Week #8
July 9, 2021 11:56 AM

https://github.com/LazarGlumac/MATB24-TUT3-Notes

start @ $12^{\underline{10}}$

MIDTERM
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def. A $n \times n$ matrix A is symmetric positive definite

$\Rightarrow A^T = A$ and $\forall x \in F^n, \vec{x} \neq \vec{0}, \ A\vec{x} \cdot \vec{x} > 0$.

1. Suppose A is symmetric positive definite,
Show all eigen values of A are positive.

Let λ be an eig. val of A with corresponding
eig vec. $\vec{v}$. $\vec{v} \neq 0$, by def.

wts $\lambda > 0$.

Consider $A\vec{v} \cdot \vec{v}$,

$A\vec{v} \cdot \vec{v} > 0$, since A is positive def.

$\Leftrightarrow v^T A^T \vec{v} > 0$, shown in A22

$\Leftrightarrow v^T \underline{A\vec{v}} > 0$, since $A^T = A$

$\Leftrightarrow \vec{v}^T \lambda \vec{v} > 0$, since $A\vec{v} = \lambda\vec{v}$

$\Leftrightarrow \lambda \vec{v}^T \vec{v} > 0$, λ is scalar

$\Leftrightarrow \lambda \vec{v}^T \vec{v} > 0$, λ is scalar

$\Leftrightarrow \lambda \|\vec{v}\|^2 > 0$, shown in A22

$\Leftrightarrow \underline{\lambda > 0}$, since $\|\vec{v}\|^2 > 0$. $\vec{v} \neq 0$.

QED

2. Let A, B be 2 $n \times n$ diag.ble matrices which share the same set of eig. vectors. Show $AB = BA$. - try it

Suppose A, B are diag.ble. which share same set of eig. vectors.

$\Rightarrow \exists P_1, P_2, P_1^{-1}, P_2^{-1}, D_1, D_2 \in M_{n \times n}(\mathbb{R})$ s.t.

$D_1 = P_1^{-1} A P_1$

$D_2 = P_2^{-1} B P_2$.

since A, B share eig. vectors, $P_1 = P_2$, $P_1^{-1} = P_2^{-1}$

Let $P = P_1 = P_2$, $P^{-1} = P_1^{-1} = P_2^{-1}$

$AB = (P D_1 \cancel{P^{-1}})(\cancel{P} D_2 P^{-1})$, by def of diag.ble.

$= P D_1 D_2 P^{-1}$ ✓

$BA = (P D_2 \cancel{P^{-1}})(\cancel{P} D_1 P^{-1})$

$= P D_2 D_1 P^{-1}$

since D_1, D_2 are diag.

$\Rightarrow D_1 = \Sigma x_i \; i=j \quad D_2 = \Sigma y_i, i=j$

$= P D_2 D_1 P^{-1}$

$= \underline{P D_1 D_2 P^{-1}}$

$= AB$, as wanted.

since D_1, D_2 are diag.

$D_1 = \begin{cases} x_i, & i=j \\ 0, & o/w \end{cases}$ $D_2 = \begin{cases} v_i, & i=j \\ 0, & o/w \end{cases}$

$D_1 D_2 = \begin{cases} x_i v_i, & i=j \\ 0, & o/w \end{cases}$

$= \begin{cases} v_i x_i, & i=j \\ 0, & o/w \end{cases}$

$= D_2 D_1$

3. Let $A^2 = I$ be $n \times n$, Find the eigen values of A.

Suppose $A^2 = I$ wts $\lambda = ?$

Let λ be an eig. value of A, with corresponding eig. vector $\vec{v}$, $\vec{v} \neq 0$.

Consider $A\vec{v}$,

$A\vec{v} = \lambda \vec{v}$, by def of eig. val/vec

$\Leftrightarrow A(A\vec{v}) = A(\lambda \vec{v})$

$\Leftrightarrow A^2 \vec{v} = \lambda (A\vec{v})$, since λ is scalar

$\Leftrightarrow I \vec{v} = \lambda(\lambda \vec{v})$, since $A\vec{v} = \lambda\vec{v}$, $A^2 = I$

$\Leftrightarrow \vec{v} = \lambda^2 \vec{v}$

$\Leftrightarrow \vec{v} - \lambda^2 \vec{v} = \vec{0}$

$\Leftrightarrow \vec{v} - \lambda^2 \vec{v} = \vec{0}$

$\Leftrightarrow \vec{v}(1-\lambda^2) = \vec{0}$

$\Leftrightarrow \vec{v}(1+\lambda)(1-\lambda) = \vec{0}$

$\Rightarrow \lambda = \pm 1$, since $\vec{v} \neq 0$.

QED.

4. Let $A \in M_{n \times n}(\mathbb{R})$. Prove the eig. values of A = eig. val of A^T.

Let $p_A(\lambda)$ be the char. poly. of A

$\Rightarrow 0 = \det(A - \lambda I)$

Let $p_{A^T}(\lambda)$ be the char. poly. of A^T

$\Rightarrow 0 = \det(A^T - \lambda I)$

$= \det((A - (\lambda I)^T)^T)$, by transpose prop.

$= \det(A - (\lambda I)^T)$, by det prop.

$= \det(A - \lambda I)$, since λI is diag. $\lambda I = (\lambda I)^T$

$= p_A(\lambda)$, as wanted.

5. Suppose $A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$.

Diagonalize. $A^3 - 5A^2 + 3A + I_{2\times2}$.

① $A^2 = PD^2P^{-1}$

② $A^3 = PD^3P^{-1}$

③ $A^3 - 5A^2 + 3A + I_{2\times2} = P(D^3 - 5D^2 + 3D + I)P^{-1}$

Consider $P^{-1}(A^3 - 5A^2 + 3A + I_{2\times2})P =$

$= \cancel{P^{-1}}(\cancel{P}(D^3 - 5D^2 + 3D + I)\cancel{P^{-1}})\cancel{P}$, by ③

$= D^3 - 5D^2 + 3D + I$

$= \begin{bmatrix} -8 & 0 \\ 0 & 16 \end{bmatrix}$ → diagonal matrix, P, P^{-1}

$\Rightarrow$ diagonalized $A^3 - 5A^2 + 3D + I_{2\times2}$

6. Show eigen values of a real, skew symmetric matrix are 0, or purely imaginary.

hint: $\overline{A\vec{v}} = \overline{\lambda}\,\overline{\vec{v}}$

Suppose A is real, skew sym. $A^T = -A$

Let λ be an eigen value of A, with corresponding eig. vector $\vec{v}$, $\vec{v} \neq 0$.

$A\vec{v} = \lambda\vec{v}$, by def

$\Rightarrow \vec{v}^T A\vec{v} = \vec{v}^T(\lambda\vec{v})$

$\vec{v}^T(A\vec{v}) = \lambda\|\vec{v}\|^2$, since λ is scalar, $\vec{v}^T\vec{v} = \|\vec{v}\|^2$.

Now,

$\vec{v}^T(A\vec{v}) = (A\vec{v})^T\,\vec{v}$, shown in A22

$= (\vec{v}^T A^T)\,\vec{v}$

$= \vec{v}^T(-A)\,\vec{v}$, since $A^T = -A$

$= -\vec{v}^T\,\overline{A\vec{v}}$

$= -\vec{v}^T\,\overline{\lambda\vec{v}}$

$= -\overline{\lambda}\,\vec{v}^T\vec{v}$

$= -\overline{\lambda}\,\|\vec{v}\|^2$

$\overline{A\vec{v}} = \overline{A}\,\overline{\vec{v}}$

$= A\overline{\vec{v}}$

$\Rightarrow \lambda\|\vec{v}\|^2 = -\overline{\lambda}\,\|\vec{v}\|^2$

$\Rightarrow \lambda = -\overline{\lambda}$, supposing $\lambda = a + ib$

$a + ib = -\overline{(a+ib)}$

$a + ib = -a + ib$

$\Rightarrow a = 0$

$\Rightarrow \lambda = ib$.

Thus, λ is 0, or purely imaginary, as wanted.