Week #9

July 16, 2021 11:55 AM

https://github.com/LazarGlumac/MATB24-TUT3-Notes

start @ $12^{\underline{10}}$

Thoughts on Midterm?

1. Suppose a real diagonal matrix A has a positive diagonal. $\langle x, y \rangle = x^T A y$ $x, y \in \mathbb{R}^n$. Prove $\langle \, , \rangle$ is an inner product.

- prove 4 properties - try it

① Let $x \in \mathbb{R}^n$. wts $\langle \vec{x}, \vec{x} \rangle \geq 0$.

$$\langle x, x \rangle = x^T A x \quad , \text{ by def of } \langle \ \rangle$$

$$= [x_1 \dots x_n] \begin{bmatrix} a_{1,1} & & 0 \\ & \ddots & \\ 0 & & a_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= [x_1 \dots x_n] \begin{bmatrix} x_1 a_{1,1} \\ \vdots \\ x_n a_{n,n} \end{bmatrix}$$

$$= x_1^2 a_{1,1} + \dots + x_n^2 a_{n,n}$$

Note: $x_i^2 \geq 0$, $a_{i,i} > 0$, since A has a positive diagonal.

$\Rightarrow x_1^2 a_{1,1} + \dots x_n^2 a_{n,n} \geq 0$.

$\Leftrightarrow \langle \vec{x}, \vec{x} \rangle \geq 0$.

$\iff \langle \vec{x}, \vec{x} \rangle \geq 0$.

(2) Let $x \in \mathbb{R}^n$. Suppose $\langle x, x \rangle = 0$.

wts $\vec{x} = 0$.

$\langle x, x \rangle = x_1^2 a_{1,1} + \ldots\ldots + x_n^2 a_{n,n}$, by above.

$0 = x_1^2 a_{1,1} + \ldots\ldots + x_n^2 a_{n,n}$

$\Rightarrow 0 = x_1^2 + \ldots\ldots x_n^2$, since $a_{i,i} > 0$.

$\Rightarrow x_i = 0$, since $x_i^2 \geq 0$.

$\Rightarrow \vec{x} = \vec{0}$.

(3) Let $u, v, w \in \mathbb{R}^n$, $r \in \mathbb{R}$.

wts: $\langle u + rv, w \rangle = \langle u, w \rangle + r \langle v, w \rangle$

$$\langle u + r\vec{v}, w \rangle = (\vec{u} + r\vec{v})^T A \vec{w}$$

$$= [u_1 + r v_1, \ldots\ldots u_n + r v_n] \begin{bmatrix} a_{1,1} & & 0 \\ & \ddots & \\ 0 & & a_{n,n} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

$$= (u_1 + r v_1)(w_1 a_{1,1}) + \ldots\ldots + (u_n + r v_n)(w_n a_{n,n})$$

$$= u_1 w_1 a_{1,1} + r v_1 w_1 a_{1,1} + \ldots\ldots u_n w_n a_{n,n} + r v_n w_n a_{n,n}$$

$$= u^T A w + r \cdot v^T A w$$

$$= \langle u, w \rangle + r \cdot \langle v, w \rangle.$$

(4) Let $v, w \in \mathbb{R}^n$, wts $\langle v, w \rangle = \overline{\langle w, v \rangle}$

$$\langle v, w \rangle = v^T A w$$

$$= v_1 w_1 a_{1,1} + \ldots\ldots v_n w_n a_{n,n}$$

$= v_1 w_1 a_{1,1} + \ldots\ldots v_n w_n a_{n,n}$

$= w_1 v_1 a_{1,1} + \ldots\ldots w_n v_n a_{n,n}$, by properties of $\mathbb{R}$.

$= \overline{w_1 v_1 a_{1,1} + \ldots\ldots w_n v_n a_{n,n}}$, since $\overline{}$ does not affect reals

$= \overline{w^T A \vec{v}}$

$= \overline{\langle w, v \rangle}$

2. Apply G.S process to $B = \left\{ \underset{\vec{b}_1}{\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}, \underset{\vec{b}_2}{\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}}, \underset{\vec{b}_3}{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}} \right\}$.

$v_1 = b_1$

$v_2 = b_2 - P_{E_1} b_2$

$= b_2 - \dfrac{\langle b_2, v_1 \rangle}{\|v_1\|^2} v_1$

$= \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} - \dfrac{-9}{9} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$v_3 = b_3 - P_{E_2} b_3$

$= b_3 - \left(\dfrac{\langle b_3, v_1 \rangle}{\|v_1\|^2} v_1 + \dfrac{\langle b_3, v_2 \rangle}{\|v_2\|^2} v_2 \right)$

$= \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \left(\dfrac{2}{9} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + \dfrac{7}{9} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right)$

$= \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} 2/9 \\ 4/9 \\ -4/9 \end{pmatrix} + \begin{pmatrix} 14/9 \\ 7/9 \\ 14/9 \end{pmatrix} \right)$

$= \begin{pmatrix} 2/9 \\ -2/9 \\ -1/9 \end{pmatrix}$

$\Rightarrow B_{G.S} = \left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2/9 \\ -2/9 \\ -1/9 \end{pmatrix} \right\}$.

3. Let $V = \mathbb{P}_3(\mathbb{R})$. $p, q \in \mathbb{P}_3(\mathbb{R})$ $\langle p, q \rangle = \int_{-1} p(x) q(x)\, dx$.

Apply G.S process to $B = \{1, x, x^2, x^3\}$.

Apply G.S process to $B = \{1, x, x^2, x^3\}$.

$v_1 = 1$

$v_2 = b_2 - P_{E_2} b_2$

$= x - \frac{\langle x, 1\rangle}{\|1\|^2} 1.$

$= x - \frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} 1\,dx} 1.$

$= x - \frac{\frac{x^2}{2}\Big|_{-1}^{1}}{x\Big|_{-1}^{1}} 1.$

$= x - \frac{0}{2} 1.$

$= x$

$v_3 = b_3 - P_{E_2} b_3$

$= x^2 - \left(\frac{\langle x^2, x\rangle}{\|x\|} x + \frac{\langle x^2, 1\rangle}{\|1\|} 1 \right)$

$= x^2 - \left(\frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx} x + \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} 1\,dx} 1 \right)$

$= x^2 - \left(\frac{\frac{x^4}{4}\Big|_{-1}^{1}}{\frac{x^3}{3}\Big|_{-1}^{1}} x + \frac{\frac{x^3}{3}\Big|_{-1}^{1}}{x\Big|_{-1}^{1}} 1 \right)$

$= x^2 - \left(\frac{0}{1/3} x + \frac{2/3}{2} 1 \right)$

$= x^2 - \frac{1}{3}$

$v_4 = b_4 - P_{E_3} b_4$

$= x^3 - \left(\frac{\langle x^3, x^2\rangle}{\|x^2\|} x^2 + \frac{\langle x^3, x\rangle}{\|x\|^2} x + \frac{\langle x^3, 1\rangle}{\|1\|^2} 1 \right)$

$= x^3 - \left(\frac{\int_{-1}^{1} x^5\,dx}{\int_{-1}^{1} x^4\,dx} x^2 + \frac{\int_{-1}^{1} x^4\,dx}{\int_{-1}^{1} x^2\,dx} x + \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} 1\,dx} 1 \right)$

$= x^3 - \left(\frac{\frac{x^6}{6}\Big|_{-1}^{1}}{\frac{x^5}{5}\Big|_{-1}^{1}} x^2 + \frac{\frac{x^5}{5}\Big|_{-1}^{1}}{\frac{x^3}{3}\Big|_{-1}^{1}} x + \frac{\frac{x^4}{4}\Big|_{-1}^{1}}{x\Big|_{-1}^{1}} 1 \right)$

$= x^3 - \left(\frac{0}{2/5} x^2 + \frac{2/5}{2/3} x + \frac{0}{2} 1 \right)$

$= x^3 - \frac{3}{5} x.$

$= x^3 - \frac{3}{5}x.$

$B_{G.S} = \{1, x, x^2 - \frac{1}{3}, x^3 - \frac{3}{5}x\}.$

4. $B = \{1, x, x^2\}$ for $U = \mathbb{P}_2(\mathbb{R})$.

$\langle p, q \rangle = \int_0^1 p(x)q(x)$

Find an <u>orthonormal</u> basis

$\boxed{\text{NOTE: some work hidden for times sake, try the calculations yourself!}}$

$u_1 = \underbrace{\frac{b_1}{\|b_1\|}}_{\text{normalizes}} = \frac{1}{\sqrt{\int_0^1 1\,dx}}$

$= 1.$

<u>orthonormal</u> $\Leftrightarrow$ <u>orthogonal</u> + <u>normal</u>

$u_2 = \frac{p_2}{\|p_2\|}, \quad p_2 = x - \left(\frac{\langle b_2, u_1 \rangle u_1}{\|u_1\| = 1} \right)$

$= x - \int_0^1 x\,dx \cdot 1$

$= x - \frac{1}{2}$

$u_2 = \frac{x - \frac{1}{2}}{\sqrt{\int_0^1 (x - \frac{1}{2})^2 dx}}$

$= 2\sqrt{3}\left(x - \frac{1}{2}\right)$

$u_3 = \frac{p_3}{\|p_3\|}, \quad p_3 = x^2 - \left(\int_0^1 x^2 dx \cdot 1 + \left(\int_0^1 2\sqrt{3}(x - \frac{1}{2}) x^2 \right)(x - \frac{1}{2}) \right)$

proj formula

$\vdots$

$= x^2 - x + \frac{1}{6}$

$$u_3 = \frac{x^2 - x + \frac{1}{6}}{\sqrt{\int_0^1 (x^2 - x + \frac{1}{6})^2 dx}}$$

$$= 6\sqrt{5}\,(x^2 - x + \tfrac{1}{6})$$

ext. Find the closest polynomial of deg 2 or less to $f(x) = xe^x$.

$$W = sp\{1, x, x^2\}$$

$$= sp\{1, 2\sqrt{3}(x - \tfrac{1}{2}), 6\sqrt{5}(x^2 - x + \tfrac{1}{6})\}$$, by prev. question

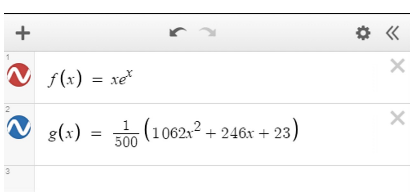
$$g(x) = \text{closest}(f)$$

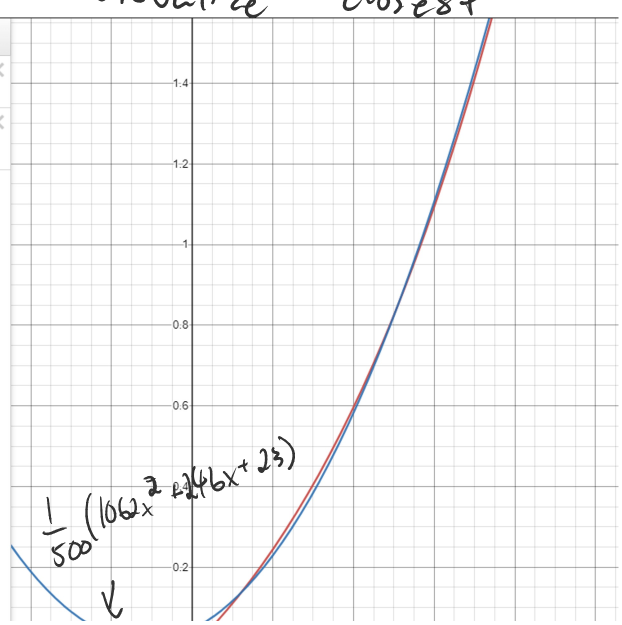
$$= \text{proj}_W f$$

$$= \langle xe^x, u_1\rangle u_1 + \langle xe^x, u_2\rangle u_2 + \langle xe^x, u_3\rangle u_3$$

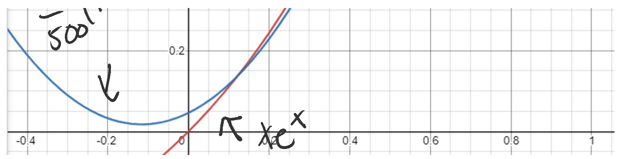
$$\approx \frac{1}{500}(1062x^2 + 246x + 23)$$

visualize "closest"





can use $g(x)$ to $\approx f(x)$ on $[0,1]$



0.2
-0.4
-0.2
0
0.4
0.6
0.8
1
xe^x