Week #12

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https://github.com/LazarGlumac/MATB24-TUT3-Notes

start @ 12^{10}

Next week schedule:
Tues. ⇒ Thurs
Wed. ⇒ Friday

1. Orthogonally diag. $A = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}$, then find all square roots of A.

i.e. find B st $B^2 = A$

ext. what are the eigen values of each B?

$$p(\lambda) = \det(A - \lambda I)$$
$$= \det \begin{pmatrix} 7-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix}$$
$$= (7-\lambda)(4-\lambda) - 4$$
$$= 28 - 11\lambda + \lambda^2 - 4$$

solve $0 = (\lambda - 8)(\lambda - 3)$

$\Rightarrow \lambda_1 = 8, \lambda_2 = 3$

for $\lambda_1 = 8$,

$(A - \lambda_1 I)\vec{x} = 0 \Leftrightarrow \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\vec{x} = 0$

$\left(\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right) \overset{R_2 \to R_2 + 2R_1}{\sim} \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

↑free

let $x_2 = s \Rightarrow x_1 = 2s$

for $\lambda_2 = 3$.

$(A - \lambda_2 I)\vec{x} = 0 \Leftrightarrow \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\vec{x} = 0$

$\left(\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right) \underset{R_1 \to \frac{1}{2}R_1}{\overset{R_2 \to R_2 - \frac{1}{2}R_1}{\sim}} \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

↑free

let $x_2 = s \Rightarrow x_1 = -\frac{1}{2}s$.

let $x_2 = s \Rightarrow x_1 = 2s$

$\Rightarrow \vec{x} = s\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

let $x_2 = s \Rightarrow x_1 = -\frac{1}{2}s$.

$\vec{x} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

$B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} \right\}$

$P_1 = \dfrac{B_1}{\|B_1\|} = \dfrac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$P_2 = \dfrac{B_2}{\|B_2\|}$, $= \dfrac{2}{\sqrt{5}}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$,

$P^* = \overline{P}^T = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$,

$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 3 \end{pmatrix}$.

Want to find B st $B^2 = A$

we have:

$A = PDP^*$, by orthogonalization.

so, we take $\sqrt{A}$, by diag prop.

$B = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}\begin{pmatrix} \pm\sqrt{8} & 0 \\ 0 & \pm\sqrt{3} \end{pmatrix}\begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$

ext. eigen values of each B?

$B_{+,-} = P\begin{pmatrix} \sqrt{8} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}P^* \Rightarrow \lambda_1 = \sqrt{8}, \lambda_2 = -\sqrt{3}$

$B_{-,-} = P\begin{pmatrix} -\sqrt{8} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}P^* \Rightarrow \lambda_1 = -\sqrt{8}, \lambda_2 = -\sqrt{3}$

$\ldots \sqrt{8} \quad 0 \Big) P^* \Rightarrow \lambda_1 = -\sqrt{8} \quad \lambda_2 = \sqrt{3}$

$B_{-,-} = P\begin{pmatrix} -\sqrt{8} & 0 \\ 0 & -\sqrt{3} \end{pmatrix} P^* \Rightarrow \lambda_1 = -\sqrt{8},\ \lambda_2$ ...

$B_{-,+} = P\begin{pmatrix} -\sqrt{8} & 0 \\ 0 & \sqrt{3} \end{pmatrix} P^* \Rightarrow \lambda_1 = -\sqrt{8},\ \lambda_2 = \sqrt{3}$

$B_{+,+} = P\begin{pmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{3} \end{pmatrix} P^* \Rightarrow \lambda_1 = \sqrt{8},\ \lambda_2 = \sqrt{3}.$

2. Prove if A is nilpotent and normal, then $A = 0$.

Suppose A is a nilpotent, normal matrix ie,

$AA^* = A^*A$, $\underline{A^k = 0}$ for some $k \in \mathbb{N}^+$

Since A is normal, we know A is diagonalizable (thm. 2.4 Chap 6.), moreover it is orthogonally diagonalizable.

$\Rightarrow A = UDU^*$ for some orthogonal U, diagonal D.

also we know D is diagonal made up of eig. val of A.

wts: eigen values of A are $= 0$.

Let λ be an eigen value of A, with eigen vector $\vec{v}$.., $\vec{v} \neq 0$.

$$A\vec{v} = \lambda\vec{v}$$

$$\underbrace{A(A \cdots\cdots (A}_{k \text{ times}}\vec{v})) = \underbrace{A(A \cdots\cdots (}_{k \text{ times}}\lambda\vec{v}))$$

$A^k\vec{v} = \lambda^k\vec{v}$, applying $A\vec{v}$ k time

$0\vec{v} = \lambda^k\vec{v}$, since A is nilpotent.

$0 = \lambda^k\vec{v}$

$\Rightarrow \lambda^k = 0$, since $\vec{v} \neq 0$

$\Rightarrow \lambda^k = 0$, since $\vec{v} \neq 0$

$\Rightarrow \lambda = 0$.

Thus, $D = 0$. since all eigen values of A are 0

$$A = UDU^*$$
$$= U(0)U^*$$
$$= 0. \quad \text{, as wanted.}$$

ie $B=0$

3. Prove if A is normal, $\underline{B \text{ is nilpotent}}$, and $A + B = I$, then $A = I$. hint: use last proof. (Suppose A normal, B nilpotent, $A+B=I$.

Wts: $B = 0$, by prev. proof, show B is normal

$$A + B = I$$
$$B = I - A$$

$$B^*B = (I-A)^*(I-A)$$
$$= (I^* - A^*)(I-A) \quad \text{, by * prop}$$
$$= (I - A^*)(I-A) \quad , I^* = I$$
$$= I - A - A^* + A^*A$$

$$BB^* = (I-A)(I-A)^*$$
$$= (I-A)(I-A^*), \quad \text{by * prop, } I^* = I$$
$$= I - A^* - A - AA^*$$
$$= I - A - A^* - A^*A \quad \text{, since } A \text{ is normal}$$

$\Rightarrow B^*B = BB^*$, so B is normal.

From Q2, B is normal + nilpotent $\Rightarrow B = 0$
Thus, from $A+B = I$, $A = I$.

4. Let A be $m \times n$ matrix. Show that

1) A^*A is self adj.
2) all eigen values of A^*A are non negative.

1). $(A^*A)^* = A^* A^{**}$, by $*$ prop.
$= A^* A$, by $*$ prop.

✓.

2) let λ be an eigen value of A^*A
with corresponding eigen vector v.

$(A^*A)\vec{v} = \lambda \vec{v}$.

Then, $\langle A^*A\vec{v}, \vec{v} \rangle = \langle \lambda \vec{v}, \vec{v} \rangle$
$= \lambda \langle \vec{v}, \vec{v} \rangle$, by $\langle , \rangle$ prop.
$= \lambda \|\vec{v}\|^2$.

Also, $\langle A^*A\vec{v}, \vec{v} \rangle = (A^*A\vec{v})^* \vec{v}$, by $\langle , \rangle$ prop.
$= \vec{v}^* A^* A^{**} \vec{v}$, by $*$ prop.
$= \vec{v}^* A^* A \vec{v}$
$= (v^* A^*)(A\vec{v})$, by matrix mult prop.
$= \langle A\vec{v}, A\vec{v} \rangle$
$= \|A\vec{v}\|^2$

$$= \langle A\vec{v}, A\vec{v} \rangle$$

$$= \|A\vec{v}\|^2$$

$$\Rightarrow \lambda \|\vec{v}\|^2 = \|A\vec{v}\|^2$$

$$\geq 0 \quad \text{by norm properties}$$

$\Rightarrow \lambda \geq 0$, since $\|\vec{v}\|^2 > 0$, since $\vec{v} \neq 0$.

QED.