Alibaba Global Mathematics Competition 2023 Qualifying Round
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1
One may test the Input cases using software such as Wolram Mathematica or Alpha. Plugging in equations, A will not work as it will never clear the horizontal asymptote $\sqrt{2}$ value and C and D will not work either as after they clear it they will actually descend down in to horizontal asymptotes not to 0 and thus the only answer which can be verified to work here is
$\boxed{\text{B Set } a = 3\text{, kick the instrument, wait for the ball lightning radius to strictly}}$
$\boxed{\text{exceed } \sqrt{2}\text{, then set } a = -\frac{1}{3}}$.

2

There exist ways to obtain $2-\epsilon$. In particular one can configure a needle with multiplicity 4 along the edges e.g. by contracting a mid-square to nearly a point and then also obtain that needle with multiplicity ≈ 8 by contracting 2 of the diagonal points on that square up and 2 of them downwards. Convexity is preserved but thus it is a super duper squiggly squiggle. So the answer ought to be $\boxed{\text{ABCD}(0,2)}$.

3

If I read the task statement quite crystal clearly technically and precisely then this first step is free from the condition but I am not too sure and can run naive Python simulations under both assumptions to spot patterns in the first 20 or so and see what I think about parity and behaviour as $n \to \infty$ and so on and so on. So I will just go ahead and assume that we are ignoring odd writing on the leading steps and thus the answer I think is $\boxed{\text{B } n = 32}$.

4

Note that we may cycle around a square until exiting with 1 straight operation costing 1 in any of the 4 directions and thus the task isomorphs in to finding the shortest path of squares which starts and ends at 2 squares sharing at least 1 vertex and this path of squares becomes adjacent to each vertex along the way. Well, each new square can add at most 2 unvisited vertices to the vertex set so assuming the initial move is free and has cost 0 it is still the case that this gives a proof that 48 is certainly impossible as the 2 new dudes each round means we can not have the 2 previous edge dudes as well as 1 of the initial 4 back in means after 48 steps I think we are doomed to end up 2 steps away or else maybe there would be a way with 49 steps where on e.g. the 48th and 49th step we merely add back in 1 final each step 1 final unvisited vertex. Closely examining the uh 5×5 case is enlightening with respect to a parity of parity of parity argumentation I think that modulo 4 in the induced sub grid like checkering there exists some argumentation that after uh 46 steps it is like uh I dunno seems to me that after 46 steps 4 more are still forced and so on and so on. I do not think that there can exist a viable path on 49 steps here just seems like impossible to me. $\boxed{\text{B } 50 \leq S < 90}$.

5

1

This is canonical. For example we may set all of the values to be equal to 1 except for $a_{1,n}$ and $a_{i+1,i}$ for all $1 \leq i \leq n-1$. Then by standard Linear Algebra manipulations we may take in the $n = 4$ case for example:

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 3 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix} = (-1)^{n-1} \cdot \begin{vmatrix} 3 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{vmatrix} =$$
$$(-1)^{2n-2} \cdot (n-1) = \boxed{n-1}$$

As desired. Where in the first step we subtracted off the 1st row from each of the other rows all whilst preserving the determinant. In the 2nd step we added each of the other rows to the 1st row. In the 3rd step we swapped the $n-1$ adjacent pairs of columns from Right to Left until that Rightmost column became the Leftmost column, and each swap induced a -1 multiplier in to the determinant. And then finally we evaluated the determinant via the usual diagonal multiplication for a Lower Left Diagonal matrix. In this particular case the 3 merely stands in as a clear function of $n-1$ as desired and needed for the claimed.

2

This is canonically known as the Hadamard Maximal Determinant Problem and for example the original papers and Online Encylopaedia Of Integer Sequences Entry A003432 immediately reveal the result, as does a simple casework bashing computational approach for example. But I think citing the Theorem presented in that entry is strong enough for full points right here in this submission as a proof rather than enumerating the casework or providing a simple written proof. 1 relatively simple argumentation can start dealing with the determinant and the volume of the underlying paralleliped for example maybe. See Theorems below. Certainly Coq or Lean could be manipulated or the C programming language to produce a rigorous machine-verifiable proof here. The $n = 2$ case is rather trivial as clearly $a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \leq 1$ by maximising the 1st term at $1 = 1 \cdot 1$ and minimising the 2nd term at 0 with a 0 in the product. The $n = 3$ case in particular follows from Hadamard and Barba and Williamson Theorem. In fact, so does the $n = 4$ case too in particular from $f(4) \leq 2^{-4} \cdot 5^{\frac{5}{2}} \approx 3.493856$ so Hadamard's Theorem works here to prove the claim as it must be an integer and hence this implies that $f(4) \leq 3$.

3

Strong induction relating matrix determinants and powers of n and powers of 2 along with exponential blowup asymptotic factors can be utilised to produce the desired matrices in a recursive construction. Frankly I think that the worked out base cases in the range of $16-31$ or $16-63$ depending upon your sources will work in conjunction with the Theorem presented as early as On The Value Of Determinants by John H. E. Cohn. There they proved that $f(n) > n^{n\left(\frac{1}{2}-\epsilon\right)} \cdot 2^{-n}$ essentially and so in particular it suffices to note that this is $> n^{\frac{n}{4}}$ in our case as $n \geq 2023 \rightarrow n^{\frac{n}{4}-\epsilon} > 2^n = \left(2^4\right)^{\frac{n}{4}} = 16^{\frac{n}{4}}$ as $n >>> 16$ and thus the claim is proven by Theorem 13 from that paper there in conjunction with the contemporary Computer Algebra solutions presented at the Online Encyclopaedia Of Integer Sequences links for example at Archives of The Hadamard Maximal Determinant Problem by William P. Orrick and B. Solomon hosted at Indiana.

`https://www.jstor.org/stable/2034278`

`https://oeis.org/A003432`

`https://web.archive.org/web/20200219170713/http://www.indiana.edu/~maxdet/`

6

1

$\boxed{\text{Yes}}$. Certainly there does exist such a nonzero real number s, namely $\boxed{s = 1}$ works! In particular it then suffices to show that in the limit as $n \to \infty$ the difference between $\left(\sqrt{2}+1\right)^n$ and its nearest positive integer goes to 0. Well this is canonical as for example irrational conjugate multiples of $\sqrt{2}$ cancel and thus $\left(\sqrt{2}+1\right)^n + \left(\sqrt{2}-1\right)^n$ is always a positive integer in the expansion with no non-integer components. Well then it suffices now to note that indeed $\lim_{n\to\infty} \left(\sqrt{2}-1\right)^n = 0$ and thus the Left hand side component in the sum will is in fact this precise quantity, the effective limit of the minimal distance under question.

2

$\boxed{\text{No}}$. In particular we may assume that there did exist such an s value. Consider the canonical results about linear recurrences, and in particular degree-2 linear recurrences such as the famous Fibonacci Sequence. Well in particular if we consider the sequence of closest-positive-integers generated via s, that is to say $\text{NearestInt}(s(3+\sqrt{2})^n)$, and the given exponentiation this sequence will eventually be generated via a similar linear recurrence and generating function. In particular $x^2 - 6x + 7$ and $g_n = 6g_{n-1} - 7g_{n-2}$. Otherwise we would have an immediate contradiction at any moment of deviation from that sequence with respect to the underlying modulo 1 argumentation. So then the fundamental integrality I mean the fact that these initial values there are integers implies that the solution for an explicit formula is, a la Alfred Binet, of the form $g_n = a(3+\sqrt{2})^n + b(3-\sqrt{2})^n$. And in particular b must be nonzero as otherwise the $n = 0$ case immediately implies a must be an integer but then the $n = 1$ case would imply a contradiction on the integer rationality of g_1 supposing that $b = 0$. Well then there is a contradiction as $\lim_{n\to\infty} \frac{g_n}{(3+\sqrt{2})^n} = a$ however this implies that $s = a$ but then in reverse producing terms outwards we get an immediate contradiction as the inequalities $b \neq 0$ and $|3-\sqrt{2}| > 1$ imply that this error term does not go to 0 namely we have that $\lim_{n\to\infty} b(3-\sqrt{2})^n \neq 0$ which implies we are doomed to deviate non trivially from that underlying integer sequence and will thus never be able to attain the desired limit of 0.

7

(a)

This follows immediately via a strict domination smoothing argument. See the following Theorems cited below in links and papers and references.

(b)

This is extremely well known and canonical in the literature known as the Secretary Problem with answer of $\boxed{\frac{1}{e}}$ being well known and understood.

https://en.wikipedia.org/wiki/Secretary_problem

(c)

$\boxed{p^{\frac{1}{1-p}}}$ is a Theorem proven in A Secretary Problem With Uncertain Employment by M. H. Smith.

https://www.jstor.org/stable/3212880