Week #11

July 30, 2021 11:53 AM

https://github.com/LazarGlumac/MATB24-TUT3-Notes

start @ 12^{10}

H/W8 marked, tut did well :)

1. Suppose A is real-skew-sym matrix, show iA is a Hermetian matrix.

$\Rightarrow (iA)^* = iA$

Consider $(iA)^* = (\overline{iA})^T$, by def of star.
$= (\overline{i}\,\overline{A})^T$, by $\overline{}$ prop.
$= ((-i)\overline{A})^T$, since $\overline{i} = -i$
$= -i(\overline{A})^T$, by T prop.
$= -i(A)^T$, since A is real, $\overline{A} = A$.
$= -i(-A)$, since A is skew-sym.
$= iA$, as wanted.

2. Orthogonally diagonalize. $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = A$, $A = PUP^{-1}$

P: orthonormal columns; U: diagonal

1. Find $p(\lambda)$ for the matrix + solve
 ↳ eigen values
2. Find eigen vectors for each λ_i
3. orthonormalize them
4. Find $\overline{P}^T = P^*$

4. Find P $= P$

1. $p(\lambda) = \det(A - \lambda I)$
$= \det\left(\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix} - \lambda I\right)$
$= \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$
$\Rightarrow \quad 0 = \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$
$= (1-\lambda)^2 - 4$
$= \left((1-\lambda)+2\right)\left((1-\lambda)-2\right)$
$= -(3-\lambda)(1+\lambda)$

$\Rightarrow \lambda_1 = 3, \ \lambda_2 = -1.$

2. for $\lambda_1 = 3$,
$(A - \lambda_1 I)\vec{x} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\vec{x} = \vec{0}$,
$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right) \underset{R_1 \to \frac{1}{2}R_1}{\overset{R_2 \to R_2 + R_1}{\sim}} \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x_2 = s \Rightarrow \vec{x} = s\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ← eigen vector

for $\lambda_2 = -1$,
$(A - \lambda_2 I)\vec{x} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\vec{x} = 0$
$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right) \underset{R_1 \to \frac{1}{2}R_1}{\overset{R_2 \to R_2 - R_1}{\sim}} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x_2 = s \Rightarrow \vec{x} = s\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ ← eigen vector.

3. if vectors not orthogonal, apply G.S
→ then normalize.

$P_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad P_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$

$P_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $\qquad P_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}.$

$\frac{1}{\|x\|}$

4. $P = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$, $P^* = \overline{P}^T$

↑ real

$= P^T$

$= \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$

$D = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$

$= \begin{pmatrix}3 & 0\\ 0 & -1\end{pmatrix}.$

$A = \frac{1}{2}\underbrace{\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}}_{P}\underbrace{\begin{pmatrix}3 & 0\\ 0 & -1\end{pmatrix}}_{D}\underbrace{\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}}_{P^{-1}}$, as wanted.

3. Show the product of unitary matrices is unitary.

– try it.

Suppose A, B are unitary.

wts: $(AB)^*(AB) = (AB)(AB)^* = I.$

since A, B unitary

$A^*A = AA^* = I$, $B^*B = BB^* = I.$

Consider $(AB)^*(AB) = (B^*A^*)(AB)$, by * properties.

ABB^*A^* $\qquad = B^*(A^*A)B$, by matrix mult. prop.

$\qquad\qquad = B^*IB$, since A is unitary

$\qquad\qquad = B^*B$

$\qquad\qquad = I$, since B is unitary.

I, since ... is unitary.

A, B are unitary,
$\Rightarrow$ square, AB must also be a square matrix,

thus, the product, AB is unitary.

4. Let U be a 2×2 orthogonal matrix, with $\det U = 1$. Show U is a rotation matrix.

Let $U = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, since U is orthogonal,

$$U = U^T = U^{-1}$$

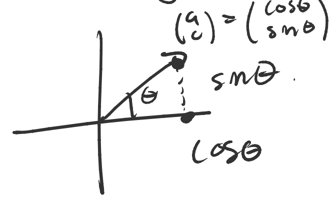
$$\underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{U^T} = \underbrace{\frac{1}{\det U}}_{1} \cdot \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{U^{-1}}$$

Thus, $a = d$, $b = -c$
so $U = \begin{bmatrix} a & -c \\ c & a \end{bmatrix}$.

Consider $\det U$

$$1 = a^2 + c^2,$$
$$= \cos^2\theta + \sin^2\theta.$$

Aside.
We can write $a = \cos\theta$, for some θ by trig iden. since $\left\|\begin{pmatrix} a \\ c \end{pmatrix}\right\| = 1$



from orthogonal matrix $\Leftrightarrow$ orthonormal columns.

Thus, $U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, as wanted.

5. Prove if A is self-adjoint, A^k, $k \in \mathbb{N}^+$ is

5. Prove if A is self-adjoint, $A^k, k \in \mathbb{N}^+$ is also self-adjoint.

Suppose A is self-adjoint, $k \in \mathbb{N}^+$

consider $(A^k)^* = (\underbrace{A \cdot A \cdots\cdots A}_{k \text{ times}})^*$

$= (A^* \cdot A^* \cdots\cdots A^*)$, by $*$ prop.

$= (A \cdot A \cdots\cdots A)$, since A is self adjoint $\Rightarrow A^* = A$.

$= A^k$.

$\Rightarrow (A^k)^* = A^k \;\wedge\; A$ is square $\Rightarrow A^k$ is square.

thus, A^k is self adjoint.

QED.