Week #13

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https://github.com/LazarGlumac/MATB24-TUT3-Notes

Start @ $12^{\underline{10}}$

1. Prove $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ is a positive semi-def operator. Solve for all values of $\vec{x}$ st $\langle A\vec{x}, \vec{x} \rangle = 0$.

$A^* = A$, so A is self adjoint,
Consider $\langle Ax, x \rangle$, for some $\vec{x} \in \mathbb{R}^2$.

$$
\begin{aligned}
\langle Ax, x \rangle &= (A\vec{x})^T \vec{x} \\
&= \vec{x}^T A^T \vec{x}, \text{ by } {}^T \text{ prop.} \\
&= x^T A \vec{x}, \text{ since } A = A^T \\
&= [x_1, x_2] \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= [x_1, x_2] \begin{bmatrix} 4x_1 + 2x_2 \\ 2x_1 + x_2 \end{bmatrix} \\
&= x_1(4x_1 + 2x_2) + x_2(2x_1 + x_2) \\
&= 4x_1^2 + 4x_1x_2 + x_2^2 \\
&= (2x_1 + x_2)^2 \\
&\geq 0
\end{aligned}
$$

equality when:

$$(2x_1 + x_2)^2 = 0$$

$$\Rightarrow \quad 0 = 2x_1 + x_2 \qquad \Rightarrow \quad \vec{x} = c\,(-1/2)$$

$(2x_1 + x_2) = 0$

$\Rightarrow \quad 0 = 2x_1 + x_2 \qquad \Rightarrow \vec{x} = s\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

$x_2 = s, \quad x_1 = -\frac{1}{2}s$

2. Prove if A and B are positive def., then so is $(A+B)$.

Suppose A and B are positive def.

$\Rightarrow A, B$ self adjoint, $\langle Ax, x\rangle > 0$, $\langle Bx, x\rangle > 0$.

$A+B$ is self adjoint, since the sum of self adj matrices is self adjoint.

Consider $\langle (A+B)\vec{x}, \vec{x}\rangle$, for some $x \in V$.

$$\begin{aligned}
\langle (A+B)\vec{x}, \vec{x}\rangle &= x^*(A+B)^*\vec{x} \\
&= \vec{x}^*(A^* + B^*)\vec{x} \quad \text{, by * prop.} \\
&= \vec{x}^*(A^*\vec{x} + B^*\vec{x}) \\
&= \underline{\vec{x}^* A^* \vec{x}} + \underline{\vec{x}^* B^* \vec{x}} \\
&= \langle Ax, x\rangle + \langle Bx, x\rangle \\
&> 0 \text{, since } A \text{ and } B \text{ are positive def.}
\end{aligned}$$

3. Prove the sum of normal operators is normal

Let N_1, N_2 be normal operators,

we know

$\ldots {}^* \ldots U \ldots$ unitary

we know

i) $N_1 = U_1 D_1 U_1^*$ $\quad$ U_i is unitary

$N_2 = U_2 D_2 U_2^*$ $\quad$ D_i is diagonal.

ii) $N_1 N_1^* = N_1^* N_1$ $\quad$ N_i is normal,

$N_2 N_2^* = N_2^* N_2$

wts: $(N_1 + N_2)^*(N_1 + N_2) = (N_1 + N_2)(N_1 + N_2)^*$

$$(N_1 + N_2)^*(N_1 + N_2) = (N_1^* + N_2^*)(N_1 + N_2)$$
$$= \underline{N_1^* N_1} + N_1^* N_2 + N_2^* N_1 + \underline{N_2^* N_2}.$$

$$(N_1 + N_2)(N_1 + N_2)^* = (N_1 + N_2)(N_1^* + N_2^*)$$
$$= N_1 N_1^* + N_1 N_2^* + N_2 N_1^* + N_2 N_2^*$$
$$= \underline{N_1^* N_1} + N_1 N_2^* + N_2 N_1^* + \underline{N_2^* N_2}, \text{ by ii)}$$

we need to show: $N_1^* N_2 + N_2^* N_1 = N_1 N_2^* + N_2 N_1^*$

$N_1^* N_2 = (U_1 D_1^* U_1^*)(U_2 D_2 U_2^*)$, by i)

$N_1 N_2^* = (U_1 D_1 U_1^*)(U_2 D_2^* U_2^*)$

$\Rightarrow$ we need to show $D_1^* U_1^* U_2 D_2 = D_1 U_1^* U_2 D_2^*$

since D_1, D_2 are diagonal, $D_1^* = \overline{D_1}$, $D_2^* = \overline{D_2}$

also, $\underline{D_1^* D_2 = D_1 D_2^*}$ since for complex numbers,

$\bar{x} \cdot y = x \cdot \bar{y}$

Expanding each entry, we can see that

$D_1^* U_1^* U_2 D_2 = D_1 U_1^* U_2 D_2^*$

Thus, $N_1^* N_2 = N_1 N_2^*$, and by extension, $N_2^* N_1 = N_2 N_1^*$

Therefore, $N_1^* N_2 + N_2^* N_1 = N_1 N_2^* + N_2 N_1^*$

$\Rightarrow$ we have the sum of normal operators is also normal.

QED.

4. Show that the number of non-zero singular values of a matrix A is equal to its rank.

from HWa $\text{rank}(A) = \text{rank}(A^* A)$,

we also have $A = W \Sigma V^*$, by SVD.

$$\Rightarrow \text{rank}(A) = \text{rank}(A^* A)$$
$$= \text{rank}\left((W \Sigma V^*)^* (W \Sigma V^*)\right)$$
$$= \text{rank}\left((V \Sigma^* W^*)(W \Sigma V^*)\right)$$
$$= \text{rank}(V \Sigma^* \Sigma V^*), \text{ since } W \text{ is orthogonal, } W^* W = I$$
$$= \text{rank}(V \Sigma^2 V^*), \text{ since } \Sigma \text{ is diagonal}$$

$= \operatorname{rank}(U \Sigma V)$, since Σ is diagonal

$= \operatorname{rank}(\Sigma^2)$, since U, U^* are orthogonal matrices, full rank.

$=$ # of non zero singular values, by def of Σ.

QED.