In-and-Out Conversions

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By an in-and-out conversion we mean that a floating-point number in one base is converted into a floating-point number in another base and then converted back to a floating-point number in the original base. For all combinations of rounding and truncation conversions the question is considered of how many significant digits are needed in the intermediate base to allow such in-and-out conversions to return the original number (when possible), or at least to cause a difference of no more than a unit in the least significant digit.

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1. Introduction

Handling floating-point data by a computer usually entails a conversion from the original decimal data to some variant of binary and a subsequent conversion back to decimal representation for output. Since numbers with a terminating expansion to one base need not have a terminating expansion to another base, an approximation can be involved, and the general question of whether or not the original number is restored after conversion into a different base followed by conversion back to the original base (which we term in-and-out conversion) is presented.

Goldberg [1] showed that a sufficient condition for the number of bits, m, needed in a floating-point binary representation to assure that a given system of n-digit floating-point decimal numbers would be identically recovered after a rounding conversion into binary followed by a rounding conversion back to decimal is that n and m satisfy the inequality $2^{m-1} > 10^n$. At the same time [2] we treated a similar problem by giving a mathematical characterization of a space of floating-point numbers with a fixed number of significant digits and a specified base, which we termed a significance space, and then stating and applying a result called the Base Conversion Theorem. This theorem, a complete proof of which is given in [3], gives the necessary *and* sufficient conditions for both truncation and rounding conversion between two significance spaces with any bases which are not variants of a common base to be: (1) a one-to-one mapping, and (2) an onto mapping. Note that two bases, β and ν, are said to be variants of a common base, γ, when $\beta = \gamma^j$, $\nu = \gamma^i$, for some positive integers i, j (e.g., octal and hexadecimal are variants of binary). It is clear that when two bases are variants of a common base, conversion may be handled in a straight forward manner by representing the numbers as blocks of digits in the common base. The results of this paper as well as in [2, 3] treat the more involved case where the bases β and ν are not variants of a common base (e.g., decimal and binary are not variants of a common base), and this is expressed in the theorems by the condition $\beta^i \neq \nu^j$ for any positive integers i, j. The procedure here is to utilize the terminology and methods of our previous work [2, 3] on conversion mappings to extend the results on in-and-out conversions.

Specifically we show that the condition $2^{m-1} > 10^n$ is necessary, as well as sufficient, for the rounding in-and-out conversion to be the identity. More generally, if we convert, by rounding, n-digit β-ary numbers (i.e., floating point numbers with n-significant digits in base β) into m-digit ν-ary numbers, where β and ν are not variants of some common base, and then convert, by rounding, the resulting numbers back to n-digit β-ary numbers, the necessary and sufficient condition for this operation to be the identity is $\nu^{m-1} > \beta^n$.

If truncation conversion is used instead of rounding conversion, then it can be shown that in-and-out conversion between significance spaces cannot yield the identity unless both bases are variants of some common base. However, for the case where β and ν are not variants of some common base, we are able to obtain the slightly weaker "near identity" result that if truncation conversion of an n-digit β-ary number into an m-digit ν-ary number is followed by truncation conversion back into an n-digit β-ary number then either the same number or its absolute predecessor must be recovered if and only if $\nu^{m-1} \geqq \beta^n - 1$.

Mixed in-and-out conversion mappings between significance spaces with bases which are not variants of a common base are also considered, so that for converting n-digit β-ary numbers to m-digit ν-ary numbers and then converting the resulting numbers back to n-digit β-ary numbers, we show that the truncation in—rounding out conversion will be the identity if and only if $\nu^{m-1} \geqq 2\beta^n - 1$, and finally, for completeness, we state that the rounding in—truncation out conversion will either return the same number or its absolute predecessor if and only if $\nu^{m-1} \geqq \frac{1}{2}(\beta^n - 1)$.

In Section 2 significance spaces and the rounding and truncation mappings are defined and some useful results from [2] are stated. Then Section 3 contains our main results on in-and-out conversion mappings.

2. Significance Spaces and Conversion Mappings

The format of representing a number by a limited number of "significant digits" and an exponent of some base has been frequently used and referred to under various names such as scientific notation, exponential notation, and floating-point notation. For preciseness, and being guided by the notion of radix representation prevalent in the "new math," we choose to define a significance space of real numbers in terms of truncated radix representations as follows [2].

Definition. For the integers $\beta \geqq 2$, called the *base*, and $n \geqq 1$, called the *significance*, let the *significance space*, $S_\beta^{\ n}$, be the following set of real numbers:

$$S_\beta^{\ n} = \{x \mid |x| = \sum_{i=1}^{n} \alpha_i \beta^{j-i}, \quad j, \alpha_i, \text{ integers}, \quad 0 \leqq \alpha_i \leqq \beta - 1 \quad \text{for all} \quad \alpha_i\}. \tag{1}$$

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Observe that by this definition, for any $x \in S_\beta{}^n$ such that $x \neq 0$, j may be chosen so that the leading digit $\alpha_1 \geqq 1$, thus $S_\beta{}^n$ may be thought of as the space of n-digit normalized β-ary numbers along with the number zero. In addition, note that $S_\beta{}^n$ is a set of real numbers, and *not* a set of symbols representing intervals of numbers. Although each number in $S_\beta{}^n$ may be realized as the image of some interval of real numbers by a conversion mapping, we choose to consider this information a property of the conversion mapping and not a property of the form of representation of the number. For clarity we point out an equivalent integral form definition of a significance space.

Alternative Definition. $x \in S_\beta{}^n$ if and only if $x = \gamma\beta^j$ for some integers γ, j where $|\gamma| < \beta^n$.

Observe that every number in $S_\beta{}^n$ other than zero has a next smallest and next largest neighbor. Furthermore, although the absolute difference between neighboring numbers in $S_\beta{}^n$ can be arbitrarily large or arbitrarily small, the relative difference is bounded both above and below. This suggests the following definitions.

Definition. For $x \in S_\beta{}^n$, $x \neq 0$, x' is the *successor* of x in $S_\beta{}^n$ if and only if

$$x' = \min \{y \mid y > x, y \in S_\beta{}^n\}. \tag{2}$$

Definition. The *gap* in $S_\beta{}^n$ at $x \in S_\beta{}^n$ for $x \neq 0$ is denoted by $\gamma_\beta{}^n(x)$ where

$$\begin{aligned} \gamma_\beta{}^n(x) &= (x' - x)/x \quad \text{for} \quad x > 0, \\ \gamma_\beta{}^n(x) &= \gamma_\beta{}^n(-x) \qquad \text{for} \quad x < 0. \end{aligned} \tag{3}$$

In [2] it is shown that the gap takes on both a minimum value, denoted by $g(S_\beta{}^n)$, and a maximum value, denoted by $G(S_\beta{}^n)$, over the space $S_\beta{}^n$ given by the following:

LEMMA.

$$g(S_\beta{}^n) = \mathit{min}\,\{\gamma_\beta{}^n(x) \mid x \in S_\beta{}^n, x \neq 0\} = 1/(\beta^n - 1), \tag{4}$$

$$G(S_\beta{}^n) = \mathit{max}\,\{\gamma_\beta{}^n(x) \mid x \in S_\beta{}^n, x \neq 0\} = 1/\beta^{n-1}. \tag{5}$$

Note that the condition $x \neq 0$ in eqs. (4) and (5) may be replaced by $x > 0$ since $\gamma_\beta{}^n(x) = \gamma_\beta{}^n(-x)$. Hence from eq. (4) we have $\max \{x/(x' - x) \mid x \in S_\beta{}^n, x > 0\} = \beta^n - 1$ and the following corollaries.

COROLLARY 1.

$$\mathit{min}\,\{(x' - x)/x' \mid x \in S_\beta{}^n, x > 0\} = 1/\beta^n. \tag{6}$$

PROOF.

$$\begin{aligned} &\min \{(x' - x)/x' \mid x \in S_\beta{}^n, x > 0\} \\ &\qquad = [1 + \max \{x/(x' - x) \mid x \in S_\beta{}^n, x > 0\}]^{-1} \\ &\qquad = [1 + \beta^n - 1]^{-1} = 1/\beta^n. \end{aligned}$$

COROLLARY 2.

$$\begin{aligned} &\mathit{min}\,\{(x' - x)/(x' + x) \mid x \in S_\beta{}^n, x > 0\} \\ &\qquad = 1/(2\beta^n - 1). \end{aligned} \tag{7}$$

PROOF.

$$\begin{aligned} &\min \{(x' - x)/(x' + x) \mid x \in S_\beta{}^n, x > 0\} \\ &\qquad = [1 + 2 \max \{x/(x' - x) \mid x \in S_\beta{}^n, x > 0\}]^{-1} \\ &\qquad = [1 + 2(\beta^n - 1)]^{-1} = 1/(2\beta^n - 1). \end{aligned}$$

The preceding Lemma and its corollaries will be needed in Section 3 to establish the sufficiency of our conditions for identity or near identity in-and-out conversion mappings. The powerful tool for showing the necessity of these conditions is the following theorem (proved in [2]), which is herein referred to as the Power Density Theorem.

POWER DENSITY THEOREM. *Let β, $\nu > 1$ be integers such that $\beta^i \neq \nu^j$ for any integers $i, j > 0$. Then for any real $\alpha \geqq 0$, $\epsilon > 0$, there exist integers $i, j > 0$ such that $\alpha < \beta^i/\nu^j < \alpha + \epsilon$.*

Conversion of a real number into a number in the significance space $S_\beta{}^n$ can be accomplished in different ways, and we shall consider in detail the following two conversion mappings usually employed in computerized data processing.

Definition. The truncation conversion mapping, $T_\beta{}^n$, and the rounding conversion mapping, $R_\beta{}^n$, of the real numbers into $S_\beta{}^n$ are defined, for $n > 1$, $\beta > 2$, by:

Truncation conversion:

$$T_\beta^n(x) = \begin{cases} \max \{y \mid y \leqq x, y \in S_\beta^n\} & \text{for} \quad x > 0, \\ \min \{y \mid y \geqq x, y \in S_\beta^n\} & \text{for} \quad x < 0, \\ 0 & \text{for} \quad x = 0. \end{cases} \tag{8}$$

Rounding conversion:

$$R_\beta^n(x) = \begin{cases} \min \{y \mid x < (y + y')/2, y \in S_\beta^n\} & \text{for}\, x > 0, \\ \min \{y \mid x \leqq (y + y')/2, y \in S_\beta^n\} & \text{for}\, x < 0, \\ 0 & \text{for}\, x = 0. \end{cases} \tag{9}$$

A distinction is made in the definition of the mapping of positive and negative numbers so that $T_\beta{}^n(-x) = -T_\beta{}^n(x)$ and $R_\beta{}^n(-x) = -R_\beta{}^n(x)$.

Either by truncation conversion or rounding conversion, every number other than zero in $S_\beta{}^n$ will be the image of some finite interval of real numbers. Thus by considering *both* the conversion mapping and the corresponding significance space of real numbers we shall be able to investigate considerations of accuracy of representation.

We close this discussion of preliminaries by stating the Base Conversion Theorem [2, 3], which will also be needed in the results of Section 3. Note that by a one-to-one mapping of one set into another, we mean that distinct elements of the original set have distinct images in the new set, and by an onto mapping we mean that each element of the new set is the image of at least one element of the original set.

BASE CONVERSION THEOREM. *If $\beta^i \neq \nu^j$ for any integers $i, j > 0$, then the truncation (rounding) conversion mapping of $S_\beta{}^n$ into $S_\nu{}^m$, i.e., $T_\nu{}^m : S_\beta{}^n \to S_\nu{}^m$ ($R_\nu{}^m : S_\beta{}^n \to S_\nu{}^m$) is (1) one-to-one if and only if $\nu^{m-1} \geqq \beta^n - 1$, and (2) onto if and only if $\beta^{n-1} \geqq \nu^m - 1$.*

3. In-and-Out Conversions

If a number $x \in S_\beta{}^n$ is converted into a number $z \in S_\nu{}^m$ and this resulting number is then converted back into a number $y \in S_\beta{}^n$, it is of interest to know what relation y has to x as a function of both the numbers n, β, m, ν and the conversion mappings employed. Let this resultant mapping obtained by successive conversion mappings be denoted by concatenation; i.e., $R_\beta^n R_\nu^m$ is the mapping such that $R_\beta^n R_\nu^m(x) = R_\beta^n(z)$ where $z = R_\nu^m(x)$.

When β and ν are both variants of some common base γ, the conditions for identity in-and-out conversion mappings may be determined by looking at the elements of S_β^n and S_ν^m in terms of their representation to base γ. Although these cases are relatively simple, one should not assume they are trivial; e.g., $R_{16}^6 R_8^8$ is not the identity on S_{16}^6, even though both S_{16}^6 and S_8^8 involve 24-bit representations, since $R_{16}^6 R_8^8(.0800001_{16}) = .0800002_{16}$. Rather than examine these cases, however, we concern ourselves here with the more difficult situation where β and ν are not variants of some common base γ. This is expressed in the following theorems by the condition $\beta^i \neq \nu^j$ for integers $i, j > 0$, which rules out *only* those cases where both bases are variants of some common base (e.g., octal and hexadecimal are variants of binary, since $8^4 = 16^3 = 2^{12}$).

THEOREM. *If $\beta^i \neq \nu^j$ for any integers $i, j > 0$, then $R_\beta^n R_\nu^m$ is the identity map on S_β^n if and only if $\nu^{m-1} > \beta^n$.*

PROOF. Let us assume $\nu^{m-1} \leqq \beta^n$ and show $R_\beta^n R_\nu^m$ is not the identity map. $\nu^{m-1} \neq \beta^n$ by assumption in the theorem, so we may write $\nu^{m-1} < \beta^n - \frac{1}{2}$. Hence

$$\beta^n \nu^{m-1} < (\beta^n - \tfrac{1}{2})(\nu^{m-1} + \tfrac{1}{2})$$

and

$$1 < (1 - \tfrac{1}{2}\beta^{-n})^{-1} < 1 + \tfrac{1}{2}\nu^{1-m}. \qquad (10)$$

By the Power Density Theorem there exists $i, j > 0$ such that

$$1 < (1 - \tfrac{1}{2}\beta^{-n})^{-1} < \beta^i/\nu^j < 1 + \tfrac{1}{2}\nu^{1-m}. \qquad (11)$$

Since $[\nu^j]' = \nu^j(1 + \nu^{1-m})$ in S_ν^m, we have

$$\nu^j < \beta^i < \nu^j(1 + \tfrac{1}{2}\nu^{1-m}) = (\nu^j + [\nu^j]')/2, \qquad (12)$$

so that $R_\nu^m(\beta^i) = \nu^j$. Also $\beta^i = [\beta^i(1 - \beta^{-n})]'$ in S_β^n, so that from inequality (11) we also have

$$\nu^j < \beta^i(1 - \tfrac{1}{2}\beta^{-n}) = (\beta^i(1 - \beta^{-n}) + [\beta^i(1 - \beta^{-n})]')/2.$$

Thus $R_\beta^n(\nu^j) < \beta^i$ and $R_\beta^n R_\nu^m(\beta^i) < \beta^i$, proving that $R_\beta^n R_\nu^m$ is not the identity on S_β^n for $\nu^{m-1} \leqq \beta^n$.

For the converse, let us assume $R_\beta^n R_\nu^m$ is not the identity on S_β^n. Since $R_\beta^n R_\nu^m(0) = 0$, there must exist $x' \in S_\beta^n$ (x' is the successor of x in S_β^n) such that $R_\beta^n R_\nu^m(x') \neq x'$. We may assume $x' > 0$; hence $R_\nu^m(x') > 0$, and two cases are possible.

CASE I. $R_\beta^n R_\nu^m(x') < x'$.

Let $z = R_\nu^m(x')$ so that $x' < (z + z')/2$. Since $R_\beta^n(z) < x'$, $z < (x + x')/2$ and $z < x'$ so that

$$z' - x' > x' - z > z - x,$$

$$(z' - z)/z > (x' - x)/x', \qquad (13)$$

$$\max\{(z' - z)/z \mid z \in S_\nu^m, z > 0\} > \min\{(x' - x)/x' \mid x \in S_\beta^n, x > 0\}$$

and from eqs. (5) and (6), $\nu^{m-1} < \beta^n$.

CASE II. $R_\beta^n R_\nu^m(x') > x'$.

Let $R_\nu^m(x') = z'$ so that $x' \geqq (z + z')/2$ and $x' > z$, and since $R_\beta^n(z') \geqq x''$ we also have $z' \geqq (x' + x'')/2$. Thus $x' - z \geqq z' - x' \geqq x'' - z'$ and again

$$(z' - z)/z > (x'' - x')/x''.$$

But as in Case I we must then have $\nu^{m-1} < \beta^n$. Thus in either case $\nu^{m-1} < \beta^n$, so that if $\nu^{m-1} > \beta^n$, then $R_\beta^n R_\nu^m$ must be the identity. Q.E.D.

Clearly if $R_\beta^n R_\nu^m$ is the identity mapping on S_β^n, then $R_\beta^m : S_\beta^n \to S_\nu^m$ is one-to-one and $R_\beta^n : S_\nu^m \to S_\beta^n$ is onto; however, the converse does not hold. Observe that from the Base Conversion Theorem the condition for $R_\nu^m : S_\beta^n \to S_\nu^m$ to be one-to-one is $\nu^{m-1} \geqq \beta^n - 1$, which is the same as the condition for $R_\beta^n : S_\nu^m \to S_\beta^n$ to be onto, but is slightly different than the condition for $R_\beta^n R_\nu^m : S_\beta^n \to S_\beta^n$ to be the identity, $\nu^{m-1} > \beta^n$. In particular, since $10^{1-1} = 2^1 - 1$, the mapping $R_2^1 R_{10}^1$ is not the identity on S_2^1 even though, by the Base Conversion Theorem, any two different one-digit binary numbers are converted by rounding into different one-digit decimal numbers and every one-digit binary number is the resulting value of the rounding conversion mapping of some one-digit decimal number. We can show that $R_2^1 R_{10}^1(2^{27}) \neq 2^{27}$ by observing that $2^{26} = 67{,}108{,}864 \in S_2^1$, and $2^{27} = 134{,}217{,}728 \in S_2^1$; hence $R_{10}^1(2^{27}) = 10^7$ and $R_2^1(10^7) = 2^{26}$, so that $R_2^1 R_{10}^1(2^{27}) = 2^{26}$ and $R_2^1 R_{10}^1$ is not the identity on S_2^1.

The previous theorem considered only rounding conversions. So we now shift our attention to truncation conversion, which is the other case of practical importance for data processing. We have just seen that if enough significance is provided in the intermediate space, the rounding in-and-out conversions will recover the original number. However, we are not so fortunate if truncation conversions are used, as it is evident that $T_\beta^n T_\nu^m$ is the identity on S_β^n only if $S_\beta^n \subset S_\nu^m$. Now it is easily shown that if p is a prime which divides β but does not divide ν, then $1/p \in S_\beta^n$ for any n, and $1/p$ has a nonterminating expansion to the base ν, so that $1/p \notin S_\nu^m$ for any m. Thus $S_\beta^n \not\subset S_\nu^m$ and $T_\beta^n T_\nu^m(1/p)$ is strictly less than $1/p$. If all distinct prime factors of β divide ν at least once, it is still true, but more difficult to show, that $S_\beta^n \not\subset S_\nu^m$ unless $\beta^i = \nu^j$ for some integers $i, j > 0$. Although $T_\beta^n T_\nu^m$ cannot be the identity on S_β^n unless β and ν are variants of a common base, the following weaker result may be derived from the Base Conversion Theorem and the Power Density Theorem.

THEOREM. *Assume $\beta^i \neq \nu^j$ for any integers $i, j > 0$; then either $T_\beta^n T_\nu^m(x) = x$ or $[|T_\beta^n T_\nu^m(x)|]' = |x|$ for all $x \in S_\beta^n$ if and only if $\nu^{m-1} \geqq \beta^n - 1$.*

PROOF. First note that $T_\beta^n T_\nu^m(0) = 0$. For elements of S_β^n other than zero: {there exists a positive $y' \in S_\beta^n$ such that $T_\beta^n T_\nu^m(y')$ is not y' nor y} $\to$ {$y > T_\nu^m(y') = T_\nu^m(y)$} $\to$ {$T_\nu^m : S_\beta^n \to S_\nu^m$ is not one-to-one} $\to$ {$\nu^{m-1} < \beta^n - 1$}.

For the converse {$\nu^{m-1} < \beta^n - 1$} $\to$ {$1 < (1 + \nu^{1-m}) \cdot (1 - \beta^{-n})$} $\to$ {there exists $i, j > 0$ such that $1 < (1 - \beta^{-n})^{-1} < \beta^i/\nu^j < 1 + \nu^{1-m}$} $\to$ {$\nu^j < \beta^i(1 - \beta^{-n}) < [\beta^i(1 - \beta^{-n})]' = \beta^i < \nu^j(1 + \nu^{1-m}) = [\nu^j]'$} $\to$ {$T_\beta^n T_\nu^m(\beta^i) < \beta^i(1 - \beta^{-n})$}. Q.E.D.

Finally we consider the case of in-and-out conversions where conversion in one direction is by truncation and in the other direction by rounding.

THEOREM. *Let $\beta^i \neq \nu^j$ for any $i, j > 0$, then $R_\beta^n T_\nu^m$ is the identity on S_β^n if and only if $\nu^{m-1} \geqq 2\beta^n - 1$.*

PROOF. Let $\nu^{m-1} < 2\beta^n - 1$; hence $0 < \nu^{1-m} - \frac{1}{2}\beta^{-n} - \frac{1}{2}\nu^{1-m}\beta^{-n}$ and

$$(1 - \tfrac{1}{2}\beta^{-n})^{-1} < 1 + \nu^{1-m}. \tag{14}$$

Using the Power Density Theorem, there exists $i, j > 0$ such that

$$1 < (1 - \tfrac{1}{2}\beta^{-n})^{-1} < \beta^i/\nu^j < 1 + \nu^{1-m}. \tag{15}$$

Therefore

$$\nu^j < \beta^i < \nu^j(1 + \nu^{1-m}) = [\nu^j]' \tag{16}$$

so that $T_\nu^m(\beta^i) = \nu^j$. Also, since $\beta^i = [\beta^i(1 - \beta^{-n})]'$ in S_β^n,

$$\nu^j < \beta^i(1 - \tfrac{1}{2}\beta^{-n}) = \{\beta^i(1 - \beta^{-n}) + [\beta^i(1 - \beta^{-n})]'\}/2,$$

so that $R_\beta^n(\nu^j) < \beta^i$. Thus $R_\beta^n T_\nu^m$ is not the identity for $\nu^{m-1} < 2\beta^n - 1$.

Conversely now, assume that $R_\beta^n T_\nu^m$ is not the identity on S_β^n. Thus there must exist a positive $x' \in S_\beta^n$ (x' the successor of x in S_β^n) such that $R_\beta^n T_\nu^m(x') < x'$. Let $T_\nu^m(x') = y$ so that $R_\beta^n(y) < x'$, and we must have $y' > x'$ and $y < (x' + x)/2$. Thus

$$(y' - y)/y > (x' - x)/(x' + x)$$

and

$$\max\{(y' - y)/y \mid y \in S_\nu^m, y > 0\}$$
$$> \min\{(x' - x)/(x' + x) \mid x \in S_\beta^n, x > 0\}$$

and by eqs. (5) and (7) we obtain $\nu^{m-1} < 2\beta^n - 1$. Hence if $\nu^{m-1} \geqq 2\beta^n - 1$, then $R_\beta^n T_\nu^m$ is the identity on S_β^n.

Thus we see that when a truncation conversion is followed by rounding conversion back to the original significance space, an identity mapping can be obtained if the significance is sufficiently large in the intermediate significance space. For a rounding conversion followed by a truncation conversion, we must be satisfied with a weaker result than the existence of an identity for conversions between bases which are not variants of some common base. This is true since from the Power Density Theorem it can be shown that for some positive $x \in S_\beta^n$, we must have $R_\nu^m(x) < x$, so that $T_\beta^n R_\nu^m(x) < x$. For completeness we shall state the following theorem, which may be proved by our previous methods.

THEOREM. *If $\beta^i \neq \nu^j$ for any $i, j > 0$, then $x = T_\beta^n R_\nu^m(x)$ or $[x] = [[T_\beta^n R_\nu^m(x)]]'$ for all $x \in S_\beta^n$ if and only if $\nu^{m-1} \geqq \frac{1}{2}(\beta^n - 1)$.*

A summary of the results for the four possible combinations of rounding and truncation in-and-out conversions are shown in Table I.

In conclusion, it is of practical interest to know the *minimum* significance needed in the intermediate space to allow an identity or near identity in-and-out conversion. Thus for the rounding in-and-out conversion, $R_\beta^n R_\nu^m$, to be the identity on S_β^n when β and ν are not variants of some common base, we desire the minimum m such that $\nu^{m-1} > \beta^n$, which is given by $[n \log \beta / \log \nu + 2]$, where $[z]$ denotes the greatest integer in z. Some values of this expression are given in Table II for decimal input ($\beta = 10$) where the intermediate space is either binary, octal, or hexidecimal ($\nu = 2, 8, 16$, resp.). Note that the minimum significance needed for an identity or near identity mapping under the other forms of in-and-out conversion would differ by no more than 1 from the values given in Table II for the rounding in-and-out conversion.

TABLE I

Summary of identity or near identity properties of in-and-out conversion mappings where the bases β and ν are not variants of some common base (i.e., $\beta^i \neq \nu^j$ for any $i, j > 0$).

In-and-out conversion	Property of mapping
Rounding in—rounding out	$R_\beta^n R_\nu^m$ is the identity on $S_\beta^n \Leftrightarrow \nu^{m-1} > \beta^n$
Truncation in—rounding out	$R_\beta^n T_\nu^m$ is the identity on $S_\beta^n \Leftrightarrow \nu^{m-1} \geqq 2\beta^n - 1$
Truncation in—truncation out	$\left\{\begin{matrix} x = T_\beta^n T_\nu^m(x) \\ \text{or} \\ [x] = [[T_\beta^n T_\nu^m(x)]]' \end{matrix}\right\}$ for all $x \in S_\beta^n \Leftrightarrow \nu^{m-1} \geqq \beta^n - 1$
Rounding in—truncation out	$\left\{\begin{matrix} x = T_\beta^n R_\nu^m(x) \\ \text{or} \\ [x] = [[T_\beta^n R_\nu^m(x)]]' \end{matrix}\right\}$ for all $x \in S_\beta^n \Leftrightarrow \nu^{m-1} \geqq \frac{1}{2}(\beta^n - 1)$

TABLE II

The minimum number of digits needed in intermediate binary, octal, or hexadecimal representations such that the rounding in-and-out conversion will identically recover any n-digit decimal number.

Number of decimal digits n	Binary min m for $R_{10}^n R_2^m$ to be the identity	Octal min m for $R_{10}^n R_8^m$ to be the identity	Hexadecimal min m for $R_{10}^n R_{16}^m$ to be the identity
1	5	3	2
2	8	4	3
3	11	5	4
4	15	6	5
5	18	7	6
6	21	8	6
7	25	9	7
8	28	10	8
9	31	11	9
10	35	13	10
11	38	14	11
12	41	15	11
13	45	16	12
14	48	17	13
15	51	18	14
16	55	19	15
17	58	20	16
18	61	21	16
19	65	23	17
20	68	24	18

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