Algebraic Identities Part 4

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Problems

A Let a, b and c be real numbers where $a \neq b$. If $c^3 = a^3 + b^3 + 3abc$, prove that $c = a + b$.

Solution

During the last lecture we learned about the following theorem and it's corollary.

Theorem 1. *Let a, b and c be real numbers.*

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

Corollary 1. *Let a, b and c be real numbers.*

If $a^3 + b^3 + c^3 = 3abc$, then either $a = b = c$ or $a + b + c = 0$ is true.

We start by rearrange the equation $c^3 = a^3 + b^3 + 3abc$ into the following form:

$$-a^3 - b^3 + c^3 = 3abc$$

This equation is now in the form of **Corollary 1**. Therefore we can conclude that either $-a = -b = -c$ or $-a - b + c = 0$ is true.

We know that $-a = -b = -c$ is false since $a \neq b$. This leaves us with $-a - b + c = 0$, which we can easily rearrange into the form of $c = a + b$, thus proving the problem.

B Let x, y and z be nonzero real numbers such that

$$x + y + z = a, \ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a}$$

Show that at least one of x, y, z is equal to a.

Solution

We begin by recognizing that either $a-x$, $a-y$ or $a-z$ is equal to zero. Thus we can rephrase the problem into proving that $(a-x)(a-y)(a-z)=0$.

Using the theorem we learned during the lesson we can expand the following expression.

$$(a-x)(a-y)(a-z)=a^3-(x+y+z)a^2+(xy+yz+zx)a-xyz$$

Substituting a for $x+y+z$, the expression a^3-a^3 cancels out.

$$(a-x)(a-y)(a-z)=(xy+yz+zx)a-xyz$$

We divide $xy+yz+zx$ by xyz and multiply it by xyz to cancel out the division.

$$(a-x)(a-y)(a-z)=xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)a-xyz$$

Now we can substitute $\frac{1}{a}$ for $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ we get $xyz\frac{a}{a}-xyz$. The fraction $\frac{a}{a}$ cancels out, leading to $xyz-xyz$ also cancelling out. This leaves us with

$$(a-x)(a-y)(a-z)=0$$

This proves that either $a-x$, $a-y$ or $a-z$ is equal to zero and therefore either a, b or c is equal to a.

C Solve the following system:

$$x+y+z=10$$
$$x^2+y^2+z^2=100$$
$$x^3+y^3+z^3=1000$$

Solution

During a past lecture we learned about the following theorem:

Theorem 2. *Let a,b and c be real numbers.*

$$(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a)$$

Let's begin by writing out **Theorem 2**.

$$(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a)$$

We can substitute 10 and 1000 for $x + y + z$ and $x^3 + y^3 + z^3$ respectively. This leaves us with $1000 = 1000 + 3(a + b)(b + c)(c + a)$, which we can simplify to $0 = (a + b)(b + c)(c + a)$.

Therefore we conclude that either $x + y = 0$ or $y + z = 0$ or $z + x = 0$ is true. Since the sum of one of the pairs of variables is equal to 0, we conclude that the third variable is equal to 10 using the fact that $x + y + z = 10$.

To find the values of the other pair of variables let's temporarily assume that z is equal to 10. Inserting this value into the equation $x^2 + y^2 + z^2 = 100$, it then simplifies to $x^2 + y^2 = 0$. Because the exponents in the terms x^2 and y^2 are both even, they are also nonnegative. Since they are nonnegative and add up to 0, they both need to be equal to 0.

From this we can conclude that there are three different solutions to the original system:

$$\begin{aligned} x &= 10 \\ y &= 0 \\ z &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} y &= 10 \\ z &= 10 \\ x &= 10 \end{aligned} \qquad \text{or} \qquad \begin{aligned} z &= 10 \\ x &= 0 \\ y &= 0 \end{aligned}$$

D Let a, b and c be real numbers such that $a+b+c = 0$. Show that $2(a^5+b^5+c^5) = 5abc(a^2 + b^2 + c^2)$.

Solution

Since $a + b + c = 0$, we can derive the following equation using Theorem **Theorem 1**, mentioned in Problem A.

$$abc = \frac{1}{3}(a^3 + b^3 + c^3) \tag{1}$$

Let's consider the expression $5abc(a^2+b^2+c^2)$. Using **Equation 1** we substitute $\frac{1}{3}(a^3+b^3+c^3)$ for abc, resulting in the expression $\frac{5}{3}(a^3 + b^3 + c^3)(a^2 + b^2 + c^2)$. After expanding and factorising we are left with the following expression.

$$\frac{5}{3}\big(a^5 + b^5 + c^5 + a^2b^2(a + b) + b^2c^2(b + c) + c^2a^2(c + a)\big)$$

Since $a + b + c = 0$, we can derive the following three equations.

$$-a = b + c$$

$$-b = c + a$$

$$-c = a + b$$

Using these equations we substitute $-a, -b$ and $-c$ into our expression to get the expression

$\frac{5}{3}(a^5+b^5+c^5-a^2b^2c-ab^2c^2-a^2bc^2)$. After factorising abc we are left with the following result.

$$\frac{5}{3}\left(a^5+b^5+c^5-abc(ab+bc+ca)\right)$$

We learned about the following theorem during a past lecture.

Theorem 3. *Let a, b and c be real numbers.*

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$$

Consider the equation $a+b+c=0$. This means that the equation $(a+b+c)^2=0$ is also true. Using **Theorem 3** we can modify the equation to get the equation $a^2+b^2+c^2+2(ab+bc+ca)=0$, which after rearranging gives us the following result.

$$ab+bc+ca = -\frac{1}{2}\left(a^2+b^2+c^2\right)$$

Returning to the previous expression we can now substitute $-\frac{1}{2}\left(a^2+b^2+c^2\right)$ for $ab+bc+ca$, giving us the following result.

$$\frac{5}{3}\left(a^5+b^5+c^5+\frac{1}{2}abc\left(a^2+b^2+c^2\right)\right)$$

As we derived this expression from the previous expression $5abc\left(a^2+b^2+c^2\right)$, the following equation holds true.

$$5abc\left(a^2+b^2+c^2\right)=\frac{5}{3}\left(a^5+b^5+c^5\right)+\frac{5}{6}abc\left(a^2+b^2+c^2\right)$$

Multiplying the preceding equation by $\frac{6}{5}$ and then simplifying it gives us the resulting equation $2(a^5+b^5+c^5)=5abc(a^2+b^2+c^2)$, which is the equation we set out to prove.

E Let a, b and c be nonzero real numbers such that $a+b+c=0$ and $a^3+b^3+c^3 = a^5+b^5+c^5$. Determine the exact value of $a^2+b^2+c^2$.

HINT: Expand $(a^2+b^2+c^2)(a^3+b^3+c^3)$

Solution

Let's do as the problem asks and begin expanding the expression $(a^2+b^2+c^2)(a^3+b^3+c^3)$. This results in the following expression.

$$a^5+b^5+c^5+a^3b^2+a^2b^3+b^3c^2+b^2c^3+c^3a^2+c^2a^3$$

After substituting $a^3+b^3+c^3$ for $a^5+b^5+c^5$ and factorising we are left with the following expression.

$$a^3+b^3+c^3+a^2b^2(a+b)+b^2c^2(b+c)+c^2a^2(c+a)$$

In the same way as we did in question D, we can rearrange $a + b + c = 0$ into $-a = b + c$, $-b = c + a$ and $-c = a + b$.

Substituting this into our previous equation and factorising out abc, we get the following result, which we will pair up with our starting expression.

$$(a^2 + b^2 + c^2)(a^3 + b^3 + c^3) = a^3 + b^3 + c^3 - abc(ab + bc + ca)$$

Using **Corollary 1** we will substitute $3abc$ for $a^3 + b^3 + c^3$ and then divide both sides by 3abc, giving us the following equation.

$$a^2 + b^2 + c^2 = 1 - \frac{ab + bc + ca}{3} \tag{2}$$

To continue we first have to find $ab+bc+ca$. But we're going to need to introduce the following theorem first.

Theorem 4. *Let a, b and x be real numbers.*

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

Using the equations established previously, we replace a, b and c with $-b-c$, $-c-a$ and $-a-b$ respectively, giving us the following equation after expanding it.

$$ab + bc + ca = a^2 + b^2 + c^2 + a(b + c) + b(a + c) + c(a + b) + ab + bc + ca$$

That after simplifying gives us this equation.

$$ab + bc + ca = -\frac{a^2 + b^2 + c^2}{2} \tag{3}$$

We can now use **Equation 3** to replace $ab + bc + ca$ in **Equation 2**, giving us the resulting equation $a^2 + b^2 + c^2 = 1 + \frac{a^2+b^2+c^2}{6}$. After rearranging this equation we are left with the following equation.

$$a^2 + b^2 + c^2 = \frac{6}{5}$$

Which is the answer we set out to find.

F Solve the following equation.

$$(x+1)^{63}+(x+1)^{62}(x-1)+(x+1)^{61}(x-1)^2+\cdots+(x+1)^2(x-1)^{61}+(x+1)(x-1)^{62}+(x-1)^{63} = 0$$

During the last lecture we learned about the following extension of the conjugate rule.

Theorem 5.

$$a^n - b^n = (a - b)\sum_{k=1}^{n} a^{n-k}b^{k-1}$$

We begin by recognizing that the left hand side of our equation is almost the same as the right hand side of **Theorem 5**, albeit with a missing multiplication by $a-b$. The expression $x+1$ and $x-1$ would take the place of a and b respectively.

As stated, the only difference is that the expression from **Theorem 5** is multiplied by $a-b$, which we can easily fix by multiplying both sides of our equation by $(x+1)-(x-1)$. Since the right hand side of our equation is equal to 0, it stays the same, giving us the following equation.

$$\big((x+1)-(x-1)\big)\Big((x+1)^{63}+(x+1)^{62}(x-1)+(x+1)^{61}(x-1)^2+\cdots+(x+1)(x-1)^{62}+(x-1)^{63}\Big)=0$$

Now that our equation is exactly the same as the one in **Theorem 5**, we can rewrite our equation into the following, more manageable, form.

$$(x+1)^{64}-(x-1)^{64}=0$$

After moving $(x-1)^{64}$ to the right hand side and taking the 64th root of both side we are left with two possibilities, either $x+1=x-1$ or $x+1=-(x-1)$. We can conclude that the first possibility has to be false since after rearranging it we are left with the contradiction $1=-1$, therefore possibility two is the only possibility. The second equation has the solution of $x=0$, thusly giving us the value asked of us in the problem.