Okay, here is a formula sheet drafted based on the provided rubric, covering Measure Theory, Functional Analysis, and Fourier Analysis.

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Analysis Formula Sheet

1. Measure Theory

(1.1) Basics of Measure Theory

* **Outer Measure** $\mu^* : \mathcal{P}(X) \to [0, \infty]$: * $\mu^*(\emptyset) = 0$ * Monotone: $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$ * Countably Subadditive: $\mu^*(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ * **Carathéodory Measurable Sets** $\mathcal{M}$: $E \in \mathcal{M}$ if $\forall A \subseteq X$, $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$. * **$\sigma$-algebra:** A collection $\mathcal{A}$ of subsets of X closed under complements and countable unions. $\mathcal{M}$ is a σ-algebra. * **Measure** μ: A countably additive set function $\mu : \mathcal{A} \to [0, \infty]$ with $\mu(\emptyset) = 0$. If μ^* is an outer measure, its restriction to $\mathcal{M}$ is a measure. * **Lebesgue Measure** m on $\mathbb{R}^n$: The completion of the unique Borel measure on $\mathbb{R}^n$ such that $m(\prod_{i=1}^{n} [a_i, b_i]) = \prod_{i=1}^{n} (b_i - a_i)$. * **Borel σ-algebra** $\mathcal{B}(X)$: Smallest σ-algebra containing all open sets in a topological space X. * **Radon Measure:** A Borel measure μ on a Hausdorff space X that is: * Finite on compact sets: $\mu(K) < \infty$ for K compact. * Outer regular: $\mu(E) = \inf\{\mu(O) : E \subseteq O, O \text{ open}\}$ for $E \in \mathcal{B}(X)$. * Inner regular on open sets: $\mu(O) = \sup\{\mu(K) : K \subseteq O, K \text{ compact}\}$ for O open. (If μ is finite, inner regular on all Borel sets). * **Approximation (Lebesgue measure m on $\mathbb{R}^n$, finite measure sets):** For $E \in \mathcal{M}$ with $m(E) < \infty$, $\forall \epsilon > 0$, $\exists O$ open, F closed such that $F \subseteq E \subseteq O$ and $m(O \setminus F) < \epsilon$.

(1.2) Measurable Functions

* **Measurability:** $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is measurable if $f^{-1}(A) \in \mathcal{M}$ for all $A \in \mathcal{N}$. For $f : X \to [-\infty, \infty]$, usually check $f^{-1}((\alpha, \infty]) \in \mathcal{M}$ for all $\alpha \in \mathbb{R}$. * **Properties:** If $f_n : X \to [-\infty, \infty]$ are measurable, then $\sup f_n$, $\inf f_n$, $\limsup f_n$, $\liminf f_n$ are measurable. If f, g are measurable real-valued, then $f + g$, fg, cf are measurable. * **Simple Functions:** $s(x) = \sum_{i=1}^{k} c_i \chi_{E_i}(x)$, where $E_i \in \mathcal{M}$. Any non-negative measurable f is the pointwise limit of an increasing sequence of non-negative simple functions. * **Lusin's Theorem:** If μ is Radon, $f : X \to \mathbb{C}$ measurable, E measurable with $\mu(E) < \infty$. Then $\forall \epsilon > 0$, $\exists F \subseteq E$ closed such that $\mu(E \setminus F) < \epsilon$ and $f|_F$ is continuous. * **Egorov's Theorem:** If $\mu(X) < \infty$, $f_n \to f$ a.e., f_n, f measurable. Then $\forall \epsilon > 0$, $\exists E \subseteq X$ such that $\mu(X \setminus E) < \epsilon$ and $f_n \to f$ uniformly on E. * **Convergence in Measure:** $f_n \to f$ in measure if $\forall \epsilon > 0$, $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$ as $n \to \infty$. If $f_n \to f$ in measure, there exists a subsequence f_{n_k} converging to f a.e. If $\mu(X) < \infty$, a.e. convergence implies convergence in measure.

(1.3) Integration

* **Integral:** $\int f d\mu = \sup\{\int s d\mu : 0 \leq s \leq f, s \text{ simple}\}$ for $f \geq 0$. $\int s d\mu = \sum c_i \mu(E_i)$. For general f, $\int f d\mu = \int f^+ d\mu - \int f^- d\mu$. f is integrable if $\int |f| d\mu < \infty$. $L^1(\mu) = \{f : \int |f| d\mu < \infty\}$. * **Bounded Convergence Theorem (BCT):** If $|f_n| \leq M$, $f_n \to f$ a.e., $\mu(X) < \infty$. Then $\int f_n d\mu \to \int f d\mu$. (General form: replace $|f_n| \leq M$ with $|f_n| \leq g \in L^1$, $\mu(X) < \infty$ not needed - Dominated Convergence Theorem, DCT). * **Fatou's Lemma:** If $f_n \geq 0$ measurable, then $\int (\liminf f_n) d\mu \leq \liminf \int f_n d\mu$. * **Monotone Convergence Theorem (MCT):** If $0 \leq f_n \uparrow f$ a.e., f_n measurable. Then $\int f_n d\mu \to \int f d\mu$. * **Absolute Continuity of the Integral:** If $f \in L^1(\mu)$, then $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $\mu(E) < \delta$, then $|\int_E f d\mu| < \epsilon$.

(1.4) Differentiation and Integration (on $\mathbb{R}$ with Lebesgue measure m)

* **Vitali Covering Lemma:** Let $E \subseteq \mathbb{R}^n$, $\mathcal{V}$ a collection of non-degenerate closed balls covering E in the sense that $\forall x \in E, \forall \epsilon > 0$, $\exists B \in \mathcal{V}$ with $x \in B$ and $\text{diam}(B) < \epsilon$. Then $\exists \{B_k\} \subseteq \mathcal{V}$ disjoint such that $m(E \setminus \cup B_k) = 0$. (Often used for $m^*(E) < \infty$). * **Monotone Functions:** If $F : [a, b] \to \mathbb{R}$ is monotone, then F is differentiable a.e. and $F' \in L^1([a, b], m)$, with $\int_a^b F'(x) dx \leq F(b) - F(a)$. * **Functions of Bounded Variation (BV):** $V_a^b(F) = \sup\{\sum_{i=1}^{k} |F(x_i) - F(x_{i-1})|\}$ over partitions $a = x_0 < \cdots < x_k = b$. $F \in BV([a, b])$ iff $F = G - H$ where G, H are monotone increasing. $F \in BV \implies F$ differentiable a.e. * **Differentiation of Indefinite Integral:** If $f \in L^1([a, b])$, let $F(x) = \int_a^x f(t) dt$. Then F is absolutely continuous (AC) and $F'(x) = f(x)$ a.e. * **Absolute Continuity (AC):** $F : [a, b] \to \mathbb{R}$ is AC if $\forall \epsilon > 0, \exists \delta > 0$ such that for any finite collection of disjoint intervals $(a_k, b_k) \subseteq [a, b]$, $\sum (b_k - a_k) < \delta \implies \sum |F(b_k) - F(a_k)| < \epsilon$. F is AC iff $F(x) = F(a) + \int_a^x g(t) dt$ for some

$g \in L^1([a,b])$. In this case $g = F'$ a.e. * **Fundamental Theorem of Calculus (Newton-Leibniz):** F is AC on $[a,b]$ iff F' exists a.e., $F' \in L^1([a,b])$, and $\int_a^b F'(x)dx = F(b) - F(a)$.

(1.5) Product Measures

* **Product σ-algebra:** $\mathcal{M} \otimes \mathcal{N}$ on $X \times Y$ is generated by measurable rectangles $A \times B$ where $A \in \mathcal{M}, B \in \mathcal{N}$. * **Product Measure:** For σ-finite measures μ, ν, there exists a unique measure $\mu \times \nu$ on $\mathcal{M} \otimes \mathcal{N}$ such that $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$. * **Tonelli's Theorem:** If $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ are σ-finite measure spaces, and $f : X \times Y \to [0, \infty]$ is measurable wrt $\mathcal{M} \otimes \mathcal{N}$. Then:

$$\int_X \left(\int_Y f(x,y) d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x,y) d\mu(x) \right) d\nu(y) = \int_{X \times Y} f d(\mu \times \nu)$$

* **Fubini's Theorem:** If $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ are σ-finite and $f \in L^1(X \times Y, \mu \times \nu)$. Then $f(x, \cdot) \in L^1(\nu)$ for a.e. x, $f(\cdot, y) \in L^1(\mu)$ for a.e. y. The functions $x \mapsto \int_Y f(x,y)d\nu(y)$ and $y \mapsto \int_X f(x,y)d\mu(x)$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively, and the conclusion of Tonelli's theorem holds (the iterated integrals equal the double integral).

(1.6) Differentiation of Measures

* **Absolute Continuity:** $\nu \ll \mu$ if $\mu(E) = 0 \implies \nu(E) = 0$. * **Mutually Singular:** $\mu \perp \nu$ if $\exists E, F$ disjoint such that $X = E \cup F$, μ is concentrated on E ($\mu(F) = 0$), and ν is concentrated on F ($\nu(E) = 0$). * **Radon-Nikodym Theorem:** Let μ, ν be σ-finite measures on $(X, \mathcal{M})$. If $\nu \ll \mu$, then $\exists! f \in L^1(\mu)$, $f \geq 0$ a.e. $[\mu]$, such that $\nu(E) = \int_E f d\mu$ for all $E \in \mathcal{M}$. We write $f = \frac{d\nu}{d\mu}$. * **Lebesgue Decomposition:** Let μ, ν be σ-finite measures. Then $\exists!$ decomposition $\nu = \nu_{ac} + \nu_{sing}$ where $\nu_{ac} \ll \mu$ and $\nu_{sing} \perp \mu$. Moreover, $\nu_{ac}(E) = \int_E \frac{d\nu}{d\mu} d\mu$. * **Signed Measures:** $\nu : \mathcal{M} \to (-\infty, \infty]$ (or $[-\infty, \infty)$ or $\mathbb{R}$) countably additive. * **Hahn Decomposition:** $\exists X = P \cup N$ disjoint, P positive set ($\nu(E) \geq 0$ for $E \subseteq P$), N negative set ($\nu(E) \leq 0$ for $E \subseteq N$). * **Jordan Decomposition:** $\nu = \nu^+ - \nu^-$, where ν^+, ν^- are measures and $\nu^+ \perp \nu^-$. Total variation: $|\nu| = \nu^+ + \nu^-$. * **Lebesgue Points:** $x \in \mathbb{R}^n$ is a Lebesgue point of $f \in L^1_{loc}(\mathbb{R}^n)$ if $\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dy = 0$. Almost every x is a Lebesgue point. * **Lebesgue-Besicovitch Differentiation Theorem:** If $f \in L^1_{loc}(\mathbb{R}^n)$, then for a.e. $x \in \mathbb{R}^n$, $\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) dy = f(x)$. (Holds for families of sets shrinking nicely to x).

(1.7) Riesz Representation Theorems

* **Dual of L^p $(1 \leq p < \infty)$:** Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. If $1 \leq p < \infty$ and $1/p + 1/q = 1$. Then for every continuous linear functional $\phi \in (L^p(\mu))^*$, there exists a unique $g \in L^q(\mu)$ such that $\phi(f) = \int_X fg d\mu$ for all $f \in L^p(\mu)$. Moreover, $\|\phi\| = \|g\|_{L^q}$. So $(L^p)^* \cong L^q$. (Note: $(L^\infty)^* \supsetneq L^1$ generally). * **Riesz-Markov(-Kakutani) Theorem (Dual of $C_c(X)$):** Let X be a locally compact Hausdorff space. For every positive linear functional $\phi : C_c(X) \to \mathbb{R}$, there exists a unique (outer regular on Borel, inner regular on open and sigma-finite sets) Radon measure μ on X such that $\phi(f) = \int_X f d\mu$ for all $f \in C_c(X)$. (Can be extended to bounded linear functionals and complex/signed Radon measures). * **Riesz-Thorin Interpolation Theorem:** Let (X, μ), (Y, ν) be measure spaces. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. Let T be a linear operator defined on simple functions on X mapping to measurable functions on Y. Suppose $\|Tf\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}}$ and $\|Tf\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}}$. For $0 < \theta < 1$, define $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Then T extends to a bounded operator $T : L^{p_\theta} \to L^{q_\theta}$ with $\|T\|_{L^{p_\theta} \to L^{q_\theta}} \leq M_0^{1-\theta} M_1^\theta$.

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2. Functional Analysis

(2.1) Hilbert Spaces

* **Hilbert Space H:** Complete inner product space. Inner product $\langle \cdot, \cdot \rangle$. Norm $\|x\| = \sqrt{\langle x, x \rangle}$. * **Orthogonality:** $x \perp y$ if $\langle x, y \rangle = 0$. $M^\perp = \{x \in H : \langle x, y \rangle = 0, \forall y \in M\}$. * **Projection Theorem:** If M is a closed subspace of H, then $H = M \oplus M^\perp$. Any $x \in H$ has a unique decomposition $x = y + z$ with $y \in M, z \in M^\perp$. $y = P_M x$ is the orthogonal projection. * **Riesz Representation Theorem (Hilbert):** For every continuous linear functional $\phi \in H^*$, there exists a unique $y \in H$ such that $\phi(x) = \langle x, y \rangle$ for all $x \in H$. Moreover, $\|\phi\|_{H^*} = \|y\|_H$. Thus $H^* \cong H$ (anti-linearly). * **Orthonormal Basis (ONB):** $\{e_\alpha\}_{\alpha \in A}$ is an ONB if $\langle e_\alpha, e_\beta \rangle = \delta_{\alpha\beta}$ and span$\{e_\alpha\}$ is dense in H. * **Parseval's Identity:** For any $x \in H$, $x = \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha$ and $\|x\|^2 = \sum_{\alpha \in A} |\langle x, e_\alpha \rangle|^2$. * **Bessel's Inequality:** For any orthonormal set $\{e_\alpha\}$, $\sum_{\alpha \in A} |\langle x, e_\alpha \rangle|^2 \leq \|x\|^2$. * Separable iff ONB is countable. *

Fourier Basis for $L^2(T)$: $T = \mathbb{R}/(2\pi\mathbb{Z})$. The set $\{e_k(x) = \frac{1}{\sqrt{2\pi}}e^{ikx} : k \in \mathbb{Z}\}$ is an ONB for $L^2(T)$. For $f \in L^2(T)$, $f = \sum_{k\in\mathbb{Z}} \hat{f}(k)e_k$ in L^2, where $\hat{f}(k) = \langle f, e_k\rangle = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} f(x)e^{-ikx}dx$. (Often normalized differently: basis e^{ikx}, $\hat{f}(k) = \frac{1}{2\pi}\int$, $f = \sum \hat{f}(k)e^{ikx}$, $\|f\|^2 = 2\pi\sum|\hat{f}(k)|^2$).

(2.2) Banach Spaces

* **Banach Space X:** Complete normed vector space. * **Continuous Linear Maps:** $T : X \to Y$ is continuous $\iff$ T is bounded ($\|Tx\|_Y \leq C\|x\|_X$ for some C). Operator norm $\|T\| = \sup_{\|x\|=1}\|Tx\|$. * **Dual Space:** $X^* = B(X, \mathbb{K})$ (space of bounded linear functionals on X). It is a Banach space. * **Direct Sums:** $X \oplus Y$ with norm e.g. $\|(x, y)\| = \|x\| + \|y\|$. * **Quotient Space:** X/M where M is a closed subspace. Norm $\|x + M\| = \inf_{m\in M}\|x + m\|$. If X is Banach, X/M is Banach. * **Hahn-Banach Theorem:** * **(Extension form):** Let M be a subspace of X. Any $\phi \in M^*$ can be extended to $\tilde{\phi} \in X^*$ such that $\tilde{\phi}|_M = \phi$ and $\|\tilde{\phi}\|_{X^*} = \|\phi\|_{M^*}$. * **(Geometric form/Separation):** Let A, B be disjoint convex subsets of X, A open. Then $\exists\phi \in X^*$, $c \in \mathbb{R}$ such that $\mathrm{Re}(\phi(a)) < c \leq \mathrm{Re}(\phi(b))$ for all $a \in A, b \in B$. * **Baire Category Theorem:** A complete metric space is not a countable union of nowhere dense sets. * **Uniform Boundedness Principle (Banach-Steinhaus):** Let X be Banach, Y normed. Let $\mathcal{F} \subseteq B(X, Y)$. If for each $x \in X$, $\{\|Tx\|_Y : T \in \mathcal{F}\}$ is bounded (pointwise bounded), then $\mathcal{F}$ is uniformly bounded ($\sup_{T\in\mathcal{F}}\|T\| < \infty$). * **Open Mapping Theorem:** If $T \in B(X, Y)$ is surjective, X, Y Banach, then T is an open map (maps open sets to open sets). * **Inverse Mapping Theorem:** If $T \in B(X, Y)$ is bijective, X, Y Banach, then $T^{-1} \in B(Y, X)$. * **Closed Graph Theorem:** If $T : X \to Y$ is linear, X, Y Banach. T is bounded $\iff$ its graph $G(T) = \{(x, Tx) : x \in X\}$ is closed in $X \times Y$.

(2.3) Topological Spaces

* **(Local) Basis:** A collection $\mathcal{B}$ is a basis for topology τ if $\tau = \{\cup\mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$. $\mathcal{N}_x$ is a local basis at x if every neighborhood of x contains some $N \in \mathcal{N}_x$. * **Convergence:** Net $x_\alpha \to x$ if for every neighborhood U of x, $\exists\alpha_0$ such that $x_\alpha \in U$ for $\alpha \geq \alpha_0$. * **Continuity:** $f : X \to Y$ is continuous if $f^{-1}(V)$ is open in X for every open $V \subseteq Y$. * **Compactness:** Every open cover has a finite subcover. (Equivalent: Tychonoff - product of compact spaces is compact. Heine-Borel in $\mathbb{R}^n$). In metric spaces: compact $\iff$ sequentially compact $\iff$ complete and totally bounded. * **Weak Topology ($\sigma(X, X^*)$):** Smallest topology on X making all $\phi \in X^*$ continuous. Basis of neighborhoods of 0: $\{x : |\phi_i(x)| < \epsilon, i = 1, \ldots, k\}$. Convergence: $x_\alpha \to x$ weakly iff $\phi(x_\alpha) \to \phi(x)$ for all $\phi \in X^*$. * **Weak-* Topology ($\sigma(X^*, X)$):** Smallest topology on X^* making all evaluation maps $\hat{x}(\phi) = \phi(x)$ (for $x \in X$) continuous. Basis of neighborhoods of 0: $\{\phi : |\phi(x_i)| < \epsilon, i = 1, \ldots, k\}$. Convergence: $\phi_\alpha \to \phi$ weak-* iff $\phi_\alpha(x) \to \phi(x)$ for all $x \in X$. * **Stone-Weierstrass Theorem:** Let X be compact Hausdorff. Let $\mathcal{A}$ be a subalgebra of $C(X, \mathbb{R})$ that separates points and contains constants. Then $\mathcal{A}$ is dense in $C(X, \mathbb{R})$ (wrt uniform norm). (Complex version: need $\mathcal{A}$ closed under conjugation). * **Banach-Alaoglu Theorem:** The closed unit ball $B^* = \{\phi \in X^* : \|\phi\| \leq 1\}$ is compact in the weak-* topology $\sigma(X^*, X)$. (If X is reflexive, the unit ball B is weakly compact).

(2.4) Locally Convex Spaces (LCS)

* **LCS:** Vector space with topology generated by a family of seminorms $\{p_\alpha\}$ separating points. Basis of neighborhoods of 0: $V = \{x : p_{\alpha_i}(x) < \epsilon, i = 1, \ldots, k\}$. * **Minkowski Gauge:** For a convex, absorbing set A, $p_A(x) = \inf\{t > 0 : x \in tA\}$. p_A is a seminorm if A is balanced. Neighborhood basis of 0 exists consisting of convex, balanced, absorbing sets. * **Hahn-Banach (LCS):** * (Analytic): Let p be a seminorm on LCS X. Let M subspace, $\phi \in M^*$ s.t. $|\phi(x)| \leq p(x)$ on M. Then $\exists\tilde{\phi} \in X^*$ s.t. $\tilde{\phi}|_M = \phi$ and $|\tilde{\phi}(x)| \leq p(x)$ on X. * (Geometric): Disjoint convex sets A, B, A open. Can be separated by a closed hyperplane (defined by a continuous linear functional). * **Metrizability:** An LCS is metrizable iff its topology can be defined by a countable family of seminorms. * **Fréchet Space:** Complete metrizable LCS. * **Banach-Steinhaus (LCS):** Let X be Fréchet (or barrelled), Y be LCS. Let $\mathcal{F} \subseteq L(X, Y)$. If $\mathcal{F}$ is pointwise bounded, then $\mathcal{F}$ is equicontinuous. * **Open Mapping Theorem (LCS):** If $T : X \to Y$ is continuous, surjective, X, Y Fréchet, then T is open.

(2.5) Distributions

* **Test Functions on $T = \mathbb{R}/(2\pi\mathbb{Z})$:** $C^\infty(T)$, space of smooth 2π-periodic functions. Topology from seminorms $\|f^{(k)}\|_\infty$. Fréchet space. * **Distributions on T:** $D'(T) = (C^\infty(T))^*$, the continuous linear functionals on $C^\infty(T)$. $u \in D'(T)$ acts as $\langle u, \phi\rangle$. * **Schwartz Functions $S(\mathbb{R}^n)$:** Smooth functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\sup_{x\in\mathbb{R}^n}|x^\alpha\partial^\beta f(x)| < \infty$ for all multi-indices α, β. Topology from these seminorms. Fréchet space. * **Tempered

Distributions $S'(\mathbb{R}^n)$:** $(S(\mathbb{R}^n))^*$, the continuous linear functionals on $S(\mathbb{R}^n)$. * **Operations:** * **Differentiation:** $\langle \partial^\alpha u, \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle$. * **Multiplication:** For $f \in C^\infty$, $\langle fu, \phi \rangle = \langle u, f\phi \rangle$. (Requires $f\phi$ to be a test function). * **Convergence:** $u_j \to u$ in $D'(T)$ (or $S'(\mathbb{R}^n)$) if $\langle u_j, \phi \rangle \to \langle u, \phi \rangle$ for all test functions ϕ. * **Examples:** * Dirac delta δ_0: $\langle \delta_0, \phi \rangle = \phi(0)$. * Principal value: $\langle \text{pv}(1/x), \phi \rangle = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{\phi(x)}{x} dx$. * $(x \pm i0)^{-1}$: $\lim_{\epsilon \to 0^+} (x \pm i\epsilon)^{-1} = \text{pv}(1/x) \mp i\pi\delta_0(x)$. These are in $S'(\mathbb{R})$. * **Support:** $\text{supp}(u) = X \setminus \cup\{O \text{ open} : u|_O = 0\}$. Smallest closed set outside which u vanishes.

(2.6) Bounded Operators on Banach Spaces

* **Adjoint Operator:** $T \in B(X,Y)$. Adjoint $T^* \in B(Y^*, X^*)$ defined by $\langle Tx, y^* \rangle_Y = \langle x, T^* y^* \rangle_X$ (or $(T^* y^*)(x) = y^*(Tx)$). $\|T^*\| = \|T\|$. If H Hilbert, $T \in B(H)$, adjoint T^* satisfies $\langle Tx, y \rangle = \langle x, T^* y \rangle$. * **Spectrum $\sigma(T)$:** $T \in B(X)$. $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } B(X)\}$. Spectrum is compact, non-empty subset of $\mathbb{C}$. * **Resolvent Set:** $\rho(T) = \mathbb{C} \setminus \sigma(T)$. For $\lambda \in \rho(T)$, $R(\lambda, T) = (T - \lambda I)^{-1}$ is the resolvent operator. * **Spectral Radius:** $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. Gelfand's formula: $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$. Always $r(T) \le \|T\|$. * **Self-Adjoint Operators on Hilbert Space:** $T \in B(H)$, $T = T^*$. Then $\sigma(T) \subseteq \mathbb{R}$. * **Banach Space Valued Integrals/Derivatives:** E.g., Bochner integral. $\frac{d}{dt} f(t)$ can be defined. * **Continuous Functional Calculus (Self-Adjoint):** For $T = T^* \in B(H)$, there is an isometric *-isomorphism $\Phi : C(\sigma(T)) \to B(H)$ such that $\Phi(\text{id}) = T$, $\Phi(1) = I$. We write $f(T) = \Phi(f)$. $\|f(T)\| = \|f\|_{C(\sigma(T))} = \sup_{\lambda \in \sigma(T)} |f(\lambda)|$. * **Spectral Measure:** For $T = T^*$, there exists a unique projection-valued measure E on $\mathcal{B}(\sigma(T))$ such that $f(T) = \int_{\sigma(T)} f(\lambda) dE(\lambda)$.

(2.7) Compact Operators on Banach Spaces

* **Compact Operator:** $T \in B(X,Y)$. T is compact if it maps bounded sets in X to pre-compact (relatively compact) sets in Y. Equivalently, if $x_n \rightharpoonup x$ weakly, then $Tx_n \to Tx$ strongly (in norm). * **Properties:** * The set of compact operators $K(X,Y)$ is a closed subspace of $B(X,Y)$. * Composition: If S is bounded, T compact, then ST and TS are compact. * Adjoint: T is compact iff T^* is compact. * **Approximation (Separable Hilbert):** If H is separable Hilbert, $T \in K(H)$ iff T is the norm limit of finite-rank operators. * **Analytic Fredholm Theorem:** Let $D \subseteq \mathbb{C}$ open, connected. Let $A(z)$ be an analytic operator-valued function $D \to K(X)$. Then either $(I - A(z))^{-1}$ exists for no $z \in D$, or it exists for all $z \in D \setminus S$, where S is a discrete subset of D. * **Riesz-Schauder Theorem (Spectrum of Compact Operators):** Let $T \in K(X)$, X infinite-dimensional Banach. * $0 \in \sigma(T)$. * $\sigma(T) \setminus \{0\}$ consists only of eigenvalues $\{\lambda_k\}$. * This set is either finite or $\lambda_k \to 0$. * Each non-zero eigenvalue has finite (geometric and algebraic) multiplicity. * **Hilbert-Schmidt Operators (Hilbert Space):** $T \in B(H)$ is Hilbert-Schmidt if $\sum \|Te_k\|^2 < \infty$ for some (hence any) ONB $\{e_k\}$. Hilbert-Schmidt operators are compact. $\|T\|_{HS}^2 = \sum \|Te_k\|^2$.

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3. Fourier Analysis

(3.1) Fourier Series on $T = \mathbb{R}/(2\pi\mathbb{Z})$

* **Fourier Coefficients:** For $f \in L^1(T)$, $\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx$, $k \in \mathbb{Z}$. * **Orthogonality:** $\{e^{ikx}\}_{k\in\mathbb{Z}}$ are orthogonal in $L^2(T)$. $\langle e^{ikx}, e^{ilx} \rangle_{L^2} = \int_0^{2\pi} e^{i(k-l)x} dx = 2\pi\delta_{kl}$. * **$L^2$ Theory (Riesz-Fischer, Parseval):** * $f \in L^2(T) \iff \{\hat{f}(k)\} \in l^2(\mathbb{Z})$. * Parseval: $\|f\|_{L^2(T)}^2 = \int_0^{2\pi} |f(x)|^2 dx = 2\pi \sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2$. * The map $f \mapsto \{\hat{f}(k)\}$ is an isomorphism $L^2(T) \to \sqrt{2\pi} l^2(\mathbb{Z})$. * Fourier series $S_N(f)(x) = \sum_{k=-N}^N \hat{f}(k) e^{ikx} \to f$ in $L^2(T)$. * **Convergence:** * If $f \in C^1(T)$, then $S_N(f) \to f$ uniformly. * If f is continuous and $\sum |\hat{f}(k)| < \infty$, then $S_N(f) \to f$ uniformly. * Pointwise convergence can fail even for continuous functions (Du Bois-Reymond). Requires conditions like Dini's test or Hölder continuity. * **Smooth Functions $C^\infty(T)$:** $f \in C^\infty(T) \iff \hat{f}(k)$ decays faster than any power of $1/|k|$ (i.e., $|\hat{f}(k)| \le C_N |k|^{-N}$ for all N). * **Distributions $D'(T)$:** $u \in D'(T) \iff \hat{u}(k) = \frac{1}{2\pi} \langle u, e^{-ikx} \rangle$ has at most polynomial growth (i.e., $|\hat{u}(k)| \le C(1+|k|)^N$ for some N). * **Operations:** * Differentiation: $\widehat{f'(k)} = ik\hat{f}(k)$. $\widehat{(\partial^\alpha u)}(k) = (ik)^\alpha \hat{u}(k)$. * Multiplication: $\widehat{(e^{ilx} f)}(k) = \hat{f}(k-l)$. * **Sobolev Spaces $H^s(T)$:** For $s \in \mathbb{R}$, $H^s(T) = \{u \in D'(T) : \|u\|_{H^s}^2 = \sum_{k\in\mathbb{Z}} (1+k^2)^s |\hat{u}(k)|^2 < \infty\}$. $L^2(T) = H^0(T)$. H^s is a Hilbert space. * **Sobolev Embedding:** If $s > r$, the inclusion $H^s(T) \hookrightarrow H^r(T)$ is compact. If $s > 1/2 + k$, then $H^s(T) \hookrightarrow C^k(T)$ (continuous embedding). * **Representation of Distributions:** Any $u \in D'(T)$ can be written as $u = \sum_{|\alpha|\le N} \partial^\alpha f_\alpha$ for some N and continuous functions f_α. Or $u = F^{(N)}$ for some

N and $F \in L^2(T)$. * **Schwartz Kernel Theorem:** Let $K : C^\infty(X) \to D'(Y)$ be continuous linear. Then there exists a unique $k \in D'(X \times Y)$ (the kernel) such that $\langle K\phi, \psi\rangle = \langle k, \phi \otimes \psi\rangle$.

(3.2) Fourier Transform on $\mathbb{R}^n$

* **Definition:** For $f \in S(\mathbb{R}^n)$, $\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx$. Inverse: $\mathcal{F}^{-1}g(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} g(\xi)e^{ix\cdot\xi}d\xi$. (Normalization varies!) * **Properties on $S(\mathbb{R}^n)$:** $\mathcal{F} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is an isomorphism. * Differentiation: $\widehat{(\partial^\alpha f)}(\xi) = (i\xi)^\alpha \hat{f}(\xi)$. * Multiplication: $\widehat{(-ix)^\alpha} f(\xi) = (\partial^\alpha \hat{f})(\xi)$. * Convolution: $\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$. (If using $(2\pi)^{-n/2}$ normalization, it's $\sqrt{2\pi}^n \hat{f}\hat{g}$). * Translation: $(\widehat{f(\cdot - a)})(\xi) = e^{-ia\cdot\xi}\hat{f}(\xi)$. Modulation: $(\widehat{e^{ia\cdot x}f(x)})(\xi) = \hat{f}(\xi - a)$. * **Fourier Transform on $S'(\mathbb{R}^n)$:** Extend by duality: $\langle \hat{u}, \phi\rangle = \langle u, \hat{\phi}\rangle$ for $u \in S'$, $\phi \in S$. $\mathcal{F} : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ is an isomorphism. Properties above extend. * **L^2 Theory (Plancherel):** $\mathcal{F}$ extends uniquely to a unitary isomorphism $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ (with $(2\pi)^{-n/2}$ normalization). $\|\hat{f}\|_{L^2} = (2\pi)^{n/2}\|f\|_{L^2}$. Parseval: $\int f\bar{g}dx = (2\pi)^{-n}\int \hat{f}\bar{\hat{g}}d\xi$. * **Compactly Supported Distributions:** If $u \in \mathcal{E}'(\mathbb{R}^n)$ (distributions with compact support), then $\hat{u}(\xi)$ is an entire analytic function given by $\hat{u}(\xi) = \langle u_x, e^{-ix\cdot\xi}\rangle$. (Paley-Wiener-Schwartz Theorem characterizes these). * **Gaussian:** $\mathcal{F}(e^{-a|x|^2/2})(\xi) = (\frac{2\pi}{a})^{n/2}e^{-|\xi|^2/(2a)}$. * **Heat Equation:** $u_t = \Delta u$, $u(0, x) = f(x)$. Solution $u(t, x) = (K_t * f)(x)$, where $K_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/(4t)}$. In Fourier space: $\hat{u}_t(t, \xi) = -|\xi|^2\hat{u}(t, \xi)$, so $\hat{u}(t, \xi) = e^{-t|\xi|^2}\hat{f}(\xi)$. $\hat{K}_t(\xi) = e^{-t|\xi|^2}$. * **Sobolev Spaces $H^s(\mathbb{R}^n)$:** $H^s(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : (1 + |\xi|^2)^{s/2}\hat{u}(\xi) \in L^2(\mathbb{R}^n)\}$. Norm $\|u\|^2_{H^s} = \int_{\mathbb{R}^n}(1 + |\xi|^2)^s|\hat{u}(\xi)|^2d\xi$. H^s is a Hilbert space. * **Sobolev Embedding:** If $s > n/2 + k$, then $H^s(\mathbb{R}^n) \hookrightarrow C^k_B(\mathbb{R}^n)$ (continuously embedded into space of k-times continuously differentiable functions with bounded derivatives).

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