1 Ball lightening

As a chief officer of a secret mission, you had the following conversation with the leading scientist.

Scientist: "Chief, we have mastered the control law of ball lightning. We found that the rate of change of the radius of ball lightning in the laboratory $v(t)$ satisfies the following equation.

$$v = ar + r^3 - r^5.$$

Here $r(t)$ represents the radius of ball lightning, and t is the time variable. At the initial moment, there is no ball lightning, that is, $r(0) = 0$. Accordingly, we also have $v(0) = 0$. And $a \in \mathbb{R}$ can be artificially controlled. You can quickly change the value of a by pulling a control lever. We set its preset value to $a = -1$."

You: "Well done, Doctor! Is a our only way of control? It doesnt seem to be able to start the ball lightning."

Scientist: "You're right, Chief. We do have another way of control, which is to kick the instrument."

You: "Doctor, are you kidding me? Kick it?"

Scientist: "Yes, if you kick it, the value of $r(t)$ will instantly increase by ε (ε is much smaller than 1)."

You: "I see. That's helpful indeed. Our test goal today is to start the ball lightning, make its radius strictly exceed $\sqrt{2}$, and then let it gradually disappear completely."

Scientist: "Yes, Chief. We have designed four control schemes for this.

What do you think of these schemes, Chief?"

You looked at these options and found that the feasible schemes are ( ).

(A). Set $a = 2$, kick the instrument, wait for the ball lightning radius to strictly exceed $\sqrt{2}$, then set $a = -\frac{1}{2}$;

(B). Set $a = 3$, kick the instrument, wait for the ball lightning radius to strictly exceed $\sqrt{2}$, then set $a = -\frac{1}{3}$;

(C). Set $a = 4$, kick the instrument, wait for the ball lightning radius to strictly exceed $\sqrt{2}$, then set $a = -\frac{1}{4}$;

(D). Set $a = 5$, kick the instrument, wait for the ball lightning radius to strictly exceed $\sqrt{2}$, then set $a = -\frac{1}{5}$.

1 Answer The answer is (B).

We introduce the following notation for the rate function

$$v = f(r; a).$$

When $v > 0$, r is increasing in time. When $v < 0$, r is decreasing in time. When $v = 0$, r remains unchanged.

We can find all the roots of $f(r, a) = 0$, which we list in the following:

$$r_1 = 0, \quad r_2 = -\frac{\sqrt{1-\sqrt{4a+1}}}{\sqrt{2}}, \quad r_3 = \frac{\sqrt{1-\sqrt{4a+1}}}{\sqrt{2}},$$

$$r_4 = -\frac{\sqrt{1+\sqrt{4a+1}}}{\sqrt{2}}, \quad r_5 = \frac{\sqrt{1+\sqrt{4a+1}}}{\sqrt{2}}.$$

When $a > 0$, we have two nonnegative roots: $r_1 = 0$ and $r_5 > 0$. Clearly, when $r \in (0, r_5)$, $v > 0$; and when $r \in (r_5, +\infty)$, $v < 0$. Thus, when $a > 0$ and if we kick the instrument, we can start the ball lightening, and its radius will grow to r_5 (but it will not exceed r_5).

To make the radius exceed $\sqrt{2}$, we need $r_5 > \sqrt{2}$. This means in the starting phase, we need $a > 2$, and thus Scheme (A) fails.

When $-\frac{1}{4} < a < 0$, we have three nonnegative roots, which satisfy $0 = r_1 < r_3 < r_5$. In particular, we have $r_5 < 1$ and when $r \in (r_5, +\infty)$, $v < 0$. This means, if we start with $r = \sqrt{2}$, the radius is getting smaller, but it will not become smaller than r_5. Therefore, the ball lightening will not vanish completely. Hence, Scheme (D) fails.

When $a = -\frac{1}{4}$, similar to the previous case, the radius will not be smaller than $r_5 = \frac{1}{\sqrt{2}}$, and the ball lightening will not vanish completely. Hence, Scheme (C) fails.

When $a < -\frac{1}{4}$, we have only one nonnegative root $r_1 = 0$. When $r > 0$, we always have $v < 0$, and thus the ball lightening will vanish completely. This means Scheme (B) works.

2 Let O_1, O_2 be two convex octahedron whose faces are all triangles, and O_1 is *inside* O_2. Let the *sum* of edge kengths of O_1(resp. O_2) be ℓ_1(resp. ℓ_2). When we calculate ℓ_1/ℓ_2, which value(s) among the following can be obtained? (Multiple Choice)

(A). 0.64

(B). 1

(C). 1.44

(D). 1.96

(E). 4

2 Answer

The answer is **(A) (B) (C) (D)**.

Comments In the 60's - 70's, the following question appeared in All-Union Math Olympiad of USSR: A tetradehron V_1 sits inside another tetrahedron V_2, prove that the sum of edge lengths of V_1 does not exceed $\frac{4}{3}$ times that of V_2. What is anti-intuitive is that, on a plane, if a triangle sits inside another triangle, then not only the area of the first triangle is strictly smaller than that of the second one, but the perimeter also is. Now in a three dimensional situation, though the "order" of volume and surface is still kept, it is not the case for the sum of edge lengths.

The "origine" of the problem is likely the following paper in Polish:

Holsztyński, W. and Kuperberg, W., *O pewnej wlasnósci czworościanów*, Wiadomości Matematyczne 6 (1962), 14-16.

They published an English version some 15 years later:

Holsztyński, W. and Kuperberg, W., *On a Property of Tetrahedra*, Alabama J. Math. 1(1977), 40 42.

Then in 1986, Carl Linderholm of the University of Alabama generalized the above result to higher dimensional Euclidean spaces:
Theorem. *Let S and T be two m-dimensional simplexes in $\mathbb{R}^n$, the first being inside the second, and $1 \leqslant r \leqslant m$. The there exists constants $B_{m,r}$, such that the sum of all r-dimensional faces of S does not exceed $B_{m,r}$ times that of T. Here $B_{m,r}$is calculated as follows: Let $m+1=(r+1)q+s$(Euclidean division), then*

$$B_{m,r} = \frac{q^{r+1-s}(q+1)^s}{m+1-r}.$$

(CARL LINDERHOLM, AN INEQUALITY FOR SIMPLICES, Geometriae Dedicata (1986) 21, 67-73.)

Now back to the current problem, the Choice (A) is trivial, so we focus on:

- why (B),(C) and (D) can be realized?
- why (E) cannot?

The mathematics that we need here is:

(A) a little geometric topology: an octahedron with all faces being triangles has $3 \times 8/2 = 12$ edges, so by Euler's Formula, the number of vertices is 6.

(B) a bit of graph theory: if one vertex has degree 5, then by a very easy argument one has another vertex with degree 5 also, and the degrees of the vertices are $(5, 5, 4, 4, 3, 3)$. The only other possibility is that every vertex has degree 4 (like that of a regular ocrahedron).

(C) a little bit of convex geometry: as we consider convex octahedron, so the maximum distance of two points on it must be attained between two vertices.

- If every vertex of the big octahedron is of degree 4,and the maximum distance $\ell$is realized between two vertices A and B that are NOT adjacent, then as the other four vertices are all adjacent to them, so ℓ_2 is at least $4\ell_2$ (and can be arbitrarily close to that valur when the other four vertices are close enough to line AB), and for the small octahedron, if every vertex is of degree 4, we can make three vertices very close to A, while the other three very close to B, so ℓ_1 would be very close to $6\ell_2$. Hence any ratio less than 1.5 is realizable. (so the Choices (A),(B) and (C))
- If the maximum distance ℓ is realized between two vertices of degree 3 in the big octahedron, then ℓ_2 is at least $3\ell_2$(and can be arbitrarily close to that valur when the other four vertices are close enough to line AB), while for the small octahedron, we can still take each vertex to be of degree 4, and three of them very close to A, while the other three very close to B, so ℓ_1 would be very close to $6\ell_2$. Hence any ratio less than 2 is realizable. (so the Choice (D))

  Actually, if the small octahedron has the some topological configuration as that of the big one, and the two vertices of degree 5 are very close to each other, while the other four vertices are very close together, then the ratio can actually approach $8/3$.
- After some easy case by case discussion, we conclude that, if the maximum distance ℓ is realized between a vertex of degree a and a vertex of degree b (whether they are adjacent or not), one has always ℓ_2 is at last $\min(a, b)\ell$, while obviously ℓ_1 cannot exceed $12\ell_2$, So (E)is impossible.

3 Two players, A and B, play a game called "draw the joker card". In the beginning, Player A has n different cards. Player B has $n+1$ cards, n of which are the same with the n cards in Player A's hand, and the rest one is a Joker (different from all other n cards). The rules are

i) Player A first draws a card from Player B, and then Player B draws a card from Player A, and then the two players take turns to draw a card from the other player.

ii) if the card that one player drew from the other one coincides with one of the cards on his/her own hand, then this player will need to take out these two identical cards and discard them.

iii) when there is only one card left (necessarily the Joker), the player who holds that card loses the game.

Assume for each draw, the probability of drawing any of the cards from the other player is the same. Which n in the following maximises Player A's chance of winning the game?

(A). $n = 31$

(B). $n = 32$

(C). $n = 999$

(D). $n = 1000$

(E). For all choices of n, A has the same chance of winning

3 Answer

The answer is **(B)**.

We denote a_n to be the probability that A wins the game when A has n cards in the beginning. So we have

$$a_1 = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} a_1. \tag{1}$$

Therefore, $a_1 = \frac{2}{3}$. In addition, we have

$$a_2 = \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{3} a_2, \tag{2}$$

so we conclude that $a_2 = \frac{3}{4}$.

Actually, we can obtain the following induction formula

$$a_n = \frac{n}{n+1} a_{n-2} + \frac{1}{n+1} \frac{1}{n+1} a_n + \frac{1}{n+1} \frac{n}{n+1} p_{n,n-1} \tag{3}$$

where the first term on the RHS is the scenario when A does not draw the joker card from B. In this case, no matter which card B draws from A, this card would match one of the cards that B has in his hand (because B holds the joker card). Then A will have $n-2$ cards and B has $n-1$ cards, with A drawing from B first and B holding the joker card. The second term on the RHS is the scenario when A first draws the joker card from B, and then B draws the joker card from A. The third term on the RHS is the scenario when A draws the joker card from B but B does not draw the joker card from A, and $p_{n,n-1}$ is the probability for A to win the game when A draws first with n cards including a joker card, B draws next with $n-1$ cards that do not include the joker card. We have

$$p_{n,n-1} = 1 - a_{n-2}, \tag{4}$$

because no matter which card A draws from B, A would have one card in hand that match this drawn card from B (because the joker card is in A's hand). Therefore, after A's drawing, A will have $n-1$ cards including the joker card, B will have $n-2$ cards without the joker card, and B draws first. In this case, the probability for B to win will be a_{n-2}, so we have $p_{n,n-1} = 1 - a_{n-2}$.

Therefore,

$$a_n = \frac{n}{n+1}a_{n-2} + \frac{1}{n+1}\frac{1}{n+1}a_n + \frac{n}{(n+1)^2} - \frac{n}{(n+1)^2}a_{n-2}, \tag{5}$$

and we can simplify the above equation to

$$\begin{aligned} a_n &= \frac{n}{n+2}a_{n-2} + \frac{1}{n+2} \\ &= \frac{n}{n+2}(\frac{n-2}{n}a_{n-4} + \frac{1}{n}) + \frac{1}{n+2} \\ &= \ldots \end{aligned} \tag{6}$$

By induction, if n is an odd number, then

$$a_n = \frac{n+3}{2(n+2)}. \tag{7}$$

On the other hand, if n is an even number, then by induction we have

$$a_n = \frac{n+4}{2(n+2)}. \tag{8}$$

Therefore, we conclude that

- $a_{31} = \frac{17}{33}$
- $a_{32} = \frac{9}{17}$
- $a_{999} = \frac{501}{1001}$
- $a_{1000} = \frac{251}{501}$.

So the correct answer is (B), and $n = 32$ initial cards will give A the biggest chance of winning.

4 There are 10 horizontal roads and 10 vertical roads in a city, and they intersect at 100 crossings. Bob drives from one crossing, passes every crossing exactly once, and return to the original crossing. At every crossing, there is no wait to turn right, 1 minute wait to go straight, and 2 minutes wait to turn left. Let S be the minimum number of total minutes on waiting at the crossings, then

(A). $S < 50$

(B). $50 \leq S < 90$

(C). $90 \leq S < 100$

(D). $100 \leq S < 150$

(E). $S \geq 150$.

4 Answer

The answer is **(C)**.

Obviously, the route of driving is a non-self-intersecting closed polyline. Regard each crossing as a vertex, then the route is regarded as a 100-gon.An interior angle may be greater than or equal to a straight angle.. By the formula of the sum of the angles of the polygon, the sum of all interior angles is $98 \times 180°$. Note that the interior angle can only be $90°$, $180°$ or $270°$, if there are a angles of $90°$, b angles of $270°$, then $90a + 270b + 180(100 - a - b) = 98 \times 180$, so $a - b = 4$. If Bob drives clockwise, then $90°$, $180°$ and $270°$ corrsponds to turn right, go straight and turn left, respectively. The total time on waiting at the crossings is $(100 - a - b) + 2b = 100 - (a - b) = 96(min)$; If Bob drives clockwise, then $90°$, $180°$ and $270°$ corrsponds to turn left, go straight and turn right, respectively. The total time on waiting at the crossings is $(100 - a - b) + 2a = 100 + (a - b) = 104(min)$. Therefore, $S = 96$, and (C) is correct.

Note: If we ignore the waiting time on the beginning/ending crossing, the total time on waiting can be decreased by 2 minutes (Bob can choose a left-turn crossing as the beginning), we have that $S = 94$, but do not affect the correct choice.

5 Let $n \geq 2$ be a given positive integer. Consider the set of $n \times n$ matrices $X = (a_{i,j})_{1 \leq i,j \leq n}$ with entries 0 and 1.

(1) show that: there exists such an X with $\det X = n - 1$.

(2) when $2 \leq n \leq 4$, show that $\det X \leq n - 1$.

(3) When $n \geq 2023$, show that there exists an X with $\det X > n^{\frac{n}{4}}$.

5 Answer

(1)If X has a zero row or two equal rows, then $\det X = 0$; if X has a row with only one nonzero entry, it reduces to $(n-1)$ matrix case; if X has a row with n nonzero entries and a row with $(n-1)$ nonzero entries, it reduces to the case that X has a row with only one nonzero entry and further reduces to $(n-1)$ matrix case. When the above all not happen, then rows of X have few possibilities and one could take a case by case verification.

(2)take $X' = (a_{i,j})_{1 \leq i,j \leq n}$ where

$$a_{i,j} = 1 - \delta_{i,j}, \quad 1 \leq i, j \leq n.$$

Then, $\det X' = (-1)^{n-1}(n-1)$. If n is odd, let $X = X'$. If n is even, get X by switching the first two rows of X'. Then, $\det X = n - 1$.

(3)when $n = 2^k - 1$, set

$$Y = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{\otimes k}.$$

Then, Y is an $(n+1) \times (n+1)$ matrix with entries ± 1 and having

$$\det Y = (\sqrt{2^k})^{2^k} = 2^{k2^{k-1}}.$$

The last row of Y is equal to $\alpha_{n+1} = (\underbrace{1, \ldots, 1}_{n+1})$. Write $t_i = \pm 1$ for the last entry of the i-th row α_i of Y $(1 \leq i \leq n)$. Put

$$\beta_i' = \frac{1}{2}(t_i \alpha_i - \alpha_{n+1}).$$

Removing the last entry of β_i' (which is 0), we get an n row vector β_i. Put

$$X' = (\beta_1, \ldots, \beta_n)^t$$

and write

$$t = \prod_{1 \leq i \leq n} t_i = \pm 1.$$

Then, X' is an $n \times n$ matrix with entries 0 and 1 and we have

$$\det X' = t 2^{(k-2)2^{k-1}+1}.$$

Switching the first two rows of X' if necessary, we get an $n\times n$ matrix X with entries $0,1$ such that $\det X = 2^{(k-2)2^{k-1}+1}$.

Assume that $2^k-1\le n<2^{k+1}-1$. When $2^k-1\le n<3\cdot 2^{k-1}$ and $n\ge 2023$, we have $k\ge 11$. There exists an $n\times n$ matrix X with entries $0,1$ such that

$$\det X\ge 2^{(k-2)2^{k-1}+1}>2^{(k-2)2^{k-1}}.$$

We have

$$n^{\frac{n}{4}}<(2^{k+1})^{3\cdot 2^{k-3}}=2^{3(k+1)\cdot 2^{k-3}}.$$

Due to $k\ge 11$, we get

$$(k-2)2^{k-1}\ge 3(k+1)\cdot 2^{k-3}.$$

Then, $\det X>n^{\frac{n}{4}}$.

When $3\cdot 2^{k-1}\le n<2^{k+1}-1$ and $n\ge 2023$, we have $k\ge 10$. There exists an $n\times n$ matrix X with entries $0,1$ such that

$$\det X\ge 2^{(k-2)2^{k-1}+1}2^{(k-3)\cdot 2^{k-2}+1}>2^{(3k-7)2^{k-2}}.$$

We have

$$n^{\frac{n}{4}}<(2^{k+1})^{2^{k-1}}=2^{(k+1)\cdot 2^{k-1}}.$$

Due to $k\ge 10$, we get

$$(3k-7)2^{k-2}>(k+1)\cdot 2^{k-1}.$$

Then, $\det X>n^{\frac{n}{4}}$.

6 For a real number r, set $||r|| = \min\{|r-n| : n \in \mathbb{Z}\}$, where $|\cdot|$ means the absolute value of a real number.

1. Is there a nonzero real number s, such that $\lim_{n\to\infty} ||(\sqrt{2}+1)^n s|| = 0$?
2. Is there a nonzero real number s, such that $\lim_{n\to\infty} ||(\sqrt{2}+3)^n s|| = 0$?

6 Answer

1. Yes. We prove that $s = 1$ has the property. Denote $(\sqrt{2}+1)^n = x_n + \sqrt{2}y_n$, where $x_n, y_n \in \mathbb{Z}$. Then $(-\sqrt{2}+1)^n = x_n - \sqrt{2}y_n$ and $x_n^2 - 2y_n^2 = (-1)^n$. It follows that $|x_n + \sqrt{2}y_n - 2x_n| = |\sqrt{2}y_n - x_n| = \frac{|2y_n^2 - x_n^2|}{\sqrt{2}y_n + x_n} \to 0$.

2. No. We prove this by contradiction. Assume that there is some real number $s \neq 0$ such that $(\sqrt{2}+3)^n s = m_n + \epsilon_n$, where $\lim_{n\to\infty} \epsilon_n = 0$. Denote $\alpha = \sqrt{2}+3, \bar{\alpha} = -\sqrt{2}+3$. Consider the power series:

$$\frac{s}{1-\alpha x} = \sum_{n=0}^{\infty} m_n x^n + \sum_{n=0}^{\infty} \epsilon_n x^n.$$

Since $(1-\alpha x)(1-\bar{\alpha}x) = 1 - 6x + 7x^2$, multiplying both sides of the above equation by $1 - 6x + 7x^2$ we get

$$s(1-\bar{\alpha}x) = (1-6x+7x^2)\sum_{n=0}^{\infty} m_n x^n + (1-6x+7x^2)\sum_{n=0}^{\infty} \epsilon_n x^n. \tag{9}$$

Denote $(1-6x+7x^2)\sum_{n=0}^{\infty} m_n x^n = \sum_{n=0}^{\infty} p_n x^n, (1-6x+7x^2)\sum_{n=0}^{\infty} \epsilon_n x^n = \sum_{n=0}^{\infty} \eta_n x^n$, where $p_n \in \mathbb{Z}, \lim_{n\to\infty} \eta_n = 0$. Because the left hand side of (9) is a polynomial of degree 1it holds that $p_n + \eta_n = 0, n \geq 2$. Since $\lim_{n\to\infty} \eta_n = 0$, we get $p_n = \eta_n = 0$ when n is large enough. As a consequnce, $(1-6x+7x^2)\sum_{n=0}^{\infty} m_n x^n$ and $(1-6x+7x^2)\sum_{n=0}^{\infty} \epsilon_n x^n$ are polynomials. So we have

$$\sum_{n=0}^{\infty} \epsilon_n x^n = \frac{G(x)}{1-6x+7x^2}.$$

Write the right hand side as $H(x) + \frac{A}{1-\bar{\alpha}x} + \frac{B}{1-\alpha x}$, where $H(x)$ is a polynomial and A, B are constants. Since $\lim_{n\to\infty} \epsilon_n = 0$, the radius of convergence of the power series in the left hand side is at least 1. While α and $\bar{\alpha}$ are larger than 1 A and B must be zero. Hence $\epsilon_n = 0$ for large n. It follows that $(\sqrt{2}+3)^n s = m_n \in \mathbb{Z}$ for large n. It's a contradiction!

7 A company has one open position available, and N candidates applied (N is known). Assume the N candidates' abilities for this position are all different from each other (in other words, there is a non-ambiguous ranking among the N candidates), and the hiring committee can observe the full relative ranking of all the candidates they have interviewed, and their observed rankings are faithful with respect to the candidates' true abilities. The hiring committee decides the following rule to select one candidate from N:

1. The committee interviews the candidates one by one, at a completely random order. They observe information on candidates' relative ranking regarding their abilities for the position. The only information available to them after interviewing m candidates is the relative ranking among these m people.
2. After each interview, the committee decides whether to offer the candidate the position or not.
3. If they decide to offer the position to the candidate just interviewed, then the candidate will accept the job with probability p, and decline the offer with probability $1-p$, independently with all other candidates. If the selected candidate accepts the offer, then he/she gets the job, and the committee stops interviewing the remaining candidates. If he/she declines the offer, then the committee proceed to interviewing the next candidate.
4. If they decide not to offer the position to the candidate just interviewed, then they proceed to interviewing the next candidate, and they can not turn back to previously interviewed candidates any more.
5. The committee continues this process until a candidate is selected and accepts the job, or until they finish interviewing all N candidates if the position has not been filled before, whichever comes first.

Since the interview order of the candidates are completely random, each ranking has equal probability among the $N!$ possibilities. The committee's mission is to maximise the probability of getting the candidate with the highest ranking (among N candidates) for the job constrained to the above selection process.

Here are the questions

(a) Fix $1 \le m \le N$, and consider the following strategy. The committee interviews the first $m-1$ candidates, and do not give offer to any of them regardless of their relative rankings. Starting from the m-th candidate, the committee offers him/her the position whenever the candidate's relative ranking is the highest among all previously interviewed candidates. If he/she declines the offer, then the committee continues the interview until the next relatively best candidate[1], and then repeat the process when applicable.

Show that for every N, there exists $m = m_N$ such that the above strategy maximises the probability of getting the best candidate among all possible strategies.

[1] "Relatively best candidate" refers to the candidate with the highest ability among all candidates who have been interviewed (including those who are offered the position and declined).

(b) Suppose $p = 1$. What is the limit of $\frac{m_N}{N}$ as $N \to +\infty$?

(c) For $p \in (0, 1)$, what is the limit of $\frac{m_N}{N}$ as $N \to +\infty$?

7 Answer For any $1 \le k \le N$, let Z_k be the expected probability of winning (getting the best candidate) when skipping the first $m-1$ candidates interviewed (not giving offers to them no matter their relative rankings), and starting optimal strategy from the m-th candidate. Then, we have

$$Z_k \ge Z_{k+1} .$$

(a) If the committee interviews the k-th candidate, and he/she is the relatively best one among the k people, then the committee will make him/her an offer. Under this event, the conditional probability that the committee gets the best person (among the N people) for the job is

$$Y_k = \frac{pk}{N} + (1-p)Z_{k+1} .$$

Hence, the committee offers the k-th candidate the position if and only if he/she is the best among the first k candidates, and

$$\frac{pk}{N} + (1-p)Z_{k+1} \ge Z_{k+1} , \tag{10}$$

which is equivalent to $\frac{k}{N} \ge Z_{k+1}$. Since $\{\frac{k}{N}\}_k$ is increasing in k while $\{Z_k\}_k$ is decreasing, and that $Z_k \le \frac{N-k+1}{N}$, there exists $k \le N-1$ such that (10) holds. We then conclude that there exists $m = m_N$ (which is the smallest k such that (10) holds) such that the above strategy maximises the probability of getting the best person.

(b) Let p_m be the probability of adopting the strategy in part (a) (offering the relatively best candidate starting from the m-th person) and getting the best person. We want to find $m = m_N$ that maximises p_m. When $p = 1$, p_m is the probability of the following disjoint union of events:

$$\bigcup_{k=m}^{N} A_k ,$$

where A_k is the event that the k-th candidate is the best among the N people and he/she is interviewed. Then, we have

$$\mathbf{P}(A_k) = \frac{1}{N} \cdot \frac{m-1}{k-1} ,$$

where $\frac{1}{N}$ is the probability that the k-th person is the best among the N, and $\frac{m-1}{k-1}$ is the probability that he/she is interviewed conditioned on him being the best. Thus, we have

$$p_m = \frac{m-1}{N} \sum_{k=m}^{N} \frac{1}{k-1} .$$

Since p_m first increases in m and then decreases, the optimal $m = m_N$ that maximises p_m should be the smallest m that satisfies

$$p_m \geq p_{m+1} ,$$

or equivalently,

$$\sum_{k=m+1}^{N} \frac{1}{k-1} \leq 1 .$$

The left hand side is approximately $\log(N/m)$ when N is large. Hence, $m_N \to \frac{1}{e}$ as $N \to +\infty$.

(c) For general $p \in (0,1) p_m$ is the probability of the following disjoint union of events:

$$\bigcup_{k=m}^{N} A_k ,$$

where A_k is the event that the k-th candidate is the best among N, he/she is interviewed, and accepts the offer. Hence, we have

$$p_m = \frac{p}{N} \sum_{k=m}^{N} q_k ,$$

where q_k is the probability that the k-th person is interviewed conditioned on him/her being the best. Thus, we have

$$q_k = \left(\frac{m-1}{m} + \frac{1-p}{m}\right)\left(\frac{m}{m+1} + \frac{1-p}{m+1}\right) \cdots \left(\frac{k-2}{k-1} + \frac{1-p}{k-1}\right) = \frac{\Gamma(m)\Gamma(k-p)}{\Gamma(k)\Gamma(m-p)} ,$$

where Γ is the standard Γ function. Hence, we get

$$p_m = \frac{p}{N} \cdot \frac{\Gamma(m)}{\Gamma(m-p)} \sum_{k=m}^{N} \frac{\Gamma(k-p)}{\Gamma(k)} .$$

Again, since p_m first increases and then decreases in m, by asymptotics of Γ functions and integral approximations to summation, we get the optimal $m = m_N$ satisfies the asymptotics

$$\frac{m_N}{N} \to p^{\frac{1}{1-p}} .$$

When $p = 1$, the limit is $\frac{1}{e}$, which agrees with part (b).