Nonlinear optimization

camera calibration and triangulation

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02504 Computer vision course lectures,
DTU Compute, Kgs. Lyngby 2800, Denmark

Welcome back to myself

About me: Morten Rieger Hannemose
Assistant prof. at DTU Compute,
Section for Visual Computing
Background in computer vision and differentiable rendering

I work with various applications within computer vision
camera calibration, 3D scanning, human pose estimation, and skin cancer.

Misc info

- Weekly quizzes
  - From 2023 exam
  - Do after weekly exercise
  - You can still do previous weeks.
- Let me know if you have any questions
  - About the course
  - About what I'm saying
  - Etc.

This lecture is being livestreamed and recorded

(hopefully)

Two feedback persons

Learning objectives

After this lecture you should be able to:

- perform non-linear optimization within computer vision
- explain the principle behind Levenberg-Marquardt
- compute the Jacobian of a function
- reason about different parameterizations of rotations in optimization

Presentation topics

- Non-linear least-squares optimization
  - Levenberg–Marquardt
- Gradients
  - Analytical gradients
  - Finite differences approximation
- Rotations in optimization
- Camera calibration

Non-linear least-squares optimization

Least-squares problems

Computer vision has many problems that can be written as

$$\min_{\boldsymbol{x}} \|g(\boldsymbol{x}) - \boldsymbol{y}\|_2^2 .$$

Make all parameters into a vector $\boldsymbol{x}$ and optimize everything!

Least-squares problems

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$$\min_{\boldsymbol{x}} \|g(\boldsymbol{x}) - \boldsymbol{y}\|_2^2 .$$

Make all parameters into a vector $\boldsymbol{x}$ and optimize everything!

- Homography estimation
- Pose estimation
- Bundle adjustment
- Camera calibration with and without lens distortion
- Triangulation
- ...

Reformulate a bit

$$
\begin{aligned}
e(\boldsymbol{x}) &= || \underbrace{g(\boldsymbol{x}) - \boldsymbol{y}}_{f(\boldsymbol{x})} ||_2^2 \\
&= \| f(\boldsymbol{x}) \|_2^2 \\
&= f(\boldsymbol{x})^\mathsf{T} f(\boldsymbol{x})
\end{aligned}
$$

Levenberg-Marquardt

We solve the problem in an iterative fashion. At the k^{th} iteration:

- Our current guess of $\boldsymbol{x}$ is $\boldsymbol{x}_k$
- Replace with f with first order approximation around $\boldsymbol{x}_k$

$$f(\boldsymbol{x}_k + \boldsymbol{\delta}) \approx f(\boldsymbol{x}_k) + \boldsymbol{J}\boldsymbol{\delta}$$

- $\boldsymbol{J}$ is the Jacobian that contains all first order derivatives of f at $\boldsymbol{x}_k$.

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How are the the Jacobians of f and g related?

They are identical!

The Jacobian

f is defined as

$$f(\boldsymbol{x}) = f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_m(\boldsymbol{x}) \end{bmatrix}$$

The Jacobian of f is given by

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Levenberg-Marquardt

The sum of squared errors is then:

$$e(\boldsymbol{x}_k + \boldsymbol{\delta}) = \|f(\boldsymbol{x}_k + \boldsymbol{\delta})\|_2^2 \approx$$
$$\tilde{e}(\boldsymbol{x}_k + \boldsymbol{\delta}) = \big(f(\boldsymbol{x}_k) + \boldsymbol{J}\boldsymbol{\delta}\big)^{\mathsf{T}} \big(f(\boldsymbol{x}_k) + \boldsymbol{J}\boldsymbol{\delta}\big)$$

Levenberg-Marquardt

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$$
\begin{aligned}
e(\boldsymbol{x}_k + \boldsymbol{\delta}) &= \|f(\boldsymbol{x}_k + \boldsymbol{\delta})\|_2^2 \approx \\
\tilde{e}(\boldsymbol{x}_k + \boldsymbol{\delta}) &= \big(f(\boldsymbol{x}_k) + \boldsymbol{J}\boldsymbol{\delta}\big)^\mathsf{T}\big(f(\boldsymbol{x}_k) + \boldsymbol{J}\boldsymbol{\delta}\big) \\
&= f(\boldsymbol{x}_k)^\mathsf{T} f(\boldsymbol{x}_k) + 2(\boldsymbol{J}\boldsymbol{\delta})^\mathsf{T} f(\boldsymbol{x}_k) + (\boldsymbol{J}\boldsymbol{\delta})^\mathsf{T}\boldsymbol{J}\boldsymbol{\delta}^\mathsf{T} \\
&= f(\boldsymbol{x}_k)^\mathsf{T} f(\boldsymbol{x}_k) + 2\boldsymbol{\delta}^\mathsf{T}\boldsymbol{J}^\mathsf{T} f(\boldsymbol{x}_k) + \boldsymbol{\delta}^\mathsf{T}\boldsymbol{J}^\mathsf{T}\boldsymbol{J}\boldsymbol{\delta}^\mathsf{T}
\end{aligned}
$$

which is a second order approximation of e using only first order derivatives of f 🙂

Levenberg-Marquardt

We can minimize our approximation $\tilde{e}$ instead of e.

Take the derivative of $\tilde{e}(\boldsymbol{x}_k + \boldsymbol{\delta})$

$$\tilde{e}(\boldsymbol{x}_k + \boldsymbol{\delta}) = f(\boldsymbol{x}_k)^\mathsf{T} f(\boldsymbol{x}_k) + 2\boldsymbol{\delta}^\mathsf{T} \boldsymbol{J}^\mathsf{T} f(\boldsymbol{x}_k) + \boldsymbol{\delta}^\mathsf{T} \boldsymbol{J}^\mathsf{T} \boldsymbol{J} \boldsymbol{\delta}^\mathsf{T}$$

$$\frac{\partial \tilde{e}(\boldsymbol{x}_k + \boldsymbol{\delta})}{\partial \boldsymbol{\delta}} = 2\boldsymbol{J}^\mathsf{T} f(\boldsymbol{x}_k) + 2\boldsymbol{J}^\mathsf{T} \boldsymbol{J} \boldsymbol{\delta}.$$

Levenberg-Marquardt

Find the optimum by setting the derivative equal to zero

$$2\boldsymbol{J}^{\mathsf{T}} f(\boldsymbol{x}_k) + 2\boldsymbol{J}^{\mathsf{T}}\boldsymbol{J}\boldsymbol{\delta} = \mathbf{0} \Leftrightarrow$$
$$\boldsymbol{J}^{\mathsf{T}}\boldsymbol{J}\boldsymbol{\delta} = -\boldsymbol{J}^{\mathsf{T}} f(\boldsymbol{x}_k).$$

Solve for $\boldsymbol{\delta}$, and then set $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{\delta}$.

Rinse and repeat!

The λ parameter

The approximation $\tilde{e}$ is better the closer we are to the minimum.

When far away, it can be less good. What can we do in this case?

The λ parameter

The approximation $\tilde{e}$ is better the closer we are to the minimum.

When far away, it can be less good. What can we do in this case? Fall back to gradient descent!

$$(\boldsymbol{J}^\mathsf{T}\boldsymbol{J} + \lambda \boldsymbol{I})\boldsymbol{\delta} = -\boldsymbol{J}^\mathsf{T} f(\boldsymbol{x}_k).$$

Decrease λ when $e(\boldsymbol{x}_{k+1}) < e(\boldsymbol{x}_k)$, otherwise increase it.

Levenberg-Marquardt summary

Levenberg-Marquardt is often well suited to the types of problems we encounter in computer vision.

It exploits the nature of the least-squares problem to get a second order method, using only first order derivatives.

Gradients

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Recall that $\boldsymbol{J}$ is just a lot of derivatives in a matrix.

- Analytical gradients
- Automatic differentiation
  - Reverse mode automatic differentiation (backpropagation)
  - Forward mode automatic differentiation (dual numbers)
- Finite differences approximation

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Analytical gradients and where to find them?

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Get out pen and paper (or Maple or Sympy) and find the exact derivative.

Analytical gradients - triangulation example

The reprojection error of a point in 3D in n cameras

$$f(\boldsymbol{Q}) = \begin{bmatrix} \Pi\big(\boldsymbol{\mathcal{P}}_1 \Pi^{-1}(\boldsymbol{Q})\big) - \tilde{\boldsymbol{q}}_1 \\ \vdots \\ \Pi\big(\boldsymbol{\mathcal{P}}_n \Pi^{-1}(\boldsymbol{Q})\big) - \tilde{\boldsymbol{q}}_n \end{bmatrix}$$

- f returns a vector of length $2n$
- $\boldsymbol{Q}$ has three parameters
- thus $\boldsymbol{J}$ is $2n \times 3$.

Analytical gradients - triangulation example

The projected point in homogeneous coordinates:

$$\boldsymbol{q} = \boldsymbol{\mathcal{P}} \underbrace{\Pi^{-1}(\boldsymbol{Q})}_{\boldsymbol{Q}_h}$$

$$= \begin{bmatrix} sx \\ sy \\ s \end{bmatrix} = \begin{bmatrix} \boldsymbol{p}^{(1)} \\ \boldsymbol{p}^{(2)} \\ \boldsymbol{p}^{(3)} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Analytical gradients - triangulation example

Let's focus on x

$$x = \frac{\boldsymbol{p}^{(1)}\boldsymbol{Q}_h}{\boldsymbol{p}^{(3)}\boldsymbol{Q}_h} = \frac{\mathscr{P}_{11}X + \mathscr{P}_{12}Y + \mathscr{P}_{13}Z + \mathscr{P}_{14}}{\mathscr{P}_{31}X + \mathscr{P}_{32}Y + \mathscr{P}_{33}Z + \mathscr{P}_{34}}$$

A single element of $\boldsymbol{J}$ is then given by:

$$\frac{d}{dX}x = \frac{\mathscr{P}_{11}}{\boldsymbol{p}^{(3)}\boldsymbol{Q}_h} - \frac{\mathscr{P}_{31}(\boldsymbol{p}^{(1)}\boldsymbol{Q}_h)}{(\boldsymbol{p}^{(3)}\boldsymbol{Q}_h)^2}$$

Analytical gradients - summary

Analytical gradients are extremely useful

- + very fast to compute
- + accurate
- − complicated to derive and implement

Many functions in OpenCV return the Jacobian in addition the values themselves.

Finite differences approximation - forward differences

A first order Taylor expansion of f

$$f(x+h) = f(x) + \frac{d}{dx} f(x)h + O(h)$$

Finite differences approximation - forward differences

A first order Taylor expansion of f

$$f(x+h) = f(x) + \frac{d}{dx}f(x)h + O(h) \Leftrightarrow$$

can be rewritten to

$$\frac{d}{dx}f(x) = \frac{f(x+h) - f(x)}{h} + O(h).$$

This is called forward differences. (2-point)

Finite differences approximation - central differences

A second order Taylor expansion of f evaluated at $x - h$ and $x + h$ can be rearranged to yield

$$\frac{d}{dx} f(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2).$$

This is called central differences, which is more accurate ($O(h^2)$), but requires two new evaluations of f.

Finite differences approximation - in practice

Principle can only be applied to one parameter at a time.

To compute $\boldsymbol{J}$ you need to evaluate f once (or twice) for each element in $\boldsymbol{x}$ (which can be a lot)

Number of evaluations can be reduced if you know the sparsity structure of $\boldsymbol{J}$, i.e. which elements of $\boldsymbol{x}$ affect which elements of $f(\boldsymbol{x})$

Finite differences approximation - h

How do we choose h?

A good choice is a fixed percentage of x.

Can this have any issues?

Finite differences approximation - h

How do we choose h?

A good choice is a fixed percentage of x.

Can this have any issues?

When x is zero, a percentage is also zero, this needs to be handled.

Finite differences approximation - summary

They only give an approximation of the gradients

- $+$ convenient
- $-$ lots of (unnecessary) computation
- $-$ has parameter h
- $-$ numeric problems

Only use when speed and robustness are not your primary concerns.

Short Break

Rotations in optimization

Rotations in optimization

- Rotations are usually 3×3 matrices, but have just three degrees of freedom.
- How can we parametrize rotations?

Rotations in optimization

- Rotations are usually 3×3 matrices, but have just three degrees of freedom.
- How can we parametrize rotations?
  - optimizing all 9 numbers can yield something that is no longer a rotation matrix

Rotations in optimization - Euler angles

Euler angles $(\theta_x, \theta_y, \theta_z)$ use only three numbers

- Suffers from gimbal lock, i.e. that two parameters do the same in certain configurations
  - unsuitable for optimization

Rotations in optimization - Euler angles

Euler angles $(\theta_x, \theta_y, \theta_z)$ use only three numbers

- Suffers from gimbal lock, i.e. that two parameters do the same in certain configurations
  - unsuitable for optimization
- If the rotation is close to identity, we are far from gimbal lock
  - no problems in this case
  - optimize over a rotation relative to the initial guess
  - $\theta_x, \theta_y, \theta_z$ will usually stay close to zero
    - only if initial guess is good
    - for finite difference start from $\theta_x = \theta_y = \theta_z = 2\pi$

Rotation Explorer

Rotations in optimization - Axis-angle

Axis-angle represents a rotation as a rotation of θ around an axis $\boldsymbol{v}$, where $\|\boldsymbol{v}\|_2 = 1$.

Store as a vector $\boldsymbol{v}/\theta$ (only three elements).

How OpenCV returns rotations (`rvec`). Can be converted to a rotation matrix with `cv2.Rodrigues`.

Has singularity at $\mathbf{0}$, but works well in most cases.

Rotations in optimization - Quaternions

Quaternions are the gold standard for representing rotations.

A quaternion uses four numbers represent a rotation (q_1, q_2, q_3, q_4) subject to $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$.

Smooth and suffers from no problems, however we have to normalize the quaternion each optimization step, to ensure it is still a valid quaternion.

Software packages

There are many implementations out there!

- Ceres (C++)
  - Dual numbers (no implementing derivatives, 🤪)
  - Quaternion parameterization
- `scipy.optimize.least_squares`
  - Easy to use
  - Finite differences or analytical gradients
- Jax

Camera calibration

Outline

More tips and tricks

- Checkerboard alternatives
- Subpixel estimation
- Overfitting
- Bundle adjustment

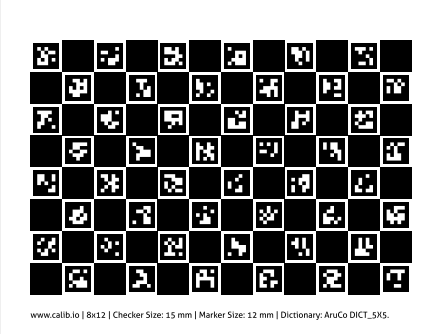
Checkerboard

OpenCV needs to see the all corners of the checkerboard in order to detect it.

Getting the entire image plane covered in detected points is hard, especially near the edges.

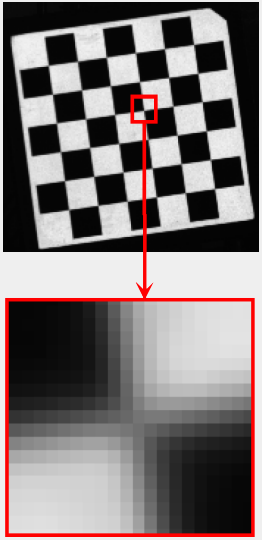
ChArUco

A ChArUco boards is a checkerboard with ArUco markers.



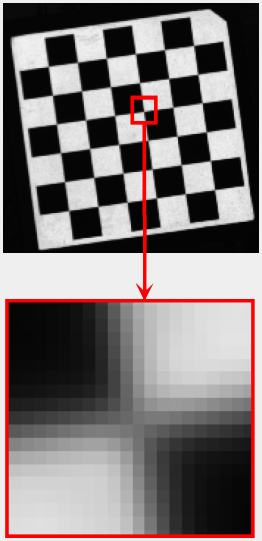
Possible to detect partial checkerboards!

Subpixel corner estimation



- Where is the corner?

Subpixel corner estimation



- Where is the corner?
- Look at a neighbourhood around a corner
- OpenCV has `cv2.cornerSubPix` but it is 💩
- What to do?

Good method for subpixel corner estimation

A good subpixel corner estimator is presented in Schops et al.[1]

Generate n random 2D vectors relative to the corner $\boldsymbol{s}_i$

$$C_{sym}(\boldsymbol{H}) = \sum_{i=1}^{n} \left(\left(I(\boldsymbol{H}(\boldsymbol{s}_i)) - I(\boldsymbol{H}(-\boldsymbol{s}_i)) \right)^2 \right)$$

$\boldsymbol{H}$ maps a point using the homography, and I interpolates the value of the image at the given point. Minimize C_{sym}.

Camera calibration - A cautionary tale

- Including more distortion parameters in your model will always give a lower re-projection error

Camera calibration - A cautionary tale

- Including more distortion parameters in your model will always give a lower re-projection error
- . . . for the images used to calibration!
  - Will not necessarily generalize.
- This can be overfitting
- Use cross-validation

Camera calibration - How to do cross validation

- For each checkerboard, split your detected corners into a validation and training set
- Do the calibration using all training corners
- Use the estimated checkerboard pose to reproject the validation corners

Bundle adjustment

- The checkerboard can have imperfections, such as not being flat
  - The checkerboards you get today are not very flat
- Optimize everything again, including the 3D positions of the corners

Learning objectives

After this lecture you should be able to:

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Info about exercise and exercise time!