Chapter 1
Fundamental Principles of Mechanics

1.1 Introduction

Mechanics is a branch of physics. In general, mechanics allows one to describe and predict the conditions of rest or movement of particles and bodies subjected to the action of forces. Aristotle[1] was among the first scholars to introduce the term mechanics. At first, the development of mechanics was related to that of knowledge about the modeling of the Universe. Plato,[2] Eudoxus,[3] and Aristotle are among the creators of the homocentric system, whereas Apollonius,[4] Hipparchus,[5] and Ptolemy[6] created the epicyclic system. The theory they developed, according to which the motionless Earth is the center of the Universe, is called the geocentric theory. As was mentioned previously, Ptolemy, an Alexandrian scholar, was the originator of this theory. He based his ideas on the works of Hipparchus, one of the greatest astronomers of antiquity who explained the complexity of the motion of planets while retaining the central location of the Earth and introducing the combination of circular motions. The geocentric theory is based on the assumptions that the immovable Earth is located at the center of the Universe and that other celestial bodies shaped like spheres revolve around Earth, moving uniformly in circular orbits.

---

[1] Aristotle (384–322 BC), Greek philosopher (Plato's student).

[2] Plato (428–348 BC), Greek philosopher and mathematician.

[3] Eudoxus of Cnidus (408–355 BC), Greek philosopher, astronomer and mathematician.

[4] Apollonius of Perga (260–190 BC), Greek mathematician and astronomer (focused on conics and the movement of the Moon).

[5] Hipparchus of Nicaea (190–120 BC), Greek astronomer, mathematician, and geographer; considered a precursor of astronomy.

[6] Claudius Ptolemaeus (100–168), Greek mathematician, astronomer, and geographer; one of the creators of the geocentric theory.

J. Awrejcewicz, *Classical Mechanics: Kinematics and Statics*, Advances in Mechanics and Mathematics 28, DOI 10.1007/978-1-4614-3791-8_1,

Aristotle was the unquestioned authority in the domain of philosophy and mechanics; nonetheless, he made a fundamental error that adversely affected the development of mechanics. First of all, he assumed that the laws governing the motion of bodies are different for the Earth than for other celestial bodies. It was only Galileo Galilei[7] who, over twenty centuries later, pointed out the incorrectness of the Aristotle's viewpoint.

The heliocentric model, in which the Sun is the center of the world, was introduced by Nicolaus Copernicus[8] in his fundamental work *De revolutionibus orbium coelestium* (*On the Revolutions of Heavenly Spheres*). This view was subsequently modified by Giordano Bruno,[9] who maintained that the solar system was but one of an infinite number of such systems in the Universe.

Problems connected with the motion of bodies were raised for the first time by Galileo Galilei, a dedicated proponent of Copernicus's theory. To Galileo is also attributed the discovery of the law of the pendulum (1583) and the law of freely falling objects (1602).

A great contribution to the development of mechanics was made by Johannes Kepler,[10] who formulated the following three laws of planetary motion on the basis of empirical observations previously made by Tycho Brahe.[11]

1. All the planets move in elliptical orbits with the Sun at one focus.
2. The position vector of any planet attached at this focus of an orbit where the Sun is located sweeps equal areas in equal times.
3. The squares of the orbital periods of the planets are proportional to the cubes of the semimajor axes of their orbits.

Kepler's three laws served as the foundation of the mechanics of Isaac Newton,[12] who assumed that space was homogeneous and isotropic and that phenomena are uniform with respect to the choice of the time instant. The equations derived by Newton are invariant with respect to Galilean transformation. Classical mechanics is also called Newtonian mechanics.

From Newton's point of view (Newtonian mechanics) *time*, *space*, and *mass* are absolute attributes that are independent of each other. These concepts cannot be a priori defined and are rather motivated by our intention and experience. The concept of mass allows us to compare the behavior of bodies. For instance, we say that two bodies have the same mass if they are attracted by the Earth in the same manner and they exhibit the same resistance to changes in translational motion.

[7]Galileo Galilei (1564–1642), Italian philosopher, astronomer, astrologer, and physicist who acknowledged the supremacy of the heliocentric theory of Copernicus.

[8]Nicolaus Copernicus (1473–1543), Polish astronomer and mathematician, creator of the heliocentric theory.

[9]Giordano Bruno (1548–1600), Italian Catholic cleric, philosopher.

[10]Johannes Kepler (1571–1630), German mathematician, astronomer, and physicist.

[11]Tycho Brahe (1546–1601), Danish astronomer.

[12]Isaac Newton (1642–1727) English physicist, mathematician, philosopher, and astronomer.

A point mass (particle) position and a body position require an introduction of the concept of space. It is necessary first to define an event. Newton also introduced the concept of force. It may depend on the mass of the body on which it acts and on changes in the velocity of the body over time. Therefore, force cannot be treated as an absolute, independent attribute of mechanics.

Mechanics can also be defined as the *science of the motion of bodies*. Instead of using real objects, mechanics makes use of their *models*. In general, the model of a given object (body) is an image reflecting only those attributes of the object that are essential to investigate the phenomena of interest for a particular branch of science. To the basic models applied in mechanics belong the following ones:

A particle (material point): A body possessing mass but having such small dimensions that it can be treated as a point in a geometric sense. However, in practice, bodies whose angular velocities are zero by assumption or whose rotational motion can be neglected are treated as particles regardless of their dimensions;

A system of particles: A collection of particles;

A rigid body: The distances between elements of such a body remain constant for arbitrarily large magnitudes of forces acting on the body.

In reality, structures, machines, and mechanisms are deformable bodies. However, usually their deformations are small, and hence in many cases their effect on the statics/dynamics of the studied bodies can be neglected.

A system of rigid bodies: A collection of rigid bodies.

The laws of mechanics introduced by Newton serve to illuminate the motions of material systems. They enable us to create a *mathematical model*, that is, to formulate *equations of motion* of particles and bodies.

The main goal of mechanics is to formulate the laws of motion suitable for the investigation of a variety of real bodies. It turns out that any real body, solid, liquid, or gaseous, can be modeled as a collection of particles. The following branches of mechanics deal with problems in the previously mentioned fields:

1. *Mechanics of rigid bodies* (statics and dynamics).
2. *Mechanics of deformable bodies* (strength of materials, elasticity theory, plastic theory, or rheology).
3. *Mechanics of fluids*: Incompressible (mechanics of liquids) and compressible (mechanics of gases, aeromechanics); the mechanics of incompressible fluids such as water is known as *hydraulics*.

In technical mechanics, during the modeling process we deal with the geometry (decomposition) of mass and the description of materials from which bodies are formed. In rigid-body mechanics, we assume that the distance between any two points of a body does not change. We can talk about a completely different problem when there is a possibility of changing the distance between the points of a body. The load of bodies in this last case leads to the change in the distance between body atoms, and interatomic forces (internal) will balance the external load. Bodies and material systems made of metal as encountered in technology have regular structures of arranged atomic networks on the order of 10^{30}. With regard to the large amount of atoms, analysis is performed on the micro scale, which leads to the averaging of anisotropy of microcrystal systems. Generally, most technical materials, after having a cubicoid cut out of them with sides of around 10^{-3} m, have the same properties irrespective of the orientation of the "cutting out," and such materials are called isotropic (of the same direction). There are also anisotropic materials (of different directions) in technology whose enduring properties depend on the orientation in which the cube of material is cut out (e.g., rolled plates, timber, fabrics, and paper).

The laws originally formulated by Newton generated a set of other fundamental laws of mechanics such as the conservation of linear momentum, the conservation of angular momentum, and the conservation of kinetic energy.

Below are the laws formulated by Newton, which are valid for particles.

First law. A body at rest not acted upon by an external force (the resultant force acting on a particle is zero) will remain at rest, and a body in motion moving at a constant speed along a straight line will remain in motion unless acted upon by an external force.

Second law. The acceleration of a particle is proportional to the net force acting on the particle; the direction and the sense of acceleration are identical to those of the force.

Third law. The mutual forces of action and reaction between two bodies are equal, opposite, and collinear.

The first two laws are true in an inertial system, whereas the third law is binding in any system. It can be shown that Newton's first law is a particular case of his second law.

It should be noted that Newton's laws are based on a concept of *force* as a vector quantity. Force appears here as a primitive notion and requires the introduction of at least two bodies. Correlation of reactions between bodies results from Newton's third law, where the action (forces) causes immediate reaction, which is graphically characterized by the description presented by Newton: "If I put pressure with a finger upon a stone with a certain force, then the stone also puts pressure upon my finger with the same force." The interaction of bodies can be implemented by the direct pressure of one body on another or by indirect reaction at a distance.

The latter case is connected with Newton's law of gravitation since, if we consider two particles of masses m_1 and m_2, the gravity force $\mathbf{F}_{12}$ with which the particle of the mass m_2 attracts the particle of the mass m_1 is given by

$$\mathbf{F}_{12} = G\frac{m_1 m_2}{r^3}\mathbf{r}_{12}, \tag{1.1}$$

where $G = 6.67 \cdot 10^{-11}\,\mathrm{Nm^2\,kg^{-2}}$, and $\mathbf{r}_{12}$ is a vector joining these two points and directed from point 1 toward point 2.

The gravitational constant G is used for describing the gravitational field and was determined for the first time by Henry Cavendish.[13] It should be noted, however, that there exists a certain arbitrariness in the definition of force. Nobel laureate Richard Feynman[14] draws attention to the fact that the definition of force in a strict sense is difficult. This is due to the approximate character of Newton's second law and generally due to the approximate character of the laws of physics.

A concept of mass can also be introduced based on Newton's second law. Let us consider an arbitrary particle and apply to it, in turn, forces of various magnitudes $\mathbf{F}_1,\ \mathbf{F}_2,\ \mathbf{F}_3, \ldots,\ \mathbf{F}_N$. Each of the forces produces motion of the particle with accelerations $\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3, \ldots,\ \mathbf{a}_N$, respectively. These accelerations, according to Newton's second law, are proportional to the magnitudes of the forces, i.e.,

$$\frac{\mathbf{F}_1}{\mathbf{a}_1} = \frac{\mathbf{F}_2}{\mathbf{a}_2} = \frac{\mathbf{F}_3}{\mathbf{a}_3} = \cdots = \frac{\mathbf{F}_N}{\mathbf{a}_N}. \tag{1.2}$$

The foregoing ratios describe the *inertia* of a body (particle) and define the *mass* of the body. Recall that the *weight* of a body is a product of body mass m and acceleration of gravity $\mathbf{g}$. The mass defined in that way is called a *gravitational mass*. Empirical research conducted by Hungarian physicist Roland Eötvös[15] proved that the *inertial* mass (defining the inertia of a particle) and the *gravitational* mass (being a measure of the gravitation) are identical. In other words, if we take a particle located on the Earth's surface, then we may use (1.1) to define the weight $\mathbf{G}$ of a particle of mass m. That is, introducing $r = R$ (R is now the Earth's radius) and introducing $g = \frac{Gm_1}{R^2}$, $m_2 = m$, the weight of a particle of mass m is $\mathbf{G} = m\mathbf{g}$. Observe that R depends on the particle elevation and on its latitude (the Earth is not perfectly spherical), and hence the value of g varies with the particle position.

Newton's second law can be formulated in the following form:

$$m\mathbf{a} = \mathbf{F}. \tag{1.3}$$

Newton's third law is also known as the *law of action and reaction*. It is valid both for bodies in contact and for bodies interacting at a distance ($\mathbf{F}_{12} = -\mathbf{F}_{21}$).

Finally, it should be noted that Newton's three laws were presented in a modified form. Newton's original text from his 1687 work *Philosophiae Naturalis Principia Mathematica* (*Mathematical Principles of Natural Philosophy*) is slightly different. For instance, Newton does not use the notion of a particle but that of a body. The concept of force was defined by him through a series of axioms and not in vector notation.

It is worth emphasizing that historically the concept of force was a very subjective notion as it was connected with the individual sensation of the exertion of muscles. Thanks to the efforts of Newton and other scholars, the concept of force

[13]Henry Cavendish (1731–1810), British physician and chemist.

[14]Richard Feynman (1918–1988), American physicist and creator of quantum electrodynamics.

[15]Roland Eötvös (1848–1919), Hungarian mathematician and physicist.

obtained its objective character. Nowadays, one can even observe certain feedback, i.e., through an objective understanding of force, scientists seek to deepen fully the notion of the so-called biological force connected with the ability of the muscular nervous system to, e.g., lift (lowering) material objects [1, 2]. In this case the force depends on the properties of fast twitch and slow twitch of muscle fibers as well as age, sex, etc.

Forces can be divided into several classes:

1. Mass (gravitational and inertial).
2. Surface and volumetric (pressure and hydrostatic pressure).
3. Electromagnetic and electrostatic.
4. Muscular (of humans or animals).
5. Contact: Compressive, acting on a surface or along a line.
6. Tensile: Such as the forces in threads, cables, strings.
7. Passive (reactive), i.e., counteracting the active forces.
8. External and internal.
9. Interaction of bodies.

Apart from the described laws, it is possible to introduce several *principles* of mechanics. While the laws describe relationships between mechanical quantities often leading to solutions (e.g., through the first integrals of *momentum, angular momentum*, or *energy*), the principles only support the formulation of equations of motion. The principles possess the value of universality since they can be applied, for example, in the theory of relativity, quantum mechanics, and some branches of physics. One can divide them into *differential principles* and *integral principles*. The principles used in classical mechanics are a part of so-called *analytical mechanics*.

The principle of *independent force of action* is a generalization of Newton's second law. If several forces act upon a particle, the acceleration of this particle is a result of a geometric sum of the accelerations produced by each of the forces acting separately (superposition principle).

Let us recall, finally, that the description of the behavior of electromagnetic fields introduced by Maxwell's[16] equations was in disagreement with Newton's idea of particle motion. It turned out that electromagnetic waves could propagate in a vacuum. This contradicts a purely mechanical approach whereby waves can propagate only in a material medium filling up space. Moreover, Maxwell's equations were invariant with respect to the Lorentz[17] transformation, whereas Newton's equations are invariant with respect to the Galilean transformation.

Albert Einstein[18] succeeded in resolving that problem thanks to the introduction of the so-called special theory of relativity in 1905. He introduced space-time as an *invariant quantity*, creating the foundations of so-called *relativistic*

[16]James Clerk Maxwell (1831–1879), Scottish mathematician and physicist.

[17]Hendrik Antoon Lorentz (1853–1928), Dutch physicist and Nobel laureate.

[18]Albert Einstein (1879–1955), distinguished German physicist, creator of the special and general theories of relativity.

mechanics. In this way, two deductive systems became unified, i.e., mechanics and electrodynamics (relativistic mechanics, like electrodynamics, is invariant with respect to the Lorentz transformation). In relativistic mechanics, space, time, and mass depend on each other and cannot be treated as *absolute independent attributes*.

Fortunately, the differences between relativistic mechanics and Newton's mechanics appear at particles speeds close to the speed of light or in the analysis of large distances. Neither of these cases will be considered in this book.

The four fundamental concepts of classical mechanics discussed so far, i.e., *space*, *time*, *mass*, and *force*, allow us to introduce the so-called *kinetic units*. However, in order to satisfy Newton's second law they cannot be taken arbitrarily, and they will be further referred to as *base units*. The remaining fourth unit will be referred to as a *derived unit*. Then the kinetic units will create the so-called *consistent system of units*. In what follows we further address only the universal system of units (SI units). In this system the base units are the units of length (meter, m), mass (kilogram, kg), and time (second, s). A meter is here defined as 1650763.73 wavelengths of orange-red light corresponding to a certain transition in an atom of krypton-86 (originally defined as one ten-millionth of the distance from the equator to either pole). A kilogram is equal to a mass of $10^{-3}\,\mathrm{m}^3$ of water, and the mass of a platinum–iridium standard kilogram is kept at the International Bureau of Weights and Measures in Sèvres in France. A second is defined as the radiation corresponding to the transition between two levels of the fundamental state of the cesium-133 atom (originally defined to represent $\frac{1}{86{,}400}$ of the mean solar day). Equation (1.3) yields the derived unit of force $1\mathrm{N} = 1\ \mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{s}^{-2}$. The *weight* of a body or the *force of gravity* is $\mathbf{G} = m\mathbf{g}$, and for a body with a mass of 1 kg, its weight is 9.81 N. There are numerous multiples and submultiples of the fundamental SI units as follows: 10^{12} (tera-T), 10^9 (giga-G), 10^6 (mega-M), 10^3 (kilo-k), 10^2 (hecto-h), 10 (deka-da), 10^{-2} (deci-d), 10^{-2} (centi-c), 10^{-3} (milli-m), 10^{-6} (micro-μ), 10^{-9} (nano-n), 10^{-12} (pico-p), 10^{-15} (femto-f), 10^{-18} (atto-a). For instance $1\,\mathrm{km} = 1{,}000\,\mathrm{m}$, $1\,\mu\mathrm{m} = 10^{-6}\,\mathrm{m}$, $1\,\mathrm{Mg} = 1{,}000\,\mathrm{kg}$, $1\,\mathrm{g} = 10^{-6}\,\mathrm{kg}$, $1\,\mathrm{MN} = 10^6\,\mathrm{N}$, etc.

In the case of time units, we have the minute (min) and the hour (h), and $1\,\mathrm{min} = 60\,\mathrm{s}$, whereas $1\,\mathrm{h} = 60\,\mathrm{min}$.

We may introduce also units of area and volume. The *square meter* (m^2) is the unit of area representing the area of a square of side 1 m. The *cubic meter* is the unit of volume equal to the volume of a cube of side 1 m.

In general, the following principal SI units are applied in mechanics:

1. Acceleration ($\frac{\mathrm{m}}{\mathrm{s}^2}$).
2. Angle [radian (rad)].
3. Angular acceleration ($\frac{\mathrm{rad}}{\mathrm{s}^2}$).
4. Angular velocity ($\frac{\mathrm{rad}}{\mathrm{s}}$).
5. Area (m^2).
6. Density ($\frac{\mathrm{kg}}{\mathrm{m}^3}$).
7. Energy and work [Joule (J)].
8. Force [Newton (N)].

9. Frequency [Hertz (Hz)].
10. Impulse [Newton · second ($\mathrm{N} \cdot \mathrm{s} = \mathrm{kg} \cdot \frac{\mathrm{m}}{\mathrm{s}}$)].
11. Length [meter (m)].
12. Mass [kilogram (kg)].
13. Moment of a force [Newton · meter (N·m)].
14. Power [Watt ($\mathrm{W} = \frac{\mathrm{J}}{\mathrm{s}}$)].
15. Pressure and stress [Pascal ($\mathrm{Pa} = \frac{\mathrm{N}}{\mathrm{m}^2}$)].
16. Time [second (s)].
17. Velocity [meter per second ($\frac{\mathrm{m}}{\mathrm{s}}$)].
18. Solid volume [cubic meter (m^3)].
19. Liquid volume [liter ($10^{-3}\,\mathrm{m}^3$)].

In general, in classical mechanics one may adhere to the following fundamental steps yielding the solution to a stated (given) problem. First, one needs to define the statement of a problem clearly and precisely. Diagrams indicating the force acting on each body considered known as *free-body diagrams* should be constructed. Then the fundamental principles and laws of mechanics should be used to derive the governing equations holding the condition of statics (rest) or dynamics (motion) of the bodies studied.

The short historical outline of the development of mechanics presented above reveals its deep roots in ancient times, and the reader will not make the mistake of thinking that only today in the field of general mechanics do many coursebooks and monographs exist. It is almost impossible to present a complete bibliography in the field of classical mechanics. Therefore, a few sources in English are given to make the book more readable, especially for students. Therefore, no attempt was made to provide an exhaustive list of references; only those works are included that were either used by the author [3–12] or are important competitors to this book [13–25].

1.2 D'Alembert's Principle

Let us consider a constrained material system (subjected to constraints) consisting of particles, described by the following equations of motion based on Newton's second law:

$$m_n \mathbf{a}_n = \mathbf{F}_n^e + \mathbf{F}_n^i + \mathbf{F}_n^R, \quad n = 1, \ldots, N, \tag{1.4}$$

where $\mathbf{F}_n^e$, $\mathbf{F}_n^i$, and $\mathbf{F}_n^R$ denote, respectively, external forces, internal forces, and reactions, which follow directly from Newton's second law. Every particle, numbered n, can be subjected to the action of forces $\mathbf{F}_n^i$ coming from other (even all) particles of the considered system of particles. The external forces $\mathbf{F}_n^e$, in turn, represent the action of the environment on our material system isolated from that environment or from other isolated system parts.

If $\mathbf{F}_n^e = \mathbf{0}$ (absence of external influence), then such a system in mechanics is known as *autonomous (isolated)*. Moreover, in a general case, a system of particles (SoP) can be *free* or *constrained*. The reaction forces $\mathbf{F}_n^R$ are reactions of the *constraints*, that is, of the restrictions imposed on the particles, i.e., on their displacements and velocities. By the free system we will understand either the SoP on which the *constraints* are not imposed or one for which the reaction of the constraints can be determined *explicitly* in the form of reaction forces, i.e., they will not require solving additional so-called *equations of constraints*, and then the forces $\mathbf{F}_n^R$ can be treated as $\mathbf{F}_n^e$. Otherwise, SoP will be called *constrained*. The forces that occur on the right-hand side of (1.4) and concerning material point n in a general case may depend on the position and velocity of other particles of the SoP as well as explicitly on time, i.e., $\mathbf{F}_n^e = \mathbf{F}_n^e(\mathbf{r}_1, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N, t)$, $\mathbf{F}_n^i = \mathbf{F}_n^i(\mathbf{r}_1, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N, t)$, $\mathbf{F}_n^R = \mathbf{F}_n^R(\mathbf{r}_1, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N, t)$.

Let every particle undergo a *virtual displacement* $\delta\mathbf{r}_n$, where $\mathbf{r}_n$ is a *radius vector* of the particle n. Multiplying (scalar product) (1.4) by $\delta\mathbf{r}_n$ and adding by sides, we obtain

$$\sum_{n=1}^{N}(\mathbf{F}_n^e + \mathbf{F}_n^i + \mathbf{F}_n^R - m_n\mathbf{a}_n) \circ \delta\mathbf{r}_n = 0. \tag{1.5}$$

Assuming that only ideal constraints are considered, which by definition satisfy the relation

$$\sum_{n=1}^{N}\mathbf{F}_n^R \circ \delta\mathbf{r}_n = 0, \tag{1.6}$$

(1.5) will take the form

$$\sum_{n=1}^{N}(\mathbf{F}_n^e + \mathbf{F}_n^i - m_n\mathbf{a}_n) \circ \delta\mathbf{r}_n = 0. \tag{1.7}$$

The equation just obtained enables us to formulate *d'Alembert's principle*, which reads:

The sum of scalar products of virtual displacements and external forces, internal forces and vectors $(-m_n\mathbf{a}_n)$ *of particles of a material system equals zero.*

One may conclude from (1.7) that d'Alembert's principle transforms a problem of dynamic equilibrium to that of a static equilibrium by adding the inertia force terms $(-m_n\mathbf{a}_n)$ and extends the *principle of virtual work* to dynamics.

Performing a projection of the vectors appearing in (1.7) on the axes of the adopted Cartesian coordinate system $(OX_1X_2X_3)$, we obtain

$$\begin{aligned}\sum_{n=1}^{N}\big[&(F_{nx_1}^e + F_{nx_1}^i - m_n\ddot{x}_{1n})\delta x_{1n} + (F_{nx_2}^e + F_{nx_2}^i - m_n\ddot{x}_{2n})\delta x_{2n}\\ &+ (F_{nx_3}^e + F_{nx_3}^i - m_n\ddot{x}_{3n})\delta x_{3n}\big] = 0,\end{aligned} \tag{1.8}$$

where $\mathbf{a}_n = \ddot{x}_{1n}\mathbf{E}_1 + \ddot{x}_{2n}\mathbf{E}_2 + \ddot{x}_{3n}\mathbf{E}_3$, and $\mathbf{E}_i$, $i = 1, 2, 3$, are *unit vectors* of the coordinate system $OX_1X_2X_3$.

The equation just obtained is often called a *general equation of mechanics*. D'Alembert's principle and the general equation of mechanics are sometimes difficult in applications because they refer to coordinates of the particles. In Hamilton's and Lagrange's mechanics the introduced scalar energy functions allow one to omit the foregoing problem. Because we are considering a free system, all virtual displacements are independent. This means that the general equation is satisfied only if the expressions in brackets equal zero.

In this way we obtain three second-order differential equations of the following form:

$$\sum_{n=1}^{N}(F^e_{nx_1} + F^i_{nx_1} - m_n\ddot{x}_{1n}) = 0,$$
$$\sum_{n=1}^{N}(F^e_{nx_2} + F^i_{nx_2} - m_n\ddot{x}_{2n}) = 0,$$
$$\sum_{n=1}^{N}(F^e_{nx_3} + F^i_{nx_3} - m_n\ddot{x}_{3n}) = 0. \tag{1.9}$$

The preceding equations are simplified even more when the sum of internal forces equals zero taking the form of

$$\sum_{n=1}^{N}(F^e_{nx_1} - m_n\ddot{x}_{1n}) = 0,$$
$$\sum_{n=1}^{N}(F^e_{nx_2} - m_n\ddot{x}_{2n}) = 0,$$
$$\sum_{n=1}^{N}(F^e_{nx_3} - m_n\ddot{x}_{3n}) = 0. \tag{1.10}$$

The three equations above can be rewritten in the vector form

$$\sum_{n=1}^{N}(\mathbf{F}_{Bn} + \mathbf{F}^e_n) = \mathbf{0}. \tag{1.11}$$

It was assumed above that the force $\mathbf{F}_{Bn} = -m_n\mathbf{a}_n$. That force is also known as inertia force or d'Alembert's force acting on a particle n. Its sense is opposite to the active force $\mathbf{F}^e_n$.

Let us recall that position vectors $\mathbf{r}_n$ determine the position of material point n measured from the origin of the coordinate system. After vector premultiplication (cross product) of (1.11) by $\mathbf{r}_n$ we obtain

$$\sum_{n=1}^{N}(\mathbf{r}_n \times \mathbf{F}_{Bn} + \mathbf{r}_n \times \mathbf{F}^e_n) = \mathbf{0}. \tag{1.12}$$

Let us note that the sums of vector products occurring above represent the main moment of force vectors of the system of external forces $\mathbf{F}_n^e$ and of the system of inertia forces $\mathbf{F}_{Bn}$, that is,

$$\mathbf{M_O} = \sum_{n=1}^{N} (\mathbf{r}_n \times \mathbf{F}_n^e), \tag{1.13}$$

$$\mathbf{M_{BO}} = \sum_{n=1}^{N} (\mathbf{r}_n \times \mathbf{F}_{Bn}). \tag{1.14}$$

Let us introduce the notions of main force vector of external forces, inertia forces and reactions, and the main moment of a force vector of reaction in the following form:

$$\mathbf{F}^e = \sum_{n=1}^{N} \mathbf{F}_n^e, \quad \mathbf{F_B} = \sum_{n=1}^{N} \mathbf{F}_{Bn},$$
$$\mathbf{F}^R = \sum_{n=1}^{N} \mathbf{F}_n^R, \quad \mathbf{M_{RO}} = \sum_{n=1}^{N} (\mathbf{R}_n \times \mathbf{F}_n^r). \tag{1.15}$$

They were introduced to the system after being released from constraints. In this way the material system remains in equilibrium under the action of inertia forces, active forces and reactions, and the torques (moments of forces) due to the aforementioned forces only if

$$\mathbf{F}^e + \mathbf{F_B} + \mathbf{F}^R = \mathbf{0}, \tag{1.16}$$

$$\mathbf{M_O} + \mathbf{M_{BO}} + \mathbf{M_{RO}} = \mathbf{0}. \tag{1.17}$$

The obtained result [(1.16) and (1.17)] is summarized in the following principle:

A system of vectors consisting of inertia forces, external forces, reactions constraining the movement of this system, and their torques is equivalent to zero.

In the case of free systems (no constraints and therefore no reactions) (1.16) and (1.17) are reduced to

$$\mathbf{F}^e + \mathbf{F_B} = \mathbf{0}, \tag{1.18}$$

$$\mathbf{M_O} + \mathbf{M_{BO}} = \mathbf{0}. \tag{1.19}$$

Thus, we obtain the following principle for free material systems: *A system of external forces and torques produced by the forces acting on particles of a free material system is in every time instant balanced by a system of inertia forces and torques produced by these inertia forces.*

1.3 Principle of Virtual Work

Let us consider a material system composed of N particles at rest. Since the system is at rest, the accelerations of all its particles equal zero. From (1.5) we obtain

$$\sum_{n=1}^{N} \left(\mathbf{F}_n^e + \mathbf{F}_n^i + \mathbf{F}_n^R\right) \circ \delta \mathbf{r}_n = 0. \tag{1.20}$$

Because the scalar product of force and virtual displacement represents a virtual work of the force, (1.20) can be interpreted in the following way:

In an equilibrium position of a material system, the sum of virtual works of all external forces, internal forces, and reactions equals zero.

The foregoing principle was formulated on the basis of equilibrium equations and is a necessary condition of equilibrium. Now, let us assume that the forces acting on the system would do the work that would result in a change of kinetic energy ΔE_{kn} of every particle of the system. The kinetic energy would be created, however, from the work of the mentioned forces, and in view of that we have

$$\sum_{n=1}^{N} \left(\mathbf{F}_n^e + \mathbf{F}_n^i + \mathbf{F}_n^R\right) \circ \delta \mathbf{r}_n = \sum_{n=1}^{N} \delta E_{kn} = 0. \tag{1.21}$$

This means that the increment of the kinetic energy of the system of particles is zero, and therefore the system is not moving. That is a sufficient condition of equilibrium. Thus, the principle of virtual work shows the necessary and sufficient condition of system equilibrium. In the case of ideal constraints (the sum of works produced by reaction forces equals zero) and rigid systems (the sum of works produced by internal forces equals zero), the stated principle is simplified and takes the form of

$$\sum_{n=1}^{N} \mathbf{F}_n^e \circ \delta \mathbf{r}_n = \sum_{n=1}^{N} \left(F_{x_{1n}}^e \delta x_{1n} + F_{x_{2n}}^e \delta x_{2n} + F_{x_{3n}}^e \delta x_{3n}\right) = 0.$$

The principle of virtual work in this case reads:

In an equilibrium position of a material system, the sum of virtual works of all external forces through the virtual displacements allowed by kinematics (compatible with the constraints) of the system equals zero.

In applications, the foregoing principle has some advantageous consequences, which are listed below:

1. Reaction forces (for smooth surfaces without friction) and internal forces can be removed from consideration (this will be shown in Example 1.1).
2. The problem of statics, after application of that principle, can be solved as a problem of kinematics.

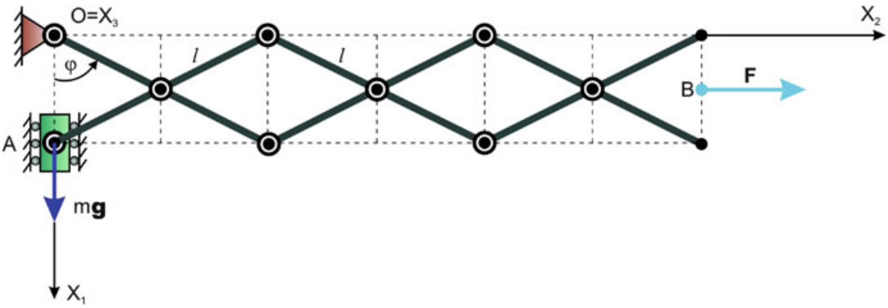

Fig. 1.1 Mechanism remaining in static equilibrium

3. The problem can be directly formulated in so-called *generalized coordinates* q_n of the form

$$\sum_{n=1}^{N} Q_n \delta q_n = 0.$$

It should be noted that in the case where the virtual work principle is related to d'Alembert's principle of the form (1.4), then item (3) is no longer valid since, in general, the latter cannot be directly formulated in terms of generalized coordinates, which makes its application much more difficult.

Example 1.1. Determine the magnitude of force $\mathbf{F}$ so that the flat mechanism depicted in Fig. 1.1 remains in static equilibrium, where the weight of slide block is denoted by $\mathbf{G} = m\mathbf{g}$.

After introducing the Cartesian coordinate system $OX_1X_2X_3$, the kinematics of points A and B is defined by the following equations:

$$x_{1A} = 2l \cos\varphi,$$

$$x_{2B} = 6l \sin\varphi.$$

According to the principle of virtual work and making use of the preceding geometric relations, we obtain

$$F\delta x_{2B} + mg\delta x_{1A} = 0.$$

Since

$$\delta x_{1A} = -2l \sin\varphi\delta\varphi,$$

$$\delta x_{2B} = 6l \cos\varphi\delta\varphi,$$

we have

$$(3F\cos\varphi - mg\sin\varphi)\delta\varphi = 0,$$

which holds true for an arbitrary $\delta\varphi$.

At the change of $\delta\varphi$ (clockwise or counterclockwise), the mechanism remains in static equilibrium when

$$F = \frac{1}{3} mg \tan\varphi. \qquad \square$$

1.4 Increment of a Function and Variation of a Function

In traditional mechanics textbooks, the presentation usually starts with a so-called *geometric* approach, based on the application of vector calculus and Newton's laws of momentum and angular momentum. Sometimes, however, it is virtually inconceivable how one should bring about the release from constraints and consider all internal and reaction forces for each single particle of a system composed of a large number of particles. Therefore, a natural question arises as to whether there exists a possibility of simplifying the problem provided that the considered system is in static or dynamic equilibrium and that internal forces in the considered system cancel each other (actions and reactions). It turns out that such a possibility exists based on the concepts of virtual work and virtual displacement, which were the subject of consideration in the previous section. In the present section, some basic information will be presented regarding a function variation in connection with the concept of virtual work, which is widely used in mechanics.

Let us first introduce the notion of virtual displacement (Fig. 1.2) after adopting the Cartesian coordinate system $OX_1X_2X_3$.

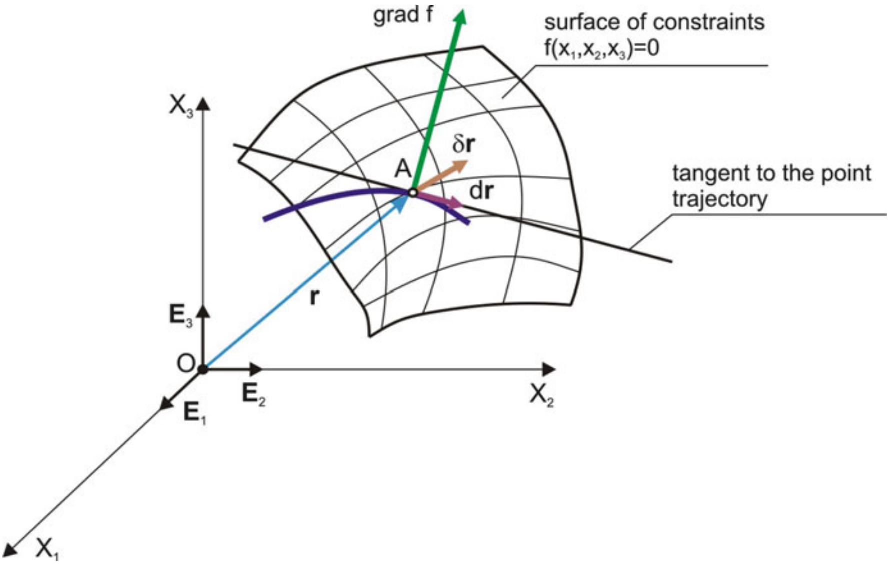

Fig. 1.2 Real ($d\mathbf{r}$) and virtual ($\delta\mathbf{r}$) displacement of a material point A

Let point A be moving in an arbitrary fashion on the surface of constraints around its position described by the radius vector $\mathbf{r}$. The arbitrariness regards both the displacement direction of point A and the length of a vector connecting the current position of point A with its position after a small displacement on the surface of constraints $f(x_1, x_2, x_3) = 0$ (a time-dependent surface can be considered as well). We shall note that the arbitrariness of the direction is associated with removal of the point motion dependency on acting forces, whereas the arbitrariness of the length $\delta\mathbf{r}$ means removing the dependency on time.

In reality, the elementary displacement of the particle takes place in the direction of the vector $d\mathbf{r}$ along the curve indicated in Fig. 1.2.

However, there exists a certain law defining the virtual displacement of point A. That point moves on the surface of constraints $f(x_1, x_2, x_3) = 0$, which means that

$$\mathrm{grad} f \circ \delta\mathbf{r} = 0, \tag{1.22}$$

where the vector

$$\mathrm{grad} f = \frac{\partial f}{\partial \mathbf{r}} = \sum_{i=1}^{3} \mathbf{E}_i \frac{\partial f}{\partial x_i} \tag{1.23}$$

is a gradient vector (normal to the surface of constraints) at the current position of the particle (for a "frozen" moment in time).

Observe that point A subjected to a virtual displacement, i.e., when its position is defined by a radius vector $\mathbf{r} + \delta\mathbf{r}$, also satisfies the equation of constraints.

Since we have

$$f(\mathbf{r} + \delta\mathbf{r}) = f(\mathbf{r}) + \frac{\partial f}{\partial \mathbf{r}} \circ \delta\mathbf{r} + O(\delta\mathbf{r})^2, \tag{1.24}$$

and because after displacement $\delta\mathbf{r}$ the point still lies on the surface of constraints, we have

$$f(\mathbf{r} + \delta\mathbf{r}) = 0. \tag{1.25}$$

Taylor's[19] expansion about point A, and on the assumption of a small variation $\delta\mathbf{r}$, showed that a point in a new position also satisfies equation of constraints (1.25). Moreover, the mentioned operation shows that $\frac{\partial f}{\partial \mathbf{r}} \equiv \mathrm{grad}\, f$ must be a vector since the result of the product $\frac{\partial f}{\partial \mathbf{r}} \circ \delta\mathbf{r}$ must be a scalar.

Let us note that a number of geometric constraints determines a number of additional conditions of the type (1.22) imposed on the considered material system.

According to the assumption that $\mathbf{E}_n$ are unit vectors of axes of the introduced coordinate system, we obtain

$$\delta\mathbf{r} = \sum_{n=1}^{3} \mathbf{E}_n \delta x_n. \tag{1.26}$$

[19]Brook Taylor (1685–1731), English mathematician.

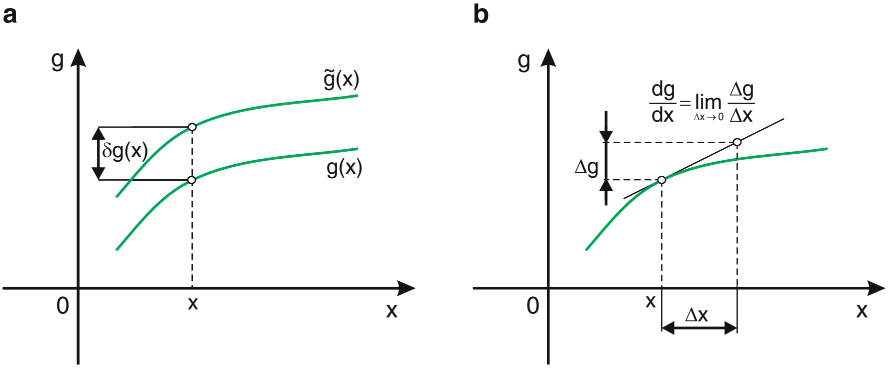

Fig. 1.3 Geometric interpretation of isochronous variation of a function (**a**) and derivative of a function (**b**) regarding point x

Let us note that $\mathbf{r} = \mathbf{r}(t)$ and the notion of virtual displacement was introduced for the "frozen" moment in time, that is, $\delta\mathbf{r}$ does not depend on time, as remarked earlier.

In a general case the notion of virtual displacement is connected with a mathematically motivated concept of function variation, which will be briefly recalled now based on the monograph [3]. At first, we will consider a so-called *isochronous* variation of a function

$$g = g(x). \tag{1.27}$$

The following function is called a variation of function $g(x)$:

$$\delta g(x) = \tilde{g}(x) - g(x) \tag{1.28}$$

where $\delta g(x) \ll 1$. It is shown that a function variation (in contrast to a function derivative) is calculated for a fixed x, whereas a function derivative about x makes use of an increment $x + \Delta x$ (Fig. 1.3).

A derivative of the variation of a function is as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[\delta g(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}[\tilde{g}(x) - g(x)] \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\tilde{g}(x) - \frac{\mathrm{d}}{\mathrm{d}x}g(x) = \tilde{g}'(x) - g'(x),\end{aligned} \tag{1.29}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}x}$.

Let us now introduce the notion of variation of derivative of $\tilde{g}'(x)$ and $g'(x)$. From the definition of isochronous variation we have

$$\delta g'(x) = \tilde{g}'(x) - g'(x) = \delta\left[\frac{\mathrm{d}\tilde{g}(x)}{\mathrm{d}x} - \frac{\mathrm{d}g(x)}{\mathrm{d}x}\right]. \tag{1.30}$$

By comparing (1.29) with (1.30) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}[\delta g(x)] = \delta\left[\frac{\mathrm{d}\tilde{g}(x)}{\mathrm{d}x} - \frac{\mathrm{d}g(x)}{\mathrm{d}x}\right]. \tag{1.31}$$

This means that the derivative of an isochronous variation of a function is equal to the isochronous variation of a derivative of a function. Let us consider now a composite function of the form

$$f \equiv f(g, g', x). \tag{1.32}$$

For a fixed x we perform variations of the functions g and g' with values δg and $\delta g'$, respectively, which means that the function f will undergo the variation δf. Owing to (1.32) we have

$$\begin{aligned} f + \delta f &= f(g + \delta g, g' + \delta g', x) \\ &= f + \frac{\partial f}{\partial g}\delta g + \frac{\partial f}{\partial g}\delta g' + O\big((\delta g)^2 + (\delta g')^2\big). \end{aligned} \tag{1.33}$$

From (1.33) we obtain

$$\delta f = \frac{\partial f}{\partial g}\delta g + \frac{\partial f}{\partial g'}\delta g'. \tag{1.34}$$

Let us go back now to our function g and assume that this time we have

$$g = g(x, t). \tag{1.35}$$

Let us introduce now the following time variation:

$$\delta t = \tilde{t} - t. \tag{1.36}$$

A total variation $\delta^* g(x,t)$ *of the function* (1.35) can be determined from the equation

$$g + \delta^* g = g + \delta g + \frac{\mathrm{d}g}{\mathrm{d}t}\delta t, \tag{1.37}$$

which means that

$$\delta^* g = \delta g + \frac{\mathrm{d}g}{\mathrm{d}t}\delta t = \delta g + \dot{g}\delta t. \tag{1.38}$$

It can be shown easily that also in the case of an independent variable such as time, we have

$$\mathrm{d}(\delta t) = \delta(\mathrm{d}t). \tag{1.39}$$

Let us calculate now the total variation of a derivative of the function g with respect to time [see (1.38), where instead of g we take $\dot{g}$]:

$$\delta^* \dot{g} = \delta \dot{g} + \frac{\mathrm{d}\dot{g}}{\mathrm{d}t}\delta t = \delta \dot{g} + \ddot{g}\delta t. \tag{1.40}$$

After differentiation of (1.38) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\delta^* g\big) = \delta \dot{g} + \ddot{g}\delta t + \dot{g}\frac{\mathrm{d}(\delta t)}{\mathrm{d}t}. \tag{1.41}$$

Equations (1.40) and (1.41) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\delta^* g\big) = \delta^* \dot{g} + \dot{g}\frac{\mathrm{d}(\delta t)}{\mathrm{d}t}, \tag{1.42}$$

which means that *total* variation is not commutative with differentiation. Although the calculations were conducted regarding a scalar function, they are also valid for a vector-valued function.

Example 1.2. Three rigid bodies of masses m_i $(i = 1, 2, 3)$ are attached to a massless inextensible cable wrapped around three pulleys of negligible masses (Fig. 1.4). The bodies are in translatory motion. Determine the accelerations of the bodies.

Let us associate the virtual displacements δx_1, δx_2, and δx_3 with the respective coordinates. According to (1.7) we have

$$\sum_{i=n}^{3}(\mathbf{F}_n^e - m_n \mathbf{a}_n) \circ \delta \mathbf{r}_n = 0.$$

In the present case the weights of the bodies play the role of external forces

$$\mathbf{F}_1 = m_1 \mathbf{g}, \quad \mathbf{F}_2 = m_2 \mathbf{g}, \quad \mathbf{F}_3 = m_3 \mathbf{g},$$

and, moreover, $\delta \mathbf{r}_n = \delta x_n \mathbf{E}_n$, $n = 1, 2, 3$.

We obtain the following *equilibrium state equation*:

$$\big(m_1 g - m_1 \ddot{x}_1\big)\delta x_1 + \big(m_2 g - m_2 \ddot{x}_2\big)\delta x_2 + +\big(m_3 g - m_3 \ddot{x}_3\big)\delta x_3 = 0.$$

Because the cable is inextensible, we have

$$x_1 + 2x_2 + x_3 = C \equiv \text{const}.$$

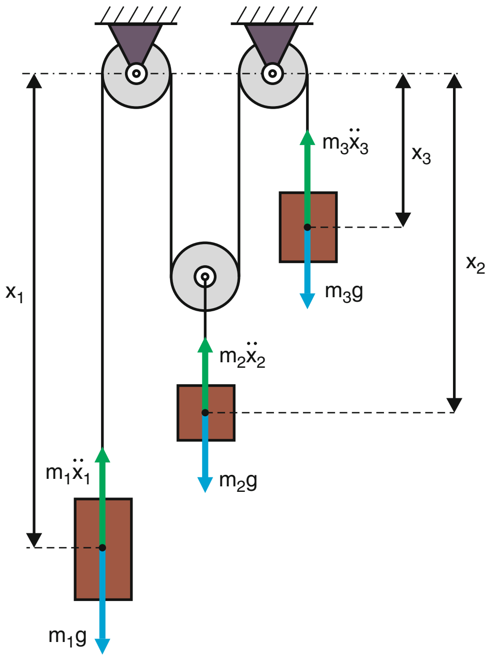

Fig. 1.4 Three bodies of masses m_i, $i = 1, 2, 3$ in translatory motion

Let us assume that the coordinates x_1 and x_3 are independent. From the last equation we obtain

$$x_2 = -\frac{1}{2}(x_1 + x_3 - C),$$

that is,

$$\ddot{x}_2 = -\frac{1}{2}(\ddot{x}_1 + \ddot{x}_3),$$

$$\delta x_2 = -\frac{1}{2}(\delta x_1 + \delta x_3).$$

Substituting the preceding equation into the equilibrium state equation we have

$$m_1(g - \ddot{x}_1)\delta x_1 - \frac{1}{2}m_2\left[g + \frac{1}{2}(\ddot{x}_1 + \ddot{x}_3)\right](\delta x_1 + \delta x_3) + m_3(g - \ddot{x}_3)\delta x_3 = 0,$$

or equivalently

$$\delta x_1\left[g\left(m_1 - \frac{1}{2}m_2\right) - \ddot{x}_1\left(m_1 + \frac{1}{4}m_2\right) - \ddot{x}_3\frac{1}{4}m_2\right]$$
$$+ \delta x_3\left[g\left(m_3 - \frac{1}{2}m_2\right) - \ddot{x}_3\left(m_3 + \frac{1}{4}m_2\right) - \ddot{x}_1\frac{1}{4}m_2\right] = 0.$$

Since the virtual displacements δx_1 and δx_3 are independent, we obtain

$$\left(m_1 + \frac{m_2}{4}\right)\ddot{x}_1 + \frac{m_2}{4}\ddot{x}_3 = g\left(-\frac{m_2}{2} + m_1\right),$$
$$\left(m_3 + \frac{m_2}{4}\right)\ddot{x}_3 + \frac{m_2}{4}\ddot{x}_1 = g\left(-\frac{m_2}{2} + m_3\right).$$

The determinant of the foregoing system of equations is equal to

$$W = \begin{vmatrix} m_1 + \frac{m_2}{4} & \frac{m_2}{4} \\ \frac{m_2}{4} & m_3 + \frac{m_2}{4} \end{vmatrix} = m_1 m_3 + \frac{m_2}{4}(m_1 + m_3),$$

and the remaining determinants have the form

$$W_{\ddot{x}_1} = \begin{vmatrix} g\left(-\frac{m_2}{2} + m_1\right) & \frac{m_2}{4} \\ g\left(-\frac{m_2}{2} + m_3\right) & m_3 + \frac{m_2}{4} \end{vmatrix}$$
$$= g\left[-\frac{3}{4}m_2 m_3 + m_1\left(m_3 + \frac{1}{4}m_2\right)\right],$$

$$W_{\ddot{x}_3} = \begin{vmatrix} m_1 + \frac{m_2}{4} & g\left(-\frac{m_2}{2} + m_1\right) \\ \frac{m_2}{4} & g\left(-\frac{m_2}{2} + m_3\right) \end{vmatrix}$$
$$= g\left[-\frac{3}{4}m_1 m_2 + m_3\left(m_1 + \frac{m_2}{4}\right)\right].$$

Eventually, the desired accelerations are as follows:

$$\ddot{x}_1 = \frac{W_{\ddot{x}_1}}{W} = \frac{g\left[-\frac{3}{4}m_2 m_3 + m_1\left(m_3 + \frac{1}{4}m_2\right)\right]}{m_1 m_3 + \frac{m_2(m_1+m_3)}{4}},$$
$$\ddot{x}_3 = \frac{W_{\ddot{x}_3}}{W} = \frac{g\left[-\frac{3}{4}m_1 m_2 + m_3\left(m_1 + \frac{1}{4}m_2\right)\right]}{m_1 m_3 + \frac{m_2(m_1+m_3)}{4}}. \qquad \square$$

Example 1.3. A rod of weight **G** and length l is hinged at point O and a pin connected to a rod has length $2l$ and weight $2\mathbf{G}$ (Fig. 1.5). Determine the configuration of rods as a result of the action of the weight forces (neglect the friction).

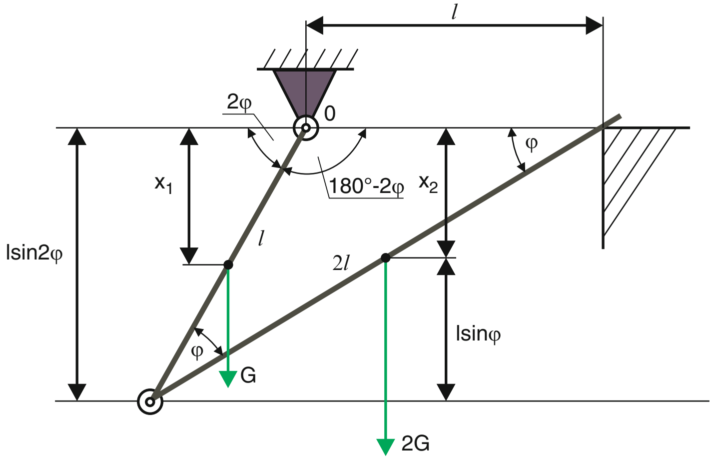

Fig. 1.5 Configuration of homogeneous rods of lengths l and $2l$ loaded with their weights

As a generalized coordinate we will take an angle φ because the analyzed system has one degree of freedom. In other words, a possible system movement can be described by only one coordinate φ. From Fig. 1.5 it follows that

$$x_1 = \frac{l}{2}\sin 2\varphi,$$

$$x_2 = l\sin 2\varphi - l\sin\varphi.$$

Because the constraints of the system are ideal, only the weight forces perform the work through the virtual displacements. The work done by the forces **G** and 2**G** through the displacements δx_n $(n = 1, 2)$ is equal to

$$G\delta x_1 + 2G\delta x_2 = 0.$$

A slight shake of the system, which remains in static equilibrium, will produce the displacements δx_1, δx_2 and change in the coordinate φ by $\delta\varphi$. We will determine the relations between δx_n and $\delta\varphi$ by applying the first two equations:

$$\delta x_1 = l\cos 2\varphi\delta\varphi, \quad \delta x_2 = (2\cos 2\varphi - \cos\varphi)l\delta\varphi,$$

and the obtained variations are substituted into the second equation, yielding

$$Gl\cos 2\varphi + 2Gl(2\cos 2\varphi - \cos\varphi) = 0.$$

Because

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi = 2\cos^2\varphi - 1,$$

our problem is reduced to the following second-order algebraic equation:

$$\cos^2 \varphi - \frac{1}{5} \cos \varphi - \frac{1}{2} = 0.$$

Solving the preceding quadratic equation we obtain

$$\cos \varphi_1 = 0.814, \quad \cos \varphi_2 = -0.614.$$

We select the physically feasible solution for which $\varphi = \varphi_1 \approx 35°30'$. □

References

1. J. Mrozowski, J. Awrejcewicz, *Introduction to Biomechanics* (The Lodz University of Technology Press, Lodz, 2004), in Polish
2. E.D. Cooper, *Mathematical Mechanics: From Particle to Muscle* (World Scientific, Singapore, 2010)
3. S. Banach, *Mechanics* (PWN, Warsaw, 1956), in Polish
4. F. Janik, *General Mechanics* (PWN, Warsaw, 1970), in Polish
5. R. Gutkowski, *Analytical Mechanics* (PWN, Warsaw, 1971), in Polish
6. Z. Osinski, *General Mechanics* (PWN, Warsaw, 1994), in Polish
7. J. Leyko, *General Mechanics* (PWN, Warsaw, 1996), in Polish
8. B. Skalmierski, *Mechanics* (BNI, Warsaw, 1998), in Polish
9. J. Niziol, *Methods of Solving Mechanical Tasks* (WNT, Warsaw, 2002), in Polish
10. J. Awrejcewicz, V.A. Krysko, Yu.V. Chebotyrevskiy, *Role of Mathematics and Mechanics in the Development of Civilization* (WNT, Warsaw, 2003), in Polish
11. W. Kurnik, *Lectures on General Mechanics* (Warsaw Technological University Press, Warsaw, 2005), in Polish
12. F.P. Beer, E.R. Johnston, *Vector Mechanics for Engineers: Statics and Dynamics*, 8th edn. (McGraw-Hill, Singapore, 2007)
13. J.L. Synge, B.A. Griffith, *Principles of Mechanics* (McGraw-Hill, Tokyo, 1970)
14. G.R. Fowles, *Analytical Mechanics*, 3rd edn. (Holt, Rinehart and Winston, New York, 1977)
15. A.D. Markeev, *Theoretical Mechanics* (Science, Moscow, 1990), in Russian
16. N.V. Butenin, N.A. Fufayev, *Introduction to Analytical Mechanics* (Nauka, Moscow, 1991), in Russian
17. V. Barger, M. Olsson, *Classical Mechanics: A Modern Perspective* (McGraw-Hill, Tokyo, 1994)
18. S.T. Thornton, J.B. Marion, *Classical Dynamics of Particles and Systems* (Saunders College Publishers, New York, 1995)
19. A.P. Arya, *Introduction to Classical Mechanics* 2nd edn. (Prentice Hall, Upper Saddle River, 1998)
20. L.N. Hand, J.D. Finch, *Analytical Mechanics* (Cambridge University Press, Cambridge, 1998)
21. H. Goldstein, C.P. Poole, J.L. Safko, *Classical Mechanics* (Addison-Wesley, Reading, 2001)
22. T.W. Kibble, F.H. Berkshire, *Classical Mechanics*, 5th edn. (Imperial College Press, London, 2004)
23. J.R. Taylor, *Classical Mechanics* (University Science Books, Colorado, 2005)
24. P.W. Johnson, *Classical Mechanics with Applications* (World Scientific, Singapore, 2010)
25. R.D. Gregory, *Classical Mechanics* (Cambridge University Press, Cambridge, 2011)