Solutions Sheet 11

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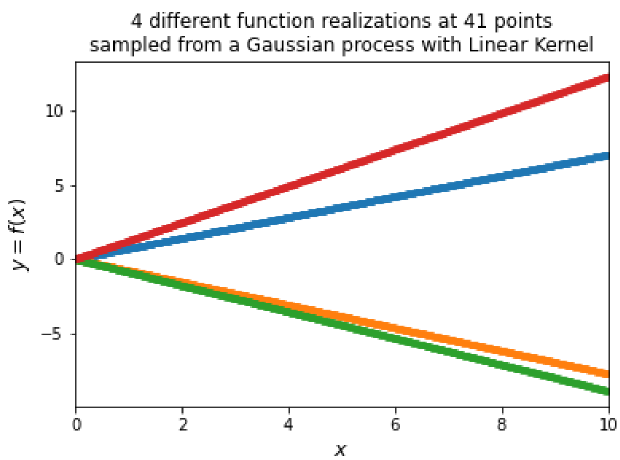
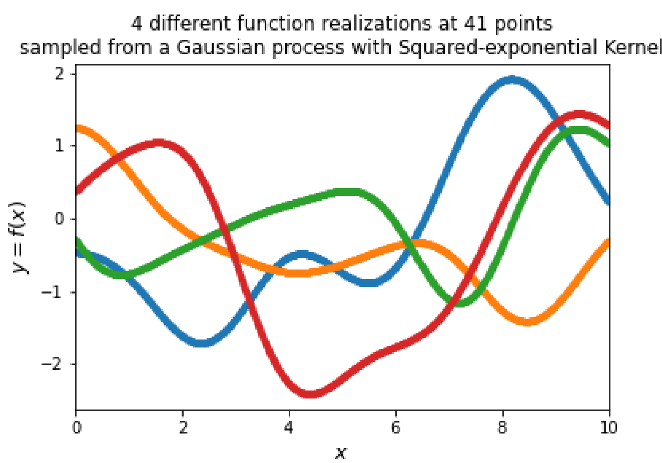
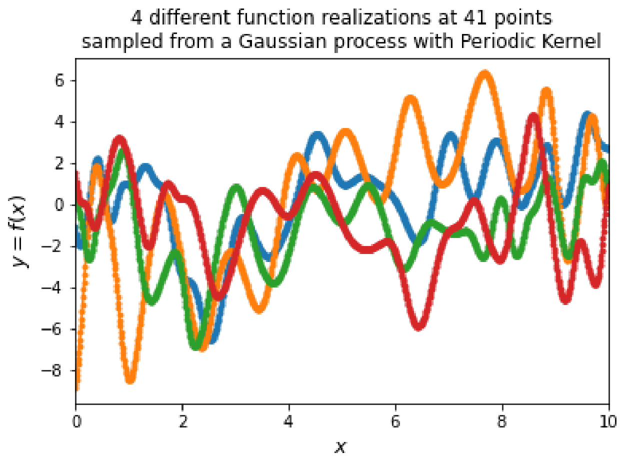
Exercise 1

For Implementation Details see the jupyter notebook. Here are the resultung plots:







Exercise 2

(a) *Proof.* We will proof that $k(x, x')$ is positive semi-definit. We will do so by prooving that :

$$\forall x_1, \cdots, x_n \in \mathbb{R} \text{ and } \forall c_1, \cdots, c_n \in \mathbb{R} : \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j k(x_i, x_j) \geq 0$$

We have:

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j k(x_i, x_j) &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j (30 x_i x_j + 1) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} (30 x_i x_j c_i c_j + c_i c_j) \\
&= 30 \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j c_i c_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \\
&= 30 (\sum_{i=1}^{n} c_i x_i)^T \sum_{j=1}^{n} c_j x_j + (\sum_{i=1}^{n} c_i)^T \sum_{j=1}^{n} c_j \\
&= 30 || \sum_{i=1}^{n} x_i c_i ||_2^2 + || \sum_{i=1}^{n} c_i ||_2^2 \geq 0
\end{aligned}$$

□

(b) With the help of the observed Data and the given Kernel function we can calculate the Covariance Matrix:

$$k_{XX} = \begin{pmatrix} 31 & 29 \\ 29 & 31 \end{pmatrix}$$

Additionally according to the lecture Λ is give by:

$$\Lambda = \sigma^2 I = I$$

We can now use the expressions from the lecture to calculate μ_* and $sigma_*^2$

$$\begin{aligned}
\mu_* &= m_{X_*} + k_{X_*X}(k_{XX} + \Lambda)^{-1}(y - m_X) \\
&= \begin{pmatrix} 31 & 29 \end{pmatrix} \begin{pmatrix} 32 & 29 \\ 29 & 32 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= 0.984
\end{aligned}$$

$$\begin{aligned}
\sigma_*^2 &= m_{X_*} + k_{X_*X}(k_{XX} + \Lambda)^{-1}(y - m_X) \\
&= k_{X_*X_*} - k_{X_*X}(k_{XX} + \Lambda)^{-1} k_{XX_*} + \sigma^2 \\
&= 31 - \begin{pmatrix} 31 & 29 \end{pmatrix} \begin{pmatrix} 32 & 29 \\ 29 & 32 \end{pmatrix}^{-1} \begin{pmatrix} 29 \\ 32 \end{pmatrix} + 1 \\
&= 4
\end{aligned}$$

Exercise 3

Our Implementation of exercise 3 can be seen in the attached notebook.