Solutions Sheet

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November 22, 2021

Exercise 1

Given: Let $m \leq n \leq k, y \in \mathbb{R}^m, b \in \mathbb{R}^k$ and $A \in \mathbb{R}^{mxn}, B \in \mathbb{R}^{kxn}$ We are considering the follwoing optimization optimization problem:

$$\min_{x \in \mathbb{R}^n} ||Ax - y||_2^2$$
$$\text{s.t.} Bx = b$$

Task: Find a matrix $P \in \mathbb{R}^{(n+k)x(n+k)}$ and a vector $p \in \mathbb{R}^{n+k}$ such that solving :

$$P \begin{bmatrix} x \\ \lambda \end{bmatrix} = p$$

gives a critical point for the optimization problem.

Proof. **Solution:** We will start by defining the Langragian function associated to this problem:

$$L(\lambda) = ||Ax - y||_2^2 + \lambda^T \cdot (Bx - b)$$

We will now search for the derivatives with respect to x and λ by using Matrix differential calculus:

- We will start by calculating $D_x L$

$$\begin{aligned} dL &= d||Ax - y||_2^2 + d\lambda^T (Bx - b) \\ &= d(Ax - y)^T (Ax - y) + \lambda^T B dx \\ &= 2(Ax - y)^T d(Ax - y) + \lambda^T B dx \\ &= 2(Ax - y)^T A dx + \lambda^T B dx \end{aligned}$$

So :

$$D_x L = 2(Ax - y)^T A + \lambda^T B$$

- We will now calculate $D_\lambda L$:

$$\begin{aligned} dL &= d||Ax-y||_2^2 + d\lambda^T(Bx-b) \\ &= (Bx-b)^T d\lambda \end{aligned}$$

So we have :

$$\nabla L = (2(Ax-y)^T A dx + \lambda^T B, (Bx-b)^T)$$

- For $\nabla L = 0$ we have :

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} (Bx-b)^T \\ 2(Ax-y)^T A + \lambda^T B \end{bmatrix} \\ \leftrightarrow \begin{bmatrix} b \\ 2y^T A \end{bmatrix} &= \begin{bmatrix} Bx \\ 2(Ax)^T A^T A + \lambda^T B \end{bmatrix} \\ \leftrightarrow \begin{bmatrix} b \\ 2(y^T A)^T \end{bmatrix} &= \begin{bmatrix} Bx \\ 2A^T Ax + B^T \lambda \end{bmatrix} \\ \leftrightarrow \begin{bmatrix} b \\ 2(y^T A)^T \end{bmatrix} &= \begin{bmatrix} B & 0^{kxn} \\ 2A^T A & B^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \end{aligned}$$

So with $p = \begin{bmatrix} b \\ 2(y^T A)^T \end{bmatrix}$ and $P = \begin{bmatrix} B & 0^{kxn} \\ 2A^T A & B^T \end{bmatrix}$ solving

$$P \begin{bmatrix} x \\ \lambda \end{bmatrix} = p$$

will give a critical point to the optimization problem.

□

Exercise 2

Exercise 3

Exercise 4