Solutions Sheet

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Exercise 1

Exercise 2

Exercise 3

(a) *Proof.* Let $k(x_1, x_2) = C$ with $C \in \mathbb{R}_{>0}$. Then for $x \in \mathbb{R}^n$ we have:

$$x^T k_{\mathbf{xx}} x = C(\sum_{i=1}^{n} x_i)(\sum_{j=1}^{n} x_j)$$

We will show that this sum is greater or equal to 0. To show that let I be the set of indices from 1 to n. Let further be :

$$P \subseteq I := \{i \in I : x_i \geq 0\}$$

$$N \subseteq I := \{i \in I : x_i < 0\}$$

Then we can write :

$$C(\sum_{i=1} x_i)(\sum_{j=1} x_j) = C(\sum_{i \in P} x_i + \sum_{j \in N} x_j)(\sum_{l \in P} x_l + \sum_{k \in N} x_k)$$

We can now distinguish two cases:

Case 1: $\sum_{i \in P} x_i \geq \sum_{j \in N} |x_j|$ Then we have

$$C(\underbrace{\sum_{i \in P} x_i + \sum_{j \in N} x_j)}_{\geq 0}(\underbrace{\sum_{l \in P} x_l + \sum_{k \in N} x_k}_{\geq 0}) \geq 0$$

Case 2: $\sum_{i \in P} x_i < \sum_{j \in N} |x_j|$ Then we have

$$C\,\underbrace{(\underbrace{\sum_{i \in P} x_i + \sum_{j \in N} x_j)}_{<0}(\underbrace{\sum_{l \in P} x_l + \sum_{k \in N} x_k}_{<0})}_{>0} > 0$$

So we have

$$x^T k_{\mathbf{xx}} x = C(\sum_{i=1}^{n} x_i)(\sum_{j=1}^{n} x_j) \geq 0$$

And k is thus positive semidefinite. □

(b) *Proof.* Let $k(x_1, x_2) = x_1 \cdot x_2$ with $X = \mathbb{R}$.

It follows:

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} c_i \cdot c_j \cdot k(x_i, x_j) &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i \cdot c_j \cdot x_i \cdot x_j \\ &= \sum_{i=1}^{n} c_i \cdot x_i \sum_{i=1}^{n} c_i \cdot x_i \qquad = (\sum_{i=1}^{n} c_i \cdot x_i)^2 \geq 0\end{aligned}$$

Thus k is positive semidefinite. □

(c) *Proof.* Let $k(x_1, x_2) = x_1 + x_2$ with $X = \mathbb{R}$. We have x $\in \mathbb{R}$. So with x=-1 it is:

$$(-1) \cdot k(-1, -1) \cdot (-1) = (-1) \cdot (-2) \cdot (-1) = -2 < 0$$

Therefore $k(x_1, x_2) = x_1 + x_2$ is not positive semidefinite and thus is not a kernel. □

(d) *Proof.* Let $k(x_1, x_2) = 5 \cdot x_1^T \cdot x_2$ with $X = \mathbb{R}^D$.

It follows:

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} c_i \cdot c_j \cdot k(x_i, x_j) &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i \cdot c_j \cdot 5 \cdot x_i^T \cdot x_j \\ &= 5 \cdot (\sum_{i=1}^{n} c_i \cdot x_i)^T \cdot (\sum_{i=1}^{n} c_i \cdot x_i) \quad = 5 \cdot \| \sum_{i=1}^{n} c_i \cdot x_i \|_2^2 \geq 0\end{aligned}$$

Thus k is positive semidefinite. □

(e) *Proof.* Let $k(x_1, x_2) = (x_1^T \cdot x_2 + 1)^2$ with $X = \mathbb{R}^N$.

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} c_i \cdot c_j \cdot k(x_i, x_j) &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i \cdot c_j \cdot (x_i \cdot x_j + 1) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i \cdot c_j \cdot x_i \cdot x_j + c_i \cdot c_j \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i \cdot x_j \cdot c_i \cdot c_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_i \cdot c_j \\
&= (\sum_{i=1}^{n} c_i \cdot x_i)^T \sum_{j=1}^{n} c_j \cdot x_j + (\sum_{i=1}^{n} c_i)^T \sum_{j=1}^{n} c_j \\
&= || \sum_{i=1}^{n} c_i \cdot x_i ||_2^2 + || \sum_{i=1}^{n} c_i ||_2^2 \geq 0
\end{aligned}$$

Thus k is positive semidefinite.

□