CHORDAL BIPARTITE GRAPHS

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A bipartite graph is chordal bipartite if it does not contain an induced cycle of length six.

1 Question

Consider chordal bipartite graphs $G(X, Y, E)$ such that for each vertex y in Y, the degree of y is at most 2 and No restriction on X.

1.1 Structural Observations

- If the $\delta(G) \geq 3$, then the X, Y partition is unique.
- The graph is a combinations of C_4s and trees.
- Graph cannot have cycles other than C_4.
- The minimum path length between any two cycles is $2n$ where $n \geq 0$.

Lemma 1. *Cycles cannot have an edge in common.*

Proof. Let us prove it by contradiction. Let $G(X, Y, E)$ be a restricted chordal bipartite graph having $C1(a, b, c, d)$, $C2(a, b, e, f)$ which are two C_4s having an edge in common. since G is a bipartite graph $\forall \left\{x, y\right\} \in E(G) \implies x \in X \ y \in Y$. since $\left\{a, b\right\}$ is the common edge for $C1$ and $C2$, $deg(a), deg(b) \geq 3$ and it contradicts the fact that $\triangle(Y) \leq 2$. □

Lemma 2. *Any two C_4s can have at most two non adjacent vertices in common.*

Proof. An argument similar to *Lemma* 1 establishes the claim and a case of three non adjacent vertices is not possible in any C_4.

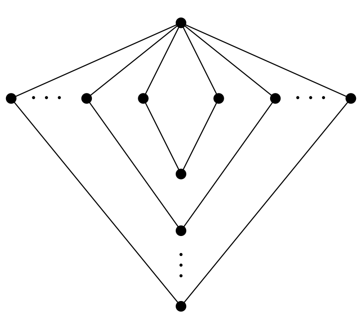


Fig 1. C_4s having 1 vertex in common

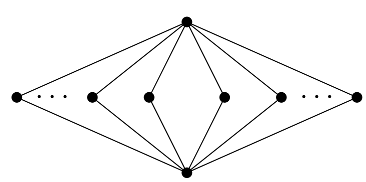


Fig 2. C_4s having 2 vertices in common

□

1.2 Graph Construction

Lemma 3. *A restricted chordal bipartite graph G has any one of the following property:*

(i) One pendent vertex.

(ii) One pendent C_4.

*(iii) One pendent pyramid.(**Fig 2**)*

Proof. We shall prove it by mathematical induction on number of vertices n of G.

Base case: For $n = 5$,
(A) G be a tree. Trivially, G has a pendent vertex as there are atleast two leaves in any tree.
(B) G is having a C_4. Then G has a C_4 and an edge having a vertex in common. So, G has a pendent vertex and a pendent C_4.
(C) G be a pyramid. By the definition G itself is a pendent pyramid.

Hypothesis: Assume that the lemma is true for all restricted bipartite graphs with $n \geq 5$.

Induction step: We shall partition the restricted chordal bipartite graphs into graphs with minimal vertex separator of size one and graphs with minimal vertex separator of size two.
Let G be any restricted chordal bipartite graph with $n \geq 6$. Let S be any minimal vertex separator(MVS) of G.
Case 1: $|S| = 1$,(**Fig 3**)
Let G_1, G_2 be any two connected components in $G \setminus S$. Let G', G'' be the graph induced on $V(G_1) \cup V(S)$ and $V(G_2) \cup V(S)$, respectively. By the induction hypothesis G', G'' have a pendent vertex or pendent C_4 or a pendent pyramid. Hence the claim.

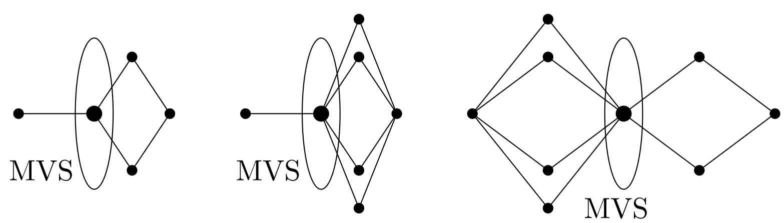


Fig 3. occurences in the graph with $|S| = 1$

Case 2: $|S| = 2$,(**Fig 2**)
The G has $|S| = 2$ if and only if the G is a pyramid of $n \geq 6$ vertices and there does not exist a minimal vertex separator of size one. So, by the definition G has a pendent pyramid. □

Theorem 1. *A graph $G(X, Y, E)$ with $\Delta(Y) \leq 2$ is Chordal Bipartite, if and only if is constructed using following rules:*

(i) A single vertex is a restricted chordal bipartite.

(ii) A C_4 is a restricted chordal bipartite.

(iii) If G is given restricted chordal bipartite, the graph G', where $V(G') = V(G) \cup \{v\}$, $E(G') = E(G) \cup \{x, v\}$ such that $x \in X$ or deg(x) = 1 if $x \in Y$.

(iv) If G is given restricted chordal bipartite, the graph G', where $V(G') = V(G) \cup \{v\}$, $E(G') = E(G) \cup \Big\{\{x_1,v\},\{x_2,v\}\Big\}$ such that $x_1,x_2 \in X$ and $N(x_1) \cap N(x_2) \neq \emptyset$.

Proof. **Sufficiency:** Let G' be a restricted chordal bipartite graph constructed using rules (i) to (iv). We shall prove it by mathematical induction on number of iterations needed to construct G'.

Basis step: G' is restricted chordal bipartite graph if G' = a single vertex or $G' = C_4$.

Hypothesis: Assume that the theorem is true for all graphs G constructed from G' by applying rules (i) to (iv) iteratively with number of iterations being $n \geq 1$.

Induction step: Let G' be obtained by rules (i) to (iv), $n \geq 1$ times iteratively. Our claim is to prove that G' is a restricted chordal bipartite graph.
Case 1: G' is obtained by rule (iii).
$V(G') = V(G) \cup \{v\}$, $E(G') = E(G) \cup \{x, v\}$ such that $x \in X$ or deg(x) = 1 if $x \in Y$ in G. By the induction hypothesis G is a restricted chordal bipartite and newly added edge $\{x, v\}$ does not create any cycle and is not violating the degree constraint on Y. Thus G' is a restricted chordal bipartite .

Case 2: G' is obtained by rule (iv).
$V(G') = V(G) \cup \{v\}$, $E(G') = E(G) \cup \Big\{\{x_1,v\},\{x_2,v\}\Big\}$ such that $x_1,x_2 \in X$ and $\mathrm{N}(x_1) \cap \mathrm{N}(x_2)$ $\neq \emptyset$. By the induction hypothesis G is a restricted chordal bipartite and the newly added vertex does not induce a cycle other than C_4, the degree constrained on Y remain satisfied since $v \in Y$ and deg$(v) = 2$. Thus G' is a restricted chordal bipartite .

Necessity: Given that G is a restricted chordal bipartite graph. By *Lemma* 3, G has a pendent vertex or pendent C_4 or a pendent pyramid, and we denote them using the label x_1 . Consider the graph $G - x_1$ obtained from G by removing the label x_1, i.e., remove a pendant vertex or pendent C_4 or a pendent pyramid. Since these restricted chordal bipartite graphs follow hereditary property, $G - x_1$ contains a label x_2 which is a pendant vertex or pendent C_4 or a pendent pyramid. Repeat the previous step by removing the label x_2 . Clearly, in at most n iterations we can get an ordering among labels which we call as *vertex cycle ordering(VCO)*. Clearly, the reverse of VCO gives the construction of the underlying restricted chordal bipartite graph. This completes the necessity. ****image**

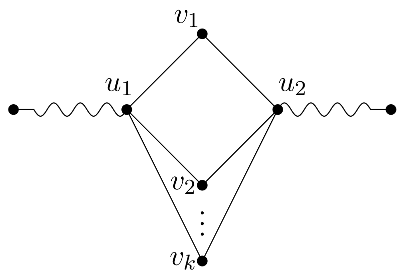
□

1.3 Minimum Dominating Set

For a graph $G(V, E)$ minimum dominating set $D \subseteq V$ is minimum set of vertices such that every vertex $v \in V$ is in D or adjacent to a vertex $u \in D$.

$\hookrightarrow$ NP-complete in general graphs, polynomial time solvable in trees.

Claim 1. *Let $G(X, Y, E)$ be the restricted chordal bipartite graph. For any pyramid$(u_1, u_2, v_1, v_2, ..., v_k)$ in G with $u_1, u_2 \in X$ and $v_1, v_2, ..., v_k \in Y$, only X vertices(i.e, u_1 or u_2 or both) will be present in the minimum dominating set D of G. for $k = 2$, pyramid becomes a C_4.*



Proof. Let us prove it using case by case analysis:

Case 1: u_1, u_2 are dominated by their adjacent vertices(other than $v_1, v_2, ..., v_k$). Now to dominate $v_1, v_2, ..., v_k$ either u_1 or u_2 is sufficient.

Case 2: Only u_1 is dominated by it's adjacent vertex(other than $v_1, v_2, ..., v_k$). Now to dominate $u_2, v_1, v_2, ..., v_k$, only one vertex i.e, u_2 is sufficient.

Case 3: Both u_1, u_2 are not dominated by it's adjacent vertices(other than $v_1, v_2, ..., v_k$). Now to dominate $u_1, u_2, v_1, v_2, ..., v_k$ we require both u_1, u_2 in the dominating set. Hence the claim. □

Claim 2. *For a restricted chordal bipartite graph G, the minimum dominating set D will not be altered by replacing each pyramid structure with a C_4.*

Proof. From Claim 1, we can say that all vertices in a pyramid can be dominated with X vertices alone irrespective of number of Y vertices in the pyramid. So without loss of generality, for finding minimum dominating set we can replace each pyramid with a C_4. □

Definition. *We define a **Leaf Book** H with corner vertices $\{u, v\}$ as follows.*

$$V(H) = \{u, v\} \cup \{u_1, u_2, ..., u_k\} \cup \{w_1, w_2, ..., w_l\} \; k \geq 2, l \geq 1.$$
$$E(H) = \Big\{\{u, u_i\}, \{v, u_i\} \mid 1 \leq i \leq k\Big\} \cup \Big\{\{v, w_j\} \mid 1 \leq j \leq l\Big\}$$

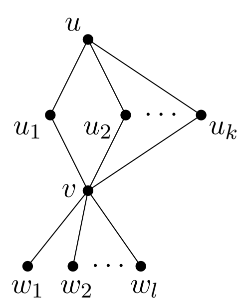


Fig 4. Leaf Book

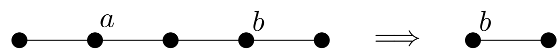
Lemma 4. *The restricted chordal bipartite graph G which is having a pendent P_3 or a pendent star or a pendent leaf book, has a minimum dominating set D of size $|D|$ if and only if $G - N[v]^s$ has a minimum dominating set of size $|D| - 1$, where v is the middle vertex for the pendent P_3 or parent for the pendent star(i.e, $deg(v) = k + 1$ and v has k pendent vertices in G) or parent of the pendent vertex in pendent leaf book.*

For pendent P_3 and pendent star, $N[v]^s = \left\{ \begin{array}{cc} N[v] & \text{if } deg(parent(v)) = 2 \\ N[v] - parent(v) & \text{if } deg(parent(v)) \geq 3 \end{array} \right\}$

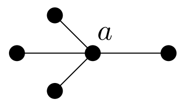
For pendent Leaf book, $N[v]^s = N[v]$

Proof. **Necessity:** We shall prove it by mathematical induction on number of vertices, n.

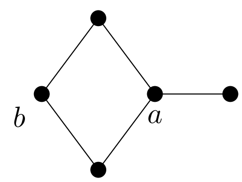
Base case: For $n = 5$,
(i) G having a pendent P_3, $D = \{a, b\}$, $|D| = 2$. for $G - N[a]^s$, $D = \{b\}$, $|D| = 1$.



(ii) G has a pendent star, $D = \{a\}$, $|D| = 1$. for $G - N[a]^s$, $D = \emptyset$, $|D| = 0$.



(iii) G has pendent leaf book, $D = \{a, b\}$, $|D| = 2$ for $G - N[a]^s$, $D = \{b\}$, $|D| = 1$.



Hypothesis: Assume that the lemma is true for all restricted chordal bipartite graphs with $n \geq 5$.

Induction step: we shall prove it is true for restricted chordal bipartite graphs $G(X, Y, E)$ with $n \geq 6$. Let D be the dominating set for G. From lemma 3, we can say that G contains a pendent vertex or a pendent C_4 or a pendent pyramid. Considering the pendent vertices,
Case 1: If pendent vertex $u \in X$, then we can say that there exists a pendent $P_3(u, v, w)$ and, since $v \in Y$ $deg(v) \leq 2$. To dominate $u, w(if deg(w) = 2)$ we need v in D irrespective of the remaining structure of the graph. The $G - N[v]^s$ does not contain $u, v, w(if deg(w) = 2)$ in the graph, which reduces the dominating set D size from $|D|$ to $|D| - 1$.

Case 2: If the pendent vertex $u \in Y$, then a pendent P_3 or pendent star(Fig 5) is possible. If there is a pendent P_3 then it is similar to the case 1.
subCase 2.1: If it is a pendent star i.e v has k pendent vertices $u_1, u_2, ..., u_k$ and all of them are dominated by v. The $G - N[v]^s$ does not contain $v, u_1, u_2, ..., u_k$ in the graph, which reduces minimum dominating set D size from $|D|$ to $|D| - 1$.

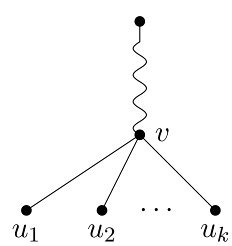


Fig 5. pendent star structure

subCase 2.2: If it is a pendent leaf book(Fig 6), v has $u_1, u_2, ..., u_k, w_1, w_2, ..., w_l$ as neighbours and all of them are dominated by v. The $G - N[v]^s$ does not contain $v, u_1, u_2, ..., u_k, w_1, w_2, ..., w_l$ in the graph, which reduces minimum dominating set D size from $|D|$ to $|D| - 1$.

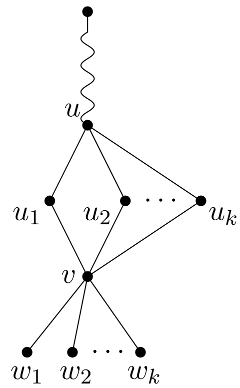


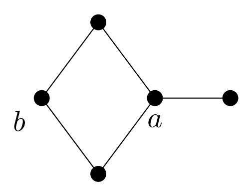
Fig 6. Pendent Leaf Book

Sufficiency: Let us consider a restricted chordal bipartite G with minimum dominating set D. Let us define $G - N[v]^s$ as G' which has minimum dominating set D' of size $|D| - 1$. where v is the middle vertex for the pendent P_3 or parent for the pendent star or parent of the pendent vertex in pendent leaf book.Now we can construct G from $G' \cup N[v]^s$ using the rules of Theorem 1. Now the minimum dominating set size of $G' \cup N[v]^s$ is $|D'| + 1$ which is equivalent to $|D| - 1 + 1$ i.e, $|D|$. $\square$

Lemma 5. *The restricted chordal bipartite graph $G(X, Y, E)$ which is having pendent C_4 has a minimum dominating set D of size $|D|$ if and only if $G - N[v]$ has a minimum dominating set of size $|D| - 1$, where $v \in X \cap C_4$, $deg(v) = 2$.*

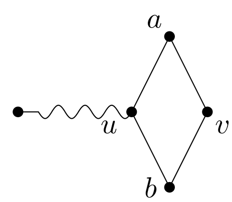
Proof. **Necessity:** We shall prove it by mathematical induction on number of vertices, n.

*Base case:*For $n = 5$, G has pendent C_4, $D = \{a, b\}$, $|D| = 2$ for $G - N[b]$, $D = \big\{a\big\}$, $|D| = 1$.



Hypothesis: Assume that the lemma is true for all restricted chordal bipartite graphs with $n \geq 5$.

Induction step: we shall prove it is true for restricted chordal bipartite graphs $G(X, Y, E)$ with $n \geq 6$. Let D be the dominating set for G. From lemma 3, we can say that G contains a pendent vertex or a pendent C_4 or a pendent pyramid. Let us consider the pendent C_4(fig below). Based on the *Claim* 1(Case 2 & 3 and Case 1 is not possible in our discussion) we can say that $v \in D$. The $G - N[v]$ does not contain u, a, b which reduces the size of minimum dominating set size from $|D|$ to $|D| - 1$.

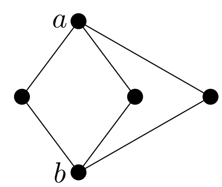


Sufficiency: Let us consider a restricted chordal bipartite G with minimum dominating set D. Let us define $G - N[v]$ as G' which has minimum dominating set D' of size $|D| - 1$. where $v \in X \cap C_4$, $deg(v) = 2$.Now we can construct G from $G' \cup N[v]$ using the rules of Theorem 1. Now the minimum dominating set size of $G' \cup N[v]$ is $|D'| + 1$ which is equivalent to $|D| - 1 + 1$ i.e, $|D|$. $\square$

Lemma 6. *The restricted chordal bipartite graph $G(X, Y, E)$ which is having pendent pyramid has a minimum dominating set D of size $|D|$ if and only if $G - N[v]$ has a minimum dominating set of size $|D| - 1$, and $\forall x \in N(v)$, $x \in pyramid$.*

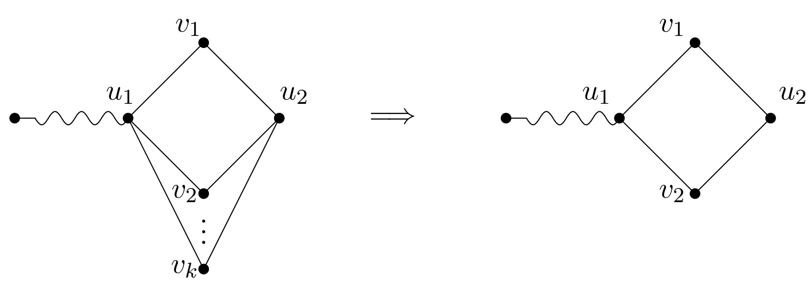
Proof. **Necessity:** We shall prove it by mathematical induction on number of vertices, n.

*Base case:*For $n = 5$, G has pendent C_4, $D = \{a, b\}$, $|D| = 2$ for $G - N[b]$, $D = \{a\}$, $|D| = 1$.



Hypothesis: Assume that the lemma is true for all restricted chordal bipartite graphs with $n \geq 5$.

Induction step: we shall prove it is true for restricted chordal bipartite graphs $G(X, Y, E)$ with $n \geq 6$. Let D be the dominating set for G. From lemma 3, we can say that G contains a pendent vertex or a pendent C_4 or a pendent pyramid. Let us consider the pendent pyramid(fig below). Based on Claim 2 we can replace each pyramid with a C_4. Now our argument is similar to Lemma 5 establishes the proof for our statement.



Sufficiency: Let us consider a restricted chordal bipartite G with minimum dominating set D. Let us define $G - N[v]$ as G' which has minimum dominating set D' of size $|D| - 1$. where $v \in X \cap C_4$, $deg(v) = 2$.Now we can construct G from $G' \cup N[v]$ using the rules of Theorem 1. Now the minimum dominating set size of $G' \cup N[v]$ is $|D'| + 1$ which is equivalent to $|D| - 1 + 1$ i.e, $|D|$. $\square$

1.3.1 ALGORITHM for MDS

Based on above conclusions, We are presenting a greedy algorithm for finding minimum dominating set for the restricted chordal bipartite.

Algorithm: Finding Minimum Dominating Set
I/P: A restricted chordal bipartite graph $G(X, Y, E)$.
O/P: Minimum dominating set D for the given graph G.

(i) Find all pyramid structures(if any) and replace each with a C_4.

(ii) Take a vertex r with $deg(r) > 2$ as root and construct the tree like structure.

(iii) Run post order traversal on the tree like structure.

(iv) If there is a pendent vertex u then,

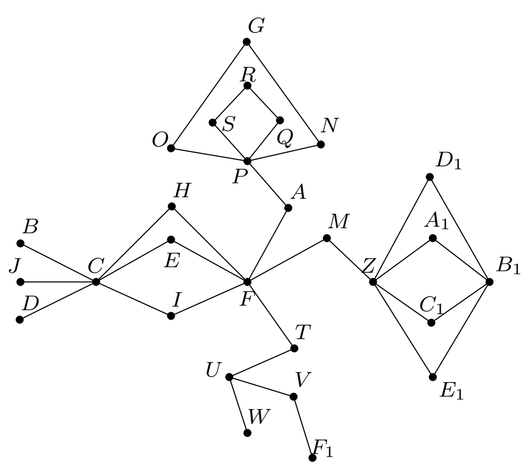
  (a) If $V = \{u\}$ and u is not dominated then $D = D \cup \{u\}$ and $G = G - N[u]$, else $G = G - N[u]$.
  (b) If $V = \{u, v\}$,$E = \{u, v\}$ and u, v are not dominated then $D = D \cup \{u\}$ and $G = G - N[u]$, else $G = G - N[u]$.
  (c) If $u \in$ Pendent $P_3(u, v, w)$, then $D = D \cup \{v\}$ and $G = G - \{u, v, w(\text{if } deg(w) \leq 2)\}$.
  (d) If $u \in$ Pendent star where v is the parent of the pendent star, then $D = D \cup \{v\}$ and $G = G - N[v]^s$(as defined in Lemma 4).
  (e) If $u \in$ Pendent leaf book where v is $parent(u)$, then $D = D \cup \{v\}$ and $G = G - N[v]$.

(v) If there is a pendent $C_4(t, u, v, w)$ such that $t, v \in X$, $u, w \in Y$ and $deg(v) = 2$, then $D = D \cup \{v\}$ and $G = G - N[v]$.

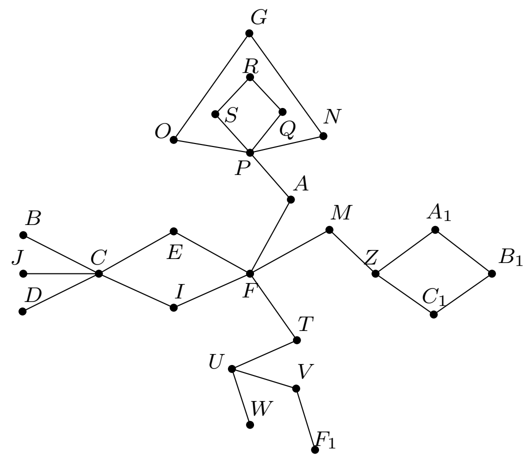
(vi) repeat the steps $(iv), (v)$ until the $X \cup Y$ is empty.

1.3.2 Example:

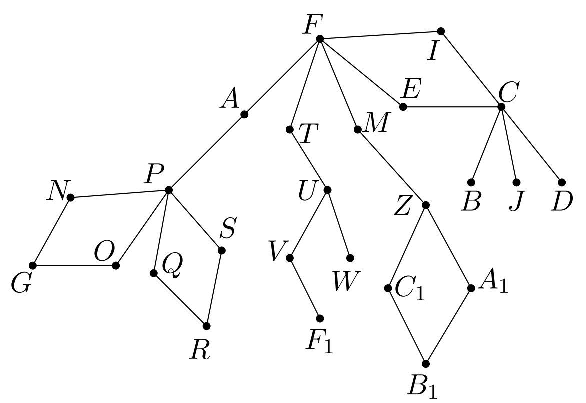
Find the minimum dominating set for the given restricted chordal bipartite graph.



Step 1: Replace each pyramid with a C_4.

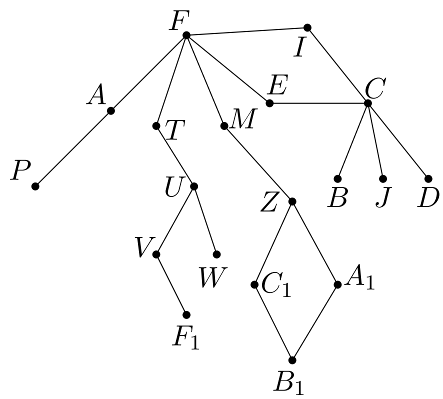


Step 2: Take F as a root vertex and construct tree like structure.
<u>Tree like structure construction:</u> For a vertex v having children $C_1, C_2, ..., C_k$ are arranged from left to right such that $depth(C_1) \geq depth(C_2) \geq ... \geq depth(C_k)$.

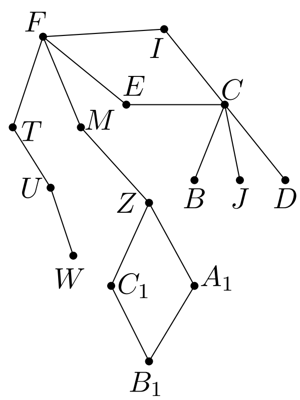


Step 3: Post order traversal
$\Big\{C_4(P,N,G,O), C_4(P,Q,R,S), P, A, F_1, V, W, U, T, C_4(Z, A_1, B_1, C_1), M, B, J, D,$
$C_4(F,E,I,C), F\Big\}$
Minimum Dominating Set $D = \emptyset$

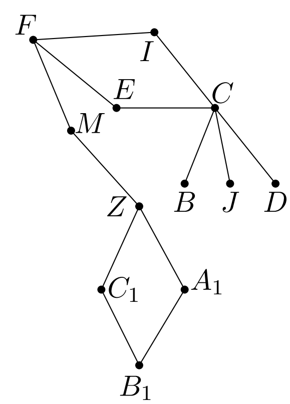
Step 4: (a) consider $C_4(P,N,G,O), C_4(P,Q,R,S)$ which follows (v).
So $D = \{G, R\}$ and $G = G - \{N, G, O, Q, R, S\}$



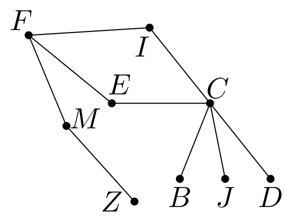
(b) Consider $P_3(P, A, F), P_3(F_1, V, U)$ which follows $[(iv).c]$.
So $D = \{G, R, A, V\}$ and $G = G - \{P, A, F_1, V\}$



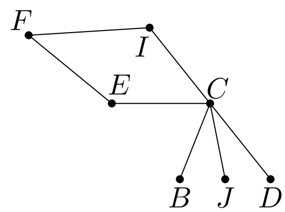
(c) Consider $P_3(W, U, T)$ which follows $[(iv).c]$
So $D = \{G, R, A, V, U\}$ and $G = G - \{W, U, T\}$



(d) Consider $C_4(Z, A_1, B_1, C_1)$ which follows (v)
So $D = \{G, R, A, V, U, B_1\}$ and $G = G - \{A_1, B_1, C_1\}$



(e) Consider $P_3(Z, M, F)$ which follows $[(iv).c]$
So $D = \{G, R, A, V, U, M\}$ and $G = G - \{Z, M\}$



(f) Consider Pendent leaf book(F, E, I, C, B, J, D) which follows $[(iv).e]$
So $D = \{G, R, A, V, U, M, C\}$ and $G = G - \{E, I, C, B, J, D\}$

(g) Consider Pendent vertex(F) which follows $[(iv).a]$
So $D = \{G, R, A, V, U, M, C\}$ and $G = G - \{F\}$

1.3.3 Analysis

1. To find all pyramid structures, for each pair $\{u, v\} \in X$ find $N[u] \cap N[v]$. If $|N[u] \cap N[v]| > 2$ then there exists a pyramid between u, v. Converting a pyramid into C_4 is simply deleting few common vertices of u, v and it takes $\mathcal{O}(n)$ time, where $n = |X \cup Y|$.
Time complexity: $\mathcal{O}(n^2(\triangle + \mathcal{O}(n))$.

2. To build the tree like structure, we need to find the depth of each child with respect to it's parent. First we traverse through the tree and when we find the leaf gadget(pendent vertex or pendent C_4) then we will update the depth of leaf gadgets as 2 for C_4 and 1 for vertex. while we back track, For an internal vertex v having children $c_1, .., c_k$, depth(v) = 1 + max$\{depth(c_1), ..., depth(c_k)\}$.
Time complexity: $\mathcal{O}(n)$.

3. For post order traversal,
Time complexity: $\mathcal{O}(n + m)$, where $m = |E|$.

4. Finally traversing through the tree like structure and updating minimum dominating set takes $\mathcal{O}(n)$.

So, the time complexity of the algorithm is $\mathcal{O}(n^2\triangle) \approx \mathcal{O}(n^3)$.

To be continued...