Week #6

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https://github.com/LazarGlumac/MATB24-TUT3-Notes

Start @ $12^{\underline{10}}$

1. An $n \times n$ matrix is called nilpotent if $A^k = O$, for some $k \in \mathbb{N}^+$.
   Show that a nilpotent matrix A, $\det A = 0$.
   <u>try it</u>

   Suppose A is nilpotent
   i.e., $A^k = O$, for $k \in \mathbb{N}^+$

   $\det(A^k) = \det(O) = 0$, since A is nilpotent

   Consider $\det(A^k)$,

   $$0 = \det(\underbrace{A \cdot A \cdot \ldots A}_{k \text{ times}})$$

   $$0 = \underbrace{\det(A) \cdot \ldots\ldots \det(A)}_{k \text{ times}}$$

   $$\sqrt[k]{0} = \sqrt[k]{\det(A)^k}$$

   $$0 = \det(A), \text{ as wanted.}$$

2. Prove if A, B are <u>similar</u>, then $\det(A) = \det(B)$

   Suppose A, B are similar
   i.e., $B = P^{-1} A P$, for some $P, P^{-1} \in M_{n \times n}(F)$

Consider $\det(B)$,

$$\det(B) = \det(P^{-1}AP) \quad \text{, since } A, B \text{ are similar}$$

$$= \det(P^{-1})\det(A)\det(P) \quad \text{, by det prop.}$$

$$= \frac{1}{\cancel{\det(P)}} \det(A) \cancel{\det(P)} \quad \text{, by det prop.}$$

$$= \det(A) \quad \text{, as wanted}$$

3. A real $n \times n$ matrix Q is called orthogonal if $Q^TQ = I$.

Prove that if Q is orthogonal, $\det(Q) = \pm 1$.

Suppose Q is orthogonal

i.e., $Q^TQ = I$

Consider $\det(Q^TQ)$,

$$\det(Q^TQ) = \det(Q^T)\det(Q) \quad \text{, by det prop}$$

$$\det(I) = \det(Q) \cdot \det(Q) \quad \text{, as proved in lec}$$

$$\sqrt{1} = \sqrt{(\det(Q))^2}$$

$$1 = |\det(Q)|$$

$$\Rightarrow \det(Q) = \pm 1 \quad \text{, as wanted.}$$

4. Show if A, C are square matrices, then $\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det A \cdot \det C$

$\boxed{\text{try it}}$

hint: using Q 3.10 on HW 5

hint: $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} I & B \\ 0 & C \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$

Suppose A, C are square matrices.

Consider $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$,

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} I & B \\ 0 & C \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

$$\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det\left(\begin{pmatrix} I & B \\ 0 & C \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} I & B \\ 0 & C \end{pmatrix} \det\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, \text{ by det prop.}$$

Q 3.10

$$\det\begin{pmatrix} I & * \\ 0 & C \end{pmatrix} = \det C$$

$$\det\begin{pmatrix} A & 0 \\ * & I \end{pmatrix} = \det A$$

CONT

$$= \det\begin{pmatrix} I & * \\ 0 & C \end{pmatrix} \cdot \det\begin{pmatrix} A & 0 \\ * & I \end{pmatrix} \quad \text{(has form)}$$

$$= \det C \cdot \det A, \text{ by } 3.10$$

as wanted.

as wanted.

5. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map s.t.

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 2y \\ \frac{1}{2}x - y \end{pmatrix}$$

Find the matrix of T w.r.t the standard basis, $S = \left\{ \underset{\vec{e}_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \underset{\vec{e}_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right\}$, and $B = \left\{ \underset{\vec{b}_1}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}, \underset{\vec{b}_2}{\begin{pmatrix} 0 \\ 3 \end{pmatrix}} \right\}$.

<u>Standard</u>.

$$T(\vec{e}_1) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1/2 \end{pmatrix}$$

$$= \underline{3\vec{e}_1 + \tfrac{1}{2}\vec{e}_2}$$

$$T(\vec{e}_2) = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= 2\vec{e}_1 + (-1)\vec{e}_2$$

$$[T]_{SS} = \begin{bmatrix} \uparrow & \uparrow \\ [T(\vec{e}_1)]_S & [T(\vec{e}_2)]_S \\ \downarrow & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 1/2 & -1 \end{bmatrix}$$

✓.

Basis B.

$$T(\vec{b}_1) = T\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}8\\0\end{pmatrix}$$

$$= 4\vec{b}_1 + \left(-\frac{4}{3}\right)\vec{b}_2$$

$$T(\vec{b}_2) = T\begin{pmatrix}0\\3\end{pmatrix} = \begin{pmatrix}6\\-3\end{pmatrix}$$

$$= 3\vec{b}_1 + (-2)\vec{b}_2$$

$$[T]_{BB} = \Big[\,[T(\vec{b}_1)]_B \quad [T(\vec{b}_2)]_B\,\Big]$$

$$= \begin{bmatrix}4 & 3\\ -4/3 & -2\end{bmatrix}.$$

✓