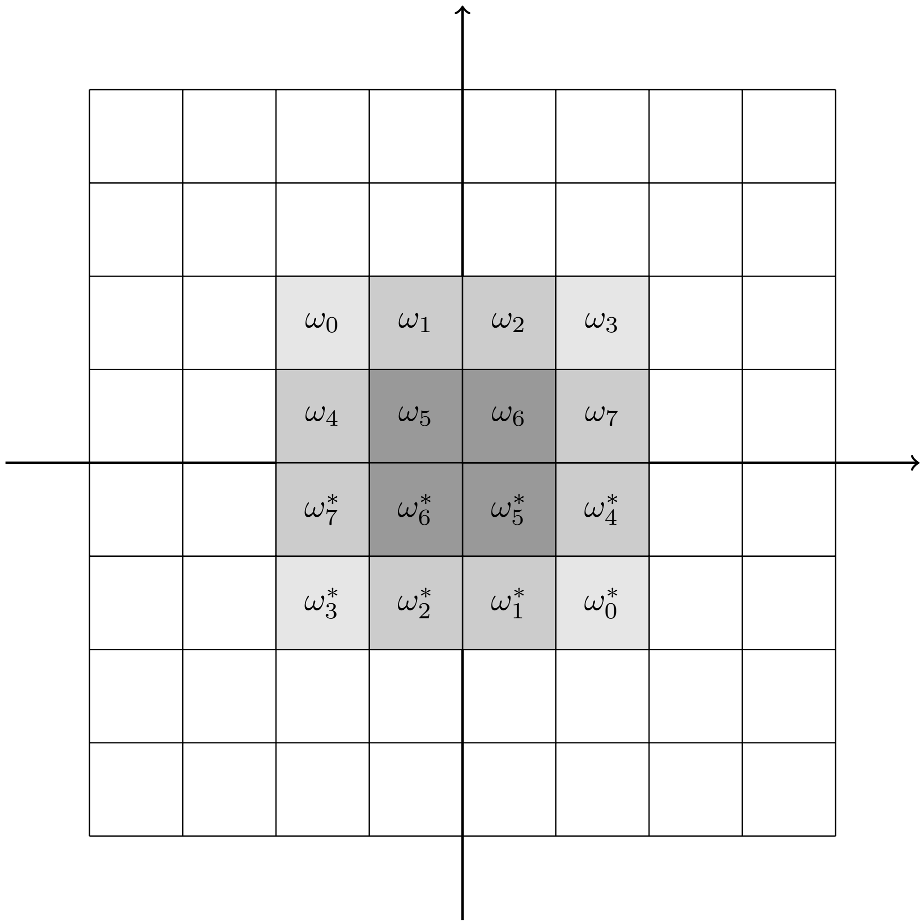


Figure 1: Caption

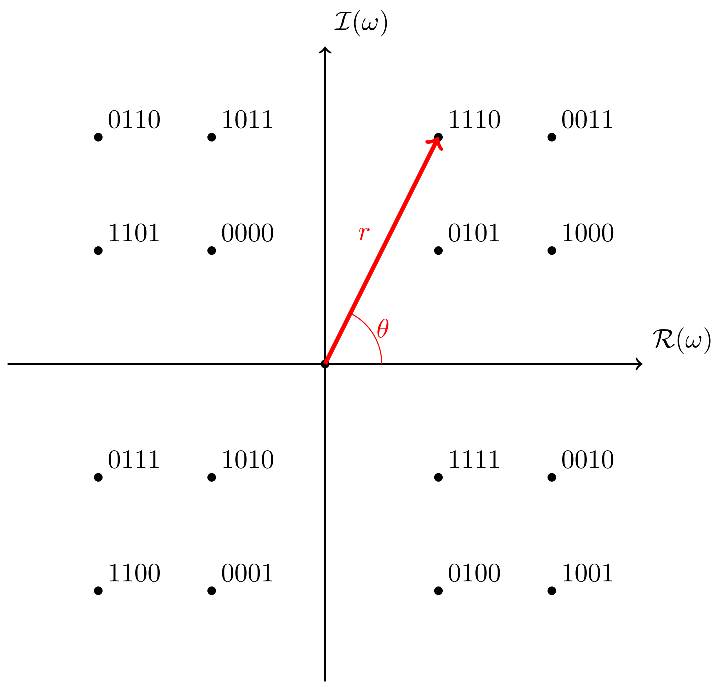
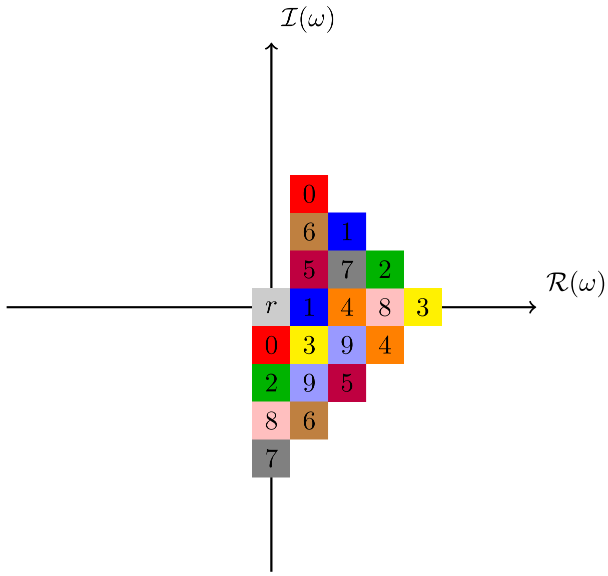


Figure 2: Encoding 4-bit words into frequencies.



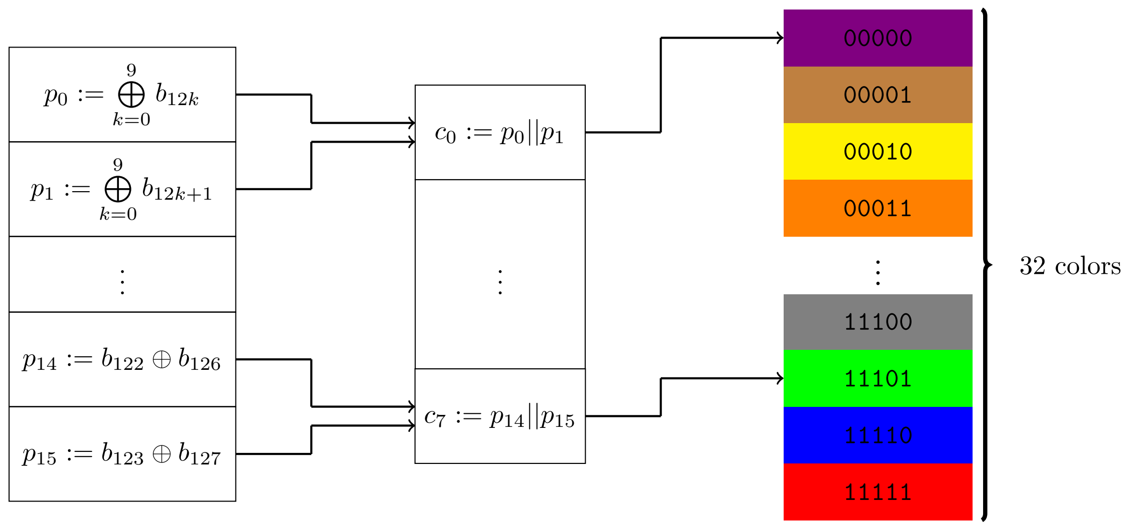
$$\mathcal{B}_m = I(f_m > 0.5),\ m \in \{1, 2, 3\}$$

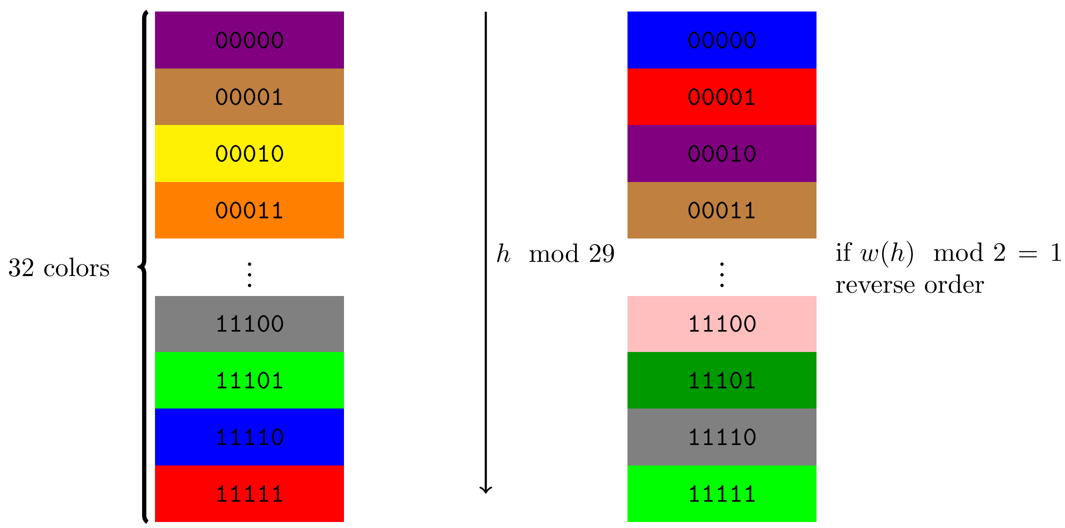
$$\mathcal{P}(\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}) \to \{Color\#1, Color\#2, \dots, Color\#2^m\}$$

$$\begin{aligned}\{Color\#1, Color\#2, \dots, Color\#2^m\} = \{&f_1 \leq 0.5 \wedge f_2 \leq 0.5 \wedge f_3 \leq 0.5,\\ &f_1 \leq 0.5 \wedge f_2 \leq 0.5 \wedge f_3 > 0.5,\\ &\dots,\\ &f_1 > 0.5 \wedge f_2 > 0.5 \wedge f_3 > 0.5\end{aligned}$$

$$h = \underbrace{\underline{1}100}_{f_1} \mid \underbrace{1000}_{f_2} \mid \underbrace{0110}_{f_3} \mid \underline{1}110 \mid 0101 \mid 1011 \mid \underline{0}100 \mid 1111 \mid 0110 \mid \underline{0}010 \mid 1001 \mid 0110 \dots \underline{0}010 \mid 1001 \mid 0101 \mid 1001 \mid 0100$$

$$p_i = \begin{cases} b_{0,i} \oplus b_{1,i} \oplus b_{2,i} \oplus \cdots \oplus b_{9,i} = \bigoplus_{k=0}^{9} b_{k,i} & 0 \leq i < 12 \\ h_{120+i-12} \oplus h_{120+i-8} & 12 \leq i < 16 \end{cases}$$





Proof that $\tilde{x} \neq x \mod 23$, where $\tilde{x}$ is x with 2 bits of the same group changed: Let ℓ and $\ell + 12m$ the indices of the 2 flipped bits, with $0 < m \leq 10$.

Ring of $\mathbb{Z}_p$ is an integral domain.

- Case $b_\ell = 1, b_{\ell+12m} = 0$:

$$
\begin{aligned}
x = \tilde{x} &= x - 2^{\ell} + 2^{\ell+12m} \mod 23 \\
\implies x &= x + 2^{\ell}\left(2^{12m} - 1\right) \mod 23 \\
\implies 0 &= 2^{\ell}\left(2^{12m} - 1\right) \mod 23 \\
\implies 0 &= 2^{12m} - 1 \mod 23 \\
\implies 1 &= 2^{12m} \mod 23
\end{aligned}
$$

which has no solution for $1 < m \leq 10$.

- Case $b_\ell = 0, b_{\ell+12m} = 1$:

$$
\begin{aligned}
x = \tilde{x} &= x - 2^{\ell} + 2^{\ell+12m} \mod 23 \\
\implies x &= x + 2^{\ell}\left(1 - 2^{12m}\right) \mod 23 \\
\implies 0 &= 2^{\ell}\left(1 - 2^{12m}\right) \mod 23
\end{aligned}
$$

which has no solution for $1 < m \leq 10$, same as the previous case.

- Case $b_\ell = 1, b_{\ell+12m} = 1$:

$$
\begin{aligned}
x = \tilde{x} &= x + 2^{\ell} + 2^{\ell+12m} \mod 23 \\
\implies x &= x + 2^{\ell}\left(1 + 2^{12m}\right) \mod 23 \\
\implies 0 &= 2^{\ell}\left(1 + 2^{12m}\right) \mod 23 \\
\implies 0 &= 1 + 2^{12m} \mod 23 \\
\implies -1 &= 2^{12m} \mod 23
\end{aligned}
$$

which has no solution for $1 < m \leq 10$.

- Case $b_\ell = 0, b_{\ell+12m} = 0$:

$$
\begin{aligned}
x = \tilde{x} &= x - 2^{\ell} - 2^{\ell+12m} \mod 23 \\
\implies x &= x + 2^{\ell}\left(-1 - 2^{12m}\right) \mod 23 \\
\implies 0 &= 2^{\ell}\left(-1 - 2^{12m}\right) \mod 23
\end{aligned}
$$

which has no solution for $1 < m \leq 10$, same as the previous case

Attacks if shift is $\mod 23$:

- Adv flips n bits of the same index, keeping same modulo 23 :
  - Effect : the same palette is used. If n is even and $n > 2$, then all colors are exactly the same as intra group parity is the same

  - "Impossible" for odd n as palette is flipped
  - Rare (I guess) for $n = 4$: maximum 8 if the ten bits are e.g. 1101011100

- Adv flips n bits, keeping same modulo 23 :
  - Effect : the same palette is used. The colors corresponding to the n indices are changed
  - "Impossible" for odd n as palette is flipped

$$\begin{aligned} x + 2^k - 2^\ell &= x \mod p \\ 2^k - 2^\ell &= 0 \mod p \\ 2^\ell \left(2^{k-\ell} - 1\right) &= 0 \mod p \\ 2^{k-\ell} &= 1 \mod p \\ k - \ell &= m \cdot |2| \mod p,\ m \in \mathbb{Z} \end{aligned}$$

Problem : 11 is $-1 \mod 12 \implies$ flipping b_i and b_{i+11} might only change 1 color.

Attacks if shift is $\sum index$:

Lots (sum of 4 indices that are for the same function divide 31)

Properties:

- Color choices : $\{(b_0 \oplus b_{12} \oplus \cdots \oplus b_{108}), (b_1 \oplus b_{13} \oplus \cdots \oplus b_{109}), \cdots, (b_{11} \oplus \cdots \oplus b_{119})\}$
- Palette shift : $h \mod 23$
- Invert palette direction : $w(h) \mod 2$
- Symmetry mode : $\sum_{i:h_i=1} i + 11 \mod 13$

Keeping same parity bits :

$$b'_0 \oplus b'_{12} \oplus \cdots \oplus b'_{108} = b_0 \oplus b_{12} \oplus \cdots \oplus b_{108}$$

$$\bigoplus_{i=0}^{9} b'_{12i} = \bigoplus_{i=0}^{10} b_{12i}$$

Implications :

- If $b'_i \neq b_i$ for an odd number of i, then the parity of the weight of h changes and the palette is flipped.
- To keep the same palette shift, we must have $h = h' \mod 23$, with $h' = h + \sum_k 2^k - \sum_\ell 2^\ell$ for all $k : b_k = 0 \wedge b'_k = 1$ and $\ell : b_\ell = 1 \wedge b'_\ell = 0$. Because of the previous point, we must have $k + \ell = 0 \mod 2$.

$$h + \sum_k 2^k - \sum_\ell 2^\ell = h \mod 23$$
$$\sum_k 2^k - \sum_\ell 2^\ell = 0 \mod 23$$

$|2| \mod 23 = 11 \implies 2^{12i} \mod 23 = \left(2^{12}\right)^i \mod 23 = \left(2 \cdot 2^{11}\right)^i \mod 23 = 2^i \mod 23$.

$2^{12i} \mod 23 = 2^i \mod 23$ for $i \in \{0, \cdots, 9\} = \{1, 2, 4, 8, 16, 9, 18, 13, 3, 6\} := M$

For $n = 2, 4, 6, 8, 10$: find n distinct elements $m_i \in M$ and a vector $\alpha \in \{-1, 1\}^n$ such that $\sum_{i=0}^{9} \alpha_i m_i = 0 \mod 23$

- $n = 2$: It is impossible to find $m_1 \neq m_2$ such that $m_1 \pm m_2 = 0 \mod 23$.
- $n = 4$: With Python, we found there are 84 possible choices for m_0, m_1, m_2, m_3 such that we can find a fitting α. For example, $m_1 = 2, m_2 = 4, m_3 = 16, m_4 = 18$, we find $\alpha = \{1, -1, -1, 1\}$.
- $n = 6$: We found there are 280 possible choices for m_0 to m_5 such that we can find a fitting α.
- $n = 8$: We found there are 255 possible choices for m_0 to m_7 such that we can find a fitting α.
- $n = 10$: Picking m_0 to m_9 as every element of M, we can find (for example) $\alpha = \{1, -1, -1, 1, -1, 1, -1, 1, 1, 1\}$ that is fitting.
- In total, 620 "collisions" with same parity and same palette shift

Introducing the symmetries: in order to keep the same symmetry, we must have $\sum_{i:h_i=1} i + 11 \mod 13 = \sum_{i:h'_i=1} i + 11 \mod 13$. That means a collision must have $\sum 12 m_i \alpha_i = 0 \mod 13$

- $n = 4$: The number of collision drops to 53 (IN TOTAL):.
- $n = 6$: We found there are 292 (IN TOTAL) possible collisions.
- $n = 8$: We found there are 230 (IN TOTAL) possible collisions.
- $n = 10$:We found there are 25 (IN TOTAL) possible collisions).
- In total, 620 "collisions" with same parity and same palette shift
- However, each requires a specific combination of bits to be possible. Some of them are mutually exclusive. $\rightarrow$ divide by $\approx$ 16, 32, 64 $\rightarrow$

If the bits are not from the same parity:
If two flipped bits of same value:
Same shift :

$$\begin{aligned}
h + 2^k + 2^\ell &= h \mod p \\
2^k + 2\ell &= 0 \mod p \\
2^\ell(2^{k-\ell} + 1) &= 0 \mod p \\
2^{k-\ell} &= -1 \mod p
\end{aligned}$$

Which is not possible as long as we pick $p \neq 3$ and $p \neq 17$
Same shift :

$$\begin{aligned}
h + 2^k - 2^\ell &= h \mod p \\
2^k - 2\ell &= 0 \mod p \\
2^\ell(2^{k-\ell} - 1) &= 0 \mod p \\
2^{k-\ell} &= 1 \mod p \\
k - \ell &= 0 \mod (|2| \mod p)
\end{aligned}$$

Same symmetry :

$$k - \ell = 0 \mod (|2| \mod q)$$

Having both yield

$$k - \ell = 0 \mod \operatorname{lcm}(|2| \mod p, |2| \mod q)$$

If we pick $p = 29, q = 23$ we have $(|2| \mod p, |2| \mod q) = (28, 11)$ and $\operatorname{lcm}(28, 11) = 308 > 256$

```
h = 101011101010101...1010011110111
```