Machine Learning Exercise Sheet 2

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Exercise 1

(a)

Given is the DAG G_1 with the following associated probabilities:

- $p(A) = 0.3$
- $p(B \mid A) = 0.2$
- $p(B \mid \neg A) = 0.4$
- $p(C \mid B) = 0.7$
- $p(C \mid \neg B) = 0.5$

From there we can deduct the following probabilities:

- $p(\neg A) = 0.7$
- $p(\neg B \mid A) = 0.8$
- $p(\neg B \mid \neg A) = 0.6$
- $p(\neg C \mid B) = 0.3$
- $p(\neg C \mid \neg B) = 0.5$

From G_1 we can deduct the joint probability distribution:

$$p(A, B, C) = p(A) \cdot p(B \mid A) \cdot p(C \mid B)$$

Using this Probabilitydistribution we can calculate P(B):

$$
\begin{aligned}
p(B) &= \sum_{A,C} p(A, B = 1, C) \\
&= p(A) \cdot p(B \mid A) \cdot p(C \mid B) \\
&+ p(\neg A) \cdot p(B \mid \neg A) \cdot p(C \mid B) \\
&+ p(\neg A) \cdot p(B \mid \neg A) \cdot p(\neg C \mid B) \\
&+ p(A) \cdot p(B \mid A) \cdot p(\neg C \mid B) \\
&= 0.3 \cdot 0.2 \cdot 0.7 \\
&+ 0.7 \cdot 0.4 \cdot 0.7 \\
&+ 0.7 \cdot 0.4 \cdot 0.3 \\
&+ 0.3 \cdot 0.2 \cdot 0.3 \\
&= 0.34
\end{aligned}
$$

and P(C):

$$
\begin{aligned}
p(C) &= \sum_{A,B} p(A, B, C = 1) \\
&= p(A) \cdot p(B \mid A) \cdot p(C \mid B) \\
&+ p(\neg A) \cdot p(B \mid \neg A) \cdot p(C \mid B) \\
&+ p(\neg A) \cdot p(\neg B \mid \neg A) \cdot p(C \mid \neg B) \\
&+ p(A) \cdot p(\neg B \mid A) \cdot p(C \mid \neg B) \\
&= 0.3 \cdot 0.2 \cdot 0.7 \\
&+ 0.7 \cdot 0.4 \cdot 0.7 \\
&+ 0.7 \cdot 0.6 \cdot 0.5 \\
&+ 0.3 \cdot 0.8 \cdot 0.5 \\
&= 0.568
\end{aligned}
$$

(b)

Given is a DAG G_2 and the associated probabilities:

- $p(A) = 0.3$
- $p(B \mid A) = 0.2$
- $p(B \mid \neg A) = 0.4$
- $p(c \mid A) = 0.7$
- $p(C \mid \neg A) = 0.6$
- $p(D \mid B, C) = 0.9$

- $p(D \mid B, \neg C) = 0.5$
- $p(D \mid \neg B, C) = 0.3$
- $p(D \mid \neg B, \neg C) = 0.3$

From there we can deductthe following probabilities:

- $p(\neg A) = 0.7$
- $p(\neg B \mid A) = 0.8$
- $p(\neg B \mid \neg A) = 0.6$
- $p(\neg c \mid A) = 0.3$
- $p(\neg C \mid \neg A) = 0.4$
- $p(\neg D \mid B, C) = 0.1$
- $p(\neg D \mid B, \neg C) = 0.5$
- $p(\neg D \mid \neg B, C) = 0.7$
- $p(\neg D \mid \neg B, \neg C) = 0.7$

From the G_2 we can deduct the following joint Probabilitydistribution:

$$p(A, B, C, D) = p(A) \cdot p(C \mid A) \cdot p(B \mid A) \cdot p(D \mid B, C)$$

Using the joint Probabilitydistribution and the probabilities from above, we can calculate $p(B)$:

$$
\begin{aligned}
p(B) = \sum_{A,C,D} & p(A, B=1, C, D) \\
&= p(A) \cdot p(C \mid A) \cdot p(B \mid A) \cdot p(D \mid B, C) \\
&+ p(A) \cdot p(C \mid A) \cdot p(B \mid A) \cdot p(\neg D \mid B, C) \\
&+ p(A) \cdot p(\neg C \mid A) \cdot p(B \mid A) \cdot p(D \mid B, \neg C) \\
&+ p(A) \cdot p(\neg C \mid A) \cdot p(B \mid A) \cdot p(\neg D \mid B, \neg C) \\
&+ p(\neg A) \cdot p(C \mid \neg A) \cdot p(B \mid \neg A) \cdot p(D \mid B, C) \\
&+ p(\neg A) \cdot p(C \mid \neg A) \cdot p(B \mid \neg A) \cdot p(\neg D \mid B, C) \\
&+ p(\neg A) \cdot p(\neg C \mid \neg A) \cdot p(B \mid \neg A) \cdot p(D \mid B, \neg C) \\
&+ p(\neg A) \cdot p(\neg C \mid \neg A) \cdot p(B \mid \neg A) \cdot p(\neg D \mid B, \neg C) \\
&= 0.7 \cdot 0.7 \cdot 0.2 \cdot 0.9 \\
&+ 0.7 \cdot 0.3 \cdot 0.8 \cdot 0.1 \\
&+ 0.7 \cdot 0.3 \cdot 0.2 \cdot 0.5 \\
&+ 0.7 \cdot 0.3 \cdot 0.2 \cdot 0.5 \\
&+ 0.3 \cdot 0.6 \cdot 0.4 \cdot 0.9 \\
&+ 0.3 \cdot 0.6 \cdot 0.4 \cdot 0.1 \\
&+ 0.3 \cdot 0.4 \cdot 0.4 \cdot 0.5 \\
&+ 0.3 \cdot 0.4 \cdot 0.4 \cdot 0.5 \\
&= 0.27
\end{aligned}
$$

and p(D):

$$
\begin{aligned}
p(D) = &\sum_{A,C,B} p(A,B,C,D=1) \\
&= p(A) \cdot p(C \mid A) \cdot p(B \mid A) \cdot p(D \mid B, C) \\
&+ p(A) \cdot p(C \mid A) \cdot p(\neg B \mid A) \cdot p(D \mid \neg B, C) \\
&+ p(A) \cdot p(\neg C \mid A) \cdot p(B \mid A) \cdot p(D \mid B, \neg C) \\
&+ p(A) \cdot p(\neg C \mid A) \cdot p(\neg B \mid A) \cdot p(D \mid \neg B, \neg C) \\
&+ p(\neg A) \cdot p(C \mid \neg A) \cdot p(B \mid \neg A) \cdot p(D \mid B, C) \\
&+ p(\neg A) \cdot p(C \mid \neg A) \cdot p\neg(B \mid \neg A) \cdot p(D \mid \neg B, C) \\
&+ p(\neg A) \cdot p(\neg C \mid \neg A) \cdot p(B \mid \neg A) \cdot p(D \mid B, \neg C) \\
&+ p(\neg A) \cdot p(\neg C \mid \neg A) \cdot p(\neg B \mid \neg A) \cdot p(D \mid \neg B, \neg C) \\
&= 0.7 \cdot 0.7 \cdot 0.2 \cdot 0.9 \\
&+ 0.7 \cdot 0.3 \cdot 0.8 \cdot 0.3 \\
&+ 0.7 \cdot 0.3 \cdot 0.2 \cdot 0.5 \\
&+ 0.7 \cdot 0.3 \cdot 0.8 \cdot 0.3 \\
&+ 0.3 \cdot 0.6 \cdot 0.4 \cdot 0.9 \\
&+ 0.3 \cdot 0.6 \cdot 0.6 \cdot 0.3 \\
&+ 0.3 \cdot 0.4 \cdot 0.4 \cdot 0.5 \\
&+ 0.3 \cdot 0.4 \cdot 0.6 \cdot 0.3 \\
&= 0.36
\end{aligned}
$$

Exercise 2

(a)

$p(A,B,C,D,E,F) = p(A) \cdot p(B \mid A) \cdot p(C \mid B) \cdot p(D \mid B) \cdot p(E \mid C, D) \cdot p(F \mid E)$

(b)

$$p(A,B,C,D,E) = p(A) \cdot p(B \mid A) \cdot p(C \mid B) \cdot p(D \mid B) \cdot p(E \mid C, D)$$

(c)

Task: Find a minimal set S that d-seperates A and F and prove that this is the case.

Beweis. We propose $S = \{B\}$ to prove that the criteria of S holds we have actually to prove two statements:

1. S d-seperates A and F
2. there is no set with fewer elements that also d-seperates A and F

We will start with the first statement. Between A and F exist two paths:

$$p_1 = A \to B \to C \to E \to F$$
$$p_2 = A \to B \to D \to E \to F$$

We have p_1 is blocked by S because with $i_k = B$ we have : $i_k \in S$ and $A \to B \to C \leftrightarrow i_{k-1} \to i_k \to i_{k+1}$ and p_2 is blocked by S because with $i_k = B$ we have : $i_k \in S$ and $A \to B \to D \leftrightarrow i_{k-1} \to i_k \to i_{k+1}$ It is left to show that there is no set with fewer elements that also d-seperates A and F. The only set that has fewer elements is the empty set but the empty set is neither blocking p_1 nor p_2 according to the definition. □

(d)

Beweis. We will proof that $C \perp\!\!\!\perp_G D \mid B$ i.e B d-seperates C and D holds. This is the case if every path is blocked by $S = \{B\}$. There are two paths from C to D:

$$p_1 = C \leftarrow B \to D$$
$$p_2 = C \to E \leftarrow D$$

p_1 is blocked by S because with $i_k = B$ we have $i_k \in S$ and $C \leftarrow B \to D \leftrightarrow i_{k-1} \leftrightarrow \leftarrow i_k \to i_{k+1}$. Also p_2 is blocked by S because with $i_k = E$ we have : $i_k \notin S$ and $C \to E \leftarrow D \leftrightarrow i_{k-1} \to i_k \leftarrow i_{k+1}$ So every path between C and D is blocked by S and therefore $C \perp\!\!\!\perp_G D \mid B$ holds. □