

# Solutions Sheet

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## Exercise 1

(a) We have the random variable  $X \sim \mathcal{U}(c, d)$ . Therefore we have the pdf :

$$p_x(X) = \begin{cases} \frac{1}{c-d} & c \leq x \leq d \\ 0 & \text{Otherwise} \end{cases}$$

We now create the new random variable  $Y(X) = aX + b$ . Therefore we can deduct the inverse function:

$$X(Y) = \frac{y-b}{a}$$

Furthermore we can deduct:

$$\frac{\delta X(Y)}{\delta Y} = \frac{1}{a}$$

Since  $Y(X)$  is an increasing monotonic function, because  $Y'(X) = a > 0$  according to theorem 7.1, we have :

$$p_y(Y) = p_x(X(Y)) \frac{\delta X(Y)}{\delta Y} = \begin{cases} \frac{1}{(c-d)a} & c \leq \frac{y-b}{a} \leq d \\ 0 & \text{Otherwise} \end{cases}$$

(b) With the same argumentation as in a we can deduct

$$p_y(Y) = \frac{1}{a(\sqrt{2\pi}\sigma^2)} \exp\left(-\frac{\left(\frac{Y-b}{a} - \mu\right)^2}{2\sigma^2}\right)$$

## Exercise 2

We have  $p(\omega) = \mathcal{N}(\omega \mid \omega_0, V_0)$  and  $p(y \mid X, \omega) = \mathcal{N}(y \mid X\omega, \Sigma)$ .

$$\begin{aligned}
p(\omega \mid X, y) &= p(y \mid X, \omega) \\
&= \mathcal{N}(y \mid X\omega, \Sigma) \cdot p(\omega) \\
&= \exp\left(-\frac{1}{2}(y - X\omega)^T \Sigma^{-1} (y - X\omega)\right) \cdot \mathcal{N}(\omega \mid \omega_0, V_0) \\
&= \exp\left(-\frac{1}{2}(y - X\omega)^T \Sigma^{-1} (y - X\omega)\right) \cdot \exp\left(-\frac{1}{2}(\omega - \omega_0)^T V_0^{-1} (\omega - \omega_0)\right) \\
&= \exp\left(-\frac{1}{2}(y - X\omega)^T \Sigma^{-1} (y - X\omega) - \frac{1}{2}(\omega - \omega_0)^T V_0^{-1} (\omega - \omega_0)\right) \\
&= \exp\left(-\frac{1}{2}((y - X\omega)^T \Sigma^{-1} (y - X\omega) + (\omega - \omega_0)^T V_0^{-1} (\omega - \omega_0))\right) \\
&= \exp\left(-\frac{1}{2}(y^T \Sigma^{-1} y - (X\omega)^T \Sigma^{-1} y - y^T \Sigma^{-1} X\omega + (X\omega)^T \Sigma^{-1} X\omega\right. \\
&\quad \left.+ \omega^T V_0^{-1} \omega - \omega_0^T V_0^{-1} \omega - \omega^T V_0^{-1} \omega_0 + \omega_0^T V_0^{-1} \omega_0)\right) \\
&= \exp\left(-\frac{1}{2}(y^T \Sigma^{-1} y + \omega_0^T V_0^{-1} \omega_0 - (2y^T \Sigma^{-1} X + 2\omega_0^T V_0^{-1})\omega\right. \\
&\quad \left.+ \omega^T (V_0^{-1} + X^T \Sigma^{-1} X)\omega)\right) \\
&\prec \exp(y^T \Sigma^{-1} X + \omega_0^T V_0^{-1})\omega - \frac{1}{2}\omega^T (V_0^{-1} + X^T \Sigma^{-1} X)\omega \\
&= \exp(\eta^T \omega - \frac{1}{2}\omega \Lambda \omega) \\
&= \mathcal{N}(\omega \mid \omega_n, V_n)
\end{aligned}$$

For the expressions  $\Lambda$  and  $\eta$  we can reordering :

$$\Lambda = V_n^{-1} = V_0^{-1} + X^T \Sigma^{-1} X \Leftrightarrow V_n = (V_0^{-1} + X^T \Sigma^{-1} X)^{-1}$$

$$\eta^T = V_n^{-1} \omega_n \Leftrightarrow \omega_n = \frac{\eta^T}{V_n^{-1}} = \eta^T V_n = (y^T \Sigma^{-1} X + \omega_0^T V_0^{-1})^T V_n$$