TUT #1

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NOTE: you must include explanation steps (in my case, verbal)

Start @ $12^{\underline{10}}$

Proving V.S

A1: $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

A2: $\vec{u}+\vec{v} = \vec{v}+\vec{u}$

A3: $\exists \vec{0}_v$ st $\vec{v}+\vec{0} = \vec{0}+\vec{v} = \vec{v}$

A4: $\vec{v}+(-\vec{v}) = \vec{0}_v$

S1: $c \in F$ $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$

S2: $c, d \in F$ $\vec{u}(c+d) = \vec{u}c+\vec{u}d$

S3: $c, d \in F$ $c(d\vec{u}) = (cd)\vec{u}$

S4: $\vec{v} = 1 \cdot \vec{v} = \vec{v}$

Prove $M_{n \times n}(\mathbb{R})$ is a real V.S (try it!)

A1 Let $X, Y, Z \in M_{n \times n}(\mathbb{R})$

$$X + (Y+Z) = (X+Y)+Z$$

$$= \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & & \\ x_{n,1} & \cdots & x_{n,n} \end{bmatrix} + \left(\begin{bmatrix} y_{1,1} & \cdots & \\ \vdots & \ddots & \\ & & \end{bmatrix} + \begin{bmatrix} z_{1,1} & \cdots & \\ \vdots & \ddots & \\ & & \end{bmatrix} \right)$$

$$= \begin{bmatrix} x_{1,1} & \cdots & \\ \vdots & \ddots & \\ & & \end{bmatrix} + \left(\begin{bmatrix} (y_{1,1}+z_{1,1}) & \cdots & \\ \vdots & \ddots & \\ & & \end{bmatrix} \right)$$

$$= \begin{bmatrix} x_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix} + \left(\begin{bmatrix} \cdots & \cdots \\ \vdots & \ddots \end{bmatrix} \right)$$

$$= \begin{bmatrix} \underline{x_{1,1} + (y_{1,1} + z_{1,1})} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$\underline{x_{i,j}, y_{i,j}, z_{i,j} \in \mathbb{R}}$

$$= \begin{bmatrix} (x_{1,1} + y_{1,1}) + z_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} (x_{1,1} + y_{1,1}) & \cdots \\ \vdots & \ddots \end{bmatrix} + \begin{bmatrix} z_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= \left(\begin{bmatrix} x_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix} + \begin{bmatrix} y_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix} \right) + Z$$

$$= (X + Y) + Z \quad \checkmark$$

$\underline{A2}$ $X + Y = Y + X$

$$X + Y = \begin{bmatrix} x_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix} + \begin{bmatrix} y_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} x_{1,1} + y_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} y_{1,1}+x_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} y_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix} + \begin{bmatrix} x_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= Y + X \quad \checkmark$$

A3 $\exists \vec{0}_m$ st $X + \vec{0}_m = X = \vec{0}_m + X$ Let $\vec{0}_m = [0]$

$$X + \vec{0}_m = \vec{0}_m + X \quad , \text{ A2,}$$

$$= \begin{bmatrix} 0 & \cdots \\ \vdots & \ddots \end{bmatrix} + \begin{bmatrix} x_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} 0+x_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} x_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= X \quad \checkmark$$

A4 $X+(-X) = (-X)+X = \vec{0}_m$ Let $-X = (-1)x_{i,j}$

$$X + (-X) = (-X) + X \quad , \text{ by A2}$$

$$= \begin{bmatrix} -x_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix} + \begin{bmatrix} x_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} -x_{1,1} + x_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= \vec{0}_n \quad \checkmark$$

$\underline{S1}$ $c \in \mathbb{R}$ $\quad c(X+Y) = cX + cY$

$$c(X+Y) = c \cdot \begin{pmatrix} (x_{1,1} + y_{1,1}) & \cdots \\ \vdots & \ddots \end{pmatrix}$$

$$= \begin{bmatrix} c(x_{1,1} + y_{1,1}) & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} cx_{1,1} + cy_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} cx_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix} + \begin{bmatrix} cy_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= cX + cY \quad \checkmark$$

$\underline{S2}$ Let $c, d \in \mathbb{R}$ $\quad (c+d)X = cX + dX$

$$(c+d)X = (c+d)\begin{bmatrix} x_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$(c+d)X = (c+d)\begin{bmatrix} \vdots & \ddots \\ \vdots & \end{bmatrix}$$

$$= \begin{bmatrix} (c+d)x_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} cx_{1,1} + dx_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= cX + cY \quad \checkmark$$

<u>S3</u> $c(dX) = (cd)X$

$$c(dX) = c\begin{bmatrix} dx_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} c(dx_{1,1}) & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} (cd)x_{1,1} & \cdots \\ \vdots & \ddots \end{bmatrix}$$

$$= (cd)X \quad \checkmark$$

$= (cd)X$ ✓

S4 $X = 1 \cdot X = X$

$$1 \cdot X = 1 \begin{bmatrix} X_{1,1} & \cdots & \\ \vdots & \ddots & \\ & & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} 1(X_{1,1}) & \cdots & \\ \vdots & \ddots & \\ & & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} X_{1,1} & \cdots & \\ \vdots & \ddots & \\ & & \ddots \end{bmatrix}$$

$= X$ ✓

$A1 - A4 \wedge S1 - S4$ ✓

Def Let V be a V.S

Let $S \subseteq V$

S is a basis for V

① S is L.I subset of V

② S is a spanning for V

Prove:

$B = \{1, x+1, x^2+x+1\}$ is basis for $\mathbb{P}_2(\mathbb{R})$

wts ① B is a spanning set

② B is L.I

$$\boxed{A = B \quad \text{iff} \quad A \subseteq B,\ B \subseteq A}$$

① wts $Sp(B) = \mathbb{P}_2(\mathbb{R})$

Take $p(x) \in Sp(B)$

$$p(x) = a_0(1) + a_1(x+1) + a_2(x^2+x+1), \quad a_i \in \mathbb{R}, \text{ by def. of lc}$$

$$= a_0 + a_1 x + a_1 + a_2 x^2 + a_2 x + a_2$$

$$= (a_0 + a_1 + a_2) + (a_1 + a_2)x + a_2 x^2$$

$$\in \mathbb{P}_2(\mathbb{R}) \qquad \Rightarrow Sp(B) \subseteq \mathbb{P}_2$$

Take $p(x) \in \mathbb{P}_2(\mathbb{R})$

$$p(x) = r_0 + r_1 x + r_2 x^2, \quad r_i \in \mathbb{R}$$

wts $p(x) \in Sp(B)$

wts $p(x) = a_0(1) + a_1(x+1) + a_2(x^2+x+1)$

ie $$r_0 + r_1 x + r_2 x^2 = a_0 + a_1(x+1) + a_2(x^2+x+1)$$

$$= (a_0 + a_1 + a_2) + (a_1 + a_2)x + a_2 x^2$$

so choose

$$\left.\begin{aligned} r_0 &= (a_0 + a_1 + a_2) \\ r_1 &= a_1 + a_2 \\ r_2 &= a_2 \end{aligned}\right\} \Rightarrow \begin{aligned} a_0 &= r_0 - (r_1 - r_2) - r_2 \in \mathbb{R} \\ a_1 &= r_1 - r_2 \in \mathbb{R} \\ a_2 &= r_2 \in \mathbb{R} \end{aligned}$$

$$\mathbb{P}_2 \subseteq sp(B)$$

$$sp(B) = \mathbb{P}_2$$

$sp(B) = \mathbb{P}_2$

(2) Suppose $\underline{a_0(1) + a_1(x+1) + a_2(x^2+x+1)} = 0$

wts $a_0 = a_1 = a_2 = 0$

$$(a_0 + a_1 + a_2) + (\underline{a_1 + a_2})x + a_2x^2 = 0 = 0(1) + 0(x) + 0x^2$$

So, $\left.\begin{aligned} a_2 &= 0 \\ a_1 + a_2 &= 0 \\ a_0 + a_1 + a_2 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} a_2 &= 0 \\ a_1 &= 0 \\ a_0 &= 0 \end{aligned}$, as wanted

$\Rightarrow$ B is a l.i set

By (1) and (2), B is a basis for $\mathbb{P}_2(\mathbb{R})$