Calculus II Notes-Techniques of Integration

Direct Integration:

$\int ax^n dx = \frac{a}{n+1}x^{n+1} + c$ for all $n \neq -1$ and $\int ax^{-1}dx = a\ln(|x|) + c$

U-substitution:

$\int f(g(x))g'(x)dx = \int f(u)du$ where $u = g(x)$ and $du = g'(x)dx$

- Undoes the chain rule
- *du* may differ by a constant factor
- Change limits of integration

Integration by Parts:

$\int udv = uv - \int vdu$ or $\int_a^b udv = uv_a^b| - \int_a^b vdu$

- Undoes the product rule
- Choose *u* following LIATE (Logs, Inverse trig, Algebra, Trig, Exponential)
- Choose *dv* so that you can integrate it to find *v* (may need to use another technique here)
- Remember $\int f(x)g(x)dx \neq \int f(x)dx \int g(x)dx$ - a product CANNOT be broken up over two integrals!

Trig Identities:

- Learn trig identities including Pythagorean and Half-angle identities
- We can use Pythag when we have even powers (odd minus one is even)
- We can use Half-angle when we only have even powers
- When either sine or cosine is odd-
    - If sine is odd, choose cosine to be *u*, use Pythag identity to turn everything into cosine except one sine term that is necessary for *du*
    - If cosine is odd, choose sine to be *u*, use Pythag identity to turn everything into sine except one cosine term that is necessary for *du*
- If both are even, use the half angle identity
- Be careful using the half angle identity to double the angle (this may happen more than once)
- Strategy for tangent and secant
    - If tangent is odd, choose *u* to be secant, save one secant and one tangent, use Pythag to change everything else into secants.
    - If secant is even, choose *u* to be tangent, save one even pair of secant and use Pythag to change everything else into tangents.
- When tangent is even and secant is odd or if we only have tangents or just secants-we have to try other ideas including parts, identities, or ingenuity. (Try separating one term, using Pythag identities, then distribute, break up integral
- Sometimes it helps to use equivalent expressions in terms of sines and cosines
- Strategy for cotangent and cosecant is similar to tangent/secant

Trig Substitution:

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, 0 \le \theta < \frac{\pi}{2} or \pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

- Choose appropriate substitution based on the table
- Don't forget $d\theta$!
- Draw the triangle before proceeding with an indefinite integral
- You may need other techniques of integration within the problem
- You may need to complete the square and make a *u* substitution before making the trig substitution
- Change limits of integration

Partial Fraction Decomposition:

- $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$
- If not proper, use long division first (note proper means smaller power on top)
- Factor the bottom into linear factors (of the form *ax*+*b*) or irreducible quadratics (discriminate ($b^2 - 4ac$) is negative) [Note: FTA says every polynomial reduces to exactly linear factors and irreducible quadratics]
- Given a rational expression (polynomial divided by a polynomial)
    - Remember that we are creating proper fractions, so on top of linear factors we use constants, on top of quadratics we use linear terms, no top will ever be greater than a linear term
    - We must account for any possible combination of bottom terms, so include a fraction for each power (when a factor has a multiplicity greater than one.)
- Example:

$$\frac{1}{x(x-1)(x+2)^3(x^2+1)(x^2+3x+4)^2} =$$

$$\frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x+2)} + \frac{D}{(x+2)^2} + \frac{E}{(x+2)^3} + \frac{Fx+G}{(x^2+1)} + \frac{Hx+I}{(x^2+3x+4)} + \frac{Jx+K}{(x^2+3x+4)^2}$$

- Solve $9x^2 - 3x + 5 = (A+B+C)x^2 + (A-2B)x + (B-C)$ by solving $9 = A+B+C$ and $-3 = A-2B$ and $5 = B-C$ simultaneously
- When we have a linear over a polynomial, we can separate it into two fractions $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$
- Mostly we integrate these to $\ln|\;|$, use u-substitution, or use the arctan identity.