Algebra & Number Theory

1. For a subset $S \subset \mathbb{R}^3$, the vanishing ideal $I(S) \subset \mathbb{R}[x, y, z]$ is the ideal of the polynomial ring $\mathbb{R}[x, y, z]$ generated by all elements $f \in \mathbb{R}[x, y, z]$ satisfying $f(x_0, y_0, z_0) = 0$ for every $(x_0, y_0, z_0) \in S$. Find the minimal integer k such that for any three distinct lines L_1, L_2, L_3 in $\mathbb{R}^3$ intersecting at one point, the ideal $I(L_1 \cup L_2 \cup L_3)$ can be generated by k elements; justify your answer.

2. Let $p > 2023$ be a prime number. Denote by $\mathcal{X}$ the set of all 2000-dimensional subspaces of the $\mathbb{F}_p$-vector space $\mathbb{F}_p^{2023}$. Find the minimal cardinality of a subset $\mathcal{Y}$ of $\mathcal{X}$ such that

$$\sum_{W \in \mathcal{Y}} (V \cap W) = V$$

holds for every $V \in \mathcal{X}$; justify your answer.

3. A number field K is totally real if for every homomorphism $\varphi\colon K \to \mathbb{C}$, the image is contained in $\mathbb{R}$. Show that for every integer $d > 1$, there exist infinitely many isomorphism classes of totally real number fields of degree d.

4. Let $n \geqslant 2$ be an integer. Consider the free $\mathbb{Z}$-module $V := \mathbb{Z}^{\oplus n}$ (written in columns) of rank n. Prove that there exist some positive integer N and some element

$$s \in V^{\otimes N} := \underbrace{V \otimes_{\mathbb{Z}} V \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} V}_{N \text{ times}},$$

such that for every prime number p, an element $g \in \mathrm{GL}_n(\mathbb{F}_p)$ is upper triangular if and only if the image of s in $V^{\otimes N} \otimes_{\mathbb{Z}} \mathbb{F}_p$ is an eigenvector of g. Here, $\mathrm{GL}_n(\mathbb{F}_p)$ diagonally acts on $V^{\otimes N} \otimes_{\mathbb{Z}} \mathbb{F}_p = (V \otimes_{\mathbb{Z}} \mathbb{F}_p)^{\otimes N}$ from the left. (There will be partial credits for the case $n = 2$.)

5. Let n be a positive integer. Denote by $\mathbb{C}[\mathbb{C}^n]$ and $\mathbb{C}(\mathbb{C}^n)$ the rings of polynomial functions and rational functions on $\mathbb{C}^n$, respectively. Let Γ be the group of automorphisms of $\mathbb{C}^n$ generated by permutations of coordinates and translations by integer points. Let $r\colon \Gamma \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[\mathbb{C}^n])$ be the homomorphism sending γ to the automorphism r_γ such that $(r_\gamma P)(x) = P(\gamma^{-1}x)$ for $P \in \mathbb{C}[\mathbb{C}^n]$ and $x \in \mathbb{C}^n$. Let R be the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathbb{C}^n])$ generated by the image of r and multiplications by elements in $\mathbb{C}[\mathbb{C}^n]$, which is in particular a $\mathbb{C}[\mathbb{C}^n]$-algebra.

   (a) Show that R is a free $\mathbb{C}[\mathbb{C}^n]$-module.

   (b) Let $T \in \mathbb{C}(\mathbb{C}^n) \otimes_{\mathbb{C}[\mathbb{C}^n]} R$ be an element, regarded as an endomorphism of $\mathbb{C}(\mathbb{C}^n)$, that preserves $\mathbb{C}[\mathbb{C}^n]$. Show that T belongs to $\mathbb{C}\langle\mathbb{C}^n\rangle \otimes_{\mathbb{C}[\mathbb{C}^n]} R$, where $\mathbb{C}\langle\mathbb{C}^n\rangle$ is the $\mathbb{C}[\mathbb{C}^n]$-subalgebra of $\mathbb{C}(\mathbb{C}^n)$ generated by the inverse of all nonzero polynomials of degree one with integer coefficients.

Geometry & Topology

1. Let A be a subset of $\mathbb{R}^3$ which consists of 2023 points. We further assume that A does not contain 5 points which are in the same sphere. (Here, sphere is in the geometric sense, namely, it is the set of points at a given distance from a given point.) Prove that there exist three spheres S_1, S_2, S_3, satisfying the following property: the complement $\mathbb{R}^3 \setminus (S_1 \cup S_2 \cup S_3)$ has 8 components, such that one component contains 252 points in A, and each of the other seven components contains 253 points.

2. Let M be an oriented compact minimal surface in $\mathbb{R}^3$ with connected boundary ∂M, prove that
$$L^2(\partial M) \geq 4\pi\sigma(M)$$
where $L(\partial M)$ is the length of ∂M and $\sigma(M)$ is the area of M.

3. Let X be a CW complex such that $H^2(X;\mathbb{Z}) = 0$, V be a real vector bundle over S^2 and $f : X \to S^2$ is a continuous map. Show that $f^*(V)$ is a trivial bundle.

4. Ali Baba knew the phrase "Open Sesame" for the thieves' cave, which contains trmendous treasures. This exotic and huge cave can be simplified as an xy-plane with $x \geq 0$ and $y \geq 0$ but without $x = y = 0$.

   Everyday Ali Baba secretly enters into the cave to take some treasures with him and left the cave to help the people in need. But one day, Ali Baba was discovered by the 40 thieves. At that time, Ali Baba was at $(4, 3)$, one group of 20 thieves was at $(2+\sqrt{3}, 0)$, another group of 20 thieves was at $(0, 1)$, the only Exit of whole cave was at $(1, 0)$ and there was a magic ring located at $(6, 5)$. The speed of Ali Baba and 40 thieves is $x + y$ if they are at the location (x, y). Either Ali Baba or each group of thieves only know their own location and location of the Exit. In addition, **Only Ali Baba** know the location of the magic ring and how to summon the genie of the ring, who can help to double Ali Baba's speed, i.e. the speed of Ali Baba will become $2(x + y)$ at (x, y).

   Question. Please determine whether Ali Baba can arrive the Exit earlier (in order to escape this cave) than two other groups of thieves or not and prove your claims.

   Remarks:

   (a) Anyone can move freely in this cave. The only possible solution to survive for Ali Baba is to arrive at Exit earlier than any group of thieves.

   (b) We assume that the 2 groups of thieves will not try to catch Ali Baba before they arrive the Exit because they do not know the location of Ali Baba and the ring. Thus they just try to arrive at the Exit as soon as possible.

   (c) We also assume that the time of summoning and gaining the ability of double speed can be neglected.

5. Let M be a complete Riemannian orbifold of dimension $n \geq 2$ with isolated singular points.

   (1) Assume that the sectional curvature is nonnegative and the volume growth at infinity is positive, namely

   $$\lim_{r\to\infty} \frac{vol(B(x,r))}{r^n} > 0. \tag{1}$$

   at some x. Show that the number of singular points is at most 1.

   (2) Assume that M is compact and the Ricci curvature is nonnegative. Assume in addition that the local group Γ_x at some singular point x acts irreducibly in the sense that it has no non-trivial invariant subspace. Show that $b_1(M) = 0$.

Here are some notions that may be used in the problem. An orbifold of dimension n is a topological space M such that at each x there is a neighborhood U_x and a homeomorphism $\phi_x : U_x \to B/\Gamma_x$ with $\phi_x(x) = 0$, where B is the unit ball in the n-dimensional Euclidean space $\mathbb{R}^n$ and Γ_x is a discrete subgroup of $O(n)$ acting linearly and effectively on $\mathbb{R}^n$. The data (U_x, Γ_x, ϕ_x) is called an orbifold chart at x. A point x is called a regular point if Γ_x is trivial in certain chart; otherwise, it is called a singular point. A Riemannian orbifold M, with isolated singular points $\Sigma = \{p_i\}_{i\in I}$, is an orbifold satisfying the followings: (a) $M\backslash\Sigma$ is a smooth manifold with a (non-complete) Riemannian metric g, (b) there is a family of charts (U_i, Γ_i, ϕ_i) at p_i such that (b1) ϕ_i is smooth on $U_i\backslash\{p_i\}$ and (b2) the metric $g = \phi_i^*(\tilde{g}_i)$ where $\tilde{g}_i$ is a smooth, Γ_i-invariant metric on B. The Riemannian orbifold is called complete if the induced metric space is complete.

Analysis & Differential Equations

1. Consider the sequence
$$a_{n+1} = a_n + \frac{a_n^2}{n^2}, \quad a_1 = \frac{2}{5}.$$
Show that $a_n < 1$ for all $n \geq 1$.

2. Let V be a n dimensional linear space with continous functions defined on the closed region $Q = [0,1] \times [0,1] \subset \mathbb{R}^2$. Prove that there exists $f \in V$ such that
$$\|f\|_{L^2(Q)} = 1, \quad \|f\|_{L^\infty(Q)} \geq \sqrt{n}.$$

3. Let $f(z)$ be an analytic function defined on the complex plane $\mathbb{C}$. Define $f^{(n+1)}(z) = f(f^{(n)}(z))$ with $f^{(1)}(z) = f(z)$. Is there analytic function $f(z)$ on $\mathbb{C}$ such that $f^{(2023)}(z) = e^{2023z}$ for all $z \in \mathbb{C}$? Prove your assersion.

4. Let $B_r = \{x \in \mathbb{R}^3 | |x| \leq r\}$ be the ball with radius r in $\mathbb{R}^3$. Suppose that u is a continuous function in $\mathbb{R}^3 \backslash B_1$ satisfying
$$\Delta u \leq -u^3, \quad \text{and} \quad u \geq 0 \text{ in } \mathbb{R}^3 \backslash B_1.$$
Show that
$$u = 0 \text{ in } \mathbb{R}^3 \backslash B_1.$$

5. Consider the function
$$F(\alpha_1, \beta_1, \cdots, \alpha_n, \beta_n) := \int_{0<s_1<t_1<\cdots<s_n<t_n<1} \prod_{j=1}^{n} e^{i(\alpha_j(t_j+s_j)+\beta_j t_j)}\, \mathrm{d}t_j \mathrm{d}s_j.$$
For any measurable set $E \subset \mathbb{R}^n$ and any $\delta > 0$, show that there exists a constant C_δ depending only on δ (in particular independent of E and n) such that
$$\int_E \mathrm{d}\beta_1 \cdots \mathrm{d}\beta_n \int_{\mathbb{R}^n} |F(\alpha_1, \beta_1, \cdots, \alpha_n, \beta_n)|\, \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n \leq C_\delta^n \cdot |E|^\delta.$$

Applied & Computational Mathematics

1. Consider the following linear system

$$Ax = b,$$

where $A = (A_{ij})_{ij} \in \mathbb{R}^{n\times n}$ is a symmetric positive semidefinite matrix with positive diagonal entries, $b = (b_i)_i \in \mathbb{R}^n$. Assume that there exists at least one solution. Now we use the Gauss-Seidel method to solve this linear system. Its iterative scheme is

$$x_i^{(k+1)} := \frac{1}{A_{ii}} \left(b_i - \sum_{j=1}^{i-1} A_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} A_{ij} x_j^{(k)} \right), \quad i = 1, \dots, n, \;\; k \in \mathbb{N}.$$

[(a)]

(a) Please prove that it converges from any initial point.

(b) In addition, if A is a rank-1 matrix, please prove that it converges after one iteration.

2. In order to align a large language model (LLM) with human preferences, it is necessary to fine-tune the LLM to adhere to rankings derived from human feedback. Given a specific prompt, let the LLM generate n responses. A human labeler will then rank these n responses from best to worst. Denote the ranking by π (for example, the $\pi(1)$-th response is considered the best). Suppose we have a function G that is strongly concave (so $-G$ is strongly convex) and increasing on the interval $[-1, 1]$. We aim to train a reward model that assigns a rating $0 \le r_i \le 1$ to the i-th response. Ideally, the rewards r_i should be the solution to the following optimization program:

$$\max_{0\le r_1,\dots,r_n \le 1} \sum_{i<j} G(r_{\pi(i)} - r_{\pi(j)}).$$

(a) Explain why

$$L(r_1, \dots, r_n) := \sum_{i<j} G(r_{\pi(i)} - r_{\pi(j)})$$

is concave but not strongly concave (you can assume sufficient smoothness of G if it is helpful). Prove that, despite this, the optimization program stated above has a *unique* solution $(r_1^\star, \dots, r_n^\star) \in [0, 1]^n$.

(b) Prove that the solution must satisfy

$$1 = r_{\pi(1)}^\star \ge r_{\pi(2)}^\star \ge \cdots \ge r_{\pi(n)}^\star = 0$$

and $r_{\pi(i)}^\star + r_{\pi(n+1-i)}^\star = 1$ for all $i = 1, \dots, n$.

(c) Now, consider the limiting version of the problem above. Assume that as $n \to \infty$, the empirical distribution of $r_1^\star, \dots, r_n^\star$ converges to a probability measure μ on the interval $[0, 1]$. The problem can then be reformulated as

$$\sup_{\mu} \mathbb{E}_{X,X' \overset{iid}{\sim} \mu} G(|X - X'|),$$

where X, X' are independent and identically distributed draws from μ. If a probability measure $\mu^\star$ maximizes $\mathbb{E}_{X,X' \overset{iid}{\sim} \mu} G(|X - X'|)$, prove that $\mathbb{E}_{X\sim\mu^\star} G(|X - c|)$ is independent of $c \in [0, 1]$.

3. Consider a one-dimensional interacting particle system represented by the following system of stochastic differential equations (SDEs):

$$\mathrm{d}X_i = -U'(X_i)\,\mathrm{d}t - \theta\left(X_i - \frac{1}{N}\sum_{k=1}^{N} X_k\right)\mathrm{d}t + \sigma\,\mathrm{d}W_t^i, \quad i = 1, 2, \cdots, N. \qquad (2)$$

Here, each particle X_i belongs to the real line $\mathbb{R}$, and the parameters θ and σ are positive. The external potential, denoted by $U(x)$, is defined as $U(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$. Additionally, note that $(W_\cdot^i)$ represents N independent standard Brownian motions in one-dimensional space.

(a) Let the joint distribution of the N particles be represented by $F_N(t,\cdot)$. Please write down the evolution equation for F_N explicitly.

(b) Assuming independent and identically distributed (i.i.d.) initial distribution, denoted by $F_N(0) = f_0^{\otimes N}$, where f_0 belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$. We define the relative entropy as

$$H(F_N(t)|f_t^{\otimes N}) = \int_{\mathbb{R}^N} F_N(t) \log \frac{F_N(t)}{f_t^{\otimes N}}\,\mathrm{d}x_1 \cdots \mathrm{d}x_N.$$

Please demonstrate that for any fixed time $T > 0$,

$$\sup_{t\in[0,T]} H(F_N(t)|f_t^{\otimes N}) \le C_T < \infty,$$

where C_T is a universal constant that only depends on T and $(f_t)_{t\in[0,T]}$ represents the solution to the following nonlinear PDE

$$\partial_t f_t + \partial_x\Big(f_t[-U'(x) - \theta\,(x - m(t))]\Big) = \frac{\sigma^2}{2}\Delta_x f_t,$$

subject to the initial condition f_0, where $m(t) = \int_{\mathbb{R}} y f(t,y)\,\mathrm{d}y$.

(c) In general, can we anticipate extending the prior result to a uniform-in-time version, as shown below?

$$\sup_{t\in[0,\infty)} \int_{\mathbb{R}^N} F_N \log \frac{F_N}{f_t^{\otimes N}}\,\mathrm{d}x_1 \cdots \mathrm{d}x_N \le C < \infty$$

Provide your conclusion along with some intuitive calculations to support your response.

4. Consider an elliptic equation on an open bounded domain Ω with C^2 boundary:

$$\begin{cases} -\nabla\cdot(a(x,u)\nabla u) = h(x)\,, & x\in\Omega\subset\mathbb{R}^d\,,\\ u(x)=g(x), \quad x\in\partial\Omega\,, \end{cases}$$

where $C \geq a(x,u(x)) \geq c > 0$ for all $x\in\Omega$. We solve this boundary value problem by finding an approximation $f(t,x;\theta)\in\mathcal{F}$, where θ are collection of approximation parameters. $\mathcal{F}$ is a function space such that for any $u\in C^2(\bar\Omega)$ and $\epsilon>0$, there exists a function $f(t,x;\theta)\in\mathcal{F}$ that satisfies

$$\max_{|\alpha|\leq 2}\sup_{x\in\bar\Omega}|\partial_x^{(\alpha)}u-\partial_x^{(\alpha)}f|\leq\epsilon\,.$$

(a) Assume that the above boundary value problem has a unique classical solution, and $a(x,u)$ and $\frac{\partial}{\partial u}a(x,u)$ are locally Lipschitz in u with at most polynomial growth, i.e.,

$$|a(x,u)-a(x,v)|\leq C(|u|^{q_1/2}+|v|^{q_2/2})|u-v|, \quad \text{uniformly in } x\,;$$
$$|\frac{\partial}{\partial u}a(x,u)-\frac{\partial}{\partial v}a(x,v)|\leq C(|u|^{q_3/2}+|v|^{q_4/2})|u-v|, \quad \text{uniformly in } x\,,$$

where q_1,q_2,q_3,q_4 are four nonnegative constants. Let

$$J(f(\cdot;\theta)) = \|\nabla\cdot(a(x,f)\nabla f)+h\|^2_{L^2(\Omega)}+\|f-g\|^2_{H^1(\partial\Omega)}\,.$$

Show that for any $\epsilon>0$, there exists a constant $C>0$ such that

$$J(f(\cdot;\theta))<C\epsilon\,.$$

(b) For a simpler case when $a(x,u)\equiv 1$, consider a sequence of approximations $f(\cdot;\theta_k)$ such that $J(f(\cdot;\theta_k))\to 0$ as $k\to\infty$. Show that $\|f(\cdot;\theta_k)-u\|_{L^2(\Omega)}\to 0$.

(c) The definition of J indicates that the boundary condition is treated as a penalty. Can you directly build the boundary condition to the approximation?

5. Kids learn how to amplify a swing's motion by adjusting their gestures to vary the swing's length. Let us consider an extremely simple mathematical model for a swing given by a linear pendulum such that

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}+\big(1+\epsilon\cos(\omega t)\big)\phi=0$$

with initial conditions $\phi(0)=0$ and $\phi'(0)=1$. Here $\epsilon\cos(\omega t)$ models the varying length of the swing and $\epsilon\ll 1$ is a small parameter.

(a) To amplify the swing's motion under the above model, what value(s) of ω should one choose?

(b) Consider a specific choice $\omega=1$, expand the solution as a series in ϵ

$$\phi^\epsilon(t)=\phi_0(t)+\epsilon\phi_1(t)+\epsilon^2\phi_2(t)+\ldots$$

Find the evolution of ϕ_0 for t up to $O(1/\epsilon^2)$.

(c) Moreover, consider the case that the swing's natural length is perturbed as well, so the model becomes

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}+\big(1-\alpha\epsilon^2+\epsilon\cos(\omega t)\big)\phi=0$$

with initial conditions $\phi(0)=0$ and $\phi'(0)=1$. Fix $\omega=1$, determine the range of α such that the pendulum is unstable.

Combinatorics & Probability

1. Let $(B_t)_{t\in\mathbb{R}}$ be such that $(B_t)_{t\geq 0}$ and $(B_{-t})_{t\geq 0}$ are two independent standard one-dimensional Brownian motions with $B_0 = 0$. (This is called a two-sided Brownian motion.) Let $X_t = B_t - |t|$ for $t \in \mathbb{R}$. Let τ be the (almost surely) unique time such that $X_\tau = \max_{t\in\mathbb{R}} X_t$. Show that $(X_{\tau+t})_{t\geq 0}$ and $(X_{\tau-t})_{t\geq 0}$ equal in law.

   Note. Partial credit will be given accordingly if a proper discrete analog of the statement above can be formulated and proved.

2. For each positive integer n, find the largest $f(n) \in \mathbb{R}$ such that the following property holds. For every $n \times n$ doubly stochastic matrix M (that is, a square matrix of non-negative real numbers, each of whose rows and columns sums to 1), there always exists a permutation π of $[n]$ such that $M_{i,\pi(i)} \geq f(n)$ for every $i \in [n]$. Here, $[n]$ denotes the set $\{1, \ldots, n\}$.

3. Let $n \geq 2$ be a positive integer and denote by $[n] = \{1, \ldots, n\}$. For $(u_1, \ldots, u_n) \in [0,1]^n$, define its S-average and its r-average by, respectively,

$$S(u_1, \ldots, u_n) = \min_{i\in[n]} \frac{n}{i} u_{(i)} \quad \text{and} \quad M_r(u_1, \ldots, u_n) = \left(\frac{1}{n} \sum_{i=1}^{n} u_i^r \right)^{1/r},$$

   where $u_{(i)}$ is the i-th smallest number in $u_1, \ldots, u_n$, and $r \in \mathbb{R} \setminus \{0\}$. A random variable U is *standard-uniform* if $\mathbb{P}(U \leq \alpha) = \alpha$ for all $\alpha \in (0,1)$ and it is *sub-uniform* if $\mathbb{P}(U \leq \alpha) \geq \alpha$ for all $\alpha \in (0,1)$. Let $U_1, \ldots, U_n$ be independent standard-uniform random variables.

   (a) Show that $S(U_1, \ldots, U_n)$ is standard-uniform.

   (b) Show that $M_r(U_1, \ldots, U_n)$ is sub-uniform if and only if $r \leq -1$.

4. The next two questions are about identification while revealing as little information as possible, but their solutions do not depend on each other.

   (a) A server stores m (not necessarily distinct) Boolean vectors from $\{0,1\}^n$, where $1 \le m \le 2^n$. You goal is to find a vector distinct from all of them (or decide that this is not possible). The only query you can make is of the type "what is the i-th bit of the j-th vector" for any $1 \le i \le n$ and $1 \le j \le m$, and the server will reveal the asked bit.

   What is the minimum number of queries you have to make, as a function of m and n? Prove that your answer is both sufficient and necessary.

   Hint. The answer changes depending on the value of m. There are three regimes.

   (b) Let $f : \{0,1\}^n \to \{-1,1\}$ be an n-dimensional function, i.e., a Boolean function with n variables. To determine the value of $f(x)$ for some $x \in \{0,1\}^n$, one may not need to know every bit of x. Consider the following "query method" $\mathcal{T}$. We reveal the bits of x one-by-one, and once the value of $f(x)$ is determined by the revealed bits so far, we stop and output the value. If $f(x)$ is not determined, the query method $\mathcal{T}$ uses the revealed bits to choose the next coordinate of x to query. This process continues until $f(x)$ is determined, and in the worst case all bits of x may be revealed.

   Now, let $X : \{1,\dots,n\} \to \{0,1\}$ be a random bit string of length n, with each bit $X_i \sim \mathrm{Ber}(q_i)$, $0 < q_i < 1$, sampled independently. Write P and for the joint law. Given a query method $\mathcal{T}$ of the function f, for each sample of X, a certain bit of X may or may not be queried in determining the value of $f(X)$. We write

   $$\delta_i(\mathcal{T}) = P[X_i \text{ is queried while executing } \mathcal{T}], \qquad i = 1,\dots,n.$$

   The influence of the i-th bit of X on the Boolean function g is defined as

   $$\mathrm{Inf}_i(g) = P[g(X) \ne g(X^{(i)})]$$

   where $X^{(i)}$ is X with its i-th bit flipped.

   Let $f, g : \{0,1\}^n \to \{-1,1\}$ be two Boolean functions, and let $\mathcal{T}$ be an arbitrary query method to determine the value of f. Show that

   $$|\mathrm{Cov}[f(X), g(X)]| \le \sum_{i=1}^{n} \delta_i(\mathcal{T}) \mathrm{Inf}_i(g).$$

   Here $\mathrm{Cov}(\cdot,\cdot)$ is the covariance.

5. Prove that for any $\varepsilon > 0$, there exists an integer n_0 such that for any $n \ge n_0$, any n-vertex simple graph with at least $n^{1+\varepsilon}$ edges contains a cycle C which has at least $|E(C)|$ chords, where $|\cdot|$ is the cardinality. (A *chord* of a cycle C is an edge connecting two vertices of C but not an edge in the edge set $E(C)$ of C.)

   Note. Partial credit will be given accordingly if a reasonable small constant $\varepsilon > 0$ is proved.