The Pigeonhole Principle

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The Pigeonhole Principle states that if there are more pigeons than holes, then at least one hole will have more than one pigeon. If there is a function f, such that $X \to Y$, or X maps to Y, and the cardinality of X is greater than the cardinality of Y, this implies that there exists x_1 and x_2 as members of the set X such that x_1 does not equal x_2 and $f(x_1)$ is equal to $f(x_2)$. See equation (1) below.

$$f : X \to Y, |X| > |Y| \Rightarrow \exists x_1, x_2 \in X : x_1 \neq x_2, \wedge f(x_1) = f(x_2) \tag{1}$$

But what does all that mean? X and Y are both sets, meaning they are similar to an array, with a list, or sequence, of elements. See example set S below. Cardinality is represented by the symbols $|$, such as $|X|$. The cardinality of a set is the number of elements in a set. The cardinality of set S below is 4 and can be represented as $|S| = 4$. Sets can also be mapped to each other, such that every value of set X maps to a value of Y. Mapping can be represented by $X \to Y$, meaning every value of X is mapped to a value of Y.

$$S = \{0, 1, 2, 3\}$$

In the Pigeonhole Principle, set X represents the pigeons and Y represents the pigeonholes. Each member of X is mapped to a hole in Y. However, if the cardinality, or number, of pigeons in set X is greater than the number of holes in set Y, then at least one member of X will have to also map to the same element as another in set Y.

The Extended Pigeonhole Principle states that in any finite sets X and Y, and any positive integer k such that $|X| > k * |Y|$, if $f : X \to Y$, then there are at least $k+1$ distinct members $x_1, ..., x_{k+1} \in X$ such that $f(x_1) = ... = f(x_{k+1})$. An alternate way of seeing this is using a ceiling and a fraction of pigeons over holes. A floor will round any number down to the closest number less than or equal to the number. An example of floor is $\lfloor 3.8 \rfloor = 3$. Ceiling is the opposite of this, so $\lceil 3.8 \rceil = 4$. Ceiling always rounds up to the closest number greater than or equal to the number. The ceiling can be used as an alternate way to the extended pigeonhole principle. The equation $\left\lceil \frac{|X|}{|Y|} \right\rceil$ shows how many members of X map to a particular member of Y.

For example, let's say there is a set of pigeons X with a cardinality of 5, and there is a set of holes Y with a cardinality of 4. Four of the pigeons can get their own hole, but one is still left and will have to go to one that already has a pigeon. We can use the alternate equation to show a hole with 2 pigeons: $\left\lceil \frac{|X|}{|Y|} \right\rceil = \left\lceil \frac{5}{4} \right\rceil = \lceil 1.25 \rceil = 2$. This demonstrates that there will be at least one hole that has two pigeons.