Algebraic Identities Part 4

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1 Problems

A Let $a, b, c \in \mathbb{R}$ where $a \neq b$. If $c^3 = b^3 + b^3 + 3abc$, prove that $c = a + b$.

Theorem 1. *lemlittle lemma This is a small lemma.*

It follows from **??** that we have thm The main result.

Theorem 2. *Let* $a, b, c \in \mathbb{R}$ *If* $a^3 + b^3 + c^3 = 3abc \Leftrightarrow a = b = c$ *or* $a + b + c = 0$

$$b^3 + b^3 + 3abc = c^3$$

$$b^3 + b^3 - c^3 = -3abc$$

$$-a^3 - b^3 + c^3 = 3abc$$

Using theorem 1 we can conclude

$$-a = -b = c \qquad \textit{or} \qquad -a - b + c = 0$$

which is false since $a \neq b$ $\qquad$ $c = a + b$

B Let x, y and z be non-zero real numbers such that

$$x + y + z = a, \ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a}$$

Show that at least one of x, y, z is equal to a.

We begin by recognizing that either $a - x$, $a - y$ or $a - z$ is equal to zero. Thus we can rephrase the problem into proving that $(a-x)(a-y)(a-z) = 0$.

Using the theorem we learned during the lesson we can expand the following expression.

$$(a-x)(a-y)(a-z) = a^3 - (x+y+z)a^2 + (xy+yz+zx)a - xyz$$

Substituting a for $x + y + z$, the expression $a^3 - a^3$ cancels out.

$$(a-x)(a-y)(a-z) = (xy+yz+zx)a - xyz$$

We divide $xy+yz+zx$ by xyz and multiply it by xyz to cancel out the division.

$$(a-x)(a-y)(a-z) = xyz\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)a - xyz$$

Now we can substitute $\frac{1}{a}$ for $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ we get $xyz\frac{a}{a} - xyz$. The fraction $\frac{a}{a}$ cancels out, leading to $xyz - xyz$ also cancelling out. This leaves us with

$$(a-x)(a-y)(a-z) = 0$$

This proves that either $a - x$, $a - y$ or $a - z$ is equal to zero and therefore either a, b or c is equal to a.

C Solve the following system:

$$x + y + z = 10$$
$$x^2 + y^2 + z^2 = 100$$
$$x^3 + y^3 + z^3 = 1000$$