Machine Learning Exercise Sheet 2

Yannick Zelle and Nina Fischer

24. Oktober 2021

Exercise 1

(a)

Given is the DAG G_1 with the following associated probabilities:

- $p(A) = 0.3$

- $p(B \mid A) = 0.2$

- $p(B \mid \neg A) = 0.4$

- $p(C \mid B) = 0.7$

- $p(C \mid \neg B) = 0.5$

From there we can deduct the following probabilities:

- $p(\neg A) = 0.7$

- $p(\neg B \mid A) = 0.8$

- $p(\neg B \mid \neg A) = 0.6$

- $p(\neg C \mid B) = 0.3$

- $p(\neg C \mid \neg B) = 0.5$

From G_1 we can deduct the joint probability distribution:

$$p(A, B, C) = p(A) \cdot p(B \mid A) \cdot p(C \mid B)$$

Using this Probabilitydistribution we can calculate P(B):

$$p(B) = \sum_{A,C} p(A, B = 1, C)$$
$$= p(A) \cdot p(B \mid A) \cdot p(C \mid B)$$
$$+ p(\neg A) \cdot p(B \mid \neg A) \cdot p(C \mid B)$$
$$+ p(\neg A) \cdot p(B \mid \neg A) \cdot p(\neg C \mid B)$$
$$+ p(A) \cdot p(B \mid A) \cdot p(\neg C \mid B)$$
$$= 0.3 \cdot 0.2 \cdot 0.7$$
$$+ 0.7 \cdot 0.4 \cdot 0.7$$
$$+ 0.7 \cdot 0.4 \cdot 0.3$$
$$+ 0.3 \cdot 0.2 \cdot 0.3$$
$$= 0.34$$

and P(C):

$$p(C) = \sum_{A,B} p(A, B, C = 1)$$
$$= p(A) \cdot p(B \mid A) \cdot p(C \mid B)$$
$$+ p(\neg A) \cdot p(B \mid \neg A) \cdot p(C \mid B)$$
$$+ p(\neg A) \cdot p(\neg B \mid \neg A) \cdot p(C \mid \neg B)$$
$$+ p(A) \cdot p(\neg B \mid A) \cdot p(C \mid \neg B)$$
$$= 0.3 \cdot 0.2 \cdot 0.7$$
$$+ 0.7 \cdot 0.4 \cdot 0.7$$
$$+ 0.7 \cdot 0.6 \cdot 0.5$$
$$+ 0.3 \cdot 0.8 \cdot 0.5$$
$$= 0.568$$

(b)

Given is a DAG G_2 and the associated probabilities:

- $p(A) = 0.3$
- $p(B \mid A) = 0.2$
- $p(B \mid \neg A) = 0.4$
- $p(c \mid A) = 0.7$
- $p(C \mid \neg A) = 0.6$
- $p(D \mid B, C) = 0.9$

- $p(D \mid B, \neg C) = 0.5$

- $p(D \mid \neg B, C) = 0.3$

- $p(D \mid \neg B, \neg C) = 0.3$

From there we can deductthe following probabilities:

- $p(\neg A) = 0.7$

- $p(\neg B \mid A) = 0.8$

- $p(\neg B \mid \neg A) = 0.6$

- $p(\neg c \mid A) = 0.3$

- $p(\neg C \mid \neg A) = 0.4$

- $p(\neg D \mid B, C) = 0.1$

- $p(\neg D \mid B, \neg C) = 0.5$

- $p(\neg D \mid \neg B, C) = 0.7$

- $p(\neg D \mid \neg B, \neg C) = 0.7$

From the G_2 we can deduct the following joint Probabilitydistribution:

$$p(A, B, C, D) = p(A) \cdot p(C \mid A) \cdot p(B \mid A) \cdot p(D \mid B, C)$$

Using the joint Probabilitydistribution and the probabilities from above, we can calculate $p(B)$:

$$p(B) = \sum_{A,C,D} p(A, B = 1, C, D)$$
$$= p(A) \cdot p(C \mid A) \cdot p(B \mid A) \cdot p(D \mid B, C)$$
$$+ p(A) \cdot p(C \mid A) \cdot p(B \mid A) \cdot p(\neg D \mid B, C)$$
$$+ p(A) \cdot p(\neg C \mid A) \cdot p(B \mid A) \cdot p(D \mid B, \neg C)$$
$$+ p(A) \cdot p(\neg C \mid A) \cdot p(B \mid A) \cdot p(\neg D \mid B, \neg C)$$
$$+ p(\neg A) \cdot p(C \mid \neg A) \cdot p(B \mid \neg A) \cdot p(D \mid B, C)$$
$$+ p(\neg A) \cdot p(C \mid \neg A) \cdot p(B \mid \neg A) \cdot p(\neg D \mid B, C)$$
$$+ p(\neg A) \cdot p(\neg C \mid \neg A) \cdot p(B \mid \neg A) \cdot p(D \mid B, \neg C)$$
$$+ p(\neg A) \cdot p(\neg C \mid \neg A) \cdot p(B \mid \neg A) \cdot p(\neg D \mid B, \neg C)$$
$$= 0.7 \cdot 0.7 \cdot 0.2 \cdot 0.9$$
$$+ 0.7 \cdot 0.3 \cdot 0.8 \cdot 0.1$$
$$+ 0.7 \cdot 0.3 \cdot 0.2 \cdot 0.5$$
$$+ 0.7 \cdot 0.3 \cdot 0.2 \cdot 0.5$$
$$+ 0.3 \cdot 0.6 \cdot 0.4 \cdot 0.9$$
$$+ 0.3 \cdot 0.6 \cdot 0.4 \cdot 0.1$$
$$+ 0.3 \cdot 0.4 \cdot 0.4 \cdot 0.5$$
$$+ 0.3 \cdot 0.4 \cdot 0.4 \cdot 0.5$$
$$= 0.27$$

and p(D):

$$p(D) = \sum_{A,C,B} p(A, B, C, D = 1)$$

$$= p(A) \cdot p(C \mid A) \cdot p(B \mid A) \cdot p(D \mid B, C)$$
$$+ p(A) \cdot p(C \mid A) \cdot p(\neg B \mid A) \cdot p(D \mid \neg B, C)$$
$$+ p(A) \cdot p(\neg C \mid A) \cdot p(B \mid A) \cdot p(D \mid B, \neg C)$$
$$+ p(A) \cdot p(\neg C \mid A) \cdot p(\neg B \mid A) \cdot p(D \mid \neg B, \neg C)$$
$$+ p(\neg A) \cdot p(C \mid \neg A) \cdot p(B \mid \neg A) \cdot p(D \mid B, C)$$
$$+ p(\neg A) \cdot p(C \mid \neg A) \cdot p\neg(B \mid \neg A) \cdot p(D \mid \neg B, C)$$
$$+ p(\neg A) \cdot p(\neg C \mid \neg A) \cdot p(B \mid \neg A) \cdot p(D \mid B, \neg C)$$
$$+ p(\neg A) \cdot p(\neg C \mid \neg A) \cdot p(\neg B \mid \neg A) \cdot p(D \mid \neg B, \neg C)$$
$$= 0.7 \cdot 0.7 \cdot 0.2 \cdot 0.9$$
$$+ 0.7 \cdot 0.3 \cdot 0.8 \cdot 0.3$$
$$+ 0.7 \cdot 0.3 \cdot 0.2 \cdot 0.5$$
$$+ 0.7 \cdot 0.3 \cdot 0.8 \cdot 0.3$$
$$+ 0.3 \cdot 0.6 \cdot 0.4 \cdot 0.9$$
$$+ 0.3 \cdot 0.6 \cdot 0.6 \cdot 0.3$$
$$+ 0.3 \cdot 0.4 \cdot 0.4 \cdot 0.5$$
$$+ 0.3 \cdot 0.4 \cdot 0.6 \cdot 0.3$$
$$= 0.36$$

Exercise 2

(a)

$$p(A, B, C, D, E, F) = p(A) \cdot p(B \mid A) \cdot p(C \mid B) \cdot p(D \mid B) \cdot p(E \mid C, D) \cdot p(F \mid E)$$

(b)

$$p(A, B, C, D, E) = p(A) \cdot p(B \mid A) \cdot p(C \mid B) \cdot p(D \mid B) \cdot p(E \mid C, D)$$

(c)

Task: Find a minimal set S that d-seperates A and F and prove that this is the case.

Beweis. We propose $S = \{B\}$ to prove that the criteria of S holds we have actually to prove two statements:

1. S d-seperates A and F

2. there is no set with fewer elements that also d-seperates A and F

5

We will start with the first statement. Between A and F exist two paths:

$$p_1 = A \to B \to C \to E \to F$$
$$p_2 = A \to B \to D \to E \to F$$

We have p_1 is blocked by S because with $i_k = B$ we have : $i_k \in S$ and $A \to B \to C \leftrightarrow i_{k-1} \to i_k \to i_{k+1}$ and p_2 is blocked by S because with $i_k = B$ we have : $i_k \in S$ and $A \to B \to D \leftrightarrow i_{k-1} \to i_k \to i_{k+1}$ It is left to show that there is no set with fewer elements that also d-seperates A and F. The only set that has fewer elements is the empty set but the empty set is neither blocking p_1 nor p_2 according to the definition. □

(d)

Beweis. We will proof that $C \perp\!\!\!\perp_G D \mid B$ i.e B d-seperates C and D holds. This is the case if every path is blocked by $S = \{B\}$. There are two paths from C to D:

$$p_1 = C \leftarrow B \to D$$
$$p_2 = C \to E \leftarrow D$$

p_1 is blocked by S because with $i_k = B$ we have $i_k \in S$ and $C \leftarrow B \to D \leftrightarrow i_{k-1} \leftrightarrow\leftarrow i_k \to i_{k+1}$. Also p_2 is blocked by S because with $i_k = E$ we have : $i_k \notin S$ and $C \to E \leftarrow D \leftrightarrow i_{k-1} \to i_k \leftarrow i_{k+1}$ So every path between C and D is blocked by S and therefore $C \perp\!\!\!\perp_G D \mid B$ holds. □

Exercise 3

(a)

We will proof that

$$A \to B \to C \implies A \perp\!\!\!\perp C \mid B$$

Beweis.

$$p(A, B, C) = p(A) \cdot p(B \mid A) \cdot p(C \mid B)$$
$$\Leftrightarrow p(A, B, C) = p(A, B) \cdot p(C \mid B)$$
$$\Leftrightarrow p(A, B, C) = p(A \mid B) \cdot p(B) \cdot p(C \mid B)$$
$$\Leftrightarrow \frac{p(A, B, C)}{p(B)} = p(A \mid B) \cdot p(C \mid B)$$
$$\Leftrightarrow p(A, C \mid B) = p(A \mid B) \cdot p(C \mid B)$$
$$\implies A \perp\!\!\!\perp C \mid B$$

□

(b)

We will proof that

$$A \leftarrow B \leftarrow C \implies A \perp\!\!\!\perp C \mid B$$

and we know from a that $p(C) \cdot p(B \mid C) \Leftrightarrow p(B) \cdot p(C \mid B)$.

Beweis.

$$p(A, B, C) = p(A \mid B) \cdot p(B \mid C) \cdot p(C)$$
$$\Leftrightarrow p(A, B, C) = p(A \mid B) \cdot p(C \mid B) \cdot p(B)$$
$$\Leftrightarrow \frac{p(A, B, C)}{p(B)} = p(A \mid B) \cdot p(C \mid B)$$
$$\Leftrightarrow p(A, C \mid B) = p(A \mid B) \cdot p(C \mid B)$$
$$\implies A \perp\!\!\!\perp C \mid B$$

$\square$

(c)

We will proof that

$$A \leftarrow B \rightarrow C \implies A \perp\!\!\!\perp C \mid B$$

Beweis.

$$p(A, B, C) = p(A \mid B) \cdot p(B) \cdot p(C \mid B)$$
$$\Leftrightarrow \frac{p(A, B, C)}{p(B)} = p(A \mid B) \cdot p(C \mid B)$$
$$\Leftrightarrow p(A, C \mid B) = p(A \mid B) \cdot p(C \mid B)$$
$$\implies A \perp\!\!\!\perp C \mid B$$

$\square$

(d)

We will proof that

$$A \rightarrow B \leftarrow C \implies A \perp\!\!\!\perp C$$

Beweis.

$\square$

Exercise 4

(a)

We will proof that $E(a \cdot X + Y) = a \cdot E(X) + E(Y)$ with the sum rule and the factor rule.

Beweis.

$$E(a \cdot X + Y) = \int a \cdot X + Y \, dx$$
$$= \int a \cdot X \, dx + \int Y \, dx$$
$$= a \cdot \int X \, dx + \int Y \, dx$$
$$= a \cdot E(X) + E(Y)$$

$\square$

(b)

We will proof that $Var(a \cdot X) = a^2 \cdot Var(X)$ with the result of a.

Beweis.

$$Var(a \cdot X) = E(a \cdot X - \mu)^2$$
$$= E(a^2 \cdot (X - \mu)^2)$$
$$= a^2 \cdot E(X - \mu)^2$$
$$= a^2 \cdot Var(X)$$

$\square$