Making truncated, irregular tetrahedra

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1 Introduction

This document briefly describes how we generate truncated, irregular tetrahedra for simulation input.

2 Math

An irregular tetrahedron consists of four points in 3D space, which we denote $\mathbf{x}_0,\ \mathbf{x}_1,\ \mathbf{x}_2$ and $\mathbf{x}_3$. Without loss of generality, we can assume that the first three points are in the xy plane, that $\mathbf{x}_0$ and $\mathbf{x}_1$ are on the x-axis, and that $\mathbf{x}_2$ is on the y-axis.

2.1 Finding the base

We limit us to the special case where the base of the tetrahedron, the triangle spanned by $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2$, is an isocles angle. If single edge is L_1 and the two legs have length L_2, then we can define

$$\mathbf{x}_0 = (-L_1/2, 0, 0)^T, \qquad , \mathbf{x}_1 = (L_1/2, 0, 0)^T, \qquad , \mathbf{x}_2 = (0, y_2, 0)^T$$

with $y_2 = \sqrt{L_2^2 - L_1^2/4}$ in order for the legs to have length L_2.

2.2 Finding the top

With the complete base defined, our problem is now limited to finding a suitable "top" $\mathbf{x}_3$. If we prescribe the tilt of two sides, then we think the third side will follow. If we describe the top in general coordinates $\mathbf{x}_3 = (x, y, z)^T$, then the projection of the top onto the xy-plane is $\mathbf{x}_{3,\perp} = (x, y, 0)^T$.

Let the edges of the base be denoted $\mathbf{b}_{01} = \mathbf{x}_1 - \mathbf{x}_0$, $\mathbf{b}_{12} = \mathbf{x}_2 - \mathbf{x}_1$, and $\mathbf{b}_{20} = \mathbf{x}_0 - \mathbf{x}_2$. On each of these edges there are points $\mathbf{h}_{01},\ \mathbf{h}_{12}$ and $\mathbf{h}_{20}$ that are closest to $\mathbf{x}_3$ and hence closest to $\mathbf{x}_{3,\perp}$. To find this point, we parametrize the edges with parameters $t_{01},\ t_{12}$ and t_{20} and minimize the squared distances.

Finding t_{01} is by far the easiest because it lies on the x-axis:

$$\|\mathbf{x}_0 + t\mathbf{b}_{01} - \mathbf{x}_{3,\perp}\|^2 = \left\| \left(-\frac{L_1}{2} + L_1 t - x, -y, 0 \right)^T \right\|^2 = y^2 + \left(L_1 \left(t - \frac{1}{2} \right) - x \right)^2$$

This expression is minimized for $t = \frac{1}{2} + x/L_1 := t_{01}$, and therefore we have $\mathbf{h}_{01} = \mathbf{x}_0 + t_{01}\mathbf{b}_{01} = (x, 0, 0)^T$. The distance between $\mathbf{h}_{01}$ and $\mathbf{x}_{3,\perp}$ is simply $(0, y, 0)^T$. The points $\mathbf{h}_{01}$, $\mathbf{x}_{3,\perp}$ and $\mathbf{x}_3$ span a right triangle whose slant is related to the tilt of this face. In particular, the angle between the long side and $-z\mathbf{e}_z$ is the *tilt* ϑ_{01}. Let ϕ_{01} be the angle between $\mathbf{x}_{3,\perp} - \mathbf{h}_{01}$ and $\mathbf{x}_3 - \mathbf{h}_{01}$. Then $\vartheta_{01} = \pi/2 - \phi_{01}$ and $\tan\phi_{01} = z/y$, so $\tan(\pi/2 - \vartheta_{01}) = z/y$. This is the first relation that fixes one of the coordinates of the top, $\mathbf{x}_3$.

To find a similar relation for the second tilt, we now move to find $\mathbf{h}_{12}$ in a similar fashion. We have $\mathbf{h}_{12} = \mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1)$ and minimize $\|\mathbf{x}_{3,\perp} - \mathbf{h}_{12}\|^2$.

$$\begin{aligned}
\frac{\partial}{\partial t}\|\mathbf{x}_{3,\perp} - \mathbf{h}_{12}\|^2 =& \frac{\partial}{\partial t}(x + L_1(1-t)t/2)^2 + (y_2 t - y)^2 = 0 \\
& (x + L_1(1-t)t/2)(-L_1/2) + (y_2 t - y)y_2 = 0 \\
t_{12} =& \frac{yy_2 - L_1 x/2}{y_2^2 - L_1^2/4}
\end{aligned}$$

Therefore the point on $\mathbf{b}_{12}$ closest to $\mathbf{x}_{3,\perp}$ is given by $\mathbf{h}_{12} = \mathbf{x}_1 + t_{12}\mathbf{b}_{12}$:

$$\begin{aligned}
\mathbf{h}_{12} &= \begin{pmatrix} L_1/2 \\ 0 \\ 0 \end{pmatrix} + \frac{yy_2 - L_1 x/2}{y_2^2 - L_1^2/4}\begin{pmatrix} -L_1/2 \\ y_0 \\ 0 \end{pmatrix} \\
&= \frac{1}{y_2^2 - \frac{L_1^2}{4}}\begin{pmatrix} \frac{L_1}{2}\left(y_2^2 - \frac{L_1^2}{4} - \left(yy_2 - \frac{L_1}{2}x\right)\right) \\ y_2\left(yy_2 - \frac{L_1}{2}x\right) 0 \end{pmatrix}
\end{aligned}$$

We can rewrite $\mathbf{h}_{12}$ entirely in vector form:

$$\mathbf{h}_{12} = (\mathbf{x}_{3,\perp} - \mathbf{b}_{12}) - \frac{\mathbf{b}_{12}\cdot\mathbf{x}_{3,\perp}}{\mathbf{b}_{12}\cdot\mathbf{b}_{12}}\mathbf{b}_{12}$$

This means that the distance between the shortest point on the edge $\mathbf{b}_{12}$ and $\mathbf{x}_{3,\perp}$ can be found as

$$\begin{aligned}
L^2 = \|\mathbf{h}_{12} - \mathbf{x}_{3,\perp}\|^2 &= \left[\mathbf{x}_{3,\perp} - \mathbf{b}_{12} - \frac{\mathbf{b}_{12}\cdot\mathbf{x}_{3,\perp}}{\mathbf{b}_{12}\cdot\mathbf{b}_{12}}\mathbf{b}_{12}\right]^2 \\
=& x^2 + y^2 + \frac{L_1^2}{4} + y_2^2 - \frac{y^2 y_2^2 + L_1^2 x^2/4 - L_1 y_2 x y}{L_1^2/4 + y_2^2} = y_2^2 + \frac{L_1^2}{4} + \frac{(\frac{1}{2}L_1 y + x y_2)^2}{L_1^2/4 + y_2^2}
\end{aligned}$$

Due to triangle geometry, we have $\tan\phi_{12} = z/L = y\tan\phi_{01}/L$, so all we need is to is solve $\tan\phi_{12} = y\tan\phi_{01}/L(x,y)$ for y and then we have fixed the second tilt. In principle we can use the x-component of $\mathbf{x}_3$ to prescribe the third tilt, but we will leave it free for now. So, we have $\tan\phi_{12} = y\tan\phi_{01}/L(x,y)$, or,

$$L^2\tan^2\phi_{12} = \left(y_2^2 + \frac{L_1^2}{4} + \frac{(\frac{1}{2}L_1 y + x y_2)^2}{L_1^2/4 + y_2^2}\right)\tan^2\phi_{12} = y^2\tan\phi_{01}$$

Solving this for y leads to the general solution

$$\begin{aligned}
y(x) =& \frac{L_1 y_2 x \tan^2\phi_{12}}{2(L_2^2\tan^2\phi_{01} - \frac{1}{4}L_1^2\tan^2\phi_{12})} \\
&\pm \frac{x^2\tan^2\phi_{12}\left(\frac{L_1}{2}y_2^2\tan^2\phi_{01} + L_2^2 y_2^2\right) + \frac{L_2^4}{4}\left(L_2^2\tan^2\phi_{01} - L_1^2\tan^2\phi_{21}\right)}{L_2^2\tan^2\phi_{01} - \frac{1}{4}L_1^2\tan^2\phi_{01}}
\end{aligned}$$

With this expression, we can fix, for a given x, the y-coordinate and subsequently the z-coordinate so that two of the three faces have a fixed angle.