Week #5

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https://github.com/LazarGlumac/MATB24-TUT3-Notes

Start @ 12 ~~10~~ 12

Feedback?

1. T / F? + Justification

a) Every vector space that is generated by a finite set has a basis. T

$sp(\{v_1 \dots v_n\}) = V \qquad \{v_1 \dots v_k\} \subseteq \{v_1 \dots v_n\}$

$sp(V) \quad l.i$

b) Every vector space has a finite basis. F

$\mathcal{F} = sp\{1, x, x^2 \dots \}$

c) A vector space cannot have >1 basis. F

$\mathbb{R}^1 \quad \{1\}, \{2\}$

d) If a vector space has a finite basis, then any number of vectors in any basis is any number of vectors in any basis is the same. $\underline{T}$

$\Rightarrow \{v_1 \dots v_n\}$ is a basis for V. $\Rightarrow \dim(V) = n$

Suppose $\{v_1 \dots v_n, v_{n+1}\}$ is a basis $\Rightarrow \dim(V) = n+1$ X

e) if $\{v_1 \dots v_n\}$ span a V.S V, then every vector in V has a unique representation. $\underline{F}$

$sp\left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\} = \mathbb{R}^2$ show $\begin{bmatrix}0\\1\end{bmatrix}$ has 2 rep.

$\begin{bmatrix}0\\1\end{bmatrix} = 1 \cdot \begin{bmatrix}0\\1\end{bmatrix} + 0\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\end{bmatrix}$ or $+ 0\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\end{bmatrix}$

f) If V is an n-dim V.S, then V has exactly 1 0 dim and n dim subspace. $\underline{T}$

0 dim - $\{\vec{0}\}$

n dim - V

2. Prove that a l.i system of vectors in a vector space V $\{v_1 \dots v_n\}$ is a basis iff $n = \dim(V)$

try it! (both ways)

$\Rightarrow$ Suppose $\{v_1 \dots v_n\}$ is a basis

WTS: $n = \dim(V)$

$\{v_1 \dots v_n\}$ is a basis

$\Leftrightarrow \{v_1 \dots v_n\}$ is generating

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$\Leftrightarrow \dim(V) = |\{v_1 \dots v_n\}|$, by def.

$\Leftarrow$ $\dim(V) = |\{v_1 \dots v_n\}|$, by def.

$= n$, as wanted

$\Leftarrow$ Suppose $n = \dim(V)$

WTS: $\{v_1, \dots, v_n\}$ is a basis

$\hookrightarrow$ $\{v_1, \dots v_n\}$ are generating

$\{v_1, \dots, v_n\}$ is a l.i. set with n vectors, and $\dim V = n$

So, by Q5.2 in HW $\{v_1 \dots v_n\}$ is generating.

Thus, $\{v_1, \dots v_n\}$ is a basis, as wanted

QED.

3. Let $\{\vec{u}, \vec{v}, \vec{w}\}$ be a basis in a V.S V.

Show $\{\vec{u}+\vec{v}+\vec{w}, \vec{v}+\vec{w}, \vec{w}\}$ is also a basis. ~~try~~ -x

wts

① l.i set

② spanning set

① Consider

$\vec{0} = c_1(\vec{u}+\vec{v}+\vec{w}) + c_2(\vec{v}+\vec{w}) + c_3(\vec{w})$, $c_{1,2,3} \in F$

$\vec{0} = c_1(\vec{u}+\vec{v}+\vec{w}) + c_2(\vec{v}+\vec{w}) + c_3(\vec{w})$, $c_{1,2,3} \in F$

$= c_1\vec{u} + c_1\vec{v} + c_1\vec{w} + c_2\vec{v} + c_2\vec{w} + c_3\vec{w}$, by Axiom X

$= c_1\vec{u} + (c_1+c_2)\vec{v} + (c_1+c_2+c_3)\vec{w}$,

$\Leftrightarrow$ $c_1 = 0$, $c_1+c_2 = 0$, $c_1+c_2+c_3 = 0$, since $\{\vec{u},\vec{v},\vec{w}\}$ is a basis. $\Leftrightarrow$ $c_1 = c_2 = c_3 = 0$

② Let $\vec{x} \in V$,

then $\exists\, c_1, c_2, c_3, a_1, a_2, a_3 \in F$ st

$c_1 = a_1$, $c_2 = a_1 + a_2$, $c_3 = a_1 + a_2 + a_3$

$\vec{x} = c_1\vec{u} + c_2\vec{v} + c_3\vec{w}$, since $\{u,v,w\}$ is basis

$= a_1\vec{u} + (a_1+a_2)\vec{v} + (a_1+a_2+a_3)\vec{w}$

$= a_1\vec{u} + a_1\vec{v} + a_2\vec{v} + a_1\vec{w} + a_2\vec{w} + a_3\vec{w}$

$= a_1(\vec{u}+\vec{v}+\vec{w}) + a_2(\vec{v}+\vec{w}) + a_3\vec{w}$

Thus, its a spanning set

by ① and ② $\{u+v+w, v+w, w\}$ is a basis

QED.

4. Find a 2×3 system st its general soln is $\begin{pmatrix}1\\1\\0\end{pmatrix} + s\begin{pmatrix}1\\2\\1\end{pmatrix}$, $s \in \mathbb{R}$.

$\Leftrightarrow$ Find $A, \vec{b}$ st $A\vec{x} = \vec{b}$ has $\vec{x} = \begin{pmatrix}1\\1\\0\end{pmatrix} + s\begin{pmatrix}1\\2\\1\end{pmatrix}$

$\Leftrightarrow$ Find A, b st $A\vec{x} = \underline{b}$ has $\vec{x} = \underline{\begin{pmatrix}1\\1\\0\end{pmatrix} + s\begin{pmatrix}1\\2\\1\end{pmatrix}}$

$$\vec{x} = \begin{pmatrix}1\\1\\0\end{pmatrix} + s\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$= \begin{pmatrix}1+s\\1+2s\\s\end{pmatrix}$$

$$\Leftrightarrow \begin{aligned} x_1 &= 1+s \\ x_2 &= 1+2s \\ x_3 &= s \end{aligned}$$

$$\Leftrightarrow \begin{aligned} x_1 &= 1+x_3 \\ x_2 &= 1+2x_3 \\ x_3 &= s \end{aligned}$$

$\Rightarrow x_3$ is free var.

$$\Leftrightarrow \begin{aligned} x_1 + 0x_2 - x_3 &= 1 \\ 0x_1 + x_2 - 2x_3 &= 1 \\ \underline{x_3 = s} \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \vec{x} = \begin{bmatrix}1\\1\end{bmatrix}$$

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