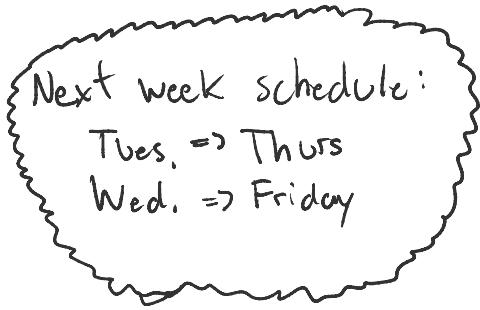


start @ 12¹⁰

1. Orthogonally diag. $A = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}$, then find all square roots of A .
i.e., find B s.t. $B^2 = A$
ext. what are the eigen values of each B ?

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} 7-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} \\ &= (7-\lambda)(4-\lambda) - 4 \\ &= 28 - 11\lambda + \lambda^2 - 4 \end{aligned}$$

$$\text{solve } 0 = (\lambda - 8)(\lambda - 3)$$

$$\Rightarrow \lambda_1 = 8, \lambda_2 = 3$$

for $\lambda_1 = 8$,

$$(A - 8I)\vec{x} = 0 \Leftrightarrow \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\vec{x} = 0$$

$$\left(\begin{array}{cc|0} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array} \right) \xrightarrow{R_2 \leftrightarrow R_2 + 2R_1} \left(\begin{array}{cc|0} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

\uparrow_{free}

$$\text{let } x_2 = s \Rightarrow x_1 = 2s$$

for $\lambda_2 = 3$.

$$(A - 3I)\vec{x} = 0 \Leftrightarrow \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\vec{x} = 0$$

$$\left(\begin{array}{cc|0} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array} \right) \xrightarrow{R_2 \rightarrow R_2 - \frac{1}{2}R_1} \left(\begin{array}{cc|0} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

\uparrow_{free}

$$\text{let } x_2 = s \Rightarrow x_1 = -\frac{1}{2}s.$$

$$\text{let } x_2 = s \Rightarrow x_1 = 2s$$

$$\Rightarrow \tilde{x} = s \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} \right\}$$

$$\text{let } x_2 = s \Rightarrow x_1 = -\frac{1}{2}s.$$

$$\tilde{x} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

$$P_1 = \frac{B_1}{\|B_1\|}$$

$$= \frac{1}{\sqrt{s}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$P_2 = \frac{B_2 - \langle B_2, P_1 \rangle P_1}{\|B_2 - 2\langle B_2, P_1 \rangle P_1\|} \quad (\text{proj + normalize})$$

$$= \frac{B_2}{\|B_2\|} \quad \text{since } 2\langle B_2, P_1 \rangle = 0$$

$$= \frac{2}{\sqrt{s}} \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 2/\sqrt{s} & -1/\sqrt{s} \\ 1/\sqrt{s} & 2/\sqrt{s} \end{pmatrix}.$$

$$P^* = \overline{P}^T = \begin{pmatrix} 2/\sqrt{s} & 1/\sqrt{s} \\ -1/\sqrt{s} & 2/\sqrt{s} \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 \\ 0 & 3 \end{pmatrix}.$$

Want to find B st $B^2 = A$

We have:

$$A = P D P^*, \quad \text{by orthogonalization.}$$

so, we take \sqrt{A} , by diag proj.

$$B = \begin{pmatrix} 2/\sqrt{s} & -1/\sqrt{s} \\ 1/\sqrt{s} & 2/\sqrt{s} \end{pmatrix} \begin{pmatrix} \pm\sqrt{8} & 0 \\ 0 & \pm\sqrt{3} \end{pmatrix} \begin{pmatrix} 2/\sqrt{s} & 1/\sqrt{s} \\ -1/\sqrt{s} & 2/\sqrt{s} \end{pmatrix}$$

ext. eigen values of each B?

$$B_{+,+} = P \begin{pmatrix} \sqrt{8} & 0 \\ 0 & -\sqrt{3} \end{pmatrix} P^* \Rightarrow \lambda_1 = \sqrt{8}, \lambda_2 = -\sqrt{3}$$

$$B_{-,-} = P \begin{pmatrix} -\sqrt{8} & 0 \\ 0 & -\sqrt{3} \end{pmatrix} P^* \Rightarrow \lambda_1 = -\sqrt{8}, \lambda_2 = -\sqrt{3}$$

$$\Rightarrow \lambda_1 = -\sqrt{8}, \lambda_2 = \sqrt{3}$$

$$B_{-, -} = P \begin{pmatrix} -\sqrt{8} & 0 \\ 0 & -\sqrt{3} \end{pmatrix} P^* \Rightarrow \lambda_1 = -\sqrt{8}, \lambda_2 = \sqrt{3}$$

$$B_{-, +} = P \begin{pmatrix} -\sqrt{8} & 0 \\ 0 & \sqrt{3} \end{pmatrix} P^* \Rightarrow \lambda_1 = -\sqrt{8}, \lambda_2 = \sqrt{3}$$

$$B_{+, +} = P \begin{pmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{3} \end{pmatrix} P^* \Rightarrow \lambda_1 = \sqrt{8}, \lambda_2 = \sqrt{3}.$$

2. Prove if A is nilpotent and normal, then $A=0$.

Suppose A is a nilpotent, normal matrix i.e,

$$AA^* = A^*A, \quad \underline{A^k = 0} \quad \text{for some } k \in \mathbb{N}^+$$

Since A is normal, we know A is diagonalizable
(thm. 2.4)
(Chap 6.), moreover it is orthogonally diagonalizable.

$$\Rightarrow A = UDU^* \quad \text{for some orthogonal } U, \text{ diagonal } D.$$

also we know D is diagonal made up of eig. val
of A .

Wtch: eigen values of $A^k = 0$.

Let λ be an eigen value of A , with eigen
vector $\vec{v} \neq 0$.

$$A\vec{v} = \lambda\vec{v}$$

$$\underbrace{A(A \dots (A}_{k \text{ times}} \underbrace{\vec{v}))}_{k \text{ times}} = \underbrace{A(A \dots (\lambda\vec{v}))}_{k \text{ times}}$$

$$A^k \vec{v} = \lambda^k \vec{v} \quad , \text{ applying } A\vec{v} \text{ } k \text{ times}$$

$$0\vec{v} = \lambda^k \vec{v} \quad , \text{ since } A \text{ is nilpotent.}$$

$$0 = \lambda^k \vec{v}$$

$$\Rightarrow \lambda^k = 0, \text{ since } \vec{v} \neq 0$$

$$\Rightarrow \lambda^k = 0, \text{ since } \vec{v} \neq 0$$

$$\Rightarrow \lambda = 0.$$

Thus, $D = 0$. Since all eigen values of A are 0

$$A = U D U^*$$

$$= U(0) U^*$$

$$= 0. \quad , \text{ as wanted.}$$

\downarrow
 $i.e. B = 0$

3. Prove if A is normal, B is nilpotent, and $A + B = I$, then

$A = I$. hint: use last proof. \angle Suppose A normal, B nilpotent,

Wts: $B = 0$, by prev. proof, show B is normal

$$A + B = I$$

$$B = I - A$$

$$B^* B = (I - A)^*(I - A)$$

$$= (I^* - A^*)(I - A), \text{ by } * \text{ prop}$$

$$= (I - A^*)(I - A), \quad I^* = I$$

$$= I - A - A^* + A^* A$$

$$B B^* = (I - A)(I - A)^*$$

$$= (I - A)(I - A^*), \text{ by } * \text{ prop, } I^* = I$$

$$= I - A^* - A - A A^*$$

$$= I - A = A^* - A^* A, \text{ since } A \text{ is normal}$$

$$\Rightarrow B^* B = B B^*, \text{ so } B \text{ is normal.}$$

From Q2, B is normal + nilpotent $\Rightarrow B = 0$

Thus, from $A+B=I$, $A=I$.

4. Let A be $m \times n$ matrix. Show that

1) A^*A is self adj.

2) all eigen values of A^*A are non negative.

$$1). (A^*A)^* = A^*A^{**} , \text{ by } * \text{ prop.}$$

$$= A^*A , \text{ by } * \text{ prop.}$$

✓

2) let λ be an eigen value of A^*A with corresponding eigen vector v .

$$(A^*A)\vec{v} = \lambda \vec{v}.$$

$$\text{Then, } \langle A^*A\vec{v}, \vec{v} \rangle = \langle \lambda \vec{v}, \vec{v} \rangle$$

$$= \lambda \langle \vec{v}, \vec{v} \rangle, \text{ by } \langle \cdot, \cdot \rangle \text{ prop.}$$

$$= \lambda \|\vec{v}\|^2.$$

$$\text{Also, } \langle A^*A\vec{v}, \vec{v} \rangle = (A^*A\vec{v})^* \vec{v} , \text{ by } \langle \cdot, \cdot \rangle \text{ prop.}$$

$$= \vec{v}^* A^* A^{**} \vec{v} , \text{ by } * \text{ prop.}$$

$$= \vec{v}^* A^* A \vec{v}$$

$$= (\vec{v}^* A^*) (A \vec{v}) , \text{ by matrix mult prop.}$$

$$= \langle A\vec{v}, A\vec{v} \rangle$$

$$= \|A\vec{v}\|^2$$

$$\begin{aligned}&= \langle A\vec{v}, \vec{v} \rangle \\&= \|A\vec{v}\|^2\end{aligned}$$

$$\Rightarrow \lambda \|\vec{v}\|^2 = \|A\vec{v}\|^2 > 0 \quad \text{by norm properties}$$

$$\Rightarrow \lambda > 0, \text{ since } \|\vec{v}\|^2 > 0, \text{ since } \vec{v} \neq 0.$$

QED.