Symplectic Alternating Algebras

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From the historical point of view, the word "algebra" means balancing and reduction. It originated from the Persian mathematician Muḥammad ibn Mūsā al-Khwārizmī's work on the subject known as Al-Jabr around 830 CE, and his work focused on solving linear and quadratic equations. As more mathematical objects were invented over the centuries, modern algebra began to focus on sets of these objects endowed with one or more mathematical operations, such as addition and multiplication. More precisely, a set equipped with a collection of operations satisfying some mathematical identities (i.e., axioms) is called an algebraic structure, and we study these structures in algebra. There are varieties of algebraic structures, such as groups, rings, fields, and algebras. A vector space is another example of an algebraic structure, but what do we mean by a vector space? Consider a line segment with an arrow that indicates the direction of the line; this object is called *a vector* in mathematics. Now, imagine these vectors can be added together, multiplied by scalars, and respect certain mathematical rules. For example, adding two vectors first and then multiplying by a scalar is the same as multiplying each vector by a scalar and then adding them - this rule is known as *distributivity*. Simply put, the collection of vectors on which certain operations can be performed is called a *vector space*. It follows from the definition that algebra focuses on algebraic properties of the addition and scalar multiplication of vectors.

Vector spaces are great tools for analyzing systems of linear equations, and for this reason, this concept is covered in any undergraduate-level linear algebra class. One of the most fascinating features is that every vector space can be generated by a number of vectors. The set of such vectors is called a *basis*. The basis vectors are not dependent on each other, and any vector in the space can be written as a linear combination of these basis vectors. For instance, the basis vectors for the three-dimensional vector space $\mathbb{R}^3$ are $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

Other than algebraic features of $\mathbb{R}^3$, we also have geometric features involving the length of a vector or the angle formed by vectors in $\mathbb{R}^3$. This geometric intuition can be carried to any vector space if we add an additional structure, called an inner product. Let u, v, and w be vectors in a vector space V and k be a scalar in a field F (for instance, $\mathbb{R}$ or $\mathbb{C}$). An *inner product* is a function $\langle\ ,\ \rangle : V \times V \to F$ which has the following properties:

1. (Positive-Definiteness) $\langle u, v\rangle \geq 0$ and $\langle u, u\rangle = 0$ if and only if $u = 0$;
2. (Conjugate-Symmetry) $\langle u, v\rangle = \overline{\langle v, u\rangle}$;
3. (Linearity in first component) $\langle ku + kv, w\rangle = k\langle u, w\rangle + k\langle v, w\rangle$.

A vector space equipped with an inner product is called an *inner product space*. As the conjugate of a real number is itself, the "conjugate-symmetry" condition becomes the "symmetry" condition for an inner product on a real vector space. It is also worth mentioning that an inner product on a real vector space is a bilinear form - it is linear in each component separately. However, an inner product on a complex vector space is not, since it is conjugate-linear in the second component. Geometrically speaking, the inner product gives us a way to measure one-dimensional notions; the lengths of vectors and angles between two vectors. Then it is natural to wonder whether there is a way of measuring two-dimensional areas in space. This can be addressed by equipping the vector space with a symplectic form instead of an inner product.

A *symplectic vector space* is a vector space V over a field F equipped with a *symplectic form* that is a function $(\ ,\ ) : V \times V \to F$ satisfying the following properties:

1. Bilinear;
2. (Alternating) $(u, u) = 0$;
3. (Non-degenerate) $(u, v) = 0$ for all $v \in V$ implies that $u = 0$.

Even though the word "symplectic" was first introduced by German mathematician and theoretical physicist Hermann Weyl in 1939, symplectic space was discovered by William Rowan Hamilton while studying dynamical systems in the 1800s. So, it is safe to say that physics found a geometric structure for mathematics to study.

There are many mathematical properties that a symplectic vector space carries, for example, every symplectic vector space is even-dimensional. Moreover, any symplectic vector space of dimension $2n$ splits as the direct sum of two isotropic subspaces of dimension n, meaning that the symplectic form vanishes on these subspaces. To be more precise, a subspace W of a symplectic vector space V is called an *isotropic subspace* if $(u, v) = 0$ for all u and v from W. Any one-dimensional subspace, Fu, is an example of an isotropic subspace. Further,

$$W \subset W^{\perp} = \{v \text{ in } V \mid (v, w) = 0 \text{ for all } w \text{ in } W\}.$$

What is a Symplectic Alternating Algebra?

Now, if we add another operation, *binary alternating product*, to a symplectic vector space that interacts with the non-degenerate form in a certain way, we get what is called Symplectic Alternating Algebra. Our goal in this expository article is to define such algebra, study some of its features, and give some examples.

A *Symplectic Alternating Algebra (SAA)* over a field F is a symplectic vector space V over F equipped with a binary alternating product " $\cdot$ ": $V \times V \to V$ such that the law

$$(u \cdot v, w) = (v \cdot w, u),$$

holds for all u, v, and w in V. We refer to this law as *SAA Property.* Notice that

$$(u \cdot v, w) = (v \cdot w, u) = -(w \cdot v, u) = (u, w \cdot v),$$

and this is known as the self-adjointness property. So, the multiplication from the right by v is *self-adjoint with respect to the alternating form*.

As V is an even-dimensional space, we can pick a basis $x_1, y_1, \ldots, x_n, y_n$ with the property that

$$(x_i, x_j) = (y_i, y_j) = 0 \text{ and}$$

$$(x_i, y_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

for $1 \leq i \leq j \leq n$. We refer to a basis of this type as a *standard basis*. Suppose we have any basis $u_1, \ldots, u_{2n}$ for V. The structure of V is then determined from

$$(u_i \cdot u_j, u_k) = \gamma_{ijk}$$

for $1 \leq i < j < k \leq 2n$ and γ_{ijk} is a scalar from the field F. We refer to such data as a *presentation* for V. Alternatively, we can describe V as follows: if we take the two isotropic subspaces, $Fx_1 + \cdots + Fx_n$ and $Fy_1 + \cdots + Fy_n$ concerning a given standard basis, then it suffices to write down only the products

$$x_i \cdot x_j,\ y_i \cdot y_j, \text{ for } 1 \leq i < j \leq n.$$

The reason for this is that having determined these products, we have determined all the triples $(u_i \cdot u_j, u_k)$ where $1 \leq i < j < k \leq 2n$ since two of those are either some x_i, x_j or some y_i, y_j, in which case the triple is determined from $x_i \cdot x_j$ or $y_i \cdot y_j$. Since

$$(x_i \cdot x_j, x_k) = (x_j \cdot x_k, x_i) = (x_k \cdot x_i, x_j)$$

and

$$(y_i \cdot y_j, y_k) = (y_j \cdot y_k, y_i) = (y_k \cdot y_i, y_j),$$

this puts some more conditions on the products $x_i \cdot x_j$ and $y_i \cdot y_j$.

Some Properties of SAAs

Let V be a symplectic alternating algebra and let u, v be in V. Now, consider the subspace generated by the element $u, uv, uvv, \ldots$ in V, (omit "$\cdot$" here), and hence any random element from this subspace is in the form of uv^r for some positive integer r. Combining this with the SAA property, we have $(uv^r, uv^s) = 0$ for some positive integers r and s. Indeed,

$$(uv^r, uv^s) = (uv^k, uv^k) = 0, \quad \text{if } r + s = 2k,$$

$$(uv^r, uv^s) = (uv^k \cdot v, uv^k) = (uv^k \cdot uv^k, v) = (0, v) = 0,$$

if $r + s = 2k + 1$. This implies that the subspace generated by $u, uv, uvv, \ldots$ is isotropic and leads us to the following lemma.

Lemma 1. *Let V be a symplectic alternating algebras. If $u, v \in V$, then the subspace generated by $u, uv, uvv, \ldots$ is isotropic.*

Next, we focus on the lower and upper central series and provide definitions similar to the ones given in group theory. The lower central series is defined recursively by

$$V^1 = V \quad \& \quad V^{n+1} = V^n \cdot V,$$

and the upper central series by

$$Z_0(V) = \{0\} \quad \& \quad Z_{n+1}(V) = \{x \text{ in } V \mid x \cdot V \text{ in } Z_n(V)\}.$$

We have the following relation between the upper and the lower central series:

Lemma 2 ([7]).

$$Z_i(V) = (V^{i+1})^{\perp}.$$

Proof. We have $a \in Z_i(V) \Leftrightarrow a\underbrace{V \cdots V}_{i-\text{many}} = 0 \Leftrightarrow 0 = (a\underbrace{V \cdots V}_{i-\text{many}}, V) = (a, V^{i+1}) \Leftrightarrow a \in (V^{i+1})^{\perp}$. □

As a consequence, we see that $\dim(Z_i(V)) + \dim(V^{i+1}) = \dim(V) = 2n$. Moreover, the upper and lower central series have another special relation:

Lemma 3.

$$Z_i(V) \cdot V^i = \{0\}.$$

Recall that in group theory, we do not have that $G/Z(G)$ is cyclic. Likewise, for any alternating algebra A, $A/Z(A)$ can not be 1-dimensional. It follows that we have the following interesting result

Theorem 4.

$$\mathit{dim}\ (V^i) \neq 1.$$

Examples

It is known that the only symplectic alternating algebra of dimension 2 is the abelian one - meaning that $u \cdot v = 0$ for all u, v in V.

In algebra, one of our main goals is to classify algebraic systems. When systems have the same algebraic structure we treat them "as same" even though they are collections of entirely different objects. This notion is known as *isomorphism*. It would be a good exercise for those interested in the subject to show that, up to isomorphism, there are two symplectic alternating algebras of dimension 4: one is abelian, whereas the other one has the following multiplication table:

$$\begin{array}{ll} x_1 \cdot y_1 = x_2 & x_1 \cdot x_2 = 0 \\ y_1 \cdot y_2 = -y_1 & x_2 \cdot y_1 = 0 \\ x_1 \cdot y_2 = -x_1 & x_2 \cdot y_2 = 0 \end{array}$$

The presentation is thus $(x_1 \cdot y_1, y_2) = 1$. Note also that the center of the non-abelian SAAs of dimension 4 is $Z(V) =< x_2 >$.

The symplectic alternating algebras of dimension 6 have been classified in [7]; when the field has three elements, there are 31 such algebras, of which 15 are simple.

As a final remark, it is worth mentioning that symplectic alternating algebras originate from a study on 2-Engel groups, and there is a one-to-one correspondence between a certain rich class of powerful 2-Engel 3-groups of exponent 27 and SAAs over the finite field with 3 elements $F_3 = \{0, 1, -1\}$. As SAA is considerably a new type of algebraic structure, many open problems still need to be explored. More information and further development for those interested in learning more about the subject can be found in [2], [3], [4], [5], [6], and [7].

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