except the one "weird" scenario where there are many fires simultaneously.)

One comment on the level method. My implementation followed the statement of the algorithm directly, so when it minimized the distance to the previous solution, the objective summed over all the first-stage variables. But a slightly easier to implement (and also correct) version would be only sum over the first-stage variables that appear in the second-stage problem (so, the z_i variables in my model). $\diamondsuit$

4. Consider the problem that you modeled and implemented in problems 5 and 6 of assignment 1. One change to the problem is that you should now assume that each arc can be expanded by at most 200 units (this ensures the first-stage feasible region is compact.) In this exercise you will implement the sample average approximation method to calculate a confidence interval on the optimality gap of a solution you generate. A new instance file, 'nd-8-4-8.pdat' is posted on the course web site. This data set no longer specifies a set of scenarios for customer demands. Instead, it specifies three new vectors, which characterize the distribution of customer demands. Customer demands are assumed to be independent of each other. For each customer $i \in C$, with some probability ρ_i, the customer has no demand. However, if the customer does have demand, then (conditional on that outcome) the demand is distributed as a normal random variable with known mean μ_i and standard deviaion σ_i. In the "starter file", the vectors 'probnodem', 'demmean', and 'demstdev' read from the data file represent the vectors ρ, μ, and σ, respctively. A python file 'hw2-q4.py' is provided for reading this new data format.

    (a) Use sample average approximation to independently estimate a lower bound and an upper bound on the optimal objective value. To estimate the lower bound, use $M = 15$ batches of size $n = 50$ scenarios. Among the solutions generated in these batches, choose one solution that achieves the best sample average objective value as the "candidate solution", and estimate the objective value of this candidate solution using an independent sample of size $N = 1000$. Report separate 95% confidence intervals for the lower and upper bounds.

(b) Now use sample average approximation to directly estimate the optimality gap of candidate solution. Generate the candidate solution by solving a SAA problem with $n = 50$ scenarios. Then use $M = 15$ batches of size $n = 50$ to estimate the optimality gap of the solution. Also estimate the objective value of the candidate solution using an independent sample of size $N = 1000$. Report a 95% confidence interval on the optimality gap of the solution, and separately a 95% confidence interval on the optimality gap of the solution. Compare the width of the confidence interval with what was obtained in part (a).

Along with the requested confidence intervals, also turn in a printout of your code in class, and the code electronically in the Dropbox folder.

Answer:
See the implementation file posted on the course web site. I summarize the results here.

(a) When independently estimating a lower and upper bound confidence interval, I obtained a 95% C.I. on the lower bound of [35696.0,37562.5], and a 95% C.I. on the upper bound of [364412.0,37920.1]. So, the overall width (upper limit of upper bound minus lower limit of lower bound) is 2224.1, which is about 5.99% of the sample average objective estimate of the feasible solution used to construct the upper bound C.I. In contructing these two-sided conficence intervals I used the critical value $\tau_{14,0.975} = 2.145$. A slightly smaller gap could be obtained if we only construct the one-sided confidence intervals (e.g., to derive an upper limit on the upper bound, and a lower limit on the lower bound), in which case $\tau_{14,0.95} = 1.761$ would be acceptable.

(b) When using common reandom numbers to obtain a one-sided 95% C.I. on the gap, I obtain the interval [0, 223.3]. The sample mean objective value of the feasible solution for which the optimality gap is estimated is 37295.3, so the width of the gap confidence interval is about 0.6% of the estimated objective value. Note that in constructing this C.I., I used the critical value $\tau_{14,0.95} = 1.761$, since this is a one-sided confidence interval. Regardless of the critical value, it is clear that using the common random numbers approach yielded a significantly better estimate of the optimality gap. The 95% C.I. on the objective value of the selected solution is [36556.2,38034.4]. It is interesting to observe that width of the *gap* C.I. is much narrower than the width of the objective value C.I., demonstrating that it may be easier to have high confidence that the solution is near-optimal than it is to have a precise estimate of its objective value.

$\Diamond$