# Solutions Sheet

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## Exercise 1

**(a)** **Question:** Given a matrix $A \in \mathbb{R}^{nxn}$ and one of its eigenvectors $v$, how do you obtain the corresponding eigenvalue $\lambda$.
**Solution:** By definition of the eigenvector we have

$$Av = \lambda v$$

Since we know $A$ and $v$ we can just plug them in and solve for $\lambda$.

**(b)** Show that scaling $v$ by a constant $c$ yields another eigenvector with the same $\lambda$

*Proof.* Let $A \in \mathbb{R}^{nxn}$ and $v$ eigenvector of $A$ with corresponding eigenvalue $\lambda$. So we have

$$Av = \lambda v$$

Let now $c \in \mathbb{R}$ so we have for the vector $cv$:

$$\begin{aligned} Acv &= (Av)c \\ &= (\lambda v)c \\ &= \lambda cv \end{aligned}$$

Thus $cv$ is another eigenvector of a with the same corresponding eigenvalue $\lambda$ □

**(c)** For symmetric $A \in \mathbb{R}^{nxn}$ with distinct eigenvalues $\lambda_1, ..., \lambda_k$ show that the corresponding eigenvectors $v_1, ...v_n$ are orthogonal to each other.

*Proof.* Let $A \in \mathbb{R}^{nxn}$ be symmetric and $v_a, v_b$ two of the eigenvectors with corresponding distinct eigenvalues $\lambda_a, \lambda_b$. With the definition of eigenvectors and the symmetrie of A we have:

$$v_a^T A v_b = \lambda_a v_a^T v_b \tag{1}$$

$$v_a^T A v_b = \lambda_b v_a^T v_b \tag{2}$$

If we subtract (2) from (1) we have :

$$0 = (\lambda_a - \lambda_b) v_a^T v_b$$

□

And since $\lambda_a$ and $\lambda_b$ are distinct we have $(\lambda_a - \lambda_b) \neq 0$. Thus we have $v_a^T v_b = 0$. So per definition $v_a$ and $v_b$ are orthogonal to each other.

## Exercise 2

**(a)** *Proof.* $\frac{1}{n} 1_n 1_n^T$ will be a matrix of the same form as $I$ with all entries $a_{ij} = \frac{1}{n}$ so H will have entries on the diagonal $a_{ii} = \frac{n-1}{n}$ and all other entries $a_{ij} = -\frac{1}{n}$. Since for a symmetric matirx to obtain the transposed all elements are mirrored on the diagonal we have :

$$H^T = H$$

□

**(c)** We will proof that $H 1_n = 0$

*Proof.*

$$H 1_n = (I_n - \frac{1}{n} 1_n 1_n^T) 1_n \quad = I_n 1_n - \frac{1}{n} 1_n 1_n^T 1_n = 1_n - \frac{1}{n} 1_n n \quad = 0$$

□

**(b)** We will show that H is idempotent.

*Proof.*

$$\begin{aligned} HH &= H(I_n - \frac{1}{n} 1_n 1_n^T) \\ &= HI_n - Hfrac1n 1_n 1_n^T \\ &= H - \frac{1}{n} H 1_n 1_n^T \\ &= H - \frac{1}{n} \cdot 0 \cdot 1_n^T \\ &= H \end{aligned}$$

Thus H is idempotent. □

**(d)** The other eigenvalue of $H$ is 1.

**(e** We will show that $\frac{1}{n} X G 1_n = 0_d$ .

*Proof.* Let $X \in \mathbb{R}^{dxn}$ then :

$$\begin{aligned} \frac{1}{n} X H 1_n &= \frac{1}{n} X 0_n \\ &= \frac{1}{N} 0_D \\ &= 0_D \end{aligned}$$

$\square$

## Exercise 3

## Exercise 4

The Resulting Plots are :

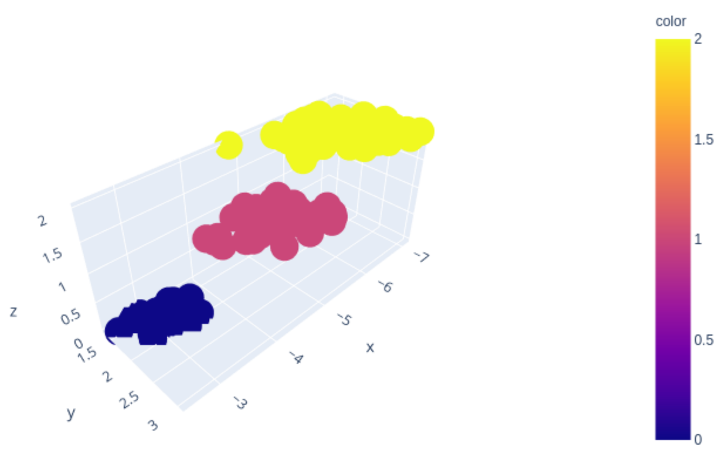


Figure 1: Without Whitening

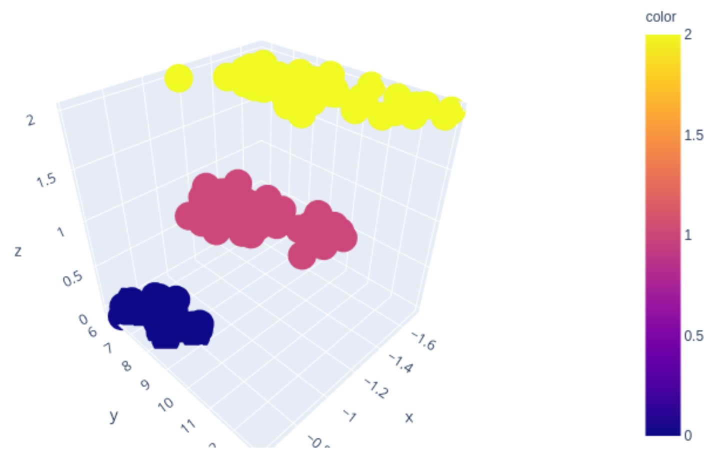


Figure 2: With Whitening

For implementation details see attached notebook

## Execrcise 5

**(a)** *Proof.*

$$
\begin{aligned}
\Phi^T(a)\Phi(b) &= a_1^2b_1^2 + a_2^2 + b_2^2 + 2a_1a_2b_1b_2 + 2a_1b_1 + 2a_2b_2 + 1 \\
&= a_1^2b_1^2 + 2a_1b_1a_2b_2 + a_2^2 + b_2^2 + 2a_1b_1 + 2a_2b_2 + 1 \\
&= (a_1b_1 + a_2b_2)^2(a_1b_1 + a_2b_2) + 1 \\
&= ((a_1b_1 + a_2b_2)^2 + 1)^2 \\
&= (a^Tb + 1)^2 \\
&= k(a, b)
\end{aligned}
$$

□

**(b)** The corresponding feature space is :

$$\Phi(a) = [a_1^3, a_2^3, \sqrt{3}a_1a_2^2, \sqrt{3}a_1^2a_2, \sqrt{3}a_1^2, \sqrt{6}a_1a_2, \sqrt{3}a_1, \sqrt{3}a_2^2, \sqrt{3}a_2, 1]^T$$

The feature space is ten diagonal

**(c)** The corresponding featurespace has $p^2 + 1$ dimensions