

Minimal CBC Casper Isabelle/HOL proofs

LayerX

March 4, 2019

Contents

1	Description of CBC Casper	3
2	Latest Message	14
3	Safety Proof	29

theory *Strict-Order*

imports *Main*

begin

notation *Set.empty* (\emptyset)

definition *strict-partial-order* $r \equiv \text{trans } r \wedge \text{irrefl } r$

definition *strict-well-order-on* $A \ r \equiv \text{strict-linear-order-on } A \ r \wedge \text{wf } r$

lemma *strict-linear-order-is-strict-partial-order* :
 $\text{strict-linear-order-on } A \ r \implies \text{strict-partial-order } r$
by (*simp add: strict-linear-order-on-def strict-partial-order-def*)

definition *upper-bound-on* $:: 'a \ \text{set} \Rightarrow 'a \ \text{rel} \Rightarrow 'a \Rightarrow \text{bool}$
where
 $\text{upper-bound-on } A \ r \ x = (\forall \ y. \ y \in A \longrightarrow (y, x) \in r \vee x = y)$

definition *maximum-on* $:: 'a \ \text{set} \Rightarrow 'a \ \text{rel} \Rightarrow 'a \Rightarrow \text{bool}$
where
 $\text{maximum-on } A \ r \ x = (x \in A \wedge \text{upper-bound-on } A \ r \ x)$

definition *minimal-on* :: 'a set \Rightarrow 'a rel \Rightarrow 'a \Rightarrow bool

where

minimal-on A r x = (x \in A \wedge (\forall y. (y, x) \in r \longrightarrow y \notin A))

definition *maximal-on* :: 'a set \Rightarrow 'a rel \Rightarrow 'a \Rightarrow bool

where

maximal-on A r x = (x \in A \wedge (\forall y. (x, y) \in r \longrightarrow y \notin A))

lemma *maximal-and-maximum-coincide-for-strict-linear-order* :

strict-linear-order-on A r \implies *maximal-on* A r x = *maximum-on* A r x

apply (simp add: *strict-linear-order-on-def* *irrefl-def* *total-on-def* *trans-def* *maximal-on-def* *maximum-on-def* *upper-bound-on-def*)

by blast

lemma *strict-partial-order-on-finite-non-empty-set-has-maximal* :

strict-partial-order r \longrightarrow finite A \longrightarrow A \neq \emptyset \longrightarrow (\exists x. *maximal-on* A r x)

proof -

have $\bigwedge n$. *strict-partial-order* r \implies (\forall A. Suc n = card A \longrightarrow finite A \longrightarrow A \neq \emptyset \longrightarrow (\exists x. *maximal-on* A r x))

proof -

assume *strict-partial-order* r

then have (\forall a. (a, a) \notin r)

by (simp add: *strict-partial-order-def* *irrefl-def*)

fix n

show \forall A. Suc n = card A \longrightarrow finite A \longrightarrow A \neq \emptyset \longrightarrow (\exists x. *maximal-on* A r x)

apply (induction n)

unfolding *maximal-on-def*

using $\langle (\forall$ a. (a, a) \notin r) \rangle

apply (metis card-eq-SucD empty-iff insert-iff)

proof -

fix n

assume \forall A. Suc n = card A \longrightarrow finite A \longrightarrow A \neq \emptyset \longrightarrow (\exists x. x \in A \wedge (\forall y. (x, y) \in r \longrightarrow y \notin A))

have \forall B. Suc (Suc n) = card B \longrightarrow finite B \longrightarrow B \neq \emptyset \longrightarrow (\exists A' b. B = A' \cup {b} \wedge card A' = Suc n \wedge b \notin A')

by (metis Un-commute add-diff-cancel-left' card-gt-0-iff card-insert-disjoint card-le-Suc-iff insert-is-Un not-le not-less-eq-eq plus-1-eq-Suc)

then have \forall B. Suc (Suc n) = card B \longrightarrow finite B \longrightarrow B \neq \emptyset \longrightarrow (\exists A' b. B = A' \cup {b} \wedge card A' = Suc n \wedge finite A' \wedge A' \neq \emptyset \wedge b \notin A')

by (metis card-gt-0-iff zero-less-Suc)

then have \forall B. Suc (Suc n) = card B \longrightarrow finite B \longrightarrow B \neq \emptyset

\longrightarrow (\exists A' b x. B = A' \cup {b} \wedge b \notin A' \wedge x \in A' \wedge (\forall y. (x, y) \in r \longrightarrow y \notin A'))

using $\langle \forall$ A. Suc n = card A \longrightarrow finite A \longrightarrow A \neq \emptyset \longrightarrow (\exists x. x \in A \wedge (\forall y. (x, y) \in r \longrightarrow y \notin A)) \rangle

by metis

then show \forall B. Suc (Suc n) = card B \longrightarrow finite B \longrightarrow B \neq \emptyset \longrightarrow (\exists x. x \in B \wedge (\forall y. (x, y) \in r \longrightarrow y \notin B))

```

    by (metis (no-types, lifting) Un-insert-right  $\langle \forall a. (a, a) \notin r \rangle$  (strict-partial-order
r) insertE insert-iff strict-partial-order-def sup-bot.right-neutral transE)
  qed
  qed
  then show ?thesis
    by (metis card.insert-remove finite.cases)
  qed
qed

```

```

lemma strict-partial-order-has-at-most-one-maximum :
  strict-partial-order r
   $\longrightarrow \{x. \text{maximum-on } A \ r \ x\} \neq \emptyset$ 
   $\longrightarrow \text{is-singleton } \{x. \text{maximum-on } A \ r \ x\}$ 
proof (rule ccontr)
  assume  $\neg (\text{strict-partial-order } r \longrightarrow \{x. \text{maximum-on } A \ r \ x\} \neq \emptyset \longrightarrow \text{is-singleton } \{x. \text{maximum-on } A \ r \ x\})$ 
  then have  $\text{strict-partial-order } r \longrightarrow \{x. \text{maximum-on } A \ r \ x\} \neq \emptyset \longrightarrow \neg \text{is-singleton } \{x. \text{maximum-on } A \ r \ x\}$ 
    by simp
  then have  $\text{strict-partial-order } r \longrightarrow \{x. \text{maximum-on } A \ r \ x\} \neq \emptyset \longrightarrow (\exists \ x1 \ x2. x1 \neq x2 \wedge \{x1, x2\} \subseteq \{x. \text{maximum-on } A \ r \ x\})$ 
    by (meson empty-subsetI insert-subset is-singletonI)
  then have  $\text{strict-partial-order } r \longrightarrow \{x. \text{maximum-on } A \ r \ x\} \neq \emptyset \longrightarrow (\exists \ x1 \ x2. x1 \neq x2 \wedge \{x1, x2\} \subseteq \{x \in A. \forall \ y. y \in A \longrightarrow (y, x) \in r \vee x = y\})$ 
    by (simp add: maximum-on-def upper-bound-on-def)
  then have  $\text{strict-partial-order } r \longrightarrow \{x. \text{maximum-on } A \ r \ x\} \neq \emptyset \longrightarrow (\exists \ x1 \ x2. x1 \neq x2 \wedge \{x1, x2\} \subseteq A \wedge (\forall \ y. y \in A \longrightarrow (y, x1) \in r \vee x1 = y) \wedge (\forall \ y. y \in A \longrightarrow (y, x2) \in r \vee x2 = y))$ 
    by auto
  then show False
    using strict-partial-order-def
    by (metis  $\langle \neg (\text{strict-partial-order } r \longrightarrow \{x. \text{maximum-on } A \ r \ x\} \neq \emptyset \longrightarrow \text{is-singleton } \{x. \text{maximum-on } A \ r \ x\}) \rangle$  insert-subset irrefl-def transE)
  qed

```

```

lemma strict-linear-order-on-finite-non-empty-set-has-one-maximum :
  strict-linear-order-on A r  $\longrightarrow \text{finite } A \longrightarrow A \neq \emptyset \longrightarrow \text{is-singleton } \{x. \text{maximum-on } A \ r \ x\}$ 
  using strict-linear-order-is-strict-partial-order strict-partial-order-on-finite-non-empty-set-has-maximal
    strict-partial-order-has-at-most-one-maximum maximal-and-maximum-coincide-for-strict-linear-order
  by fastforce
end

```

1 Description of CBC Casper

```

theory CBCCasper

```

```
imports Main HOL.Real Libraries/Strict-Order Libraries/Restricted-Predicates Li-  
braries/LaTeXsugar
```

```
begin
```

```
notation Set.empty ( $\emptyset$ )
```

```
typedecl validator
```

```
typedecl consensus-value
```

```
datatype message =  
  Message consensus-value * validator * message list
```

```
type-synonym state = message set
```

```
fun sender :: message  $\Rightarrow$  validator  
  where  
    sender (Message  $(-, v, -)$ ) = v
```

```
fun est :: message  $\Rightarrow$  consensus-value  
  where  
    est (Message  $(c, -, -)$ ) = c
```

```
fun justification :: message  $\Rightarrow$  state  
  where  
    justification (Message  $(-, -, s)$ ) = set s
```

```
fun  
   $\Sigma\text{-}i$  :: (validator set  $\times$  consensus-value set  $\times$  (message set  $\Rightarrow$  consensus-value  
  set))  $\Rightarrow$  nat  $\Rightarrow$  state set and  
   $M\text{-}i$  :: (validator set  $\times$  consensus-value set  $\times$  (message set  $\Rightarrow$  consensus-value  
  set))  $\Rightarrow$  nat  $\Rightarrow$  message set  
  where  
     $\Sigma\text{-}i$  (V, C,  $\varepsilon$ ) 0 =  $\{\emptyset\}$   
    |  $\Sigma\text{-}i$  (V, C,  $\varepsilon$ ) n =  $\{\sigma \in \text{Pow } (M\text{-}i \text{ } (V, C, \varepsilon) \text{ } (n - 1)). \text{finite } \sigma \wedge (\forall m. m \in \sigma$   
     $\longrightarrow \text{justification } m \subseteq \sigma)\}$   
    |  $M\text{-}i$  (V, C,  $\varepsilon$ ) n =  $\{m. \text{est } m \in C \wedge \text{sender } m \in V \wedge \text{justification } m \in (\Sigma\text{-}i$ 
```

$(V, C, \varepsilon) \ n) \wedge \text{est } m \in \varepsilon \ (\text{justification } m)\}$

locale *Params* =

fixes $V :: \text{validator set}$

and $W :: \text{validator} \Rightarrow \text{real}$

and $t :: \text{real}$

fixes $C :: \text{consensus-value set}$

and $\varepsilon :: \text{message set} \Rightarrow \text{consensus-value set}$

begin

definition $\Sigma = (\bigcup_{i \in \mathbb{N}} \Sigma\text{-}i \ (V, C, \varepsilon) \ i)$

definition $M = (\bigcup_{i \in \mathbb{N}} M\text{-}i \ (V, C, \varepsilon) \ i)$

definition $\text{is-valid-estimator} :: (\text{state} \Rightarrow \text{consensus-value set}) \Rightarrow \text{bool}$

where

$\text{is-valid-estimator } e = (\forall \sigma \in \Sigma. \ e \ \sigma \in \text{Pow } C - \{\emptyset\})$

lemma $\Sigma\text{-}i\text{-subset-Mi}: \Sigma\text{-}i \ (V, C, \varepsilon) \ (n + 1) \subseteq \text{Pow } (M\text{-}i \ (V, C, \varepsilon) \ n)$

by *force*

lemma $\Sigma\text{-}i\text{-subset-to-Mi}: \Sigma\text{-}i \ (V, C, \varepsilon) \ n \subseteq \Sigma\text{-}i \ (V, C, \varepsilon) \ (n+1) \Longrightarrow M\text{-}i \ (V, C, \varepsilon)$

$n \subseteq M\text{-}i \ (V, C, \varepsilon) \ (n+1)$

by *auto*

lemma $M\text{-}i\text{-subset-to-}\Sigma\text{-}i: M\text{-}i \ (V, C, \varepsilon) \ n \subseteq M\text{-}i \ (V, C, \varepsilon) \ (n+1) \Longrightarrow \Sigma\text{-}i \ (V, C, \varepsilon)$

$(n+1) \subseteq \Sigma\text{-}i \ (V, C, \varepsilon) \ (n+2)$

by *auto*

lemma $\Sigma\text{-}i\text{-monotonic}: \Sigma\text{-}i \ (V, C, \varepsilon) \ n \subseteq \Sigma\text{-}i \ (V, C, \varepsilon) \ (n+1)$

apply *(induction n)*

apply *simp*

apply *(metis Mi-subset-to-Σi Suc-eq-plus1 Σi-subset-to-Mi add commute add-2-eq-Suc)*

done

lemma $M\text{-}i\text{-monotonic}: M\text{-}i \ (V, C, \varepsilon) \ n \subseteq M\text{-}i \ (V, C, \varepsilon) \ (n+1)$

apply *(induction n)*

defer

using $\Sigma\text{-}i\text{-monotonic}$ $\Sigma\text{-}i\text{-subset-to-Mi}$ **apply** *blast*

apply *auto*

done

lemma $\Sigma\text{-}i\text{-monotonicity}: \forall \ m \in \mathbb{N}. \ \forall \ n \in \mathbb{N}. \ m \leq n \longrightarrow \Sigma\text{-}i \ (V, C, \varepsilon) \ m \subseteq \Sigma\text{-}i$

$(V, C, \varepsilon) \ n$

using $\Sigma\text{-}i\text{-monotonic}$

by *(metis Suc-eq-plus1 lift-Suc-mono-le)*

lemma $M\text{-}i\text{-monotonicity}: \forall \ m \in \mathbb{N}. \ \forall \ n \in \mathbb{N}. \ m \leq n \longrightarrow M\text{-}i \ (V, C, \varepsilon) \ m \subseteq$

$M\text{-}i \ (V, C, \varepsilon) \ n$

using $M\text{-}i\text{-monotonic}$

by (*metis Suc-eq-plus1 lift-Suc-mono-le*)

lemma *message-is-in-M-i* :

$\forall m \in M. \exists n \in \mathbb{N}. m \in M-i (V, C, \varepsilon) (n - 1)$

apply (*simp add: M-def Σ -i.elims*)

by (*metis Nats-1 Nats-add One-nat-def diff-Suc-1 plus-1-eq-Suc*)

lemma *state-is-in-pow-M-i* :

$\forall \sigma \in \Sigma. (\exists n \in \mathbb{N}. \sigma \in Pow (M-i (V, C, \varepsilon) (n - 1)) \wedge (\forall m \in \sigma. justification\ m \subseteq \sigma))$

apply (*simp add: Σ -def*)

apply *auto*

proof –

fix $y :: nat$ **and** $\sigma :: message\ set$

assume $a1: \sigma \in \Sigma-i (V, C, \varepsilon) y$

assume $a2: y \in \mathbb{N}$

have $\sigma \subseteq M-i (V, C, \varepsilon) y$

using $a1$ **by** (*meson Params. Σ i-monotonic Params. Σ i-subset-Mi Pow-iff contra-subsetD*)

then have $\exists n. n \in \mathbb{N} \wedge \sigma \subseteq M-i (V, C, \varepsilon) (n - 1)$

using $a2$ **by** (*metis (no-types) Nats-1 Nats-add diff-Suc-1 plus-1-eq-Suc*)

then show $\exists n \in \mathbb{N}. \sigma \subseteq \{m. est\ m \in C \wedge sender\ m \in V \wedge justification\ m \in \Sigma-i (V, C, \varepsilon) (n - Suc\ 0) \wedge est\ m \in \varepsilon (justification\ m)\}$

by *auto*

next

show $\bigwedge y\ \sigma\ m\ x. y \in \mathbb{N} \implies \sigma \in \Sigma-i (V, C, \varepsilon) y \implies m \in \sigma \implies x \in justification\ m \implies x \in \sigma$

using *Params. Σ i-monotonic* **by** *fastforce*

qed

lemma *message-is-in-M-i-n* :

$\forall m \in M. \exists n \in \mathbb{N}. m \in M-i (V, C, \varepsilon) n$

by (*smt Mi-monotonic Suc-diff-Suc add-leE diff-add diff-le-self message-is-in-M-i neg0-conv plus-1-eq-Suc subsetCE zero-less-diff*)

lemma *message-in-state-is-valid* :

$\forall \sigma\ m. \sigma \in \Sigma \wedge m \in \sigma \longrightarrow m \in M$

apply (*rule, rule, rule*)

proof –

fix $\sigma\ m$

assume $\sigma \in \Sigma \wedge m \in \sigma$

have

$\exists n \in \mathbb{N}. m \in M-i (V, C, \varepsilon) n$

$\implies m \in M$

using *M-def* **by** *blast*

then show

$m \in M$

apply (*simp add: M-def*)
by (*smt M-i.simps Params.Σi-monotonic PowD Suc-diff-Suc (σ ∈ Σ ∧ m ∈ σ)*
add-leE diff-add diff-le-self grOI mem-Collect-eq plus-1-eq-Suc state-is-in-pow-M-i
subsetCE zero-less-diff)
qed

lemma *state-is-subset-of-M* : $\forall \sigma \in \Sigma. \sigma \subseteq M$
using *message-in-state-is-valid* **by** *blast*

lemma *state-is-finite* : $\forall \sigma \in \Sigma. \text{finite } \sigma$
apply (*simp add: Σ-def*)
using *Params.Σi-monotonic* **by** *fastforce*

lemma *justification-is-finite* : $\forall m \in M. \text{finite } (\text{justification } m)$
apply (*simp add: M-def*)
using *Params.Σi-monotonic* **by** *fastforce*

lemma *Σ-is-subseteq-of-pow-M* : $\Sigma \subseteq \text{Pow } M$
by (*simp add: state-is-subset-of-M subsetI*)

lemma *M-type* : $\bigwedge m. m \in M \implies \text{est } m \in C \wedge \text{sender } m \in V \wedge \text{justification } m \in \Sigma$
unfolding *M-def Σ-def*
by *auto*

end

locale *Protocol* = *Params* +
assumes *V-type* : $V \neq \emptyset$
and *W-type* : $\bigwedge w. w \in \text{range } W \implies w > 0$
and *t-type* : $0 \leq t \wedge t < \text{Sum } (W \text{ ' } V)$
and *C-type* : $\text{card } C > 1$
and *ε-type* : *is-valid-estimator* ε

lemma (**in** *Protocol*) *estimates-are-non-empty* : $\bigwedge \sigma. \sigma \in \Sigma \implies \varepsilon \sigma \neq \emptyset$
using *is-valid-estimator-def ε-type* **by** *auto*

lemma (**in** *Protocol*) *estimates-are-subset-of-C* : $\bigwedge \sigma. \sigma \in \Sigma \implies \varepsilon \sigma \subseteq C$
using *is-valid-estimator-def ε-type* **by** *auto*

lemma (**in** *Params*) *empty-set-exists-in-Σ-0* : $\emptyset \in \Sigma\text{-i } (V, C, \varepsilon)$
by *simp*

lemma (**in** *Params*) *empty-set-exists-in-Σ* : $\emptyset \in \Sigma$
apply (*simp add: Σ-def*)
using *Nats-0 Σ-i.simps(1)* **by** *blast*

lemma (**in** *Params*) *Σ-i-is-non-empty* : $\Sigma\text{-i } (V, C, \varepsilon) \neq \emptyset$

```

apply (induction n)
using empty-set-exists-in- $\Sigma$ -0 by auto

lemma (in Params)  $\Sigma$ -is-non-empty:  $\Sigma \neq \emptyset$ 
using empty-set-exists-in- $\Sigma$  by blast

lemma (in Protocol) estimates-exists-for-empty-set :
 $\varepsilon \emptyset \neq \emptyset$ 
by (simp add: empty-set-exists-in- $\Sigma$  estimates-are-non-empty)

lemma (in Protocol) non-justifying-message-exists-in-M-0:
 $\exists m. m \in M\text{-}i (V, C, \varepsilon) \ 0 \wedge \text{justification } m = \emptyset$ 
apply auto
proof –
  have  $\varepsilon \emptyset \subseteq C$ 
    using Params.empty-set-exists-in- $\Sigma$   $\varepsilon$ -type is-valid-estimator-def by auto
  then show  $\exists m. \text{est } m \in C \wedge \text{sender } m \in V \wedge \text{justification } m = \emptyset \wedge \text{est } m \in \varepsilon$ 
    ( $\text{justification } m$ )  $\wedge \text{justification } m = \emptyset$ 
    by (metis V-type all-not-in-conv est.simps estimates-exists-for-empty-set justi-
      fication.simps sender.simps set-empty subsetCE)
qed

lemma (in Protocol) M-i-is-non-empty:  $M\text{-}i (V, C, \varepsilon) n \neq \emptyset$ 
apply (induction n)
using non-justifying-message-exists-in-M-0 apply auto
using Mi-monotonic empty-iff empty-subsetI by fastforce

lemma (in Protocol) M-is-non-empty:  $M \neq \emptyset$ 
using non-justifying-message-exists-in-M-0 M-def Nats-0 by blast

lemma (in Protocol) C-is-not-empty :  $C \neq \emptyset$ 
using C-type by auto

lemma (in Params)  $\Sigma$ i-is-subset-of- $\Sigma$  :
 $\forall n \in \mathbb{N}. \Sigma\text{-}i (V, C, \varepsilon) n \subseteq \Sigma$ 
by (simp add:  $\Sigma$ -def SUP-upper)

lemma (in Protocol) message-justifying-state-in- $\Sigma$ -n-exists-in-M-n :
 $\forall n \in \mathbb{N}. (\forall \sigma. \sigma \in \Sigma\text{-}i (V, C, \varepsilon) n \longrightarrow (\exists m. m \in M\text{-}i (V, C, \varepsilon) n \wedge$ 
 $\text{justification } m = \sigma))$ 
apply auto
proof –
  fix n  $\sigma$ 
  assume  $n \in \mathbb{N}$ 
  and  $\sigma \in \Sigma\text{-}i (V, C, \varepsilon) n$ 
  then have  $\sigma \in \Sigma$ 
    using  $\Sigma$ i-is-subset-of- $\Sigma$  by auto
  have  $\varepsilon \sigma \neq \emptyset$ 
    using estimates-are-non-empty  $\langle \sigma \in \Sigma \rangle$  by auto

```


have *finite* σ
using *state-is-finite* $\langle \sigma \in \Sigma \rangle$ **by** *auto*
moreover have $\exists m. \text{sender } m \in V \wedge \text{est } m \in \varepsilon \sigma \wedge \text{justification } m = \sigma$
using *est.simps sender.simps justification.simps V-type* $\langle \varepsilon \sigma \neq \emptyset \rangle \langle \text{finite } \sigma \rangle$
by (*metis all-not-in-conv finite-list*)
moreover have $\varepsilon \sigma \subseteq C$
using *estimates-are-subset-of-C* $\Sigma i\text{-is-subset-of-}\Sigma \langle n \in \mathbb{N} \rangle \langle \sigma \in \Sigma\text{-}i (V, C, \varepsilon) \rangle$
 $n \rangle$ **by** *blast*
ultimately show $\exists m. \text{est } m \in C \wedge \text{sender } m \in V \wedge \text{justification } m \in \Sigma\text{-}i (V, C, \varepsilon) n \wedge \text{est } m \in \varepsilon (\text{justification } m) \wedge \text{justification } m = \sigma$
using *Nats-1 One-nat-def*
using $\langle \sigma \in \Sigma\text{-}i (V, C, \varepsilon) n \rangle$ **by** *blast*
qed

lemma (*in Protocol*) $\Sigma\text{-type: } \Sigma \subset \text{Pow } M$
proof –
obtain m **where** $m \in M\text{-}i (V, C, \varepsilon) 0 \wedge \text{justification } m = \emptyset$
using *non-justifying-message-exists-in-M-0* **by** *auto*
then have $\{m\} \in \Sigma\text{-}i (V, C, \varepsilon) (\text{Suc } 0)$
using *Params.Σi-subset-Mi* **by** *auto*
then have $\exists m'. m' \in M\text{-}i (V, C, \varepsilon) (\text{Suc } 0) \wedge \text{justification } m' = \{m\}$
using *message-justifying-state-in-Σ-n-exists-in-M-n Nats-1 One-nat-def* **by** *metis*
then obtain m' **where** $m' \in M\text{-}i (V, C, \varepsilon) (\text{Suc } 0) \wedge \text{justification } m' = \{m\}$
by *auto*
then have $\{m'\} \in \text{Pow } M$
using *M-def*
by (*metis Nats-1 One-nat-def PowD PowI Pow-bottom UN-I insert-subset*)
moreover have $\{m'\} \notin \Sigma$
using *Params.state-is-in-pow-M-i Protocol-axioms* $\langle m' \in M\text{-}i (V, C, \varepsilon) (\text{Suc } 0) \wedge \text{justification } m' = \{m\} \rangle$ **by** *fastforce*
ultimately show *?thesis*
using *Σ-is-subseteq-of-pow-M* **by** *auto*
qed

lemma (*in Protocol*) $M\text{-type-counterexample:}$
 $(\forall \sigma. \varepsilon \sigma = C) \implies M = \{m. \text{est } m \in C \wedge \text{sender } m \in V \wedge \text{justification } m \in \Sigma\}$
apply (*simp add: M-def*)
apply *auto*
using *Σi-is-subset-of-Σ* **apply** *blast*
by (*simp add: Σ-def*)

definition *observed* :: *message set* \Rightarrow *validator set*
where
 $\text{observed } \sigma = \{\text{sender } m \mid m. m \in \sigma\}$

lemma (in *Protocol*) *observed-type* :
 $\forall \sigma \in \text{Pow } M. \text{observed } \sigma \in \text{Pow } V$
using *Params.M-type Protocol-axioms observed-def* **by** *fastforce*

lemma (in *Protocol*) *observed-type-for-state* :
 $\forall \sigma \in \Sigma. \text{observed } \sigma \subseteq V$
using *Params.M-type Protocol-axioms observed-def state-is-subset-of-M* **by** *fastforce*

fun *is-future-state* :: (state * state) \Rightarrow bool
where
is-future-state ($\sigma 1, \sigma 2$) = ($\sigma 1 \subseteq \sigma 2$)

lemma (in *Params*) *state-difference-is-valid-message* :
 $\forall \sigma \sigma'. \sigma \in \Sigma \wedge \sigma' \in \Sigma$
 $\longrightarrow \text{is-future-state}(\sigma, \sigma')$
 $\longrightarrow \sigma' - \sigma \subseteq M$
using *state-is-subset-of-M* **by** *blast*

definition *justified* :: message \Rightarrow message \Rightarrow bool
where
justified $m1\ m2$ = ($m1 \in \text{justification } m2$)

definition *equivocation* :: (message * message) \Rightarrow bool
where
equivocation =
 $(\lambda(m1, m2). \text{sender } m1 = \text{sender } m2 \wedge m1 \neq m2 \wedge \neg (\text{justified } m1\ m2) \wedge$
 $\neg (\text{justified } m2\ m1))$

definition *is-equivocating* :: state \Rightarrow validator \Rightarrow bool
where
is-equivocating $\sigma\ v$ = ($\exists m1 \in \sigma. \exists m2 \in \sigma. \text{equivocation } (m1, m2) \wedge \text{sender } m1 = v$)

definition *equivocating-validators* :: state \Rightarrow validator set
where
equivocating-validators σ = $\{v \in \text{observed } \sigma. \text{is-equivocating } \sigma\ v\}$

lemma (in *Protocol*) *equivocating-validators-type* :
 $\forall \sigma \in \Sigma. \text{equivocating-validators } \sigma \subseteq V$
using *observed-type-for-state equivocating-validators-def* **by** *blast*

definition (in *Params*) *equivocating-validators-paper* :: state \Rightarrow validator set
where
equivocating-validators-paper σ = $\{v \in V. \text{is-equivocating } \sigma\ v\}$

lemma (in *Protocol*) *equivocating-validators-is-equivalent-to-paper* :

$\forall \sigma \in \Sigma. \text{equivocating-validators } \sigma = \text{equivocating-validators-paper } \sigma$
by (*smt Collect-cong Params.equivocating-validators-paper-def equivocating-validators-def*
is-equivocating-def mem-Collect-eq observed-type-for-state observed-def subsetCE)

definition (*in Params*) *equivocation-fault-weight* :: *state* \Rightarrow *real*
where
equivocation-fault-weight $\sigma = \text{sum } W \text{ (equivocating-validators } \sigma)$

definition (*in Params*) *is-faults-lt-threshold* :: *state* \Rightarrow *bool*
where
is-faults-lt-threshold $\sigma = (\text{equivocation-fault-weight } \sigma < t)$

definition (*in Protocol*) Σt :: *state set*
where
 $\Sigma t = \{\sigma \in \Sigma. \text{is-faults-lt-threshold } \sigma\}$

lemma (*in Protocol*) Σt -is-subset-of- Σ : $\Sigma t \subseteq \Sigma$
using Σt -def **by** *auto*

type-synonym *state-property* = *state* \Rightarrow *bool*

type-synonym *consensus-value-property* = *consensus-value* \Rightarrow *bool*

definition (*in Params*) *message-justification* :: *message rel*
where
message-justification = $\{(m1, m2). \{m1, m2\} \subseteq M \wedge \text{justified } m1 \ m2\}$

lemma (*in Protocol*) *transitivity-of-justifications* :
trans message-justification
apply (*simp add: trans-def message-justification-def justified-def*)
by (*meson Params.M-type Params.state-is-in-pow-M-i Protocol-axioms contra-subsetD*)

lemma (*in Protocol*) *irreflexivity-of-justifications* :
irrefl message-justification
apply (*simp add: irrefl-def message-justification-def justified-def*)
apply (*simp add: M-def*)
apply *auto*
proof –
fix *n m*

assume $est\ m \in C$
assume $sender\ m \in V$
assume $justification\ m \in \Sigma\text{-}i\ (V, C, \varepsilon)\ n$
assume $est\ m \in \varepsilon\ (justification\ m)$
assume $m \in justification\ m$
have $m \in M\text{-}i\ (V, C, \varepsilon)\ (n - 1)$
by (*smt* $M\text{-}i.simps\ One\text{-}nat\text{-}def\ Params.\Sigma i\text{-}subset\text{-}Mi\ Pow\text{-}iff\ Suc\text{-}pred\ \langle est\ m \in C \rangle \langle est\ m \in \varepsilon\ (justification\ m) \rangle \langle justification\ m \in \Sigma\text{-}i\ (V, C, \varepsilon)\ n \rangle \langle m \in justification\ m \rangle \langle sender\ m \in V \rangle add.\textit{right-neutral}\ add.\textit{Suc-right}\ diff\text{-}is\text{-}0\text{-}eq'\ diff\text{-}le\text{-}self\ diff\text{-}zero\ mem\text{-}Collect\text{-}eq\ not\text{-}gr0\ subsetCE$)
then have $justification\ m \in \Sigma\text{-}i\ (V, C, \varepsilon)\ (n - 1)$
using $M\text{-}i.simps$ **by** *blast*
then have $justification\ m \in \Sigma\text{-}i\ (V, C, \varepsilon)\ 0$
apply (*induction* n)
apply *simp*
by (*smt* $M\text{-}i.simps\ One\text{-}nat\text{-}def\ Params.\Sigma i\text{-}subset\text{-}Mi\ Pow\text{-}iff\ Suc\text{-}pred\ \langle m \in justification\ m \rangle add.\textit{right-neutral}\ add.\textit{Suc-right}\ diff\text{-}Suc\text{-}1\ mem\text{-}Collect\text{-}eq\ not\text{-}gr0\ subsetCE\ subsetCE$)
then have $justification\ m \in \{\emptyset\}$
by *simp*
then show *False*
using $\langle m \in justification\ m \rangle$ **by** *blast*
qed

lemma (*in Protocol*) *message-cannot-justify-itself* :
 $(\forall\ m \in M. \neg\ justified\ m\ m)$
proof –
have *irrefl message-justification*
using *irreflexivity-of-justifications* **by** *simp*
then show *?thesis*
by (*simp* $add:\textit{irreflexivity-of-justifications}\ \textit{irrefl-def}\ \textit{message-justification-def}$)
qed

lemma (*in Protocol*) *justification-is-strict-partial-order-on-M* :
 $\textit{strict-partial-order}\ \textit{message-justification}$
apply (*simp* $add:\textit{strict-partial-order-def}$)
by (*simp* $add:\textit{irreflexivity-of-justifications}\ \textit{transitivity-of-justifications}$)

lemma (*in Protocol*) *monotonicity-of-justifications* :
 $\forall\ m\ m'\ \sigma. m \in M \wedge \sigma \in \Sigma \wedge justified\ m'\ m \longrightarrow justification\ m' \subseteq justification\ m$
apply *simp*
by (*meson* $M\text{-}type\ justified\text{-}def\ \textit{message-in-state-is-valid}\ \textit{state-is-in-pow-M-i}$)

lemma (*in Protocol*) *strict-monotonicity-of-justifications* :
 $\forall\ m\ m'\ \sigma. m \in M \wedge \sigma \in \Sigma \wedge justified\ m'\ m \longrightarrow justification\ m' \subset justification\ m$
by (*metis* $M\text{-}type\ \textit{message-cannot-justify-itself}\ \textit{justified-def}\ \textit{message-in-state-is-valid}\ \textit{monotonicity-of-justifications}\ \textit{psubsetI}$)

lemma (in *Protocol*) *justification-implies-different-messages* :
 $\forall m m'. m \in M \wedge m' \in M \longrightarrow \text{justified } m' m \longrightarrow m \neq m'$
 using *message-cannot-justify-itself* by auto

lemma (in *Protocol*) *only-valid-message-is-justified* :
 $\forall m \in M. \forall m'. \text{justified } m' m \longrightarrow m' \in M$
 apply (simp add: *justified-def*)
 using *Params.M-type message-in-state-is-valid* by blast

lemma (in *Protocol*) *justified-message-exists-in-M-i-n-minus-1* :
 $\forall n m m'. n \in \mathbb{N}$
 $\longrightarrow \text{justified } m' m$
 $\longrightarrow m \in M\text{-}i (V, C, \varepsilon) n$
 $\longrightarrow m' \in M\text{-}i (V, C, \varepsilon) (n - 1)$

proof –

have $\forall n m m'. \text{justified } m' m$
 $\longrightarrow m \in M\text{-}i (V, C, \varepsilon) n$
 $\longrightarrow m \in M \wedge m' \in M$
 $\longrightarrow m' \in M\text{-}i (V, C, \varepsilon) (n - 1)$
 apply (rule, rule, rule, rule, rule, rule)
proof –
 fix $n m m'$
 assume *justified* $m' m$
 assume $m \in M\text{-}i (V, C, \varepsilon) n$
 assume $m \in M \wedge m' \in M$
 then have *justification* $m \in \Sigma\text{-}i (V, C, \varepsilon) n$
 using *M-i.simps* $\langle m \in M\text{-}i (V, C, \varepsilon) n \rangle$ by blast
 then have *justification* $m \in \text{Pow } (M\text{-}i (V, C, \varepsilon) (n - 1))$
 by (metis (no-types, lifting) *Suc-diff-Suc* $\Sigma\text{-}i.\text{simps}(1)$ $\Sigma\text{-}i.\text{subset-Mi}$ $\langle \text{justified } m' m \rangle$ *add-leE* *diff-add* *diff-le-self* *empty-iff* *justified-def* *neq0-conv* *plus-1-eq-Suc* *singletonD* *subsetCE*)
 show $m' \in M\text{-}i (V, C, \varepsilon) (n - 1)$
 using $\langle \text{justification } m \in \text{Pow } (M\text{-}i (V, C, \varepsilon) (n - 1)) \rangle \langle \text{justified } m' m \rangle$
justified-def by auto
qed
 then show ?thesis
 by (metis (no-types, lifting) *M-def* *UN-I* *only-valid-message-is-justified*)
qed

lemma (in *Protocol*) *monotonicity-of-card-of-justification* :
 $\forall m m'. m \in M$
 $\longrightarrow \text{justified } m' m$
 $\longrightarrow \text{card } (\text{justification } m') < \text{card } (\text{justification } m)$
 by (meson *M-type* *Protocol.strict-monotonicity-of-justifications* *Protocol-axioms* *justification-is-finite* *psubset-card-mono*)

lemma (in *Protocol*) *justification-is-well-founded-on-M* :

```

    wfp-on justified M
  proof (rule ccontr)
    assume  $\neg$  wfp-on justified M
    then have  $\exists f. \forall i. f\ i \in M \wedge \text{justified}\ (f\ (\text{Suc}\ i))\ (f\ i)$ 
      by (simp add: wfp-on-def)
    then obtain f where  $\forall i. f\ i \in M \wedge \text{justified}\ (f\ (\text{Suc}\ i))\ (f\ i)$  by auto
    have  $\forall i. \text{card}\ (\text{justification}\ (f\ i)) \leq \text{card}\ (\text{justification}\ (f\ 0)) - i$ 
      apply (rule)
    proof -
      fix i
      have  $\text{card}\ (\text{justification}\ (f\ (\text{Suc}\ i))) < \text{card}\ (\text{justification}\ (f\ i))$ 
      using  $\langle \forall i. f\ i \in M \wedge \text{justified}\ (f\ (\text{Suc}\ i))\ (f\ i) \rangle$  by (simp add: monotonicity-of-card-of-justification)
      show  $\text{card}\ (\text{justification}\ (f\ i)) \leq \text{card}\ (\text{justification}\ (f\ 0)) - i$ 
        apply (induction i)
        apply simp
        using  $\langle \text{card}\ (\text{justification}\ (f\ (\text{Suc}\ i))) < \text{card}\ (\text{justification}\ (f\ i)) \rangle$ 
        by (smt Suc-diff-le  $\langle \forall i. f\ i \in M \wedge \text{justified}\ (f\ (\text{Suc}\ i))\ (f\ i) \rangle$  diff-Suc-Suc
            diff-is-0-eq le-iff-add less-Suc-eq-le less-imp-le monotonicity-of-card-of-justification
            not-less-eq-eq trans-less-add1)
      qed
      then have  $\exists i. i = \text{card}\ (\text{justification}\ (f\ 0)) + \text{Suc}\ 0 \wedge \text{card}\ (\text{justification}\ (f\ i))$ 
 $\leq \text{card}\ (\text{justification}\ (f\ 0)) - i$ 
        by blast
      then show False
        using le-0-eq le-simps(2) linorder-not-le monotonicity-of-card-of-justification
            nat-diff-split order-less-imp-le
        by (metis  $\langle \forall i. f\ i \in M \wedge \text{justified}\ (f\ (\text{Suc}\ i))\ (f\ i) \rangle$  add.right-neutral add-Suc-right)
      qed
end

lemma (in Protocol) subset-of-M-have-minimal-of-justification :
   $\forall S \subseteq M. S \neq \emptyset \longrightarrow (\exists m\text{-min} \in S. \forall m. \text{justified}\ m\ m\text{-min} \longrightarrow m \notin S)$ 
  by (metis justification-is-well-founded-on-M wfp-on-imp-has-min-elt wfp-on-mono)
end

```

2 Latest Message

```

theory LatestMessage

imports Main CBCCasper Libraries/LaTeXsugar

begin

```

definition *later* :: (message * message set) \Rightarrow message set

where

later = $(\lambda(m, \sigma). \{m' \in \sigma. \text{justified } m \ m'\})$

lemma (in *Protocol*) *later-type* :

$\forall \sigma \ m. \sigma \in \text{Pow } M \wedge m \in M \longrightarrow \text{later } (m, \sigma) \subseteq M$

apply (simp add: *later-def*)

by *auto*

lemma (in *Protocol*) *later-type-for-state* :

$\forall \sigma \ m. \sigma \in \Sigma \wedge m \in M \longrightarrow \text{later } (m, \sigma) \subseteq M$

apply (simp add: *later-def*)

using *state-is-subset-of-M* **by** *auto*

definition *from-sender* :: (validator * message set) \Rightarrow message set

where

from-sender = $(\lambda(v, \sigma). \{m \in \sigma. \text{sender } m = v\})$

lemma (in *Protocol*) *from-sender-type* :

$\forall \sigma \ v. \sigma \in \text{Pow } M \wedge v \in V \longrightarrow \text{from-sender } (v, \sigma) \in \text{Pow } M$

apply (simp add: *from-sender-def*)

by *auto*

lemma (in *Protocol*) *from-sender-type-for-state* :

$\forall \sigma \ v. \sigma \in \Sigma \wedge v \in V \longrightarrow \text{from-sender } (v, \sigma) \subseteq M$

apply (simp add: *from-sender-def*)

using *state-is-subset-of-M* **by** *auto*

lemma (in *Protocol*) *messages-from-observed-validator-is-non-empty* :

$\forall \sigma \ v. \sigma \in \Sigma \wedge v \in \text{observed } \sigma \longrightarrow \text{from-sender } (v, \sigma) \neq \emptyset$

apply (simp add: *observed-def* *from-sender-def*)

by *auto*

lemma (in *Protocol*) *messages-from-validator-is-finite* :

$\forall \sigma \ v. \sigma \in \Sigma \wedge v \in V \longrightarrow \text{finite } (\text{from-sender } (v, \sigma))$

by (simp add: *from-sender-def* *state-is-finite*)

definition *from-group* :: (validator set * message set) \Rightarrow state

where

from-group = $(\lambda(v\text{-set}, \sigma). \{m \in \sigma. \text{sender } m \in v\text{-set}\})$

lemma (in *Protocol*) *from-group-type* :

$\forall \sigma \ v. \sigma \in \text{Pow } M \wedge v\text{-set} \subseteq V \longrightarrow \text{from-group } (v\text{-set}, \sigma) \in \text{Pow } M$

apply (simp add: *from-group-def*)

by *auto*

lemma (in *Protocol*) *from-group-type-for-state* :
 $\forall \sigma v. \sigma \in \Sigma \wedge v\text{-set} \subseteq V \longrightarrow \text{from-group } (v\text{-set}, \sigma) \subseteq M$
apply (simp add: *from-group-def*)
using *state-is-subset-of-M* **by** *auto*

definition *later-from* :: (message * validator * message set) \Rightarrow message set
where
 $\text{later-from} = (\lambda(m, v, \sigma). \text{later } (m, \sigma) \cap \text{from-sender } (v, \sigma))$

lemma (in *Protocol*) *later-from-type* :
 $\forall \sigma v m. \sigma \in \text{Pow } M \wedge v \in V \wedge m \in M \longrightarrow \text{later-from } (m, v, \sigma) \in \text{Pow } M$
apply (simp add: *later-from-def*)
using *later-type from-sender-type* **by** *auto*

lemma (in *Protocol*) *later-from-type-for-state* :
 $\forall \sigma v m. \sigma \in \Sigma \wedge v \in V \wedge m \in M \longrightarrow \text{later-from } (m, v, \sigma) \subseteq M$
apply (simp add: *later-from-def*)
using *later-type-for-state from-sender-type-for-state* **by** *auto*

definition *latest-messages* :: message set \Rightarrow (validator \Rightarrow message set)
where
 $\text{latest-messages } \sigma v = \{m \in \text{from-sender } (v, \sigma). \text{later-from } (m, v, \sigma) = \emptyset\}$

lemma (in *Protocol*) *latest-messages-type* :
 $\forall \sigma v. \sigma \in \text{Pow } M \wedge v \in V \longrightarrow \text{latest-messages } \sigma v \in \text{Pow } M$
apply (simp add: *latest-messages-def later-from-def*)
using *from-sender-type* **by** *auto*

lemma (in *Protocol*) *latest-messages-type-for-state* :
 $\forall \sigma v. \sigma \in \Sigma \wedge v \in V \longrightarrow \text{latest-messages } \sigma v \subseteq M$
apply (simp add: *latest-messages-def later-from-def*)
using *from-sender-type-for-state* **by** *auto*

lemma (in *Protocol*) *latest-messages-from-non-observed-validator-is-empty* :
 $\forall \sigma v. \sigma \in \Sigma \wedge v \in V \wedge v \notin \text{observed } \sigma \longrightarrow \text{latest-messages } \sigma v = \emptyset$
by (simp add: *latest-messages-def observed-def later-def from-sender-def*)

definition *observed-non-equivocating-validators* :: state \Rightarrow validator set
where
 $\text{observed-non-equivocating-validators } \sigma = \text{observed } \sigma - \text{equivocating-validators } \sigma$

lemma (in *Protocol*) *observed-non-equivocating-validators-type* :
 $\forall \sigma \in \Sigma. \text{observed-non-equivocating-validators } \sigma \in \text{Pow } V$
apply (simp add: *observed-non-equivocating-validators-def*)
using *observed-type-for-state equivocating-validators-type* **by** *auto*

lemma (in *Protocol*) *justification-is-well-founded-on-messages-from-validator*:
 $\forall \sigma \in \Sigma. (\forall v \in V. \text{wfp-on justified (from-sender (v, } \sigma))})$
using *justification-is-well-founded-on-M from-sender-type-for-state wfp-on-subset*
by *blast*

lemma (in *Protocol*) *justification-is-total-on-messages-from-non-equivocating-validator*:
 $\forall \sigma \in \Sigma. (\forall v \in V. v \notin \text{equivocating-validators } \sigma \longrightarrow \text{Relation.total-on (from-sender (v, } \sigma)) \text{ message-justification})$

proof –
have $\forall m1\ m2\ \sigma\ v. v \in V \wedge \sigma \in \Sigma \wedge \{m1, m2\} \subseteq \text{from-sender (v, } \sigma) \longrightarrow \text{sender } m1 = \text{sender } m2$
by (*simp add: from-sender-def*)
then have $\forall \sigma \in \Sigma. (\forall v \in V. v \notin \text{equivocating-validators } \sigma \longrightarrow (\forall m1\ m2. \{m1, m2\} \subseteq \text{from-sender (v, } \sigma) \longrightarrow m1 = m2 \vee \text{justified } m1\ m2 \vee \text{justified } m2\ m1))$
apply (*simp add: equivocating-validators-def is-equivocating-def equivocation-def from-sender-def observed-def*)
by *blast*
then show *?thesis*
apply (*simp add: Relation.total-on-def message-justification-def*)
using *from-sender-type-for-state* **by** *blast*
qed

lemma (in *Protocol*) *justification-is-strict-linear-order-on-messages-from-non-equivocating-validator*:
 $\forall \sigma \in \Sigma. (\forall v \in V. v \notin \text{equivocating-validators } \sigma \longrightarrow \text{strict-linear-order-on (from-sender (v, } \sigma)) \text{ message-justification})$
by (*simp add: strict-linear-order-on-def justification-is-total-on-messages-from-non-equivocating-validator*
irreflexivity-of-justifications transitivity-of-justifications)

lemma (in *Protocol*) *justification-is-strict-well-order-on-messages-from-non-equivocating-validator*:
 $\forall \sigma \in \Sigma. (\forall v \in V. v \notin \text{equivocating-validators } \sigma \longrightarrow \text{strict-linear-order-on (from-sender (v, } \sigma)) \text{ message-justification } \wedge \text{wfp-on justified (from-sender (v, } \sigma))})$
using *justification-is-well-founded-on-messages-from-validator*
justification-is-strict-linear-order-on-messages-from-non-equivocating-validator
by *blast*

lemma (in *Protocol*) *latest-message-is-maximal-element-of-justification* :
 $\forall \sigma\ v. \sigma \in \Sigma \wedge v \in V \longrightarrow \text{latest-messages } \sigma\ v = \{m. \text{maximal-on (from-sender (v, } \sigma)) \text{ message-justification } m\}$
apply (*simp add: latest-messages-def later-from-def later-def message-justification-def maximal-on-def*)
using *from-sender-type-for-state* **apply** *auto*
apply (*metis (no-types, lifting) IntI empty-iff from-sender-def mem-Collect-eq prod.simps(2)*)

by *blast*

lemma (in *Protocol*) *observed-non-equivocating-validators-have-one-latest-message*:
 $\forall \sigma \in \Sigma. (\forall v \in \text{observed-non-equivocating-validators } \sigma. \text{is-singleton } (\text{latest-messages } \sigma \ v))$
apply (*simp add: observed-non-equivocating-validators-def*)
proof –
have $\forall \sigma \in \Sigma. (\forall v \in \text{observed } \sigma - \text{equivocating-validators } \sigma. \text{is-singleton } \{m. \text{maximal-on } (\text{from-sender } (v, \sigma)) \text{ message-justification } m\})$
using
messages-from-observed-validator-is-non-empty
messages-from-validator-is-finite
observed-type-for-state
equivocating-validators-def
justification-is-strict-linear-order-on-messages-from-non-equivocating-validator
strict-linear-order-on-finite-non-empty-set-has-one-maximum
maximal-and-maximum-coincide-for-strict-linear-order
by (*smt Collect-cong DiffD1 DiffD2 set-mp*)
then show $\forall \sigma \in \Sigma. \forall v \in \text{observed } \sigma - \text{equivocating-validators } \sigma. \text{is-singleton } (\text{latest-messages } \sigma \ v)$
using *latest-message-is-maximal-element-of-justification*
observed-non-equivocating-validators-def observed-non-equivocating-validators-type
by *fastforce*
qed

definition *latest-estimates* :: *state* \Rightarrow *validator* \Rightarrow *consensus-value set*
where

latest-estimates $\sigma \ v = \{\text{est } m \mid m. m \in \text{latest-messages } \sigma \ v\}$

lemma (in *Protocol*) *latest-estimates-type* :
 $\forall \sigma \ v. \sigma \in \Sigma \wedge v \in V \longrightarrow \text{latest-estimates } \sigma \ v \subseteq C$
using *M-type Protocol.latest-messages-type-for-state Protocol-axioms latest-estimates-def*
by *fastforce*

lemma (in *Protocol*) *latest-estimates-from-non-observed-validator-is-empty* :
 $\forall \sigma \ v. \sigma \in \Sigma \wedge v \in V \wedge v \notin \text{observed } \sigma \longrightarrow \text{latest-estimates } \sigma \ v = \emptyset$
using *latest-estimates-def latest-messages-from-non-observed-validator-is-empty*
by *auto*

definition *latest-messages-from-non-equivocating-validators* :: *state* \Rightarrow *validator*
 \Rightarrow *message set*

where

latest-messages-from-non-equivocating-validators σ v = (if *is-equivocating* σ v
then \emptyset else *latest-messages* σ v)

lemma (in *Protocol*) *latest-messages-from-non-equivocating-validators-type* :

$\forall \sigma v. \sigma \in \Sigma \wedge v \in V \longrightarrow \text{latest-messages-from-non-equivocating-validators } \sigma v$
 $\subseteq M$

by (*simp add: latest-messages-type-for-state latest-messages-from-non-equivocating-validators-def*)

definition *latest-estimates-from-non-equivocating-validators* :: *state* \Rightarrow *validator*
 \Rightarrow *consensus-value set*

where

latest-estimates-from-non-equivocating-validators σ v = {*est* m | $m. m \in$
latest-messages-from-non-equivocating-validators σ v }

lemma (in *Protocol*) *latest-estimates-from-non-equivocating-validators-type* :

$\forall \sigma v. \sigma \in \Sigma \wedge v \in V \longrightarrow \text{latest-estimates-from-non-equivocating-validators } \sigma v$
 $\in \text{Pow } C$

using *Protocol.latest-estimates-type Protocol-axioms latest-estimates-def latest-estimates-from-non-equivocating-validators-def* **by** *auto*

lemma (in *Protocol*) *latest-estimates-from-non-equivocating-validators-from-non-observed-validator-is-empty*
:

$\forall \sigma v. \sigma \in \Sigma \wedge v \in V \wedge v \notin \text{observed } \sigma \longrightarrow \text{latest-estimates-from-non-equivocating-validators}$
 $\sigma v = \emptyset$

by (*simp add: latest-estimates-from-non-equivocating-validators-def latest-messages-from-non-equivocating-validators-def latest-messages-from-non-observed-validator-is-empty*)

end

theory *StateTransition*

imports *Main CBCCasper*

begin

definition (in *Params*) *state-transition* :: *state rel*

where

state-transition = {($\sigma 1, \sigma 2$). { $\sigma 1, \sigma 2$ } $\subseteq \Sigma \wedge \text{is-future-state}(\sigma 1, \sigma 2)$ }

lemma (in *Params*) *reflexivity-of-state-transition* :
reft-on Σ *state-transition*
apply (simp add: *state-transition-def reft-on-def*)
by *auto*

lemma (in *Params*) *transitivity-of-state-transition* :
trans *state-transition*
apply (simp add: *state-transition-def trans-def*)
by *auto*

lemma (in *Params*) *state-transition-is-preorder* :
preorder-on Σ *state-transition*
by (simp add: *preorder-on-def reflexivity-of-state-transition transitivity-of-state-transition*)

lemma (in *Params*) *antisymmetry-of-state-transition* :
antisym *state-transition*
apply (simp add: *state-transition-def antisym-def*)
by *auto*

lemma (in *Params*) *state-transition-is-partial-order* :
partial-order-on Σ *state-transition*
by (simp add: *partial-order-on-def state-transition-is-preorder antisymmetry-of-state-transition*)

definition (in *Protocol*) *minimal-transitions* :: (state * state) set
where
minimal-transitions $\equiv \{(\sigma, \sigma') \mid \sigma \sigma'. \sigma \in \Sigma t \wedge \sigma' \in \Sigma t \wedge \text{is-future-state } (\sigma, \sigma') \wedge \sigma \neq \sigma' \wedge (\nexists \sigma''. \sigma'' \in \Sigma \wedge \text{is-future-state } (\sigma, \sigma'') \wedge \text{is-future-state } (\sigma'', \sigma') \wedge \sigma \neq \sigma'' \wedge \sigma'' \neq \sigma')\}$

definition *immediately-next-message* **where**
immediately-next-message = $(\lambda(\sigma, m). \text{justification } m \subseteq \sigma \wedge m \notin \sigma)$

lemma (in *Protocol*) *state-transition-by-immediately-next-message-of-same-depth-non-zero*:

$\forall n \geq 1. \forall \sigma \in \Sigma\text{-}i(V, C, \varepsilon) n. \forall m \in M\text{-}i(V, C, \varepsilon) n. \text{immediately-next-message } (\sigma, m) \longrightarrow \sigma \cup \{m\} \in \Sigma\text{-}i(V, C, \varepsilon) (n+1)$

apply (*rule, rule, rule, rule, rule*)

proof–

fix $n \sigma m$

assume $1 \leq n \sigma \in \Sigma\text{-}i(V, C, \varepsilon) n m \in M\text{-}i(V, C, \varepsilon) n \text{immediately-next-message } (\sigma, m)$

have $\exists n'. n = \text{Suc } n'$

using $\langle 1 \leq n \rangle \text{old.nat.exhaust}$ **by** *auto*

hence *si*: $\Sigma\text{-}i(V, C, \varepsilon) n = \{\sigma \in \text{Pow } (M\text{-}i(V, C, \varepsilon) (n - 1)). \text{finite } \sigma \wedge (\forall m. m \in \sigma \longrightarrow \text{justification } m \subseteq \sigma)\}$

by force

hence $\Sigma\text{-}i(V, C, \varepsilon)(n+1) = \{\sigma \in \text{Pow}(M\text{-}i(V, C, \varepsilon) n). \text{finite } \sigma \wedge (\forall m. m \in \sigma \longrightarrow \text{justification } m \subseteq \sigma)\}$
 by force

have $\text{justification } m \subseteq \sigma$
 using immediately-next-message-def
 by (metis (no-types, lifting) immediately-next-message (σ, m) case-prod-conv)
 hence $\text{justification } m \subseteq \sigma \cup \{m\}$
 by blast

moreover have $\bigwedge m'. \text{finite } \sigma \wedge m' \in \sigma \implies \text{justification } m' \subseteq \sigma$
 using $\langle \sigma \in \Sigma\text{-}i(V, C, \varepsilon) n \rangle$ si by blast
 hence $\bigwedge m'. \text{finite } \sigma \wedge m' \in \sigma \implies \text{justification } m' \subseteq \sigma \cup \{m\}$
 by auto

ultimately have $\bigwedge m'. m' \in \sigma \cup \{m\} \implies \text{justification } m \subseteq \sigma$
 using $\langle \text{justification } m \subseteq \sigma \rangle$ by blast

have $\{m\} \in \text{Pow}(M\text{-}i(V, C, \varepsilon) n)$
 using $\langle m \in M\text{-}i(V, C, \varepsilon) n \rangle$ by auto
 moreover have $\sigma \in \text{Pow}(M\text{-}i(V, C, \varepsilon)(n-1))$
 using $\langle \sigma \in \Sigma\text{-}i(V, C, \varepsilon) n \rangle$ si by auto
 hence $\sigma \in \text{Pow}(M\text{-}i(V, C, \varepsilon) n)$
 using Mi-monotonic
 by (metis (full-types) PowD PowI Suc-eq-plus1 $\exists n'. n = \text{Suc } n'$ diff-Suc-1 subset-iff)

ultimately have $\sigma \cup \{m\} \in \text{Pow}(M\text{-}i(V, C, \varepsilon) n)$
 by blast

show $\sigma \cup \{m\} \in \Sigma\text{-}i(V, C, \varepsilon)(n+1)$
 using $\langle \bigwedge m'. \text{finite } \sigma \wedge m' \in \sigma \implies \text{justification } m' \subseteq \sigma \cup \{m\} \rangle$ $\langle \sigma \cup \{m\} \in \text{Pow}(M\text{-}i(V, C, \varepsilon) n) \rangle$ $\langle \text{justification } m \subseteq \sigma \cup \{m\} \rangle$
 $\langle \sigma \in \Sigma\text{-}i(V, C, \varepsilon) n \rangle$ si by auto

qed

lemma (in Protocol) state-transition-by-immediately-next-message-of-same-depth:

$\forall \sigma \in \Sigma\text{-}i(V, C, \varepsilon) n. \forall m \in M\text{-}i(V, C, \varepsilon) n. \text{immediately-next-message}(\sigma, m) \longrightarrow \sigma \cup \{m\} \in \Sigma\text{-}i(V, C, \varepsilon)(n+1)$
 apply (cases n)
 apply auto[1]
 using state-transition-by-immediately-next-message-of-same-depth-non-zero
 by (metis le-add1 plus-1-eq-Suc)

lemma (in Params) past-state-exists-in-same-depth :

$\forall \sigma \sigma'. \sigma' \in \Sigma\text{-}i(V, C, \varepsilon) n \longrightarrow \sigma \subseteq \sigma' \longrightarrow \sigma \in \Sigma \longrightarrow \sigma \in \Sigma\text{-}i(V, C, \varepsilon) n$
 apply (rule, rule, rule, rule, rule)
 proof (cases n)
 case 0

```

  show  $\bigwedge \sigma \sigma'. \sigma' \in \Sigma\text{-}i(V, C, \varepsilon) \ n \implies \sigma \subseteq \sigma' \implies \sigma \in \Sigma \implies n = 0 \implies \sigma \in \Sigma\text{-}i(V, C, \varepsilon) \ n$ 
  by auto
next
  case (Suc nat)
  show  $\bigwedge \sigma \sigma' \text{ nat}. \sigma' \in \Sigma\text{-}i(V, C, \varepsilon) \ n \implies \sigma \subseteq \sigma' \implies \sigma \in \Sigma \implies n = \text{Suc nat} \implies \sigma \in \Sigma\text{-}i(V, C, \varepsilon) \ n$ 
  proof -
    fix  $\sigma \sigma'$ 
    assume  $\sigma' \in \Sigma\text{-}i(V, C, \varepsilon) \ n$ 
    and  $\sigma \subseteq \sigma'$ 
    and  $\sigma \in \Sigma$ 
    have  $n > 0$ 
    by (simp add: Suc)
    have finite  $\sigma \wedge (\forall m. m \in \sigma \longrightarrow \text{justification } m \subseteq \sigma)$ 
    using  $\langle \sigma \in \Sigma \rangle \text{ state-is-finite state-is-in-pow-M-i}$  by blast
    moreover have  $\sigma \in \text{Pow } (M\text{-}i(V, C, \varepsilon) (n - 1))$ 
    using  $\langle \sigma \subseteq \sigma' \rangle$ 
    by (smt Pow-iff Suc-eq-plus1  $\Sigma\text{-}i\text{-monotonic}$   $\Sigma\text{-}i\text{-subset-Mi}$   $\langle \sigma' \in \Sigma\text{-}i(V, C, \varepsilon) \ n \rangle \text{ add-diff-cancel-left' add-eq-if diff-is-0-eq diff-le-self plus-1-eq-Suc subset-iff}$ )
    ultimately have  $\sigma \in \{\sigma \in \text{Pow } (M\text{-}i(V, C, \varepsilon) (n - 1)). \text{finite } \sigma \wedge (\forall m. m \in \sigma \longrightarrow \text{justification } m \subseteq \sigma)\}$ 
    by blast
    then show  $\sigma \in \Sigma\text{-}i(V, C, \varepsilon) \ n$ 
    by (simp add: Suc)
  qed
qed

```

lemma (in Protocol) *immediately-next-message-exists-in-same-depth*:
 $\forall \sigma \in \Sigma. \forall m \in M. \text{immediately-next-message } (\sigma, m) \longrightarrow (\exists n \in \mathbb{N}. \sigma \in \Sigma\text{-}i(V, C, \varepsilon) \ n \wedge m \in M\text{-}i(V, C, \varepsilon) \ n)$
 apply (simp add: immediately-next-message-def M-def $\Sigma\text{-}def$)
 using past-state-exists-in-same-depth
 using $\Sigma\text{-}i\text{-is-subset-of-}\Sigma$ by blast

lemma (in Protocol) *state-transition-by-immediately-next-message*:
 $\forall \sigma \in \Sigma. \forall m \in M. \text{immediately-next-message } (\sigma, m) \longrightarrow \sigma \cup \{m\} \in \Sigma$
 apply (rule, rule, rule)
 proof -
 fix $\sigma \ m$
 assume $\sigma \in \Sigma$
 and $m \in M$
 and immediately-next-message (σ, m)
 then have $(\exists n \in \mathbb{N}. \sigma \in \Sigma\text{-}i(V, C, \varepsilon) \ n \wedge m \in M\text{-}i(V, C, \varepsilon) \ n)$
 using immediately-next-message-exists-in-same-depth $\langle \sigma \in \Sigma \rangle \langle m \in M \rangle$
 by blast
 then have $\exists n \in \mathbb{N}. \sigma \cup \{m\} \in \Sigma\text{-}i(V, C, \varepsilon) \ (n + 1)$
 using state-transition-by-immediately-next-message-of-same-depth
 using $\langle \text{immediately-next-message } (\sigma, m) \rangle$ by blast

show $\sigma \cup \{m\} \in \Sigma$
apply (*simp add: Σ -def*)
by (*metis Nats-1 Nats-add Un-insert-right $\langle \exists n \in \mathbb{N}. \sigma \cup \{m\} \in \Sigma-i (V, C, \varepsilon)$*
 $(n + 1)$ sup-bot.right-neutral)
qed

lemma (*in Protocol*) *state-transition-imps-immediately-next-message*:
 $\forall \sigma \in \Sigma. \forall m \in M. \sigma \cup \{m\} \in \Sigma \wedge m \notin \sigma \longrightarrow \text{immediately-next-message } (\sigma, m)$
proof –
have $\forall \sigma \in \Sigma. \forall m \in M. \sigma \cup \{m\} \in \Sigma \longrightarrow (\forall m' \in \sigma \cup \{m\}. \text{justification } m' \subseteq \sigma \cup \{m\})$
using *state-is-in-pow-M-i* **by** *blast*
then have $\forall \sigma \in \Sigma. \forall m \in M. \sigma \cup \{m\} \in \Sigma \longrightarrow \text{justification } m \subseteq \sigma \cup \{m\}$
by *auto*
then have $\forall \sigma \in \Sigma. \forall m \in M. \sigma \cup \{m\} \in \Sigma \wedge m \notin \sigma \longrightarrow \text{justification } m \subseteq \sigma$
using *justification-implies-different-messages justified-def* **by** *fastforce*
then show *?thesis*
by (*simp add: immediately-next-message-def*)
qed

lemma (*in Protocol*) *state-transition-only-made-by-immediately-next-message*:
 $\forall \sigma \in \Sigma. \forall m \in M. \sigma \cup \{m\} \in \Sigma \wedge m \notin \sigma \longleftrightarrow \text{immediately-next-message } (\sigma, m)$
using *state-transition-imps-immediately-next-message state-transition-by-immediately-next-message*
apply (*simp add: immediately-next-message-def*)
by *blast*

lemma (*in Protocol*) *state-transition-is-immediately-next-message*:
 $\forall \sigma \in \Sigma. \forall m \in M. \sigma \cup \{m\} \in \Sigma \longleftrightarrow \text{justification } m \subseteq \sigma$
using *state-transition-only-made-by-immediately-next-message*
apply (*simp add: immediately-next-message-def*)
using *insert-Diff state-is-in-pow-M-i* **by** *fastforce*

lemma (*in Protocol*) *strict-subset-of-state-have-immediately-next-messages*:
 $\forall \sigma \in \Sigma. \forall \sigma'. \sigma' \subset \sigma \longrightarrow (\exists m \in \sigma - \sigma'. \text{immediately-next-message } (\sigma', m))$
apply (*simp add: immediately-next-message-def*)
apply (*rule, rule, rule*)
proof –
fix $\sigma \sigma'$
assume $\sigma \in \Sigma$
assume $\sigma' \subset \sigma$
show $\exists m \in \sigma - \sigma'. \text{justification } m \subseteq \sigma'$
proof (*rule ccontr*)
assume $\neg (\exists m \in \sigma - \sigma'. \text{justification } m \subseteq \sigma')$
then have $\forall m \in \sigma - \sigma'. \exists m' \in \text{justification } m. m' \in \sigma - \sigma'$
using $\langle \neg (\exists m \in \sigma - \sigma'. \text{justification } m \subseteq \sigma') \rangle$ *state-is-in-pow-M-i $\langle \sigma' \subset \sigma \rangle$*
by (*metis Diff-iff $\langle \sigma \in \Sigma \rangle$ subset-eq*)
then have $\forall m \in \sigma - \sigma'. \exists m'. \text{justified } m' m \wedge m' \in \sigma - \sigma'$
using *justified-def* **by** *auto*
then have $\forall m \in \sigma - \sigma'. \exists m'. \text{justified } m' m \wedge m' \in \sigma - \sigma' \wedge m \neq m'$

```

    using justification-implies-different-messages state-difference-is-valid-message
    message-in-state-is-valid  $\langle \sigma' \subset \sigma \rangle$ 
    by (meson DiffD1  $\langle \sigma \in \Sigma \rangle$ )
  have  $\sigma - \sigma' \subseteq M$ 
    using  $\langle \sigma \in \Sigma \rangle \langle \sigma' \subset \sigma \rangle$  state-is-subset-of-M by auto
  then have  $\exists m\text{-min} \in \sigma - \sigma'. \forall m. \text{justified } m \ m\text{-min} \longrightarrow m \notin \sigma - \sigma'$ 
    using subset-of-M-have-minimal-of-justification  $\langle \sigma' \subset \sigma \rangle$ 
    by blast
  then show False
    using  $\langle \forall m \in \sigma - \sigma'. \exists m'. \text{justified } m' \ m \wedge m' \in \sigma - \sigma' \rangle$  by blast
qed
qed

lemma (in Protocol) union-of-two-states-is-state :
   $\forall \sigma 1 \in \Sigma. \forall \sigma 2 \in \Sigma. (\sigma 1 \cup \sigma 2) \in \Sigma$ 
  apply (rule, rule)
proof -
  fix  $\sigma 1 \ \sigma 2$ 
  assume  $\sigma 1 \in \Sigma$  and  $\sigma 2 \in \Sigma$ 
  show  $\sigma 1 \cup \sigma 2 \in \Sigma$ 
  proof (cases  $\sigma 1 \subseteq \sigma 2$ )
    case True
    then show ?thesis
      by (simp add: Un-absorb1  $\langle \sigma 2 \in \Sigma \rangle$ )
  next
    case False
    then have  $\neg \sigma 1 \subseteq \sigma 2$  by simp
    have  $\forall \sigma \in \Sigma. \forall \sigma' \in \Sigma. \neg \sigma \subseteq \sigma' \longrightarrow (\exists m \in \sigma - (\sigma \cap \sigma'). \text{immediately-next-message}(\sigma \cap \sigma', m))$ 
      by (metis Int-subset-iff psubsetI strict-subset-of-state-have-immediately-next-messages subsetI)
    then have  $\forall \sigma \in \Sigma. \forall \sigma' \in \Sigma. \neg \sigma \subseteq \sigma' \longrightarrow (\exists m \in \sigma - (\sigma \cap \sigma'). \text{immediately-next-message}(\sigma', m))$ 
      by (metis Int-subset-iff psubsetI strict-subset-of-state-have-immediately-next-messages subsetI)
    immediately-next-message( $\sigma', m$ )
    apply (simp add: immediately-next-message-def)
    by blast
    then have  $\forall \sigma \in \Sigma. \forall \sigma' \in \Sigma. \neg \sigma \subseteq \sigma' \longrightarrow (\exists m \in \sigma - \sigma'. \sigma' \cup \{m\} \in \Sigma)$ 
      using state-transition-by-immediately-next-message
      by (metis DiffD1 DiffD2 DiffI IntI message-in-state-is-valid)
    have  $\forall \sigma \in \Sigma. \forall \sigma' \in \Sigma. \neg \sigma \subseteq \sigma' \longrightarrow \sigma \cup \sigma' \in \Sigma$ 
    proof -
      have  $\forall \sigma \in \Sigma. \forall \sigma' \in \Sigma. \neg \sigma \subseteq \sigma' \longrightarrow \text{card } (\sigma - \sigma') > 0$ 
        by (meson Diff-eq-empty-iff card-0-eq finite-Diff gr0I state-is-finite)
      have  $\forall n. \forall \sigma \in \Sigma. \forall \sigma' \in \Sigma. \neg \sigma \subseteq \sigma' \wedge \text{Suc } n = \text{card } (\sigma - \sigma') \longrightarrow \sigma \cup \sigma' \in \Sigma$ 
      apply (rule)
    proof -
      fix n
      show  $\forall \sigma \in \Sigma. \forall \sigma' \in \Sigma. \neg \sigma \subseteq \sigma' \wedge \text{Suc } n = \text{card } (\sigma - \sigma') \longrightarrow \sigma \cup \sigma' \in \Sigma$ 
      apply (induction n)

```



```

    apply (rule, rule, rule)
  proof -
    fix  $\sigma \sigma'$ 
    assume  $\sigma \in \Sigma$  and  $\sigma' \in \Sigma$  and  $\neg \sigma \subseteq \sigma' \wedge \text{Suc } 0 = \text{card } (\sigma - \sigma')$ 
    then have is-singleton  $(\sigma - \sigma')$ 
      by (simp add: is-singleton-altdef)
    then have  $\{ \text{the-elem } (\sigma - \sigma') \} \cup \sigma' \in \Sigma$ 
      using  $\langle \forall \sigma \in \Sigma. \forall \sigma' \in \Sigma. \neg \sigma \subseteq \sigma' \longrightarrow (\exists m \in \sigma - \sigma'. \sigma' \cup \{m\} \in \Sigma) \rangle \langle \sigma \in \Sigma \rangle \langle \sigma' \in \Sigma \rangle$ 
      by (metis Un-commute  $\langle \neg \sigma \subseteq \sigma' \wedge \text{Suc } 0 = \text{card } (\sigma - \sigma') \rangle$ 
is-singleton-the-elem singletonD)
    then show  $\sigma \cup \sigma' \in \Sigma$ 
      by (metis Un-Diff-cancel2 is-singleton  $(\sigma - \sigma')$  is-singleton-the-elem)

  next
    show  $\bigwedge n. \forall \sigma \in \Sigma. \forall \sigma' \in \Sigma. \neg \sigma \subseteq \sigma' \wedge \text{Suc } n = \text{card } (\sigma - \sigma') \longrightarrow \sigma \cup \sigma' \in \Sigma \implies \forall \sigma \in \Sigma. \forall \sigma' \in \Sigma. \neg \sigma \subseteq \sigma' \wedge \text{Suc } (\text{Suc } n) = \text{card } (\sigma - \sigma') \longrightarrow \sigma \cup \sigma' \in \Sigma$ 
      apply (rule, rule, rule)
    proof -
      fix  $n \sigma \sigma'$ 
      assume  $\forall \sigma \in \Sigma. \forall \sigma' \in \Sigma. \neg \sigma \subseteq \sigma' \wedge \text{Suc } n = \text{card } (\sigma - \sigma') \longrightarrow \sigma \cup \sigma' \in \Sigma$  and  $\sigma \in \Sigma$  and  $\sigma' \in \Sigma$  and  $\neg \sigma \subseteq \sigma' \wedge \text{Suc } (\text{Suc } n) = \text{card } (\sigma - \sigma')$ 
      have  $\forall m \in \sigma - \sigma'. \neg \sigma \subseteq \sigma' \cup \{m\} \wedge \text{Suc } n = \text{card } (\sigma - (\sigma' \cup \{m\}))$ 
        using  $\langle \neg \sigma \subseteq \sigma' \wedge \text{Suc } (\text{Suc } n) = \text{card } (\sigma - \sigma') \rangle$ 
        by (metis Diff-eq-empty-iff Diff-insert Un-insert-right  $\langle \sigma \in \Sigma \rangle$ 
add-diff-cancel-left' card-0-eq card-Suc-Diff1 finite-Diff nat.simps(3) plus-1-eq-Suc
state-is-finite sup-bot.right-neutral)
      have  $\exists m \in \sigma - \sigma'. \sigma' \cup \{m\} \in \Sigma$ 
        using  $\langle \forall \sigma \in \Sigma. \forall \sigma' \in \Sigma. \neg \sigma \subseteq \sigma' \longrightarrow (\exists m \in \sigma - \sigma'. \sigma' \cup \{m\} \in \Sigma) \rangle \langle \sigma \in \Sigma \rangle \langle \sigma' \in \Sigma \rangle \langle \neg \sigma \subseteq \sigma' \wedge \text{Suc } (\text{Suc } n) = \text{card } (\sigma - \sigma') \rangle$ 
        by blast
      then have  $\exists m \in \sigma - \sigma'. \sigma' \cup \{m\} \in \Sigma \wedge \neg \sigma \subseteq \sigma' \cup \{m\} \wedge \text{Suc } n = \text{card } (\sigma - (\sigma' \cup \{m\}))$ 
        using  $\langle \forall m \in \sigma - \sigma'. \neg \sigma \subseteq \sigma' \cup \{m\} \wedge \text{Suc } n = \text{card } (\sigma - (\sigma' \cup \{m\})) \rangle$ 
        by simp
      then show  $\sigma \cup \sigma' \in \Sigma$ 
        using  $\langle \forall \sigma \in \Sigma. \forall \sigma' \in \Sigma. \neg \sigma \subseteq \sigma' \wedge \text{Suc } n = \text{card } (\sigma - \sigma') \longrightarrow \sigma \cup \sigma' \in \Sigma \rangle$ 
        by (smt Un-Diff-cancel Un-commute Un-insert-right  $\langle \sigma \in \Sigma \rangle$ 
insert-absorb2 mk-disjoint-insert sup-bot.right-neutral)
    qed
  qed
  qed
  then show ?thesis
    by (meson  $\langle \forall \sigma \in \Sigma. \forall \sigma' \in \Sigma. \neg \sigma \subseteq \sigma' \longrightarrow (\exists m \in \sigma - \sigma'. \sigma' \cup \{m\} \in \Sigma) \rangle$ 
card-Suc-Diff1 finite-Diff state-is-finite)
  qed
  then show ?thesis

```

using *False* $\langle \sigma 1 \in \Sigma \rangle \langle \sigma 2 \in \Sigma \rangle$ by *blast*
 qed
 qed

lemma (in *Protocol*) *union-of-finite-set-of-states-is-state* :

$\forall \sigma\text{-set} \subseteq \Sigma. \text{finite } \sigma\text{-set} \longrightarrow \bigcup \sigma\text{-set} \in \Sigma$
 apply *auto*
proof –
 have $\forall n. \forall \sigma\text{-set} \subseteq \Sigma. n = \text{card } \sigma\text{-set} \longrightarrow \text{finite } \sigma\text{-set} \longrightarrow \bigcup \sigma\text{-set} \in \Sigma$
 apply (rule)
proof –
 fix *n*
 show $\forall \sigma\text{-set} \subseteq \Sigma. n = \text{card } \sigma\text{-set} \longrightarrow \text{finite } \sigma\text{-set} \longrightarrow \bigcup \sigma\text{-set} \in \Sigma$
 apply (induction *n*)
 apply (rule, rule, rule, rule)
 apply (simp add: *empty-set-exists-in-Σ*)
 apply (rule, rule, rule, rule)
proof –
 fix *n* $\sigma\text{-set}$
 assume $\forall \sigma\text{-set} \subseteq \Sigma. n = \text{card } \sigma\text{-set} \longrightarrow \text{finite } \sigma\text{-set} \longrightarrow \bigcup \sigma\text{-set} \in \Sigma$ and
 $\sigma\text{-set} \subseteq \Sigma$ and $\text{Suc } n = \text{card } \sigma\text{-set}$ and *finite* $\sigma\text{-set}$
 then have $\forall \sigma \in \sigma\text{-set}. \sigma\text{-set} - \{\sigma\} \subseteq \Sigma \wedge \bigcup (\sigma\text{-set} - \{\sigma\}) \in \Sigma$
 using $\langle \sigma\text{-set} \subseteq \Sigma \rangle \langle \text{Suc } n = \text{card } \sigma\text{-set} \rangle \langle \forall \sigma\text{-set} \subseteq \Sigma. n = \text{card } \sigma\text{-set} \longrightarrow$
finite $\sigma\text{-set} \longrightarrow \bigcup \sigma\text{-set} \in \Sigma \rangle$
 by (metis (*mono-tags*, *lifting*) *Suc-inject* *card.remove* *finite-Diff* *insert-Diff* *insert-subset*)
 then have $\forall \sigma \in \sigma\text{-set}. \sigma\text{-set} - \{\sigma\} \subseteq \Sigma \wedge \bigcup (\sigma\text{-set} - \{\sigma\}) \in \Sigma \wedge \bigcup (\sigma\text{-set} - \{\sigma\}) \cup \sigma \in \Sigma$
 using *union-of-two-states-is-state* $\langle \sigma\text{-set} \subseteq \Sigma \rangle$ by *auto*
 then show $\bigcup \sigma\text{-set} \in \Sigma$
 by (metis *Sup-bot-conv*(1) *Sup-insert* *Un-commute* *empty-set-exists-in-Σ* *insert-Diff*)
 qed
 qed
 then show $\bigwedge \sigma\text{-set}. \sigma\text{-set} \subseteq \Sigma \implies \text{finite } \sigma\text{-set} \implies \bigcup \sigma\text{-set} \in \Sigma$
 by *blast*
 qed

lemma (in *Protocol*) *state-differences-have-immediately-next-messages*:

$\forall \sigma \in \Sigma. \forall \sigma' \in \Sigma. \text{is-future-state } (\sigma, \sigma') \wedge \sigma \neq \sigma' \longrightarrow (\exists m \in \sigma' - \sigma. \text{immediately-next-message } (\sigma, m))$
 using *strict-subset-of-state-have-immediately-next-messages*
 by (simp add: *psubsetI*)

lemma *non-empty-non-singleton-implies-two-elements* :

$A \neq \emptyset \implies \neg \text{is-singleton } A \implies \exists a1\ a2. a1 \neq a2 \wedge \{a1, a2\} \subseteq A$
 by (metis *inf.orderI* *inf-bot-left* *insert-subset* *is-singletonI*)

lemma (in *Protocol*) *minimal-transition-implies-recieving-single-message* :

$\forall \sigma \sigma'. (\sigma, \sigma') \in \text{minimal-transitions} \longrightarrow \text{is-singleton } (\sigma' - \sigma)$

proof (rule *ccontr*)

assume $\neg (\forall \sigma \sigma'. (\sigma, \sigma') \in \text{minimal-transitions} \longrightarrow \text{is-singleton } (\sigma' - \sigma))$

then have $\exists \sigma \sigma'. (\sigma, \sigma') \in \text{minimal-transitions} \wedge \neg \text{is-singleton } (\sigma' - \sigma)$

by blast

have $\forall \sigma \sigma'. (\sigma, \sigma') \in \text{minimal-transitions} \longrightarrow$
 $(\nexists \sigma''. \sigma'' \in \Sigma \wedge \text{is-future-state } (\sigma, \sigma'') \wedge \text{is-future-state } (\sigma'', \sigma') \wedge \sigma$
 $\neq \sigma'' \wedge \sigma'' \neq \sigma')$

by (*simp add: minimal-transitions-def*)

have $\forall \sigma \sigma'. (\sigma, \sigma') \in \text{minimal-transitions} \wedge \neg \text{is-singleton } (\sigma' - \sigma)$
 $\longrightarrow (\exists m1 m2. \{m1, m2\} \subseteq M \wedge m1 \in \sigma' - \sigma \wedge m2 \in \sigma' - \sigma \wedge m1 \neq m2 \wedge$
immediately-next-message $(\sigma, m1))$

apply (rule, rule, rule)

proof –

fix $\sigma \sigma'$

assume $(\sigma, \sigma') \in \text{minimal-transitions} \wedge \neg \text{is-singleton } (\sigma' - \sigma)$

then have $\sigma' - \sigma \neq \emptyset$

apply (*simp add: minimal-transitions-def*)

by blast

have $\sigma' \in \Sigma \wedge \sigma \in \Sigma \wedge \text{is-future-state } (\sigma, \sigma')$

using $\langle (\sigma, \sigma') \in \text{minimal-transitions} \wedge \neg \text{is-singleton } (\sigma' - \sigma) \rangle$

by (*simp add: minimal-transitions-def Σ t-def*)

then have $\sigma' - \sigma \subseteq M$

using *state-difference-is-valid-message* **by auto**

then have $\exists m1 m2. \{m1, m2\} \subseteq M \wedge m1 \in \sigma' - \sigma \wedge m2 \in \sigma' - \sigma \wedge m1$
 $\neq m2$

using *non-empty-non-singleton-imps-two-elements*
 $\langle (\sigma, \sigma') \in \text{minimal-transitions} \wedge \neg \text{is-singleton } (\sigma' - \sigma) \rangle \langle \sigma' - \sigma \neq \emptyset \rangle$

by (*metis (full-types) contra-subsetD insert-subset subsetI*)

then show $\exists m1 m2. \{m1, m2\} \subseteq M \wedge m1 \in \sigma' - \sigma \wedge m2 \in \sigma' - \sigma \wedge m1$
 $\neq m2 \wedge \text{immediately-next-message } (\sigma, m1)$

using *state-differences-have-immediately-next-messages*

by (*metis Diff-iff $\langle \sigma' \in \Sigma \wedge \sigma \in \Sigma \wedge \text{is-future-state } (\sigma, \sigma') \rangle$ insert-subset*
message-in-state-is-valid)

qed

have $\forall \sigma \sigma'. (\sigma, \sigma') \in \text{minimal-transitions} \wedge \neg \text{is-singleton } (\sigma' - \sigma) \longrightarrow$
 $(\exists \sigma''. \sigma'' \in \Sigma \wedge \text{is-future-state } (\sigma, \sigma'') \wedge \text{is-future-state } (\sigma'', \sigma') \wedge \sigma$
 $\neq \sigma'' \wedge \sigma'' \neq \sigma')$

apply (rule, rule, rule)

proof –

fix $\sigma \sigma'$

assume $(\sigma, \sigma') \in \text{minimal-transitions} \wedge \neg \text{is-singleton } (\sigma' - \sigma)$

then have $\exists m1 m2. \{m1, m2\} \subseteq M \wedge m1 \in \sigma' - \sigma \wedge m2 \in \sigma' - \sigma \wedge m1 \neq$
 $m2 \wedge \text{immediately-next-message } (\sigma, m1)$

using $\langle \forall \sigma \sigma'. (\sigma, \sigma') \in \text{minimal-transitions} \wedge \neg \text{is-singleton } (\sigma' - \sigma) \longrightarrow$
 $\longrightarrow (\exists m1 m2. \{m1, m2\} \subseteq M \wedge m1 \in \sigma' - \sigma \wedge m2 \in \sigma' - \sigma \wedge m1 \neq m2 \wedge$

immediately-next-message ($\sigma, m1$)
 by *simp*
 then obtain $m1\ m2$ where $\{m1, m2\} \subseteq M \wedge m1 \in \sigma' - \sigma \wedge m2 \in \sigma' - \sigma \wedge m1 \neq m2 \wedge$ *immediately-next-message* ($\sigma, m1$)
 by *auto*
 have $\sigma \in \Sigma \wedge \sigma' \in \Sigma$
 using $\langle (\sigma, \sigma') \in \text{minimal-transitions} \wedge \neg \text{is-singleton} (\sigma' - \sigma) \rangle$
 by (*simp add: minimal-transitions-def* $\Sigma t\text{-def}$)
 then have $\sigma \cup \{m1\} \in \Sigma$
 using $\langle \{m1, m2\} \subseteq M \wedge m1 \in \sigma' - \sigma \wedge m2 \in \sigma' - \sigma \wedge m1 \neq m2 \wedge$ *immediately-next-message* ($\sigma, m1$)
state-transition-by-immediately-next-message
 by *simp*
 have *is-future-state* ($\sigma, \sigma \cup \{m1\}$) \wedge *is-future-state* ($\sigma \cup \{m1\}, \sigma'$)
 using $\langle (\sigma, \sigma') \in \text{minimal-transitions} \wedge \neg \text{is-singleton} (\sigma' - \sigma) \rangle \langle \{m1, m2\} \subseteq M \wedge m1 \in \sigma' - \sigma \wedge m2 \in \sigma' - \sigma \wedge m1 \neq m2 \wedge$ *immediately-next-message* ($\sigma, m1$)
minimal-transitions-def by *auto*
 have $\sigma \neq \sigma \cup \{m1\} \wedge \sigma \cup \{m1\} \neq \sigma'$
 using $\langle \{m1, m2\} \subseteq M \wedge m1 \in \sigma' - \sigma \wedge m2 \in \sigma' - \sigma \wedge m1 \neq m2 \wedge$ *immediately-next-message* ($\sigma, m1$)
 by *auto*
 then show $\exists \sigma''. \sigma'' \in \Sigma \wedge \text{is-future-state} (\sigma, \sigma'') \wedge \text{is-future-state} (\sigma'', \sigma') \wedge \sigma \neq \sigma'' \wedge \sigma'' \neq \sigma'$
 using $\langle \sigma \cup \{m1\} \in \Sigma \rangle \langle \text{is-future-state} (\sigma, \sigma \cup \{m1\}) \wedge \text{is-future-state} (\sigma \cup \{m1\}, \sigma') \rangle$
 by *auto*
 qed
 then show *False*
 using $\langle \forall \sigma \sigma'. (\sigma, \sigma') \in \text{minimal-transitions} \longrightarrow (\nexists \sigma''. \sigma'' \in \Sigma \wedge \text{is-future-state} (\sigma, \sigma'') \wedge \text{is-future-state} (\sigma'', \sigma') \wedge \sigma \neq \sigma'' \wedge \sigma'' \neq \sigma') \rangle \langle \neg (\forall \sigma \sigma'. (\sigma, \sigma') \in \text{minimal-transitions} \longrightarrow \text{is-singleton} (\sigma' - \sigma)) \rangle$ by *blast*
 qed

lemma (in *Protocol*) *minimal-transitions-reconstruction* :
 $\forall \sigma \sigma'. (\sigma, \sigma') \in \text{minimal-transitions} \longrightarrow \sigma \cup \{\text{the-elem} (\sigma' - \sigma)\} = \sigma'$
 apply (*rule, rule, rule*)
 proof –
 fix $\sigma \sigma'$
 assume $(\sigma, \sigma') \in \text{minimal-transitions}$
 then have *is-singleton* ($\sigma' - \sigma$)
 using *minimal-transitions-def* *minimal-transition-implies-recieving-single-message*
 by *auto*
 then have $\sigma \subseteq \sigma'$
 using $\langle (\sigma, \sigma') \in \text{minimal-transitions} \rangle$ *minimal-transitions-def* by *auto*
 then show $\sigma \cup \{\text{the-elem} (\sigma' - \sigma)\} = \sigma'$
 by (*metis* *Diff-partition* $\langle \text{is-singleton} (\sigma' - \sigma) \rangle$ *is-singleton-the-elem*)
 qed
 end

3 Safety Proof

theory *ConsensusSafety*

imports *Main CBCCaspar StateTransition Libraries/LaTeXsugar*

begin

definition (*in Protocol*) *futures* :: *state* \Rightarrow *state set*
where
futures $\sigma = \{\sigma' \in \Sigma t. \text{is-future-state } (\sigma, \sigma')\}$

lemma (*in Protocol*) *monotonic-futures* :
 $\forall \sigma' \sigma. \sigma' \in \Sigma t \wedge \sigma \in \Sigma t$
 $\longrightarrow \sigma' \in \text{futures } \sigma \longleftrightarrow \text{futures } \sigma' \subseteq \text{futures } \sigma$
apply (*simp add: futures-def*) **by** *auto*

theorem (*in Protocol*) *two-party-common-futures* :
 $\forall \sigma 1 \sigma 2. \sigma 1 \in \Sigma t \wedge \sigma 2 \in \Sigma t$
 $\longrightarrow \text{is-faults-lt-threshold } (\sigma 1 \cup \sigma 2)$
 $\longrightarrow \text{futures } \sigma 1 \cap \text{futures } \sigma 2 \neq \emptyset$
apply (*simp add: futures-def Σt -def*) **using** *union-of-two-states-is-state*
by *blast*

theorem (*in Protocol*) *n-party-common-futures* :
 $\forall \sigma\text{-set}. \sigma\text{-set} \subseteq \Sigma t$
 $\longrightarrow \text{finite } \sigma\text{-set}$
 $\longrightarrow \text{is-faults-lt-threshold } (\bigcup \sigma\text{-set})$
 $\longrightarrow \bigcap \{\text{futures } \sigma \mid \sigma. \sigma \in \sigma\text{-set}\} \neq \emptyset$
apply (*simp add: futures-def Σt -def*) **using** *union-of-finite-set-of-states-is-state*
by *blast*

lemma (*in Protocol*) *n-party-common-futures-exists* :
 $\forall \sigma\text{-set}. \sigma\text{-set} \subseteq \Sigma t$
 $\longrightarrow \text{finite } \sigma\text{-set}$
 $\longrightarrow \text{is-faults-lt-threshold } (\bigcup \sigma\text{-set})$
 $\longrightarrow (\exists \sigma \in \Sigma t. \sigma \in \bigcap \{\text{futures } \sigma \mid \sigma. \sigma \in \sigma\text{-set}\})$
apply (*simp add: futures-def Σt -def*) **using** *union-of-finite-set-of-states-is-state*
by *blast*

definition (in *Protocol*) *state-property-is-decided* :: (*state-property* * *state*) \Rightarrow *bool*
where
state-property-is-decided = ($\lambda(p, \sigma). (\forall \sigma' \in \text{futures } \sigma . p \sigma')$)

lemma (in *Protocol*) *forward-consistency* :
 $\forall \sigma' \sigma. \sigma' \in \Sigma t \wedge \sigma \in \Sigma t$
 $\longrightarrow \sigma' \in \text{futures } \sigma$
 $\longrightarrow \text{state-property-is-decided } (p, \sigma)$
 $\longrightarrow \text{state-property-is-decided } (p, \sigma')$
apply (*simp add: futures-def state-property-is-decided-def*)
by *auto*

fun *state-property-not* :: *state-property* \Rightarrow *state-property*
where
state-property-not *p* = ($\lambda\sigma. (\neg p \sigma)$)

lemma (in *Protocol*) *backward-consistency* :
 $\forall \sigma' \sigma. \sigma' \in \Sigma t \wedge \sigma \in \Sigma t$
 $\longrightarrow \sigma' \in \text{futures } \sigma$
 $\longrightarrow \text{state-property-is-decided } (p, \sigma')$
 $\longrightarrow \neg \text{state-property-is-decided } (\text{state-property-not } p, \sigma)$
apply (*simp add: futures-def state-property-is-decided-def*)
by *auto*

theorem (in *Protocol*) *two-party-consensus-safety* :
 $\forall \sigma 1 \sigma 2. \sigma 1 \in \Sigma t \wedge \sigma 2 \in \Sigma t$
 $\longrightarrow \text{is-faults-lt-threshold } (\sigma 1 \cup \sigma 2)$
 $\longrightarrow \neg (\text{state-property-is-decided } (p, \sigma 1) \wedge \text{state-property-is-decided } (\text{state-property-not } p, \sigma 2))$
apply (*simp add: state-property-is-decided-def*)
using *two-party-common-futures*
by (*metis Int-emptyI*)

definition (in *Protocol*) *state-properties-are-inconsistent* :: *state-property set* \Rightarrow *bool*
where
state-properties-are-inconsistent *p-set* = ($\forall \sigma \in \Sigma. \neg (\forall p \in p\text{-set}. p \sigma)$)

definition (in *Protocol*) *state-properties-are-consistent* :: *state-property set* \Rightarrow *bool*
where
state-properties-are-consistent *p-set* = ($\exists \sigma \in \Sigma. \forall p \in p\text{-set}. p \sigma$)

definition (in *Protocol*) *state-property-decisions* :: *state* \Rightarrow *state-property set*

where

state-property-decisions $\sigma = \{p. \text{state-property-is-decided } (p, \sigma)\}$

theorem (in *Protocol*) *n-party-safety-for-state-properties* :

$\forall \sigma\text{-set}. \sigma\text{-set} \subseteq \Sigma t$

\longrightarrow *finite* $\sigma\text{-set}$

\longrightarrow *is-faults-lt-threshold* $(\bigcup \sigma\text{-set})$

\longrightarrow *state-properties-are-consistent* $(\bigcup \{\text{state-property-decisions } \sigma \mid \sigma. \sigma \in \sigma\text{-set}\})$

apply *rule+*

proof –

fix $\sigma\text{-set}$

assume $\sigma\text{-set}: \sigma\text{-set} \subseteq \Sigma t$

and *finite* $\sigma\text{-set}$

and *is-faults-lt-threshold* $(\bigcup \sigma\text{-set})$

hence $\exists \sigma \in \Sigma t. \sigma \in \bigcap \{\text{futures } \sigma \mid \sigma. \sigma \in \sigma\text{-set}\}$

using *n-party-common-futures-exists* **by** *simp*

hence $\exists \sigma \in \Sigma t. \forall s \in \sigma\text{-set}. \sigma \in \text{futures } s$

by *blast*

hence $\exists \sigma \in \Sigma t. (\forall s \in \sigma\text{-set}. \sigma \in \text{futures } s) \wedge (\forall s \in \sigma\text{-set}. \sigma \in \text{futures } s \longrightarrow (\forall p.$

state-property-is-decided $(p, s) \longrightarrow \text{state-property-is-decided } (p, \sigma))$

by (*simp add: subset-eq state-property-is-decided-def futures-def*)

hence $\exists \sigma \in \Sigma t. \forall s \in \sigma\text{-set}. (\forall p. \text{state-property-is-decided } (p, s) \longrightarrow \text{state-property-is-decided } (p, \sigma))$

by *blast*

hence $\exists \sigma \in \Sigma t. \forall s \in \sigma\text{-set}. (\forall p \in \text{state-property-decisions } s. \text{state-property-is-decided } (p, \sigma))$

by (*simp add: state-property-decisions-def*)

hence $\exists \sigma \in \Sigma t. \forall p \in \bigcup \{\text{state-property-decisions } \sigma \mid \sigma. \sigma \in \sigma\text{-set}\}. \text{state-property-is-decided } (p, \sigma)$

proof –

obtain σ **where** $\sigma \in \Sigma t \forall s \in \sigma\text{-set}. (\forall p \in \text{state-property-decisions } s. \text{state-property-is-decided } (p, \sigma))$

using $\langle \exists \sigma \in \Sigma t. \forall s \in \sigma\text{-set}. \forall p \in \text{state-property-decisions } s. \text{state-property-is-decided } (p, \sigma) \rangle$ **by** *blast*

have $\forall p \in \bigcup \{\text{state-property-decisions } \sigma \mid \sigma. \sigma \in \sigma\text{-set}\}. \text{state-property-is-decided } (p, \sigma)$

using $\langle \forall s \in \sigma\text{-set}. \forall p \in \text{state-property-decisions } s. \text{state-property-is-decided } (p, \sigma) \rangle$ **by** *fastforce*

thus *?thesis*

using $\langle \sigma \in \Sigma t \rangle$ **by** *blast*

qed

hence $\exists \sigma \in \Sigma t. \forall p \in \bigcup \{\text{state-property-decisions } \sigma \mid \sigma. \sigma \in \sigma\text{-set}\}. \forall \sigma' \in \text{futures } \sigma. p \sigma'$

by (*simp add: state-property-decisions-def futures-def state-property-is-decided-def*)

show *state-properties-are-consistent* $(\bigcup \{\text{state-property-decisions } \sigma \mid \sigma. \sigma \in \sigma\text{-set}\})$

unfolding *state-properties-are-consistent-def*

by (*metis* (*mono-tags*, *lifting*) $\Sigma t\text{-def}$ $\langle \exists \sigma \in \Sigma t. \forall p \in \bigcup \{ \text{state-property-decisions} \}$
 $\sigma \mid \sigma. \sigma \in \sigma\text{-set} \rangle. \forall \sigma' \in \text{futures } \sigma. p \sigma' \rangle$ *mem-Collect-eq* *monotonic-futures order-refl*)
qed

definition (*in Protocol*) *naturally-corresponding-state-property* :: *consensus-value-property*
 \Rightarrow *state-property*

where

naturally-corresponding-state-property $q = (\lambda \sigma. \forall c \in \varepsilon \sigma. q \ c)$

definition (*in Protocol*) *consensus-value-properties-are-consistent* :: *consensus-value-property*
 $\text{set} \Rightarrow \text{bool}$

where

consensus-value-properties-are-consistent $q\text{-set} = (\exists c \in C. \forall q \in q\text{-set}. q \ c)$

lemma (*in Protocol*) *naturally-corresponding-consistency* :

$\forall q\text{-set}. \text{state-properties-are-consistent } \{ \text{naturally-corresponding-state-property } q \mid q. q \in q\text{-set} \}$

$\longrightarrow \text{consensus-value-properties-are-consistent } q\text{-set}$

apply (*rule*, *rule*)

proof –

fix $q\text{-set}$

have

state-properties-are-consistent $\{ \text{naturally-corresponding-state-property } q \mid q. q \in q\text{-set} \}$

$\longrightarrow (\exists \sigma \in \Sigma. \forall p \in \{ \lambda \sigma'. \forall c \in \varepsilon \sigma'. q \ c \mid q. q \in q\text{-set} \}. p \ \sigma)$

by (*simp add: naturally-corresponding-state-property-def state-properties-are-consistent-def*)

moreover have

$(\exists \sigma \in \Sigma. \forall p \in \{ \lambda \sigma'. \forall c \in \varepsilon \sigma'. q \ c \mid q. q \in q\text{-set} \}. p \ \sigma)$

$\longrightarrow (\exists \sigma \in \Sigma. \forall q' \in q\text{-set}. (\lambda \sigma'. \forall c \in \varepsilon \sigma'. q' \ c) \ \sigma)$

by (*metis* (*mono-tags*, *lifting*) *mem-Collect-eq*)

moreover have

$(\exists \sigma \in \Sigma. \forall q \in q\text{-set}. (\lambda \sigma'. \forall c \in \varepsilon \sigma'. q \ c) \ \sigma)$

$\longrightarrow (\exists \sigma \in \Sigma. \forall q' \in q\text{-set}. \forall c \in \varepsilon \sigma. q' \ c)$

by *blast*

moreover have

$(\exists \sigma \in \Sigma. \forall q \in q\text{-set}. \forall c \in \varepsilon \sigma. q \ c)$

$\longrightarrow (\exists \sigma \in \Sigma. \forall c \in \varepsilon \sigma. \forall q' \in q\text{-set}. q' \ c)$

by *blast*

moreover have

$(\exists \sigma \in \Sigma. \forall c \in \varepsilon \sigma. \forall q \in q\text{-set}. q \ c)$

$\longrightarrow (\exists \sigma \in \Sigma. \exists c \in \varepsilon \sigma. \forall q' \in q\text{-set}. q' \ c)$

by (*meson all-not-in-conv estimates-are-non-empty*)

moreover have

$(\exists \sigma \in \Sigma. \exists c \in \varepsilon \sigma. \forall q \in q\text{-set}. q \ c)$

$\longrightarrow (\exists c \in C. \forall q' \in q\text{-set}. q' c)$
using *is-valid-estimator-def* ε -type **by** *fastforce*
ultimately show
 $state\text{-}properties\text{-}are\text{-}consistent \{naturally\text{-}corresponding\text{-}state\text{-}property\ q \mid q. q \in q\text{-}set\}$
 $\implies consensus\text{-}value\text{-}properties\text{-}are\text{-}consistent\ q\text{-}set$
by (*simp add: consensus-value-properties-are-consistent-def*)
qed

definition (*in Protocol*) *consensus-value-property-is-decided* :: (*consensus-value-property* * *state*) \Rightarrow *bool*
where
 $consensus\text{-}value\text{-}property\text{-}is\text{-}decided$
 $= (\lambda(q, \sigma). state\text{-}property\text{-}is\text{-}decided\ (naturally\text{-}corresponding\text{-}state\text{-}property\ q, \sigma))$

definition (*in Protocol*) *consensus-value-property-decisions* :: *state* \Rightarrow *consensus-value-property set*
where
 $consensus\text{-}value\text{-}property\text{-}decisions\ \sigma = \{q. consensus\text{-}value\text{-}property\text{-}is\text{-}decided\ (q, \sigma)\}$

theorem (*in Protocol*) *n-party-safety-for-consensus-value-properties* :
 $\forall \sigma\text{-}set. \sigma\text{-}set \subseteq \Sigma t$
 $\longrightarrow finite\ \sigma\text{-}set$
 $\longrightarrow is\text{-}faults\text{-}lt\text{-}threshold\ (\bigcup \sigma\text{-}set)$
 $\longrightarrow consensus\text{-}value\text{-}properties\text{-}are\text{-}consistent\ (\bigcup \{consensus\text{-}value\text{-}property\text{-}decisions\ \sigma \mid \sigma. \sigma \in \sigma\text{-}set\})$
apply (*rule, rule, rule, rule*)
proof –
fix $\sigma\text{-}set$
assume $\sigma\text{-}set \subseteq \Sigma t$
and *finite* $\sigma\text{-}set$
and *is-faults-lt-threshold* $(\bigcup \sigma\text{-}set)$
hence $state\text{-}properties\text{-}are\text{-}consistent\ (\bigcup \{state\text{-}property\text{-}decisions\ \sigma \mid \sigma. \sigma \in \sigma\text{-}set\})$
using $\langle \sigma\text{-}set \subseteq \Sigma t \rangle$ *n-party-safety-for-state-properties* **by** *auto*
hence $state\text{-}properties\text{-}are\text{-}consistent\ \{p \in \bigcup \{state\text{-}property\text{-}decisions\ \sigma \mid \sigma. \sigma \in \sigma\text{-}set\}. \exists q. p = naturally\text{-}corresponding\text{-}state\text{-}property\ q\}$
unfolding *naturally-corresponding-state-property-def* *state-properties-are-consistent-def*
apply (*simp*)
by *meson*
hence $state\text{-}properties\text{-}are\text{-}consistent\ \{naturally\text{-}corresponding\text{-}state\text{-}property\ q \mid q. naturally\text{-}corresponding\text{-}state\text{-}property\ q \in \bigcup \{state\text{-}property\text{-}decisions\ \sigma \mid \sigma. \sigma \in \sigma\text{-}set\}\}$
by (*smt Collect-cong*)

hence *consensus-value-properties-are-consistent* $\{q. \text{naturally-corresponding-state-property } q \in \bigcup \{ \text{state-property-decisions } \sigma \mid \sigma. \sigma \in \sigma\text{-set} \} \}$
using *naturally-corresponding-consistency*
proof –
show *?thesis*
by (*metis* (*no-types*) *Setcompr-eq-image* $\langle \forall q\text{-set}. \text{state-properties-are-consistent } \{ \text{naturally-corresponding-state-property } q \mid q. q \in q\text{-set} \} \longrightarrow \text{consensus-value-properties-are-consistent } q\text{-set} \rangle \langle \text{state-properties-are-consistent } \{ \text{naturally-corresponding-state-property } q \mid q. \text{naturally-corresponding-state-property } q \in \bigcup \{ \text{state-property-decisions } \sigma \mid \sigma. \sigma \in \sigma\text{-set} \} \} \rangle \text{setcompr-eq-image}$)
qed
hence *consensus-value-properties-are-consistent* $(\bigcup \{ \text{consensus-value-property-decisions } \sigma \mid \sigma. \sigma \in \sigma\text{-set} \})$
apply (*simp* *add: consensus-value-property-decisions-def consensus-value-property-is-decided-def state-property-decisions-def consensus-value-properties-are-consistent-def*)
by (*metis* *mem-Collect-eq*)
thus
consensus-value-properties-are-consistent $(\bigcup \{ \text{consensus-value-property-decisions } \sigma \mid \sigma. \sigma \in \sigma\text{-set} \})$
by *simp*
qed

fun *consensus-value-property-not* :: *consensus-value-property* \Rightarrow *consensus-value-property*
where
consensus-value-property-not $p = (\lambda c. (\neg p \ c))$

theorem (**in** *Protocol*) *n-party-safety* :
 $\forall \sigma\text{-set}. \sigma\text{-set} \subseteq \Sigma t$
 $\longrightarrow \text{finite } \sigma\text{-set}$
 $\longrightarrow \text{is-faults-lt-threshold } (\bigcup \sigma\text{-set})$
 $\longrightarrow (\forall \sigma \sigma' p. \{ \sigma, \sigma' \} \subseteq \sigma\text{-set} \wedge p \in \text{consensus-value-property-decisions } \sigma \longrightarrow \text{consensus-value-property-not } p \notin \text{consensus-value-property-decisions } \sigma')$
apply (*rule, rule, rule, rule, rule, rule, rule, rule*)
proof –
fix $\sigma\text{-set } \sigma \sigma' p$
assume $\sigma\text{-set} \subseteq \Sigma t$ **and** *finite* $\sigma\text{-set}$ **and** *is-faults-lt-threshold* $(\bigcup \sigma\text{-set})$ **and** $\{ \sigma, \sigma' \} \subseteq \sigma\text{-set} \wedge p \in \text{consensus-value-property-decisions } \sigma$
hence $\exists \sigma. \sigma \in \Sigma t \wedge \sigma \in \bigcap \{ \text{futures } \sigma \mid \sigma. \sigma \in \sigma\text{-set} \}$
using *n-party-common-futures-exists* **by** *meson*
then obtain σ'' **where** $\sigma'' \in \Sigma t \wedge \sigma'' \in \bigcap \{ \text{futures } \sigma \mid \sigma. \sigma \in \sigma\text{-set} \}$ **by** *auto*
hence *state-property-is-decided* (*naturally-corresponding-state-property* p, σ'')
using $\langle \{ \sigma, \sigma' \} \subseteq \sigma\text{-set} \wedge p \in \text{consensus-value-property-decisions } \sigma \rangle \text{consensus-value-property-decisions-def consensus-value-property-is-decided-def}$
using $\langle \sigma\text{-set} \subseteq \Sigma t \rangle$ *forward-consistency* **by** *fastforce*
have $\sigma'' \in \text{futures } \sigma'$
using $\langle \sigma'' \in \Sigma t \wedge \sigma'' \in \bigcap \{ \text{futures } \sigma \mid \sigma. \sigma \in \sigma\text{-set} \} \rangle \langle \{ \sigma, \sigma' \} \subseteq \sigma\text{-set} \wedge p \in \text{consensus-value-property-decisions } \sigma \rangle$
by *auto*
hence $\neg \text{state-property-is-decided } (\text{state-property-not } (\text{naturally-corresponding-state-property } p, \sigma'))$

$p), \sigma')$

```

using backward-consistency  $\langle \text{state-property-is-decided } (\text{naturally-corresponding-state-property } p, \sigma') \rangle$ 
using  $\langle \sigma'' \in \Sigma t \wedge \sigma'' \in \bigcap \text{-Collect } (\text{futures } \sigma) (\sigma \in \sigma\text{-set}) \rangle \langle \sigma\text{-set} \subseteq \Sigma t \rangle \langle \{\sigma, \sigma'\} \subseteq \sigma\text{-set} \wedge p \in \text{consensus-value-property-decisions } \sigma \rangle$  by auto
then show consensus-value-property-not  $p \notin \text{consensus-value-property-decisions } \sigma'$ 
apply (simp add: consensus-value-property-decisions-def consensus-value-property-is-decided-def naturally-corresponding-state-property-def state-property-is-decided-def)
using  $\Sigma t\text{-def estimates-are-non-empty futures-def}$  by fastforce
qed

end
theory SafetyOracle

```

imports *Main CBCCasper LatestMessage StateTransition*

begin

```

fun latest-justifications-from-non-equivocating-validators :: state  $\Rightarrow$  validator  $\Rightarrow$  state set
where
  latest-justifications-from-non-equivocating-validators  $\sigma v =$ 
     $\{\text{justification } m \mid m. m \in \text{latest-messages-from-non-equivocating-validators } \sigma v\}$ 

```

```

lemma (in Protocol) latest-justifications-from-non-equivocating-validators-type :
   $\forall \sigma v. \sigma \in \Sigma \wedge v \in V \longrightarrow \text{latest-justifications-from-non-equivocating-validators } \sigma v \subseteq \Sigma$ 
using M-type latest-messages-from-non-equivocating-validators-type by auto

```

```

fun agreeing-validators :: (consensus-value-property * state)  $\Rightarrow$  validator set
where
  agreeing-validators  $(p, \sigma) = \{v \in \text{observed-non-equivocating-validators } \sigma. \forall c \in \text{latest-estimates-from-non-equivocating-validators } \sigma v. p\ c\}$ 

```

lemma (in *Protocol*) *agreeing-validators-type* :

$\forall \sigma \in \Sigma. \text{agreeing-validators } (p, \sigma) \subseteq V$

apply (simp add: *observed-non-equivocating-validators-def*)

using *observed-type-for-state* **by** *auto*

fun *disagreeing-validators* :: (consensus-value-property * state) \Rightarrow validator set

where

$\text{disagreeing-validators } (p, \sigma) = \{v \in \text{observed-non-equivocating-validators } \sigma. \exists c \in \text{latest-estimates-from-non-equivocating-validators } \sigma v. \neg p c\}$

lemma (in *Protocol*) *disagreeing-validators-type* :

$\forall \sigma \in \Sigma. \text{disagreeing-validators } (p, \sigma) \subseteq V$

apply (simp add: *observed-non-equivocating-validators-def*)

using *observed-type-for-state* **by** *auto*

definition (in *Params*) *weight-measure* :: validator set \Rightarrow real

where

$\text{weight-measure } v\text{-set} = \text{sum } W v\text{-set}$

fun (in *Params*) *is-majority* :: (validator set * state) \Rightarrow bool

where

$\text{is-majority } (v\text{-set}, \sigma) = (\text{weight-measure } v\text{-set} > (\text{weight-measure } V - \text{weight-measure } (\text{equivocating-validators } \sigma)) \text{ div } 2)$

definition (in *Protocol*) *is-majority-driven* :: consensus-value-property \Rightarrow bool

where

$\text{is-majority-driven } p = (\forall \sigma c. \sigma \in \Sigma \wedge c \in C \wedge \text{is-majority } (\text{agreeing-validators } (p, \sigma), \sigma) \longrightarrow (\forall c \in \varepsilon \sigma. p c))$

definition (in *Protocol*) *is-max-driven* :: consensus-value-property \Rightarrow bool

where

$\text{is-max-driven } p =$

$(\forall \sigma c. \sigma \in \Sigma \wedge c \in C \wedge \text{weight-measure } (\text{agreeing-validators } (p, \sigma)) > \text{weight-measure } (\text{disagreeing-validators } (p, \sigma)) \longrightarrow c \in \varepsilon \sigma \wedge p c)$

fun *later-disagreeing-messages* :: (consensus-value-property * message * validator * state) \Rightarrow message set

where

$\text{later-disagreeing-messages } (p, m, v, \sigma) = \{m' \in \text{later-from } (m, v, \sigma). \neg p (\text{est } m')\}$

lemma (in *Protocol*) *later-disagreeing-messages-type* :

$\forall p \sigma v m. \sigma \in \Sigma \wedge v \in V \wedge m \in M \longrightarrow \text{later-disagreeing-messages } (p, m, v, \sigma) \subseteq M$
 using *later-from-type-for-state* by *auto*

fun *is-clique* :: (*validator set* * *consensus-value-property* * *state*) \Rightarrow *bool*
where
 is-clique (*v-set*, *p*, σ) =
 ($\forall v \in v\text{-set}. v\text{-set} \subseteq \text{agreeing-validators } (p, \text{the-elem } (\text{latest-justifications-from-non-equivocating-validators } \sigma v))$
 $\wedge (\forall v' \in v\text{-set}. \text{later-disagreeing-messages } (p, \text{the-elem } (\text{latest-messages-from-non-equivocating-validators } (\text{the-elem } (\text{latest-justifications-from-non-equivocating-validators } \sigma v)) v'), v', \sigma) = \emptyset)$)

lemma (*in Protocol*) *later-from-not-affected-by-minimal-transitions* :

$\forall \sigma \sigma' m m' v. (\sigma, \sigma') \in \text{minimal-transitions}$
 $\longrightarrow m' = \text{the-elem } (\sigma' - \sigma)$
 $\longrightarrow v \in V - \{\text{sender } m'\}$
 $\longrightarrow \text{later-from } (m, v, \sigma) = \text{later-from } (m, v, \sigma')$
apply (*rule*, *rule*, *rule*, *rule*, *rule*, *rule*, *rule*, *rule*)

proof–

fix $\sigma \sigma' m m' v$
assume $(\sigma, \sigma') \in \text{minimal-transitions}$
assume $m' = \text{the-elem } (\sigma' - \sigma)$
assume $v \in V - \{\text{sender } m'\}$

have $\text{later-from } (m, v, \sigma) = \{m'' \in \sigma. \text{sender } m'' = v \wedge \text{justified } m m''\}$

apply (*simp add: later-from-def from-sender-def later-def*)

by *auto*

also have $\dots = \{m'' \in \sigma. \text{sender } m'' = v \wedge \text{justified } m m''\} \cup \emptyset$

by *auto*

also have $\dots = \{m'' \in \sigma. \text{sender } m'' = v \wedge \text{justified } m m''\} \cup \{m'' \in \{m'\}.$

sender $m'' = v\}$

proof–

have $\{m'' \in \{m'\}. \text{sender } m'' = v\} = \emptyset$

using $\langle v \in V - \{\text{sender } m'\} \rangle$ **by** *auto*

thus *?thesis*

by *blast*

qed

also have $\dots = \{m'' \in \sigma. \text{sender } m'' = v \wedge \text{justified } m m''\} \cup \{m'' \in \{m'\}.$

```

sender m'' = v ∧ justified m m'
proof-
  have sender m' = v ⇒ justified m m'
    using ⟨v ∈ V - {sender m'}⟩ by auto
  thus ?thesis
    by blast
qed
also have ... = {m'' ∈ σ ∪ {m'}. sender m'' = v ∧ justified m m''}
  by auto
also have ... = {m'' ∈ σ'. sender m'' = v ∧ justified m m''}
proof -
  have σ' = σ ∪ {m'}
  using ⟨(σ, σ') ∈ minimal-transitions⟩ ⟨m' = the-elem (σ' - σ)⟩ minimal-transitions-reconstruction
by auto
  then show ?thesis
    by auto
qed
then have ... = later-from (m, v, σ')
  apply (simp add: later-from-def from-sender-def later-def)
  by auto
then show later-from (m, v, σ) = later-from (m, v, σ')
  using {m'' ∈ σ ∪ {m'}. sender m'' = v ∧ justified m m''} = {m'' ∈ σ'. sender
m'' = v ∧ justified m m''} calculation by auto
qed

```

```

fun (in Params) gt-threshold :: (validator set * state) ⇒ bool
where
  gt-threshold (v-set, σ)
    = (weight-measure v-set > (weight-measure v-set) div 2 + t - weight-measure
(equivocating-validators σ))

```

```

fun (in Params) is-clique-oracle :: (validator set * state * consensus-value-property)
⇒ bool
where
  is-clique-oracle (v-set, σ, p)
    = (is-clique (v-set - (equivocating-validators σ), p, σ) ∧ gt-threshold (v-set
- (equivocating-validators σ), σ))

```

```

end
theory TFGCasper

```

```

imports Main HOL.Real CBCCasper LatestMessage SafetyOracle

```

```

begin

```

type-synonym $block = consensus\text{-}value$

locale $GhostParams = Params +$

fixes $B :: block\ set$
fixes $genesis :: block$

and $prev :: block \Rightarrow block$

fun (**in** $GhostParams$) $n\text{-cestor} :: block * nat \Rightarrow block$
where
 $n\text{-cestor } (b, 0) = b$
 $| n\text{-cestor } (b, n) = n\text{-cestor } (prev\ b, n-1)$

fun (**in** $GhostParams$) $blockchain\text{-}membership :: block \Rightarrow block \Rightarrow bool$ (**infixl** \downarrow 70)
where
 $b1 \downarrow b2 = (\exists\ n. n \in \mathbb{N} \wedge b1 = n\text{-cestor } (b2, n))$

notation ($ASCII$)
 $comp$ (**infixl** $blockchain\text{-}membership$ 70)

definition (**in** $GhostParams$) $score :: state \Rightarrow block \Rightarrow real$
where
 $score\ \sigma\ b = sum\ W\ \{v \in observed\ \sigma. \exists\ b' \in B. b' \in (latest\text{-}estimates\text{-}from\text{-}non\text{-}equivocating\text{-}validators\ \sigma\ v) \wedge (b \downarrow b')\}$

definition (**in** $GhostParams$) $children :: block * state \Rightarrow block\ set$
where
 $children = (\lambda(b, \sigma). \{b' \in est\ ' \sigma. b = prev\ b'\})$

definition (**in** $GhostParams$) $best\text{-}children :: block * state \Rightarrow block\ set$
where
 $best\text{-}children = (\lambda(b, \sigma). \{arg\text{-}max\text{-}on\ (score\ \sigma)\ (children\ (b, \sigma))\})$

function (**in** $GhostParams$) $GHOST :: (block\ set * state) \Rightarrow block\ set$
where
 $GHOST\ (b\text{-}set, \sigma) =$
 $(\bigcup\ b \in \{b \in b\text{-}set. children\ (b, \sigma) \neq \emptyset\}. GHOST\ (best\text{-}children\ (b, \sigma), \sigma))$
 $\cup \{b \in b\text{-}set. children\ (b, \sigma) = \emptyset\}$
by $auto$

definition (in *GhostParams*) *GHOST-estimator* :: *state* \Rightarrow *block set*
where
GHOST-estimator $\sigma = \text{GHOST} (\{\text{genesis}\}, \sigma) \cup (\bigcup b \in \text{GHOST} (\{\text{genesis}\}, \sigma). \text{children } (b, \sigma))$

abbreviation (in *GhostParams*) *P* :: *consensus-value-property set*
where
 $P \equiv \{p. \exists! b \in B. \forall b' \in B. (b \downarrow b' \longrightarrow p \ b' = \text{True}) \wedge \neg (b \downarrow b' \longrightarrow p \ b' = \text{False})\}$

locale *Ghost* = *GhostParams* + *Protocol* +
assumes *prev-type* : $\forall b. b \in B \longleftrightarrow \text{prev } b \in B$
and *block-is-consensus-value* : $B = C$
and *ghost-is-estimator* : $\varepsilon = \text{GHOST-estimator}$
and *genesis-type* : $\text{genesis} \in C$

lemma (in *Ghost*) *children-type* :
 $\forall b \ \sigma. b \in B \wedge \sigma \in \Sigma \longrightarrow \text{children } (b, \sigma) \subseteq B$
apply (*simp add: children-def*)
using *Ghost-axioms Ghost-axioms-def Ghost-def* **by** *auto*

lemma (in *Ghost*) *best-children-type* :
 $\forall b \ \sigma. b \in B \wedge \sigma \in \Sigma \longrightarrow \text{best-children } (b, \sigma) \subseteq B$
apply (*simp add: best-children-def arg-max-on-def arg-max-def is-arg-max-def*)
apply *auto*
oops

lemma (in *Ghost*) *GHSOT-type* :
 $\forall \sigma \ b\text{-set}. \sigma \in \Sigma \wedge b\text{-set} \subseteq B \longrightarrow \text{GHOST}(b\text{-set}, \sigma) \subseteq B$
oops

lemma (in *GhostParams*) *GHOST-is-valid-estimator* :
 $(\forall b. b \in B \longleftrightarrow \text{prev } b \in B) \wedge B = C \wedge \text{genesis} \in C$
 $\implies \text{is-valid-estimator } \text{GHOST-estimator}$
apply (*simp add: is-valid-estimator-def GhostParams.GHOST-estimator-def*)
oops

lemma (in *Ghost*) *block-membership-property-is-majority-driven* :
 $\forall p \in P. \text{is-majority-driven } p$
apply (*simp add: is-majority-driven-def*)
oops

lemma (in *Ghost*) *block-membership-property-is-max-driven* :
 $\forall p \in P. \text{is-max-driven } p$
apply (*simp add: is-max-driven-def*)

oops
end