Solutions Sheet 3

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Exercise 1

Task: Describe the maximum likelihood estimator for the following distributions:

(a)

$$\mathcal{N}(x \mid \mu, \sigma^2)$$

We denote with f_μ the PDF of the normal distribution. By using the Lemma 6.2.3 we can write the log-likelihood function as:

$$\begin{aligned}
l(\mu) &= ln(\prod_{i=1}^{n} f_\mu) \\
&= ln(\prod_{i=1}^{n} e^{a+\eta x_i - \frac{1}{2}\lambda^2 x_i^2}) \\
&= \sum_{i=1}^{n} ln(e^{a+\eta x_i - \frac{1}{2}\lambda^2 x_i^2}) \\
&= \sum_{i=1}^{n} -\frac{1}{2}(log(2\pi) - log(\lambda^2) + \frac{\mu^2}{\sigma^2}) + \frac{\mu}{\sigma^2} x_i - \frac{1}{2}\lambda^2 x_i^2
\end{aligned}$$

To find μ which maximizes $l(\mu)$ we calculate:

$$\begin{aligned}
\frac{\delta l(\mu)}{\delta \mu} &= \sum_{i=1}^{n} -\frac{\mu}{\sigma^2} + \frac{1}{\sigma^2} x_i \\
&= -n\frac{\mu}{\sigma^2} + \frac{1}{\sigma^2} \sum_{i=1}^{n} x_i
\end{aligned}$$

$l(\mu)$ is maximal if :

$$l(\mu) = 0 \leftrightarrow 0 = -n\frac{\mu}{\sigma^2} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i$$
$$\leftrightarrow \mu = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Since $\frac{\delta^2 l(\mu)}{\delta^2 \mu} = -\frac{1}{\sigma^2} < 0$, with $\mu_o := \frac{1}{n}\sum_{i=1}^{n} x_i$ we obtain a global maximum for $l(\mu)$ and we chose μ_0 therefore as our estimator.

(b)

Exponential distribution with the densitity function $f(x|\lambda) = \lambda e^{-\lambda x}$ with $\lambda > 0$ and observations $x_i \geq 0$. We have :

$$f(x|\lambda) = \lambda e^{-\lambda x}$$
$$= e^{-\lambda x + ln(\lambda)}$$

and the likelihood function:

$$L(\lambda) = \prod_{i=1}^{n} f(x_i|\lambda)$$

Therefore the log-likelihood is giveb by:

$$l(\lambda) = ln(\prod_{i=1}^{n} f(x_i|\lambda))$$
$$= \sum_{i=1}^{n} ln(f(x_i|\lambda))$$
$$= \sum_{i=1}^{n} ln(e^{-\lambda x_i + ln(\lambda)})$$
$$= sum_{i=1}^{n} - \lambda x_i + ln(\lambda)$$

With the same argument as in (a) we derive:

$$\frac{\delta l(\lambda)}{\delta \lambda} = \sum_{i=1}^{n} x_i + \frac{1}{\lambda}$$
$$= \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

Then we have :

$$0 = \frac{\delta l(\lambda)}{\delta \lambda} \leftrightarrow \lambda = \frac{n}{\sum_{i=1}^{n} x_i}$$

We also have

$$\frac{\delta^2 l(\lambda)}{\delta^2 \lambda} = -\frac{1}{\lambda^2} < 0$$

therefore we chose $\lambda_0 = \frac{n}{\sum_{i=1}^{n} x_i}$ as our estimator.

Exercise 2

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