Solutions Sheet

Nina Fischer and Yannick Zelle

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Exercise 1

(a) We have the random variable $X \sim \mathcal{U}(c, d)$. Therefore we have the pdf :

$$p_x(X) = \begin{cases} \frac{1}{c-d} & c \leq x \leq d \\ 0 & \text{Otherwise} \end{cases}$$

We now create the new random variable $Y(X) = aX + b$. Therefore we can deduct the inverse function:

$$X(Y) = \frac{y-b}{a}$$

Furthermore we can deduct:

$$\frac{\delta X(Y)}{\delta Y} = \frac{1}{a}$$

Since Y(X) is an increasing monotonic function, because $Y'(X) = a > 0$ according to therorem 7.1, we have :

$$p_y(Y) = p_x(X(Y))\frac{\delta X(Y)}{\delta Y} = \begin{cases} \frac{1}{(c-d)a} & c \leq \frac{y-b}{a} \leq d \\ 0 & \text{Otherwise} \end{cases}$$

(b) With the same argumentation as in a we can deduct

$$p_y(Y) = \frac{1}{a(\sqrt{2\pi\sigma^2})} exp(-\frac{(\frac{Y-b}{a} - \mu)^2}{2\sigma^2})$$

Exercise 2

We have $p(\omega) = \mathcal{N}(\omega \mid \omega_0, V_0)$ and $p(y \mid X, \omega) = \mathcal{N}(y \mid X\omega, \sum)$.

$$
\begin{aligned}
&p(\omega \mid X, y)\\
&= p(y \mid X, \omega)\\
&= \mathcal{N}(y \mid X\omega, \sum) \cdot p(\omega)\\
&= exp(-\frac{1}{2}(y - X\omega)^T \sum^{-1}(y - X\omega)) \cdot \mathcal{N}(\omega \mid \omega_0, V_0)\\
&= exp(-\frac{1}{2}(y - X\omega)^T \sum^{-1}(y - X\omega)) \cdot exp(-\frac{1}{2}(\omega - \omega_0)^T V_0^{-1}(\omega - \omega_0))\\
&= exp(-\frac{1}{2}(y - X\omega)^T \sum^{-1}(y - X\omega) - \frac{1}{2}(\omega - \omega_0)^T V_0^{-1}(\omega - \omega_0))\\
&= exp - \frac{1}{2}((y - X\omega)^T \sum^{-1}(y - X\omega) + (\omega - \omega_0)^T V_0^{-1}(\omega - \omega_0))\\
&= exp - \frac{1}{2}(y^T \sum^{-1} y - (X\omega)^T \sum^{-1} y - y^T \sum^{-1} X\omega + (X\omega^T \sum^{-1} X\omega\\
&+ w^T V_0^{-1}\omega - w_0^T V_0^{-1}\omega - \omega^T V_0^{-1}\omega_0 + \omega_0^T V_0^{-1}\omega_0)\\
&= exp - \frac{1}{2}(y^T \sum^{-1} y + \omega_0^T V_0^{-1}\omega_0 - (2y^T \sum^{-1} X + 2\omega_0^T V_0^{-1})\omega\\
&+ \omega^T(V_0^{-1} + X^T \sum^{-1} X)\omega)\\
&\prec exp(y^T \sum^{-1} X + \omega_0^T V_0^{-1})\omega - \frac{1}{2}\omega^T(V_0^{-1} + X^T \sum^{-1} X)\omega)\\
&= exp(\eta^T\omega - \frac{1}{2}\omega\Lambda\omega)\\
&= \mathcal{N}(\omega \mid \omega_n, V_n)
\end{aligned}
$$

For the expressions Λ and η we can reorder :

$$\Lambda = V_n^{-1} = V_0^{-1} + X^T \sum^{-1} X \Leftrightarrow V_n = (V_0^{-1} + X^T \sum^{-1} X)^{-1}$$

$$\eta^T = V_n^{-1}\omega_n \Leftrightarrow \omega_n = \frac{\eta^T}{V_n^{-1}} = \eta^T V_n = (y^T \sum^{-1} X + \omega_0^T V_0^{-1})^T V_n$$