Eighth International Olympiad, 1966

19661.

In a mathematical contest, three problems, A, B, C were posed. Among the participants there were 25 students who solved at least one problem each. Of all the contestants who did not solve problem A, the number who solved B was twice the number who solved C. The number of students who solved only problem A was one more than the number of students who solved A and at least one other problem. Of all students who solved just one problem, half did not solve problem A. How many students solved only problem B?

1966/2.

Let a, b, c be the lengths of the sides of a triangle, and α, β, γ, respectively, the angles opposite these sides. Prove that if

$$a + b = \tan \frac{\gamma}{2}(a \tan \alpha + b \tan \beta),$$

the triangle is isosceles.

1966/3.

Prove: The sum of the distances of the vertices of a regular tetrahedron from the center of its circumscribed sphere is less than the sum of the distances of these vertices from any other point in space.

1966/4.

Prove that for every natural number n, and for every real number $x \neq k\pi/2^t$ $(t = 0, 1, \ldots, n; k$ any integer)

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \cdots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x.$$

1966/5.

Solve the system of equations

$$\begin{aligned}
|a_1 - a_2|x_2 + |a_1 - a_3|x_3 + |a_1 - a_4|x_4 &= 1\\
|a_2 - a_1|x_1 + |a_2 - a_3|x_3 + |a_2 - a_4|x_4 &= 1\\
|a_3 - a_1|x_1 + |a_3 - a_2|x_2 + |a_3 - a_4|x_4 &= 1\\
|a_4 - a_1|x_1 + |a_4 - a_2|x_2 + |a_4 - a_3|x_3 &= 1
\end{aligned}$$

where a_1, a_2, a_3, a_4 are four different real numbers.

1966/6.

In the interior of sides BC, CA, AB of triangle ABC, any points K, L, M, respectively, are selected. Prove that the area of at least one of the triangles AML, BKM, CLK is less than or equal to one quarter of the area of triangle ABC.