

Characterization of Algebraically closed fields.

Prop Let F be a field. Then F is algebraically closed if one of the following holds:

- (1) Any irreducible $f(x) \in F[x]$ has the property that $\deg(f) = 1$.
- (2) \exists a root in F for any poly. $f(x) \in F[x]$, with $\deg(f) \geq 1$.
- (3) F is the only algebraic extension of F .

Proof. (1) \Rightarrow (2): Let $f(x) \in F[x]$ with $\deg(f) = 1 \Rightarrow f(x) = ax + b$ s.t. $a, b \in F$
 $a \neq 0$
but then $(-1) \cdot (b) \cdot (a^{-1})$ is a root of $f(x)$ in F .

Hence all polys of degree 1 have a root in F . Notice that these however are all the irreducible polys in $F[x]$. As $F[x]$ is a UFD, any non constant poly. will have a root in F . \square

(3) \Rightarrow (1): Let $f(x) \in F[x]$ be an irreducible poly $\Rightarrow \frac{F[x]}{\langle f(x) \rangle}$ is a finite alg. extension of F . By assumption, $\deg(f)$ must be 1. \square

(2) \Rightarrow (3): Argue by contra. and assume that $E (\neq F)$ is an algebraic ext. of F .

Then \exists algebraic elt $\alpha \in E$ with $f(x) \in F[x]$; $f(\alpha) = 0$.

By assumption, \exists a root of f in F ; say β .

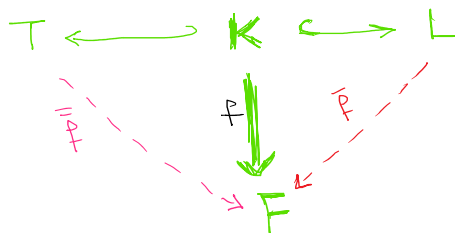
As f is irreducible + monic, $f(x) = x - \beta$. But since $f(\alpha) = 0$, we arrive at $f(x) = x - \beta = 0 \Rightarrow \alpha = \beta \in F$. Hence $E = F$. \square

Ex. The fundamental thm of algebra states that \mathbb{C} is alg.-closed.

- $\mathbb{R}, \mathbb{Q}, \mathbb{Z}_p$ are not algebraically closed. ($\mathbb{R}, \mathbb{Q} \hookrightarrow \mathbb{C}$)

Thm Every homomorphism of a field K into an algebraically closed field (F)
($f: K \rightarrow F$)

can be extended to a homomorphism from any algebraic extension of F .
(\bar{f}, \bar{f}).



(f, f) .

\vec{F}

★ V. NIG thm Since it settled the situation of extending homomorphisms into fields.

★ pf depends on Zorn's lemma (p. 166).

Lemma Every field K has an algebraic extension that contains a root of every non-constant poly. with coefficients in K .

Thm Every field K has an algebraic extension \bar{K} that is algebraically closed. Moreover, \bar{K} is unique up to K -isomorphism.

Df (algebraic closure of a field K) is an algebraic extension \bar{K} of K , that is algebraically closed.)

As All algebraic closure of K are K -isomorphic \Rightarrow "the" alg. closure of K

Prop Every K -endomorphism of \bar{K} is a K -automorphism.

Pf. Let $\sigma: \bar{K} \rightarrow \bar{K}$ be a K -homomorphism. Then $\text{Im } \sigma \cong \bar{K}$ is algebraically closed and \bar{K} is algebraic over $\text{Im } \sigma$. Hence, $\bar{K} = \text{Im } \sigma$ and so σ is a K -automorphism.

Prop If $K \subseteq E \subseteq \bar{K}$ is an algebraic extension of K , then every K -homomorphism of E into \bar{K} extends to a K -automorphism of \bar{K} .

$$f: E \rightarrow \bar{K}$$

Pf Let $f: E \rightarrow \bar{K}$ be a K -homomorphism. Then by (1), f can be extended to a K -homomorphism $g: \bar{K} \rightarrow \bar{K}$. It follows that g is a K -automorphism of \bar{K} .

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