Week #10

July 23, 2021 11:52 AM

https://github.com/LazarGlumac/MATB24-TUT3-Notes

Start @ 12^{10}

1. Find all vectors $\in \mathbb{R}^4$ orthogonal to

$$\underbrace{\left\{\begin{pmatrix}1\\1\\2\\4\end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix}1\\3\\2\\1\end{pmatrix}}_{v_2}\right\}.$$

$\vec{x}\cdot v_1 = 0$

$0 = x_1 + x_2 + 2x_3 + 4x_4$

$\vec{x}\cdot v_2 = 0$

$0 = x_1 + 3x_2 + 2x_3 + x_4$

let $x \in \mathbb{R}^4$, $x = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$

$$\Rightarrow \begin{bmatrix}1&1&2&4\\1&3&2&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}1&1&2&4&0\\1&3&2&1&0\end{array}\right] \overset{R_2 \leftrightarrow R_2 - R_1}{\sim} \left[\begin{array}{cccc|c}1&1&2&4&0\\0&2&0&-3&0\end{array}\right]$$

Free.

$x_3 = s$, $x_4 = t$, $s,t \in \mathbb{R}$.

$$\Rightarrow \left.\begin{aligned} x_1 + x_2 + 2s + 4t &= 0 \\ 2x_2 - 3t &= 0 \end{aligned}\right\} \quad \begin{aligned} x_1 &= -2s - 4t - x_2 \\ x_2 &= \tfrac{3}{2}t \end{aligned}$$

$$\Rightarrow x_1 = -2s - \frac{11}{2}t$$

$$\vec{x} = \begin{pmatrix}-2s - \frac{11}{2}t\\ \frac{3}{2}t\\ s\\ t\end{pmatrix} = s\begin{pmatrix}-2\\0\\1\\0\end{pmatrix} + t\begin{pmatrix}-\frac{11}{2}\\ 3/2\\ 0\\ 1\end{pmatrix}$$

So, $\vec{x} \in \operatorname{sp}\left\{\begin{pmatrix}-2\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-11/2\\3/2\\0\\1\end{pmatrix}\right\}$.

2. Find the L.S soln of the system $Ax=b$,

$$A=\begin{bmatrix}1&0\\1&1\\1&2\end{bmatrix},\quad b=\begin{bmatrix}2\\1\\2\end{bmatrix}.$$

$\Rightarrow$ solve solutions of $A^*A\vec{x}=A^*\vec{b}$

$$A^*=\bar{A}^T=\begin{bmatrix}1&1&1\\0&1&2\end{bmatrix},\quad A\vec{x}=\begin{bmatrix}1&0\\1&1\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\x_1+x_2\\x_1+2x_2\end{bmatrix}\qquad \vec{x}=(A^*A)^{-1}A^*b$$

$$A^*Ax=\begin{bmatrix}1&1&1\\0&1&2\end{bmatrix}\begin{bmatrix}x_1\\x_1+x_2\\x_1+2x_2\end{bmatrix}\qquad A^*b=\begin{bmatrix}1&1&1\\0&1&2\end{bmatrix}\begin{bmatrix}2\\1\\2\end{bmatrix}$$

$$=\begin{bmatrix}x_1+x_1+x_2+x_1+2x_2\\x_1+x_2+2(x_1+2x_2)\end{bmatrix}\qquad =\begin{bmatrix}5\\5\end{bmatrix}$$

$$=\begin{bmatrix}3x_1+3x_2\\3x_1+5x_2\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}3&3\\3&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}5\\5\end{bmatrix}$$

$$\left[\begin{array}{cc|c}3&3&5\\3&5&5\end{array}\right]\overset{R_2\to R_2-R_1}{\sim}\left[\begin{array}{cc|c}3&3&5\\0&2&0\end{array}\right]$$

$$\overset{R_1\to R_1-3/2R_2}{\sim}\left[\begin{array}{cc|c}3&0&5\\0&2&0\end{array}\right]\quad\Rightarrow x=\begin{bmatrix}5/3\\0\end{bmatrix}\ \checkmark$$

$A^*A\vec{x}$ $\uparrow$ $\vec{b}$

3. Fit a plane $z=a+bx+cy$ to the points

$$\left\{\begin{pmatrix}1\\1\\3\end{pmatrix},\begin{pmatrix}0\\3\\6\end{pmatrix},\begin{pmatrix}2\\1\\5\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$$

$$\left\{ \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}3\\6\end{pmatrix}, \begin{pmatrix}1\\5\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix} \right\}$$

$\Rightarrow$ solve L.S soln system using points,

$$\left.\begin{aligned} a + (1)b + (1)c &= 3 \\ a + (0)b + (3)c &= 6 \\ a + (2)b + (1)c &= 5 \\ a + (0)b + (0)c &= 0 \end{aligned}\right\} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ 6 \\ 5 \\ 0 \end{bmatrix}}_{\vec{b}}.$$

$$A^* = \bar{A}^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \end{bmatrix}, \qquad A\vec{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b+c \\ a+3c \\ a+2b+c \\ a \end{bmatrix}$$

$$A^* A\vec{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} a+b+c \\ a+3c \\ a+2b+c \\ a \end{bmatrix}$$

$$= \begin{bmatrix} (a+b+c) + (a+3c) + (a+2b+c) + (a) \\ (a+b+c) + 2(a+2b+c) \\ (a+b+c) + 3(a+3c) + (a+2b+c) \end{bmatrix}$$

$$= \begin{bmatrix} 4a + 3b + 5c \\ 3a + 5b + 3c \\ 5a + 3b + 11c \end{bmatrix}$$

$$A^*b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 14 \\ 13 \\ 26 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} 4 & 3 & 5 \\ 3 & 5 & 3 \\ 5 & 3 & 11 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 26 \end{bmatrix}$$

$$\Rightarrow \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3/25 \\ 0 & 1 & 0 & 73/50 \\ 0 & 0 & 1 & 101/50 \end{array}\right].$$

thus, $a = \frac{-3}{25}$, $b = 23/50$, $c = 101/50$.

$\Rightarrow z = -\frac{3}{25} + \frac{23}{50}x + \frac{101}{50}y$. is the eqn of the plane.

4. Show that for a square matrix A,

$\det(A^*) = \overline{\det(A)}$, $A_{n \times n}$

By Strong Induc.

Base case: $n = 1$,

Base case: $n=1$,

$A=[a+ib]$ $\quad \det(A^*)=a-ib$

$A^*=[a-ib]$. $\quad\quad = \overline{\det(A)}$

Let $n>1$, Suppose $A_{j\times j}$ $\det(A^*)=\overline{\det(A)}$

holds whenever $1\le j<n$. (IH)

WTS: $A_{n\times n}$ it holds.

IS: Suppose $A_{n\times n}$, $n>1$.

$$\det(A^*)=\det(\overline{A}^T)$$

$$=\det(\overline{A}), \quad \text{by det. prop.}$$

$$=\sum_{j=1}^{n}\overline{a_{i,j}}\cdot(-1)^{i+j}\cdot\det(\overline{M_{i,j}}), \quad 1\le i\le n, \ M_{i,j} \text{ is } i,j \text{ minor of } A.$$

$$=\sum_{j=1}^{n}\overline{a_{i,j}}\cdot(-1)^{i+j}\cdot\det(\overline{M_{i,j}}^T), \quad \text{by det prop.}$$

$$=\sum_{j=1}^{n}\overline{a_{i,j}}\cdot(-1)^{i+j}\cdot\det(M_{i,j}^*)$$

$$=\sum_{j=1}^{n}\overline{a_{i,j}}\cdot(-1)^{i+j}\cdot\overline{\det(M_{i,j})}, \quad \text{by IH, since } M_{i,j} \text{ is } (n-1)\times(n-1).$$

$$=\sum_{j=1}^{n}\overline{a_{i,j}}\,\overline{(-1)^{i+j}}\cdot\overline{\det(M_{i,j})}, \quad \text{since } (-1)^{i+j}\in\mathbb{R}.$$

$$=\sum_{j=1}^{n}\overline{a_{i,j}(-1)^{i+j}\cdot\det(M_{i,j})}, \quad \text{by } \overline{} \text{ prop.}$$

$$= \sum_{j=1}^{n} \overline{a_{i,j}(-1)^{i+j} \cdot \det(M_{i,j})}, \quad \text{by } \overline{} \text{ prop.} \quad \overline{z+w} = \overline{z} + \overline{w}$$

$$= \overline{\sum_{j=1}^{n} a_{i,j}(-1)^{i+j} \cdot \det(M_{i,j})}$$

$$= \overline{\det(A)},$$ as wanted.