Algebraic Identities Part 4

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1 Problems

A Let $a, b, c \in \mathbb{R}$ where $a \neq b$. If $c^3 = b^3 + b^3 + 3abc$, prove that $c = a + b$.

Proof.

Theorem 1. *Let $a, b, c \in \mathbb{R}$ If $a^3 + b^3 + c^3 = 3abc \Leftrightarrow a = b = c$ or $a + b + c = 0$*

$$b^3 + b^3 + 3abc = c^3$$
$$b^3 + b^3 - c^3 = -3abc$$
$$-a^3 - b^3 + c^3 = 3abc$$

Using theorem 1 we can conclude

$$-a = -b = c \quad \text{which is false since} \quad a \neq b \qquad or \qquad -a - b + c = 0 \quad c = a + b$$

$\square$

B Let x, y and z be non-zero real numbers such that

$$x + y + z = a, \ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a}$$

Show that at least one of x, y, z is equal to a.

We begin by recognizing that either $a - x$, $a - y$ or $a - z$ is equal to zero. Thus we can rephrase the problem into proving that $(a - x)(a - y)(a - z) = 0$.

Using the theorem we learned during the lesson we can expand the following expression.

$$(a - x)(a - y)(a - z) = a^3 - (x + y + z)a^2 + (xy + yz + zx)a - xyz$$

Substituting a for $x+y+z$, the expression a^3-a^3 cancels out.

$$(a-x)(a-y)(a-z) = (xy+yz+zx)a - xyz$$

We divide $xy+yz+zx$ by xyz and multiply it by xyz to cancel out the division.

$$(a-x)(a-y)(a-z) = xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)a - xyz$$

Now we can substitute $\frac{1}{a}$ for $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ we get $xyz\frac{a}{a} - xyz$. The fraction $\frac{a}{a}$ cancels out, leading to $xyz - xyz$ also cancelling out. This leaves us with

$$(a-x)(a-y)(a-z) = 0$$

This proves that either $a-x$, $a-y$ or $a-z$ is equal to zero and therefore either a, b or c is equal to a.