Characterization of Algebraically closed Fields.

Prop Let F be a field. Then F is algebraically closed iff one of the following holds:

(1) Any irreducible $f(x) \in F[x]$ has the property that $\deg(f) = 1$.

(2) $\exists$ a root in F, for any poly. $f(x) \in F[x]$, with $\deg(f) \geq 1$.

(3) F is the only algebraic extension of F.

Proof. $(1) \Rightarrow (2)$: Let $f(x) \in F[x]$ with $\deg(f) = 1 \Rightarrow f(x) = ax + b$ s.t. $a, b \in F$, $a \neq 0$

but then $(-1)\cdot(b)\cdot(a^{-1})$ is a root of $f(x)$ in F.

Hence all polys of degree 1 have a root in F. Notice that these however are all the irreducible polys in $F[x]$. As $F[x]$ is a UFD, any non constant poly. will have a root in F ∎

$(3) \Rightarrow (1)$: Let $f(x) \in F[x]$ be an irreducible poly $\Rightarrow \dfrac{F[x]}{\langle f(x) \rangle}$ is a finite alg. extension of F. By assumption, $\deg(f)$ must be 1. ∎

$(2) \Rightarrow (3)$: Argue by contra. and assume that $E \ (\neq F)$ is an algebraic ext. of F.

Then $\exists$ algebraic elt $\alpha \in E$ with $f(x) \in F[x]$; $f(\alpha) = 0$.

By assumption, $\exists$ a root of f in F; say β.

As f is irreducible + monic, $f(x) = x - \beta$. But since $f(\alpha) = 0$,

we arrive at $f(\alpha) = \alpha - \beta = 0 \Rightarrow \alpha = \beta \in F$. Hence $E = F$ ∎

Ex. • The fundamental thm of algebra state that $\mathbb{C}$ is alg. closed.

• $\mathbb{R}, \mathbb{Q}, \mathbb{Z}_p$ are not algebraically closed. ($\mathbb{R}, \mathbb{Q} \hookrightarrow \mathbb{C}$)

Thm Every homomorphism of a field K into an algebraically closed field (F) $(f: K \to F)$

can be extended to a homomorphism from any algebraic extension of F. $(\bar{f}, \bar{\bar{f}})$.

$$\begin{tikzcd} T & K \arrow[l] \arrow[r, hook] \arrow[d, "f"] & L \arrow[dl, dashed, "\bar{f}"] \\ & F \arrow[from=1-1, dashed, "\bar{\bar{f}}"'] & \end{tikzcd}$$

[illegible]

(f, f).

★ V. IMP thm since it settled the situation of extending homomorphisms into fields.

★ Pf depends on Zorn's lemma (p. 166).

Lemma Every field K has an algebraic extension that contains a root of every non-constant poly. with coefficients in K.

Thm Every field K has an algebraic extension $\overline{K}$ that is algebraically closed. Moreover, $\overline{K}$ is unique up to K-isomorphism.

Df (algebraic closure of a field K) is an algebraic extension $\overline{K}$ of K, that is algebraically closed.)

As all algebraic closure of K are K-isomorphic $\Longrightarrow$ "the" alg. closure of K

Prop Every K-endomorphism of $\overline{K}$ is a K-automorphism.

Pf. Let $\varphi : \overline{K} \longrightarrow \overline{K}$ be a K-homomorphism. Then $\operatorname{im}\varphi \cong \overline{K}$ is algebraically closed and $\overline{K}$ is algebraic over $\operatorname{Im}\varphi$. Hence, $\overline{K} = \operatorname{Im}\varphi$ and so φ is a K-automorphism.

Prop If $K \subseteq E \subseteq \overline{K}$ is an algebraic extension of K, then every K-homomorphism of E into $\overline{K}$ extends to a K-automorphism of $\overline{K}$.

$f : E \longrightarrow \overline{K}$

Pf Let $f : E \longrightarrow \overline{K}$ be a K-homomorphism. Then by (↑), f can be extended to a K-homomorphism $g : \overline{K} \longrightarrow \overline{K}$. It follows that g is a K-automorphism of $\overline{K}$. ∎

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