62/100p

Solutions Sheet

Nina Fischer and Yannick Zelle

November 30, 2021

Exercise 1

(a) **Question:** Given a matrix $A \in \mathbb{R}^{nxn}$ and one of its eigenvectors v, how do you obtain the corresponding eigenvalue λ.
Solution: By definition of the eigenvector we have

$$Av = \lambda v$$

Since we know A and v we can just plug them in and solve for λ.

so what is λ if you solve it?

(b) Show that scaling v by a constant c yields another eigenvector with the same λ

Proof. Let $A \in \mathbb{R}^{nxn}$ and v eigenvector of A with corresponding eigenvalue λ. So we have

$$Av = \lambda v$$

Let now $c \in \mathbb{R}$ so we have for the vector cv:

$$\begin{aligned} Acv &= (Av)c \\ &= (\lambda v)c \\ &= \lambda cv \end{aligned}$$

Thus cv is another eigenvector of a with the same corresponding eigenvalue λ □

(c) For symmetric $A \in \mathbb{R}^{nxn}$ with distinct eigenvalues $\lambda_1, ..., \lambda_k$ show that the corresponding eigenvectors $v_1, ...v_n$ are orthogonal to each other.

Proof. Let $A \in \mathbb{R}^{nxn}$ be symmetric and v_a, v_b two of the eigenvectors with corresponding distinct eigenvalues λ_a, λ_b. With the definition of eigenvectors and the symmetrie of A we have:

$$v_a^T A v_b = \lambda_a v_a^T v_b \tag{1}$$
$$v_a^T A v_b = \lambda_b v_a^T v_b \tag{2}$$

If we subtract (2) from (1) we have :

$$0 = (\lambda_a - \lambda_b) v_a^T v_b$$

□

And since λ_a and λ_b are distinct we have $(\lambda_a - \lambda_b) \neq 0$. Thus we have $v_a^T v_b = 0$. So per definition v_a and v_b are orthogonal to each other.

10/15P

Exercise 2

(a) *Proof.* $\frac{1}{n} 1_n 1_n^T$ will be a matrix of the same form as I with all entries $a_{ij} = \frac{1}{n}$ so H will have entries on the diagonal $a_{ii} = \frac{n-1}{n}$ and all other entries $a_{ij} = -\frac{1}{n}$. Since for a symmetric matirx to obtain the transposed all elements are mirrored on the diagonal we have :

$$H^T = H$$

□

(c) We will proof that $H 1_n = 0$

Proof.

$$H 1_n = (I_n - \frac{1}{n} 1_n 1_n^T) 1_n \quad = I_n 1_n - \frac{1}{n} 1_n 1_n^T 1_n = 1_n - \frac{1}{n} 1_n n \quad = 0$$

□

(b) We will show that H is idempotent.

Proof.

$$\begin{aligned} HH &= H(I_n - \frac{1}{n} 1_n 1_n^T) \\ &= HI_n - H frac1n 1_n 1_n^T \\ &= H - \frac{1}{n} H 1_n 1_n^T \\ &= H - \frac{1}{n} \cdot 0 \cdot 1_n^T \\ &= H \end{aligned}$$

?

Thus H is idempotent. □

(d) The other eigenvalue of H is 1. how did you get it?

(e We will show that $\frac{1}{n} X G 1_n = 0_d$.

Proof. Let $X \in \mathbb{R}^{dxn}$ then :

$$\begin{aligned} \frac{1}{n} X H 1_n &= \frac{1}{n} X 0_n \\ &= \frac{1}{N} 0_D \\ &= 0_D \end{aligned}$$

17/20P

□

Exercise 3

0/15 P

Exercise 4

The Resulting Plots are :

25/30P

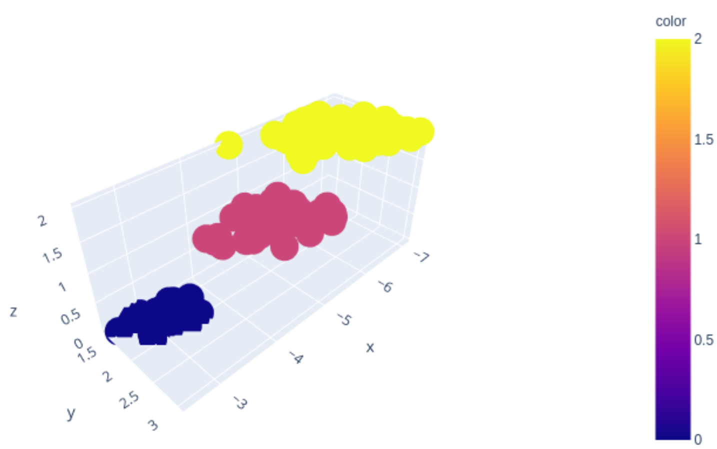

Figure 1: Without Whitening

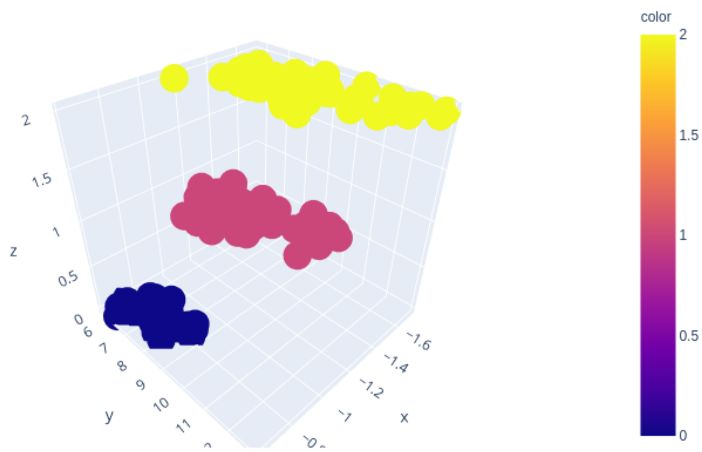

Figure 2: With Whitening

For implementation details see attached notebook

Execrcise 5

(a) *Proof.*

$$
\begin{aligned}
\Phi^T(a)\Phi(b) &= a_1^2b_1^2 + a_2^2 + b_2^2 + 2a_1a_2b_1b_2 + 2a_1b_1 + 2a_2b_2 + 1 \\
&= a_1^2b_1^2 + 2a_1b_1a_2b_2 + a_2^2 + b_2^2 + 2a_1b_1 + 2a_2b_2 + 1 \\
&= (a_1b_1 + a_2b_2)^2(a_1b_1 + a_2b_2) + 1 \\
&= ((a_1b_1 + a_2b_2)^2 + 1)^2 \\
&= (a^Tb + 1)^2 \\
&= k(a, b)
\end{aligned}
$$

□

(b) The corresponding feature space is :

$$\Phi(a) = [a_1^3, a_2^3, \sqrt{3}a_1a_2^2, \sqrt{3}a_1^2a_2, \sqrt{3}a_1^2, \sqrt{6}a_1a_2, \sqrt{3}a_1, \sqrt{3}a_2^2, \sqrt{3}a_2, 1]^T$$

The feature space is ten diagonal

dim = 10

(c) The corresponding featurespace has $p^2 + 1$ dimensions

f

10/20 P