Week #3

May 28, 2021 12:02 PM

https://github.com/LazarGlumac/MATB24-TUT3-Notes

TUT notes

start @ $12^{\underline{10}}$

HWK 1

- Formal proof structure
- Real vs Complex V.S
- Justification on steps

$\mathbb{C}$ complex $a+ib$

$\{1\}$

real - $\{1, i\}$

Q.

1. Let $T: V \to W$ be an isomorphism, $\{v_1, \ldots, v_n\}$ be linearly indep. Show $\{T(v_1), \ldots, T(v_n)\}$ is also linearly indep.

try it - 5 mins (might be useful for HWK Q ☺)

Suppose T is an isomorphism,
$\{v_i\}_1^n$ are l.i

Let $S = T^{-1}$, T is iso. $\Rightarrow$ T invert.

Consider

$$0 = a_1 T(v_1) + \dots a_n T(v_n)$$

$$0 = T(a_1 v_1 + \dots a_n v_n), \quad T \text{ is linear}$$

$$S(0) = S(T(a_1 v_1 + \dots + a_n v_n))$$

$$0 = a_1 v_1 + \dots + a_n v_n, \quad \text{since } S \text{ injective}$$

$\Leftrightarrow a_i = 0 \quad \forall i$, since $\{v_i\}_1^n$ l.i

$\therefore \{T(v_i)\}_1^n$ is also l.i

2. $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_2 \\ x_3 - x_1 \end{pmatrix}$

Show T is invertible.

try it - 5 mins

WTS: $\exists S \; \underset{\mathbb{R}^3 \to \mathbb{R}^3}{W \to V}$ st $T \circ S = S \circ T = I$

Let $S\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_3 \\ x_2 \\ x_1 \end{pmatrix}$

$$T \circ S\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = T\begin{pmatrix} x_1 - x_3 \\ x_2 \\ x_1 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ x_2 \\ x_1 - (x_1 - x_3) \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$S \circ T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = S\begin{pmatrix} x_3 \\ x_2 \\ x_3 - x_1 \end{pmatrix}$$

$$= \begin{pmatrix} x_3 - (x_3 - x_1) \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

QED.

3. Give examples of matrices A, B st

i) $A+B$ is not invert. but $A \wedge B$ invert

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

ii) $A+B$ is invert. but $A \wedge B$ are not invert

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

iii) $A+B \wedge A \wedge B$ are invertible

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A+B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$B = A$$

4. Let T_θ be rotation by θ ~~deg~~ radians in $\mathbb{R}^2$

Show $T_\theta T_{-\theta} = T_{-\theta} T_\theta = I$

try it

$$T_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$T_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$T_{-\theta} = \begin{bmatrix} \cdots & \cdots \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & \sin\theta \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}, \quad \cos(-x) = \cos x, \ \sin(-x) = -\sin x$$

$$T_\theta T_{-\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \underline{\cos^2\theta + \sin^2\theta} & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \underline{\sin\theta\cos\theta - \cos\theta\sin\theta} & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

since $\cos^2\theta + \sin^2\theta = 1$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

($T_{-\theta} T_\theta$ similar, try it)

5. Let A, AB be invertible. Show B is also invertible.

try it

AB invert. $\Leftrightarrow \exists C \in M_{n \times n}$ st $C(AB) = I$

$I = C(AB)$

$I = (\underline{CA}) B$, commutative

$\underline{AB = BA = I}$

$\Rightarrow$ B's left inverse is $= CA$. Note, since A, AB are invert. $\Rightarrow A, AB$ are square. $\therefore CA$ is also square also square

$\Rightarrow$ B's inverse is CA

QED.