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Solutions Sheet 1

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Exercise 1

First, we will assign values to $p(M), p(m), p(e), P(n \mid M), P(n \mid m), P(n \mid e)$, so that $p(M) + p(m) + p(e) = 1$:

$$p(M) := 0.01$$
$$p(m) := 0.1$$
$$p(e) := 0.89$$

$$p(n \mid M) = 0.9$$
$$p(n \mid m) = 0.08$$
$$p(n \mid e) = 0.02$$

The next step to apply Bayes' rule is now to calculate the probability of a noise:

$$p(n) = p(M) \cdot p(n \mid M) + p(m) \cdot p(n \mid m) + p(e) \cdot p(n \mid e)$$

$$= 0.01 \cdot 0.9 + 0.1 \cdot 0.08 + 0.89 \cdot 0.02$$

$$= 0.0348$$

We can now calculate the probability of a monster given noise by applying Bayes' rule:

$$p(M \mid n) = \frac{p(n \mid M) \cdot p(M)}{p(n)}$$
$$= \frac{0.9 \cdot 0.01}{0.034}$$
$$\approx 0.26$$

These results got confirmed by the jupyter notebook presented in the lecture:

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In [5]:  1 # what is the probability of having noise
         2 p_n = p_nM + p_nm + p_ne      # p(n) = p(n,M) + p(n,m) + p(n,e)
         3 print(f"p(n) = {p_n}")

         p(n) = 0.0348

In [7]:  1 # what is the (posterior) probability of having a ... given that we hear noise
         2 p_M_n = Bayes_rule(p_M, p_n_M, p_n)      # p(M|n) = p(M) p(n|M) / p(n)
         3 p_m_n = Bayes_rule(p_m, p_n_m, p_n)      # p(m|n) = p(m) p(n|m) / p(n)
         4 p_e_n = Bayes_rule(p_e, p_n_e, p_n)      # p(e|n) = p(e) p(n|e) / p(n)
         5 print(f"p(M|n) = {p_M_n}")
         6 print(f"p(m|n) = {p_m_n}")
         7 print(f"p(e|n) = {p_e_n}")

         p(M|n) = 0.2586206896551725
         p(m|n) = 0.22988850574712644
         p(e|n) = 0.5114942528735632
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Exercise 2

Given:

- The plausibility of B is $p(B)$.

- $p(B \mid A) \geq p(B)$

Show: (a) $p(B \mid A) \geq p(B)$

Proof. Is equivalent to second assumption.   □

(b) $p(B \mid \neg A) \leq p(B)$

Proof. By assumption:

$$p(B|A) \geq p(B)$$

$$\Leftrightarrow \frac{p(A \mid B) \cdot p(B)}{p(A)} \geq p(B)$$

$$\Leftrightarrow \frac{(1 - p(\neg A \mid B)) \cdot p(B)}{1 - p(\neg A)} \geq p(B)$$

$$\Leftrightarrow (1 - p(\neg A \mid B)) \cdot p(B) \geq p(B) \cdot (1 - p(\neg A))$$

$$\Leftrightarrow 1 - p(\neg A \mid B) \geq 1 - p(\neg A)$$

$$\Leftrightarrow -p(\neg A \mid B) \geq -p(\neg A)$$

$$\Leftrightarrow p(\neg A \mid B) \leq p(\neg A)$$

$$\Leftrightarrow p(\neg A \mid B) \leq p(\neg A) \cdot \frac{p(B)}{p(B)}$$

$$\Leftrightarrow \frac{p(\neg A \mid B) \cdot p(B)}{p(\neg A)} \leq p(B)$$

$$\Leftrightarrow p(B \mid \neg A) \leq p(B)$$

□

(c) $p(A \mid B) \geq p(A)$

Proof. By assumption:

$$p(B|A) \geq p(B)$$
$$\Leftrightarrow \frac{p(A \mid B) \cdot p(B)}{p(A)} \geq p(B)$$
$$\Leftrightarrow p(A \mid B) \geq p(A)$$

□

(d) $p(A \mid \neg B) \leq p(A)$

Proof. By assumption:

$$p(B|A) \geq p(B)$$
$$\overset{(c)}{\Leftrightarrow} p(A \mid B) \geq p(A)$$
$$\overset{(b)}{\Leftrightarrow} p(A \mid \neg B) \leq p(A)$$

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Exercise 3

For this exercise we will formalise the given facts as follows:

1. $C = $ "The Patient has Cancer"

2. $R = $ "A Patient's Mammogram is positive"

We can therefore deduct from the given facts:

1. $P(C) = 0.008$

2. $P(R \mid C) = 0.9$

3. $P(R \mid \neg C) = 0.07$

4. $P(\neg C) = 1 - P(C) = 0.992$

5. $P(R) = P(C) \cdot P(R \mid C) + P(\neg C) \cdot P(R \mid \neg C) = 0.008 \cdot 0.9 + 0.992 \cdot 0.07 = 0.08$

Given these deductions, we can now calculate the adressed probability that a patient has breast cancer given a positive Mammogram $P(C \mid R)$ by applying Bayes' Rule:

$$P(C \mid R)$$
$$= \frac{P(R \mid C) \cdot P(C)}{P(R)}$$
$$= \frac{0.9 \cdot 0.008}{0.08}$$
$$= 0.09$$

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Exercise 4

For this exercise we will formulate the given facts as follows follows :

- $W_A = $ "The warden announces A"

- $W_B = $ "The warden announces B"

- $W_C = $ "The warden announces C"

- $A = $ "A is pardonned"

- $B = $ "B is pardonned"

- $C = $ "C is pardonned"

Before any announcement is made by the warden we obviously have :

$$P(A) = P(B) = P(C) = \frac{1}{3}$$

No Matter what the warden will eather announce A or B, which brings us to the conclusion:

$$P(A) = P(W_B \mid A) + P(W_C \mid A) = \frac{1}{3}$$

Because in this case the warden flips a coin we know:

$$P(W_B \mid A) = P(W_C \mid A) = \frac{1}{6}$$

We know that $P(W_B \mid B) = 0$ and we can therefore also deduct that :

$$P(W_B) = P(A) \cdot P(W_B \mid A) + P(C) \cdot P(W_B \mid C) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

Now we can apply Bayes' formula to calulate the updatet probability after the Warden's announcement:

$$P(A \mid W_B) = \frac{P(W_B \mid A) \cdot P(A)}{P(W_B)}$$
$$= \frac{\frac{1}{6} \cdot \frac{1}{3}}{\frac{1}{2}}$$
$$= \frac{1}{3}$$

We also now that $P(W_B \mid C) = 1$. It follows:

$$P(C \mid W_B) = \frac{P(W_B \mid C) \cdot P(C)}{P(W_B)}$$

$$= \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}}$$

$$= \frac{2}{3}$$

Therefore C's argument is right.

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