Solutions Sheet

Nina Fischer and Yannick Zelle

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Exercise 1

Let us consider z_n to be se from a previous previous iterration step. So we have :

$$J = \sum_n \sum_i z_n^i ||x_- \mu_i||^{\hat{A}2}$$

We will now show that this Loss function converges and therefore the k-means algorithm converges

Proof. **(a)** The reassiment step is give by

$$z_n^{i\,\prime} = \begin{cases} 1, & \text{if } i = argmin_j ||x_n - u_l||^2 \\ 0, & \text{otherwise} \end{cases}$$

So

$$\begin{aligned} J' &= \sum_n \sum_i z_n^{i\,\prime} ||x_n - u_i||^2 \\ &= \sum_n \min_i ||x_n - u_i||^2 \end{aligned}$$

So we have :

$$J' \leq J$$

Thus the e-step minimizes J

b For the reassigment of the mean we reassine the $\mu_i' s$ by definition with teh average of those x, which already minimizes J, so by construction if we reassign :

$$\mu_i' = \frac{\sum_n z_n^i x_n}{\sum_n z_n^i}$$

and

$$J'' = \sum_n \sum_i z_n^{i\prime} ||x_n - u_i'||^2$$

we have

$$0 \leq J'' \leq J' \leq J$$

(c) We thus conclude that the distortion measure or i.e. the loss function of the k-means algo converges towards 0 and thus at a given point there will be no more need to reassign the z_n^i or expressed differently k-means is guaranteed to converge.

□

Exercise 2

Exercise 3

Exercise 4