Solutions Sheet 1

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October 19, 2021

Exercise 1

First, we will assign values to $p(M), p(m), p(e), P(n \mid M), P(n \mid m), P(n \mid e)$, so that $p(M) + p(m) + p(e) = 1$:

$$p(M) := 0.01$$
$$p(m) := 0.1$$
$$p(e) := 0.89$$

$$p(n \mid M) = 0.9$$
$$p(n \mid m) = 0.08$$
$$p(n \mid e) = 0.02$$

The next step to apply Bayes' rule is now to calculate the probability of a noise:

$$p(n) = p(M) \cdot p(n \mid M) + p(m) \cdot p(n \mid m) + p(e) \cdot p(n \mid e)$$

$$= 0.01 \cdot 0.9 + 0.1 \cdot 0.08 + 0.89 \cdot 0.02$$

$$= 0.0348$$

We can now calculate the probability of a monster given noise by applying Bayes' rule:

$$p(M \mid n) = \frac{p(n \mid M) \cdot p(M)}{p(n)}$$
$$= \frac{0.9 \cdot 0.01}{0.034}$$
$$\approx 0.26$$

These results got confirmed by the jupyter notebook presented in the lecture:

In [5]:

```
# what is the probability of having noise
p_n = p_nM + p_nm + p_ne     # p(n) = p(n,M) + p(n,m) + p(n,e)
print(f"p(n) = {p_n}")
```

```
p(n) = 0.0348
```

In [7]:

```
# what is the (posterior) probability of having a ... given that we hear noise
p_M_n = Bayes_rule(p_M, p_n_M, p_n)     # p(M|n) = p(M) p(n|M) / p(n)
p_m_n = Bayes_rule(p_m, p_n_m, p_n)     # p(m|n) = p(m) p(n|m) / p(n)
p_e_n = Bayes_rule(p_e, p_n_e, p_n)     # p(e|n) = p(e) p(n|e) / p(n)
print(f"p(M|n) = {p_M_n}")
print(f"p(m|n) = {p_m_n}")
print(f"p(e|n) = {p_e_n}")
```

```
p(M|n) = 0.2586206896551725
p(m|n) = 0.2298850574712644
p(e|n) = 0.5114942528735632
```

Exercise 2

Given:

- The plausibility of B is $p(B)$.
- $p(B \mid A) \geq p(B)$

Show: **(a)** $p(B \mid A) \geq p(B)$

Proof. Is equivalent to second assumption. □

(b) $p(B \mid \neg A) \leq p(B)$

Proof. By assumption:

$$
\begin{aligned}
& p(B|A) \geq p(B) \\
& \Leftrightarrow \frac{p(A \mid B) \cdot p(B)}{p(A)} \geq p(B) \\
& \Leftrightarrow \frac{(1 - p(\neg A \mid B)) \cdot p(B)}{1 - p(\neg A)} \geq p(B) \\
& \Leftrightarrow (1 - p(\neg A \mid B)) \cdot p(B) \geq p(B) \cdot (1 - p(\neg A)) \\
& \Leftrightarrow 1 - p(\neg A \mid B) \geq 1 - p(\neg A) \\
& \Leftrightarrow -p(\neg A \mid B) \geq -p(\neg A) \\
& \Leftrightarrow p(\neg A \mid B) \leq p(\neg A) \\
& \Leftrightarrow p(\neg A \mid B) \leq p(\neg A) \cdot \frac{p(B)}{p(B)} \\
& \Leftrightarrow \frac{p(\neg A \mid B) \cdot p(B)}{p(\neg A)} \leq p(B) \\
& \Leftrightarrow p(B \mid \neg A) \leq p(B)
\end{aligned}
$$

□

(c) $p(A \mid B) \geq p(A)$

Proof. By assumption:

$$\begin{aligned} & p(B|A) \geq p(B) \\ & \Leftrightarrow \frac{p(A \mid B) \cdot p(B)}{p(A)} \geq p(B) \\ & \Leftrightarrow p(A \mid B) \geq p(A) \end{aligned}$$

□

(d) $p(A \mid \neg B) \leq p(A)$

Proof. By assumption:

$$\begin{aligned} & p(B|A) \geq p(B) \\ & \overset{(c)}{\Leftrightarrow} p(A \mid B) \geq p(A) \\ & \overset{(b)}{\Leftrightarrow} p(A \mid \neg B) \leq p(A) \end{aligned}$$

□

Exercise 3

For this exercise we will formalise the given facts as follows:

1. $C =$ "The Patient has Cancer"
2. $R =$ "A Patient's Mammogram is positive"

We can therefore deduct from the given facts:

1. $P(C) = 0.008$
2. $P(R \mid C) = 0.9$
3. $P(R \mid \neg C) = 0.07$
4. $P(\neg C) = 1 - P(C) = 0.992$
5. $P(R) = P(C) \cdot P(R \mid C) + P(\neg C) \cdot P(R \mid \neg C) = 0.008 \cdot 0.9 + 0.992 \cdot 0.07 = 0.08$

Given these deductions, we can now calculate the adressed probability that a patient has breast cancer given a positive Mammogram $P(C \mid R)$ by applying Bayes' Rule:

$$\begin{aligned} & P(C \mid R) \\ & = \frac{P(R \mid C) \cdot P(C)}{P(R)} \\ & = \frac{0.9 \cdot 0.008}{0.08} \\ & = 0.09 \end{aligned}$$

Exercise 4

For this exercise we will formulate the given facts as follows follows :

- $W_A =$ "The warden announces A"
- $W_B =$ "The warden announces B"
- $W_C =$ "The warden announces C"
- $A =$ "A is pardonned"
- $B =$ "B is pardonned"
- $C =$ "C is pardonned"

Before any announcement is made by the warden we obviously have :

$$P(A) = P(B) = P(C) = \frac{1}{3}$$

No Matter what the warden will eather announce A or B, which brings us to the conclusion:

$$P(A) = P(W_B \mid A) + P(W_C \mid A) = \frac{1}{3}$$

Because in this case the warden flips a coin we know:

$$P(W_B \mid A) = P(W_C \mid A) = \frac{1}{6}$$

We know that $P(W_B \mid B) = 0$ and we can therefore also deduct that :

$$P(W_B) = P(A) \cdot P(W_B \mid A) + P(C) \cdot P(W_B \mid C) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

Now we can apply Bayes' formula to calulate the updatet probability after the Warden's announcement:

$$\begin{aligned} P(A \mid W_B) &= \frac{P(W_B \mid A) \cdot P(A)}{P(W_B)} \\ &= \frac{\frac{1}{6} \cdot \frac{1}{3}}{\frac{1}{2}} \\ &= \frac{1}{3} \end{aligned}$$

We also now that $P(W_B \mid C) = 1$. It follows:

$$
\begin{aligned}
P(C \mid W_B) &= \frac{P(W_B \mid C) \cdot P(C)}{P(W_B)} \\
&= \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}} \\
&= \frac{2}{3}
\end{aligned}
$$

Therefore C's argument is right.