

# Solutions Sheet 1

Yannick Zelle and Nina Fischer

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## Exercise 1

First, we will assign values to  $p(M), p(m), p(e), P(n | M), P(n | m), P(n | e)$ , so that  $p(M) + p(m) + p(e) = 1$ :

$$p(M) := 0.01$$

$$p(m) := 0.1$$

$$p(e) := 0.89$$

$$p(n | M) = 0.9$$

$$p(n | m) = 0.08$$

$$p(n | e) = 0.02$$

The next step to apply Bayes' rule is now to calculate the probability of a noise:

$$p(n) = p(M) \cdot p(n | M) + p(m) \cdot p(n | m) + p(e) \cdot p(n | e)$$

$$= 0.01 \cdot 0.9 + 0.1 \cdot 0.08 + 0.89 \cdot 0.02$$

$$= 0.0348$$

We can now calculate the probability of a monster given noise by applying Bayes' rule:

$$\begin{aligned} p(M | n) &= \frac{p(n | M) \cdot p(M)}{p(n)} \\ &= \frac{0.9 \cdot 0.01}{0.034} \\ &\approx 0.26 \end{aligned}$$

These results got confirmed by the jupyter notebook presented in the lecture:

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In [5]: 1 # what is the probability of having noise
        2 p_n = p_nm + p_nm + p_ne # p(n) = p(n,M) + p(n,m) + p(n,e)
        3 print(f"p(n) = {p_n}")
        p(n) = 0.0348

In [7]: 1 # what is the (posterior) probability of having a ... given that we hear noise
        2 p_M_n = Bayes_rule(p_M, p_n_M, p_n) # p(M|n) = p(M) p(n|M) / p(n)
        3 p_m_n = Bayes_rule(p_m, p_n_m, p_n) # p(m|n) = p(m) p(n|m) / p(n)
        4 p_e_n = Bayes_rule(p_e, p_n_e, p_n) # p(e|n) = p(e) p(n|e) / p(n)
        5 print(f"p(M|n) = {p_M_n}")
        6 print(f"p(m|n) = {p_m_n}")
        7 print(f"p(e|n) = {p_e_n}")

p(M|n) = 0.2586206896551725
p(m|n) = 0.22985859574712644
p(e|n) = 0.5114942528735632

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## Exercise 2

**Given:**

- The plausibility of  $B$  is  $p(B)$ .
- $p(B \mid A) \geq p(B)$

**Show: (a)**  $p(B \mid A) \geq p(B)$

*Proof.* Is equivalent to second assumption. □

**(b)**  $p(B \mid \neg A) \leq p(B)$

*Proof.* By assumption:

$$\begin{aligned}
 p(B \mid A) &\geq p(B) \\
 \Leftrightarrow \frac{p(A \mid B) \cdot p(B)}{p(A)} &\geq p(B) \\
 \Leftrightarrow \frac{(1 - p(\neg A \mid B)) \cdot p(B)}{1 - p(\neg A)} &\geq p(B) \\
 \Leftrightarrow (1 - p(\neg A \mid B)) \cdot p(B) &\geq p(B) \cdot (1 - p(\neg A)) \\
 \Leftrightarrow 1 - p(\neg A \mid B) &\geq 1 - p(\neg A) \\
 \Leftrightarrow -p(\neg A \mid B) &\geq -p(\neg A) \\
 \Leftrightarrow p(\neg A \mid B) &\leq p(\neg A) \\
 \Leftrightarrow p(\neg A \mid B) &\leq p(\neg A) \cdot \frac{p(B)}{p(B)} \\
 \Leftrightarrow \frac{p(\neg A \mid B) \cdot p(B)}{p(\neg A)} &\leq p(B) \\
 \Leftrightarrow p(B \mid \neg A) &\leq p(B)
 \end{aligned}$$

□

**(c)**  $p(A \mid B) \geq p(A)$

*Proof.* By assumption:

$$\begin{aligned}
 p(B|A) &\geq p(B) \\
 \Leftrightarrow \frac{p(A|B) \cdot p(B)}{p(A)} &\geq p(B) \\
 \Leftrightarrow p(A|B) &\geq p(A)
 \end{aligned}$$

□

(d)  $p(A | \neg B) \leq p(A)$

*Proof.* By assumption:

$$\begin{aligned}
 p(B|A) &\geq p(B) \\
 \stackrel{(c)}{\Leftrightarrow} p(A|B) &\geq p(A) \\
 \stackrel{(b)}{\Leftrightarrow} p(A|\neg B) &\leq p(A)
 \end{aligned}$$

□

### Exercise 3

For this exercise we will formalise the given facts as follows:

1.  $C$  = "The Patient has Cancer"
2.  $R$  = "A Patient's Mammogram is positive"

We can therefore deduct from the given facts:

1.  $P(C) = 0.008$
2.  $P(R | C) = 0.9$
3.  $P(R | \neg C) = 0.07$
4.  $P(\neg C) = 1 - P(C) = 0.992$
5.  $P(R) = P(C) \cdot P(R | C) + P(\neg C) \cdot P(R | \neg C) = 0.008 \cdot 0.9 + 0.992 \cdot 0.07 = 0.08$

Given these deductions, we can now calculate the addressed probability that a patient has breast cancer given a positive Mammogram  $P(C | R)$  by applying Bayes' Rule:

$$\begin{aligned}
 P(C | R) &= \frac{P(R | C) \cdot P(C)}{P(R)} \\
 &= \frac{0.9 \cdot 0.008}{0.08} \\
 &= 0.09
 \end{aligned}$$

## Exercise 4

For this exercise we will formulate the given facts as follows :

- $W_A$  = "The warden announces A"
- $W_B$  = "The warden announces B"
- $W_C$  = "The warden announces C"
- $A$  = "A is pardoned"
- $B$  = "B is pardoned"
- $C$  = "C is pardoned"

Before any announcement is made by the warden we obviously have :

$$P(A) = P(B) = P(C) = \frac{1}{3}$$

No Matter what the warden will either announce  $A$  or  $B$ , which brings us to the conclusion:

$$P(A) = P(W_B | A) + P(W_C | A) = \frac{1}{3}$$

Because in this case the warden flips a coin we know:

$$P(W_B | A) = P(W_C | A) = \frac{1}{6}$$

We know that  $P(W_B | B) = 0$  and we can therefore also deduct that :

$$P(W_B) = P(A) \cdot P(W_B | A) + P(C) \cdot P(W_B | C) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

Now we can apply Bayes' formula to calculate the updated probability after the Warden's announcement:

$$\begin{aligned} P(A | W_B) &= \frac{P(W_B | A)P(A)}{P(W_B)} \\ &= \frac{\frac{1}{6} \cdot \frac{1}{3}}{\frac{1}{2}} \\ &= \frac{1}{3} \end{aligned}$$

We also now that  $P(B | W_B) = 0$  and one of them has to be pardoned for sure. It follows  $P(C | W_B) = \frac{2}{3}$  Therefore C's argument is right.