An Elementary Treatise on Fourier's Series and Spherical, Cylindrical, and Ellipsoidal Harmonics, with Applications to Problems in Mathematical Physics

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PREFACE

About ten years ago I gave a course of lectures on Trigonometric Series, following closely the treatment of that subject in Riemann's "Partielle Differentialgleichungen," to accompany a short course on The Potential Function, given by Professor B. O. Peirce.

My course has been gradually modified and extended until it has become an introduction to Spherical Harmonics and Bessel's and Lamé's Functions.

Two years ago my lecture notes were lithographed by my class for their own use and were found so convenient that I have prepared them for publication, hoping that they may prove useful to others as well as to my own students. Meanwhile, Professor Peirce has published his lectures on "The Newtonian Potential Function" (Boston, Ginn & Co.), and the two sets of lectures form a course (Math. 10) given regularly at Harvard, and intended as a partial introduction to modern Mathematical Physics.

Students taking this course are supposed to be familiar with so much of the infinitesimal calculus as is contained in my "Differential Calculus" (Boston, Ginn & Co.) and my "Integral Calculus" (second edition, same publishers), to which I refer in the present book as "Dif. Cal." and "Int. Cal." Here, as in the "Calculus," I speak of a "derivative" rather than a "differential coefficient," and use the notation D_x instead of $\frac{\partial}{\partial x}$ for "partial derivative with respect to x."

The course was at first, as I have said, an exposition of Riemann's "Partielle Differentialgleichungen." In extending it, I drew largely from Ferrer's "Spherical Harmonics" and Heine's "Kugelfunctionen." and was somewhat indebted to Todhunter ("Functions of Laplace, Bessel, and Lamé"), Lord Rayleigh ("Theory of Sound"), and Forsyth ("Differential Equations").

In preparing the notes for publication, I have been greatly aided by the criticisms and suggestions of my colleagues, Professor B. O. Peirce and Dr. Maxime Bôcher, and the latter has kindly contributed the brief historical sketch contained in Chapter IX.

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ANALYTICAL TABLE OF CONTENTS

Chapter I
INTRODUCTION

1. In many important problems in mathematical physics we are obliged to deal with *partial differential equations* of a comparatively simple form.

For example, in the Analytical Theory of Heat we have for the change of temperature of any solid due to the flow of heat within the solid, the equation

$$D_t u = a^2(D_x^2 u + D_y^2 u + D_z^2 u),^{1} \tag{I}$$

where u represents the temperature at any point of the solid and t the time.

In the simplest case, that of a slab of infinite extent with parallel plane faces, where the temperature can be regarded as a function of one coordinate, (I) reduces to

$$D_t u = a^2 D_x^2 u, \tag{II}$$

a form of considerable importance in the consideration of the problem of the cooling of the earth's crust.

In the problem of the permanent state of temperatures in a thin rectangular plate, the equation (I) becomes

$$D_x^2 + D_y^2 = 0. \tag{III}$$

In *polar* or *spherical coordinates* (I) is less simple, it is

$$D_t u = \frac{a^2}{r^2}\left[D_r(r^2 D_r u) + \frac{1}{\sin\theta} D_\theta(\sin\theta D_\theta u) + \frac{1}{\sin^2\theta} D_\phi^2 u\right]. \tag{IV}$$

In the case where the solid in question is a sphere and the temperature at any point depends merely on the distance of the point from the centre IV reduces to

$$D_t(ru) = a^2 D_r^2(ru). \tag{V}$$

In *cylindrical coordinates* (I) becomes

$$D_t u = a^2[D_r^2 u + \frac{1}{r} D_r u + \frac{1}{r^2} D_\phi^2 u + D_z^2 u]. \tag{VI}$$

[1] For the sake of brevity we shall often use the symbol ∇^2 for the operation $D_x^2 + D_y^2 + D_z^2$; and with this notation equation I would be written $D_t u = a^2 \nabla^2 u$.

In considering the flow of heat in a cylinder when the temperature at any point depends merely on the distance r of the point from the axis (VI) becomes

$$D_t u = a^2(D_r^2 u + \frac{1}{r} D_r u). \tag{VII}$$

In Acoustics in several problems we have the equation

$$D_t^2 y = a^2 D_x^2 y; \tag{VIII}$$

for instance, in considering the transverse or the longitudinal vibrations of a stretched elastic string, or the transmission of plane sound waves through the air.

If in considering the transverse vibrations of a stretched string we take account of the resistance of the air (VIII) is replaced by

$$D_t^2 y + 2k D_t y = a^2 D_x^2 y. \tag{IX}$$

In dealing with the vibrations of a stretched elastic membrane, we have the equation

$$D_t^2 z = c^2(D_x^2 z + D_y^2 z), \tag{X}$$

or in *cylindrical coordinates*

$$D_t^2 z = c^2(D_r^2 z + \frac{1}{r} D_r z + \frac{1}{r^2} D_\phi^2 z). \tag{XI}$$

In the theory of *Potential* we constantly meet Laplace's Equation

$$D_x^2 V + D_y^2 V + D_z^2 V = 0 \tag{XII}$$

or

$$\nabla^2 V = 0$$

which in *spherical coordinates* becomes

$$\frac{1}{r^2}\left[r D_r^2(rV) + \frac{1}{\sin\theta} D_\theta(\sin\theta D_\theta V) + \frac{1}{\sin^2\theta} D_\phi^2 V = 0\right], \tag{XIII}$$

and in *cylindrical coordinates*

$$D_r^2 V + \frac{1}{r} D_r V + \frac{1}{r^2} D_\phi^2 V + D_z^2 V = 0. \tag{XIV}$$

In *curvilinear coordinates* it is

$$h_1 h_2 h_3 \left[D_{\rho_1}\left(\frac{h_1}{h_2 h_3} D_{\rho_1} V\right) + D_{\rho_2}\left(\frac{h_2}{h_3 h_1} D_{\rho_2} V\right) + D_{\rho_3}\left(\frac{h_3}{h_1 h_2} D_{\rho_3} V\right)\right] = 0; \tag{XV}$$

where

$$f_1(x, y, z) = \rho_1,\ f_2(x, y, z) = \rho_2,\ f_3(x, y, z) = \rho_3$$

represent a set of surfaces which cut one another at right angles, no matter what values are given to $\rho_1,\ \rho_2$, and ρ_3; and where

$$\begin{aligned} h_1^2 &= (D_x\rho_1)^2 + (D_y\rho_1)^2 + (D_z\rho_1)^2 \\ h_2^2 &= (D_x\rho_2)^2 + (D_y\rho_2)^2 + (D_z\rho_2)^2 \\ h_3^2 &= (D_x\rho_3)^2 + (D_y\rho_3)^2 + (D_z\rho_3)^2, \end{aligned}$$

and, of course, must be expressed in terms of $\rho_1,\ \rho_2$, and ρ_3.

If if happens that $\nabla^2\rho_1 = 0, \nabla^2\rho_2 = 0$, and $\nabla^2\rho_3 = 0$, then Laplace's Equation (XV) assumes the very simple form

$$h_1^2 D_{\rho_1}^2 V + h_2^2 D_{\rho_2}^2 V + h_3^2 D_{\rho_3}^2 V = 0. \tag{XVI}$$

2. A *differential equation* is an equation containing derivatives or differentials with or without the primitive variables from which they are derived.

The *general solution* of a differential equation is the equation expressing the most general relation between the primitive variables which is consistent with the given differential equation and which does not involved differentials or derivatives. A general solution will always contain arbitrary (*i.e.*, undetermined) *constants* or *arbitrary functions.*

A *particular solution* of a differential equation is a relation between the primitive variables which is consistent with the given differential equation, but which is less general than the general solution, although included in it.

Theoretically, every particular solution can be obtained from the general solution by substituting in the general solution particular values for the arbitrary constants or particular functions for the arbitrary functions; but in practice it is often easy to obtain particular solutions directly from the differential equation when it would be difficult or impossible to obtain the general solution.

3. If a problem requiring for its solution the solving of a differential equation is *determinate*, there must always be given in addition to the differential equation enough outside conditions for the determination of all the arbitrary constants or arbitrary functions that enter into the general solution of the equation; and in dealing with such a problem, if the differential equation can be readily solved the natural method of procedure is to obtain its general solution, and then to determine the constants or functions by the aid of the given conditions.

It often happens, however, that the general solution of the differential equation in question cannot be obtained, and then, since hte problem *if determinate* will be solved if by any means a solution of the equation can be found which will also satisfy the given outside conditions, it is worth while to try to get *particular solutions* and so to combine them as to form a result which shall satisfy the given conditions without ceasing to satisfy the differential equation.

4. A differential equation is *linear* when it would be of the first degree if the dependent variable and all its derivatives were regarded as algebraic unknown quantities. If it is linear and contains no term which does not involve the dependent variable or one of its derivatives, it is said to be linear and *homogeneous.*

All the differential equations collected in Art. 1 are linear and homogeneous.

5. *if a value of the dependent variable has been found which satisfies a given homogeneous, linear, differential equation, the product formed by multiplying this value by any constant will also be a value of the dependent variable which will satisfy the equation.*

For if all the terms of the given equation are transposed to the first member, the substitution of the first-named value must reduce that member to zero; substituting the second value is equivalent to multiplying each term of the result of the first substitution by the same constant factor, which therefore may be taken out as a factor of the whole first number. The remaining factor being zero, the product is zero and the equation is satisfied.

If several values of the dependent variable have been found each of which satisfies the given differential equation, their sum will satisfy the equation; for it the sum of the values in question is substituted in the equation each term of the sum will give rise to a set of terms which must be equal to zero, and therefore the sum of these sets must be zero.

6. It is generally possible to get by some simple device *particular solutions* of such differential equations as those we have collected in Art. 1. The object of the branch of mathematics with which we are about to deal is to find methods of so combining these particular solutions as to satisfy any given conditions which are consistent with the nature of the problem in question.

This often requires us to be able to develop any given functions of the variables which enter into the expression of these conditions in terms of *normal forms* suited to the problem with which we happen to be dealing, and suggested by the form of particular solution that we are able to obtain for the differential equation.

These normal forms are frequently sines and cosines, but they are often much more complicated functions known as *Legendre's Coefficients,* or *Zonal Harmonics; Laplace's Coefficients,* or *Spherical Harmonics: Bessel's Functions,* or *Cylindrical Harmonics; Lamé's Functions,* or *Ellipsoidal Harmonics*, &c.

7. As an illustration, let us take Fourier's problem of the permanent state of temperatures in a thin rectangular plate of breadth π and of infinite length whose faces are impervious to heat. We shall suppose that the two long edges of the plate are kept at the constant temperature zero, that one of the short edges, which we shall call the base of the plate, is kept at the temperature unity, and that the temperatures of points in the plate decrease indefinitely as we recede from the base; we shall attempt to find the temperature at any point of the plate.

Let us take the base as the axis of X and one end of the base as the origin. Then to solve the problem we are to find the temperature u of any point from the equation

$$D_x^2 u + D_y^2 u = 0 \qquad \text{(III Art. 1)}$$

subject to the conditions

$$\begin{aligned} u &= 0 \quad \text{when} \quad x = 0 && (1)\\ u &= 0 \quad \text{"} \quad x = \pi && (2)\\ u &= 0 \quad \text{"} \quad y = \infty && (3)\\ u &= 1 \quad \text{"} \quad y = 0. && (4) \end{aligned}$$

We shall begin by getting a particular solution of (III Art. 1), and we shall use a device which always succeeds when the equation is *linear* and *homogeneous* and has *constant coefficients.*

Assume[2] $u = e^{\alpha y + \beta x}$, where α and β are constants, substitute in (III Art. 1) and divide by $e^{\alpha y + \beta x}$, and we have $\alpha^2 + \beta^2 = 0$. If, then, this condition is satisfied $u = e^{\alpha y + \beta x}$ is a solution.

Hence $u = e^{\alpha y \pm \beta x i 3}$ is a solution of (III Art. 1), no matter what value may be given to α.

This form is objectionable, since it involves an imaginary. We can, however, readily improve it.

Take $u = e^{\alpha y} e^{\alpha x i}$, a solution of (III Art. 1), and $u = e^{\alpha y} e^{-\alpha x i}$, another solution of (III Art. 1); add these values of u and divide the sum by 2 and we have $a^{\alpha y} \cos \alpha x$. (v. Int. Cal. Art. 35, [1].) Therefore by Art. 5

$$u = e^{\alpha y} \cos \alpha x \qquad (5)$$

is a solution of (III Art. 1). Take $u = e^{\alpha y} e^{\alpha x i}$ and $u = e^{\alpha y} e^{-\alpha x i}$, subtract the second value of u from the first and divide by $2i$ and we have $e^{\alpha y} \sin \alpha x$. (v. Int. Cal. Art. 35, [2]). Therefore by Art. 5

$$u = e^{\alpha y} \sin \alpha x \qquad (6)$$

is a solution of (III Art. 1).

Let us now see if out of these particular solutions we can build up a solution which will satisfy the conditions (1), (2), (3), and (4).

Consider $$u = e^{\alpha y} \sin \alpha x. \qquad (6)$$
It is zero when $x = 0$ for all values of α. It is zero when $x = \pi$ if α is a whole number. It is zero when $y = \infty$ is negative. If, then, we write u equal to a sum of terms of the form

[2] This assumption mst be regarded as purely tentative. It must be tested by substituting in the equation, and is justified if it leads to a solution.

[3] We shall regularly use the symbol i for $\sqrt{-1}$.

$Ae^{-my}\sin mx$, where m is a positive integer, we shall have a solution of (III Art. 1) which satisfies conditions (1), (2), and (3). Let this solution be

$$u = A_1 e^{-y}\sin x + A_2 e^{-2y}\sin 2x + A_3 e^{-3y}\sin 3x + A_4 e^{-4y}\sin 4x + \dots \tag{7}$$

$A_1,\ A_2,\ A_3,\ A_4$, &c., being undetermined constants.

When $y = 0$ (7) reduces to

$$u = A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + A_4 \sin 4x + \dots . \tag{8}$$

If now it is possible to develop unity into a series of the form (8), our problem is solved; we have only to substitute the coefficients of that series for $A_1,\ A_2,\ A_3$. and &c. in (7).

It will be proved later that

$$1 = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \dots\right)$$

for all values of x between 0 and π; hence our required solution is

$$u = \frac{4}{\pi}\left[e^{-y}\sin x + \frac{1}{3}e^{-3y}\sin 3x + \frac{1}{5}e^{-5y}\sin 5x + \frac{1}{7}e^{-7y}\sin 7x + \dots\right] \tag{9}$$

for this satisfies the differential equation and all the given conditions.

If the given temperature of the base of the plate instead of being unity is a function of x, we can solve the problem as before if we can express the given function of x as a sum of terms of the form $A\sin mx$, where m is a whole number.

The problem of finding the value of the *potential function* at any point of a long, thin, rectangular conducting sheet, of breadth π, through which an electric current is flowing, when the two long edges are kept at potential zero, and one short edge at potential unity, is mathematically identical with the problem we have just solved.

Example.

Taking the temperature of the base of the plate described above as 100°centigrade, and that of the sides of the plate 0°, compute the temperatures of the points

$$(a)\ (\frac{\pi}{6}, 1);\ (b)\ (\frac{\pi}{3}, 2);\ (c)\ (\frac{\pi}{2}, 3),$$

correct to the nearest degree. *Ans. (a) 26°;(b) 15°; (c) 6°.*

8. As another illustration, we shall take the problem of the transverse vibrations of a stretched string fastened at the ends, initially distorted into some given curve and then allowed to swing.

Let the length of the string be l. Take the position of equilibrium of the string as the axis of X, and one of the ends as the origin, and suppose the string initially distorted into a curve whose equation $y = f(x)$ is given.

We have then to find an expression for y which will be a solution of the equation

$$D_t^2 y = a^2 D_x^2 y \qquad \text{(VIII Art. 1,)}$$

while satisfying the conditions

$$\begin{aligned} y &= 0 \quad \text{when} \quad x = 0 && (1)\\ y &= 0 \quad \text{“} \quad x = l && (2)\\ y &= f(x) \quad \text{“} \quad t = 0 && (3)\\ D_t y &= 0 \quad \text{“} \quad t = 0, && (4)\end{aligned}$$

the last condition meaning merely that the string starts from rest.

As in the last problem let[4] $y = e^{\alpha x + \beta t}$ and substitute in (VIII Art. 1,). Divide by ${}^{\alpha x+\beta t}$ and we have $\beta^2 = a^2\alpha^2$ as the condition that our assumed value of y shall satisfy the equation.

$$y = e^{\alpha x \pm a\alpha t} \qquad (5)$$

is, then, a solution of (VIII Art. 1,) whatever the value of α.

It is more convenient to have a trigonometric than an exponential form to deal with, and we can readily obtain one by using an imaginary value for α in (5). Replace α by $-\alpha i$ and (5) becomes $y = e^{-(x \pm at)\alpha i}$, another solution of (VIII Art. 1,). Add these values of y and divide by 2 and we have $\cos\alpha(x \pm at)$. Subtract the second value of y from the first and divide by $2i$ and we have $\sin\alpha(x \pm at)$.

$$\begin{aligned} y &= \cos\alpha(x + at)\\ y &= \cos\alpha(x - at)\\ y &= \sin\alpha(x + at)\\ y &= \sin\alpha(x - at)\end{aligned}$$

and, then, solutions of (VIII Art. 1,). Writing y successivly equal to half the sum of the first pair of values, helf their difference, half the sum of the last pair of values, and half their difference, we get the very convenient particular solutions of (VIII Art. 1,).

$$\begin{aligned} y &= \cos\alpha x \cos\alpha at\\ y &= \sin\alpha x \sin\alpha at\\ y &= \sin\alpha x \cos\alpha at\\ y &= \cos\alpha x \sin\alpha at.\end{aligned}$$

If we take the third form

$$y = \sin\alpha x \cos\alpha at$$

[4] See note on page 5

it will satisfy conditions (1) and (4), no matter what value may be given to α, and it will satisfy (2) if $\alpha = \frac{m\pi}{l}$ where m is an integer.

If then we take

$$y = A_1 \sin \frac{\pi x}{l} \cos \frac{\pi a t}{l} + A_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi a t}{l} + A_3 \sin \frac{3\pi x}{l} \cos \frac{3\pi a t}{l} + \dots \tag{6}$$

where A_1, A_2, A_3 ... are undetermined constsants, we shall have a solution of (VIII Art. 1,) which satisfies (1), (2), and (4). When $t = 0$ it reduces to

$$y = A_1 \sin \frac{\pi x}{l} + A_2 \sin \frac{2\pi x}{l} + A_3 \sin \frac{3\pi x}{l} + \dots \tag{7}$$

If now it is possible to develop $f(x)$ into a series of the form (7), we can solve our problem completely. We have only to take the coefficients of this series as values of A_1, A_2, A_3 ... in (6), and we shall have a solution of (VIII Art. 1,) which satisfies all our given conditions.

In each of the preceding problems the *normal function*, in terms of which a given function has to be expressed, is the sine of a simple multiple of the variable. It would be easy to modify the problem so that the *normal form* shold be a cosine.

We shall now take a couple of problems which are more complicated and where the normal function is an unfamiliar one.

9. Let it be required to find the potential function due to a circular wire ring of small cross section and of given radius c, supposing the matter of the ring to attract according to the law of nature.

We can readily find, by direct integration, the value of the potential function at any point of the axis of the ring. We get for it

$$V = \frac{M}{\sqrt{c^2 + x^2}} \tag{1}$$

where M is the mass of the ring, and x the distance of the point from the centre of the ring.

Let us use spherical coordinates, taking the centre of the ring as origin and the axis of the ring as the polar axis.

To obtain the value of the potential function at any point in space, we must satisfy the equation

$$rD_r^2(rV) + \frac{1}{\sin\theta} D_\theta(\sin\theta D_\theta V) + \frac{1}{\sin^2\theta} D_\phi^2 V = 0, \qquad \text{(XIII Art. 1)}$$

subject to the condition

$$V = \frac{M}{(c^2 + r^2)^{\frac{1}{2}}} \quad \text{when} \quad \theta = 0. \tag{1}$$

From the symmetry of the ring, it is clear that the value of the potential function must be independent of ϕ, so that (XIII Art. 1) will reduce to

$$rD_r^2(rV) + \frac{q}{\sin\theta} D_\theta(\sin\theta D_\theta V) = 0. \tag{2}$$

We must now try to get particular solutions of (2), and as the coefficients are not constant, we are driven to a new device.

Let[5] $V = r^m P$, where P is a function of θ only, and m is a positive integer, and substitute in (II), which becomes

$$m(m+1)r^m P + \frac{r^m}{\sin\theta} D_\theta(\sin\theta D_\theta P) = 0.$$

Divide by r^m and use the notation of ordinary derivatives since P depends upon θ only, and we have the equation

$$m(m+1)P + \frac{1}{\sin\theta}\frac{d\left(\sin\theta\frac{dP}{d\theta}\right)}{d\theta} = 0, \tag{3}$$

from which to obtain P.

Equation (3) can be simplified by changing the independent variable. Let $x = \cos\theta$ and (3) becomes

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + m(m+1)P = 0. \tag{4}$$

Assume[6] now that P can be expressed as a sum or a series of terms involving whole powers of x multiplied by constant coefficients.

Let $P = \sum a_n x^n$ and substitute this value of P in (4). We get

$$\sum[n(n-1)a_n x^{n-2} - n(n+1)a_n x^n + m(m+1)a_n x^n] = 0, \tag{5}$$

where the symbol $\sum$ indicates that we are to form all the terms we can by taking successive whole numbers for n.

As (5) must be true no matter what the value of x, the coefficient of any given power of x, as for instance x^k, must vanish. Hence

$$(k+2)(k+1)a_{k+2} - k(k+1)a_k + m(m+1)a_k = 0 \tag{6}$$

and

$$a_{k+2} = -\frac{m(m+1) - k(k+1)}{(k+1)(k+2)}a_k. \tag{7}$$

If now any set of coefficients satisfying the relation (7) be taken, $P = \sum a_k x^k$ will be a solution of (4).

If $$k = m, \quad a_{k+2} = 0, \quad a_{k+4} = 0, \quad \&c.$$

Since it will answer our purpose if we pick out the simplest set of coefficients that will obey the condition (7), we can take a set including a_m.

Let us rewrite (7) in the form

$$a_k = -\frac{(k+2)(k+1)}{(m-k)(m+k+1)}a_{k+2}. \tag{8}$$

[5]See note on page 5.

[6]See note on page 5

We get from (8), beginning with $k = m - 2$,

$$\begin{aligned}
a_{m-2} &= -\frac{m(m-1)}{2\cdot(2m-1)}a_{k+2}.\\
a_{m-4} &= -\frac{m(m-1)(m-2)(m-3)}{2\cdot 4\cdot(2m-1)(2m-3)}a_m\\
a_{m-6} &= -\frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{2\cdot 4\cdot 6\cdot(2m-1)(2m-3)(2m-5)}a_m, \quad \&\text{c.}
\end{aligned}$$

If m is even we see that the set will end with a_0, if m is odd, with a_1.

$$P = a_m\left[x^m - \frac{m(m-1)}{2\cdot(2m-1)}x^{m-2} + \frac{m(m-1)(m-2)(m-3)}{2\cdot 4\cdot(2m-1)(2m-3)}x^{m-4} - \dots\right]$$

where a_m is entirely arbitrary, is, then, a solution of (4). It is found convenient to take a_m equal to

$$\frac{(2m-1)(2m-3)\dots 1}{m!}$$

and it can be shown that with this value of a_m $P = 1$ when $x = 1$.

P is a function of x and contains no higher powers of x than x^m. It is usual to write it as $P_m(x)$.

We proceed to compute a few values of $P_m(x)$ from the formula

$$\begin{aligned}
P_m(x) = \frac{(2m-1)(2m-3)\dots 1}{m!}&\left[x^m - \frac{m(m-1)}{2\cdot(2m-1)}x^{m-2}\right.\\
&\left.+\frac{m(m-1)(m-2)(m-3)}{2\cdot 4\cdot(2m-1)(2m-3)}x^{m-4} - \dots\right].
\end{aligned} \tag{9}$$

We have:

$$\begin{array}{lll}
P_0(x) = 1 & \text{or} & P_0(\cos\theta) = 1\\
P_1(x) = x & \text{“} & P_1(\cos\theta) = \cos\theta\\
P_2(x) = \frac{1}{2}(3x^2-1) & \text{“} & P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)\\
P_3(x) = \frac{1}{2}(5x^3-3x) & \text{“} & P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)\\
P_4(x) = \frac{1}{8}(35x^4-30x^2+3) & \text{or} & P_4(\cos\theta) = \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3)\\
P_5(x) = \frac{1}{8}(63x^5-70x^3+15x) & \text{or} & P_5(\cos\theta) = \frac{1}{8}(63\cos^5\theta - 70\cos^3\theta + 15\cos\theta).
\end{array} \tag{10}$$

We have obtained $P = P_m(x)$ as a particular solution of (4) and $P = P_m(\cos\theta)$ as a particular solution of (3). $P_m(x)$ or $P_m(\cos\theta)$ is a new function, known as a *Legendre's Coefficient*, or as a *Surface Zonal Harmonic*, and occurs as a normal form in many important problems.

$V = r^m P_m(\cos\theta)$ is a particular solution of (2) and $r^m P_m(\cos\theta)$ is sometimes called a *Solid Zonal Harmonic.*

We can now proceed to the solution of our original problem.

$$V = A_0 r^0 P_0(\cos\theta) + A_1 r^1 P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta) + A_3 r^3 P_3(\cos\theta) + \dots \qquad (11)$$

where $A_0,\ A_1,\ A_2$, &c., are entirely arbitrary, is a solution of (2) (v. Art. 5). When $\theta = 0$ (11) reduces to

$$V = A_0 + A_1 r + A_2 r^2 + A_3 r^3 + \dots,$$

since, as we have said, $P_m(x) = 1$ when $x = 1$, or $P_m(\cos\theta) = 1$ when $\theta = 0$.

By our condition (1)

$$V = \frac{M}{(c^2 + r^2)^{\frac{1}{2}}}$$

when $\theta = 0$.

By the Binomial Theorem

$$\frac{M}{(c^2 + r^2)^{\frac{1}{2}}} = \frac{M}{c}\left[1 - \frac{1}{2}\frac{r^2}{c^2} + \frac{1\cdot 3}{2\cdot 4}\frac{r^4}{c^4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{r^6}{c^6} + \dots\right]$$

provided $r < c$. Hence

$$V \quad = \quad \frac{M}{c}\left[P_\theta(\cos\theta) - \frac{1}{2}\frac{r^2}{c^2}P_2(\cos\theta) + \frac{1\cdot 3}{2\cdot 4}\frac{r^4}{c^4}P_4(\cos\theta) - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{r^6}{c^6}P_6(\cos\theta) + \dots\right] \quad (12)$$

is our required solution if $r < c$; for it is a solution of equation (2) and satisfies condition (1).

Example.

Taking the mass of the ring as one pound and the radius of the ring as one foot, compute to two decimal places the value of the potential function due to the ring at the points

(a) $(r = .2,\ \theta = 0)$; (d) $(r = .6,\ \theta = 0)$; (f) $(r = .6,\ \theta = \frac{\pi}{3})$;
(b) $(r = .2,\ \theta = \frac{\pi}{4})$; (e) $(r = .6,\ \theta = \frac{\pi}{6})$; (g) $(r = .6,\ \theta = \frac{\pi}{2})$;
(c) $(r = .2,\ \theta = \frac{\pi}{2})$;

Ans. (a) .98; *(b)* .99; *(c)* 1.01; *(d)* .86; *(e)* .90; *(f)* 1.00; *(g)* 1.10.
The unit used is the potential due to a pound of mass concentrated at a point and attracting a second pound of mass concentrated at a point, the two points being a foot apart.

10. A slightly different problem calling for development in terms of Zonal Harmonics is the following:

Required the permanent temperatures within a solid sphere of radius 1, one half of the surface being kept at the constant temperature zero, and the other half at the constant temperature unity.

Let us take the diameter perpendicular to the plane separating the unequally heated surfaces as our axis and let use use spherical coordinates. As in he last problem, we must solve the equation

$$rD_r^2(ru) + \frac{1}{\sin\theta}D_\theta(\sin\theta D_\theta u) + \frac{1}{\sin^2\theta}D_\phi^2 u = 0 \qquad \text{(XIII Art. 1)}$$

which as before reduces to

$$rD_r^2(ru) + \frac{1}{\sin\theta}D_\theta(\sin\theta D_\theta u) = 0 \tag{1}$$

from the consideration that the temperatures must be independent of ϕ.

Our equation of condition is

$$u = 1 \text{ from } \theta = 0 \text{ to } \theta = \frac{\pi}{2} \text{ and } u = 0 \text{ from } \theta = \frac{\pi}{2} \text{ to } \theta = \pi, \tag{2}$$

when $r = 1$.

As we have seen $u = r^m P_m(\cos\theta)$ is a particular solution of (1), m being any positive whole number, and

$$u = A_0 r^0 P_0(\cos\theta) + A_1 r^1 P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta) + A_3 r^3 P_3(\cos\theta) + \dots \tag{3}$$

where A_0, A_1, A_2, A_3 ... are undetermined constants, is a solution of (1).

When $r = 1$ (3) reduces to

$$u = A_0 P_0(\cos\theta) + A_1 P_1(\cos\theta) + A_2 P_2(\cos\theta) + A_3 P_3(\cos\theta) + \dots \tag{4}$$

If then we can develop our function of θ which enters into equation (2) in a series of the form (4), we have only to take the coefficients of that series as the values of A_0, A_1, A_2, &c., in (3) and we shall have our required solution.

11. As a last example we shall take the problem of the vibration of a stretched circular membrane fastened at the circumference, that is, of an ordinary drumhead. We shall suppose the membrane initially distorted into any given form which has circular symmetry[7] about an axis through the centre perpendicular to the plane of the boundary, and then allowed to vibrate.

[7]A function of the coordinates of a point has *circular symmetry* about an axis when its value is not affected by rotating the point through any angle about the axis. A surface has circular symmetry about an axis when it is a surface of revolution about the axis.

Here we have to solve

$$D_t^2 z = c^2 \left(D_r^2 z + \frac{1}{r} D_r z + \frac{1}{r^2} d_\phi^2 z \right) \qquad \text{([XI] Art. 1)}$$

subject to the conditions

$$\begin{aligned} z = f(r) &\quad \text{when} \quad t = 0 && (1) \\ D_t z = 0 &\quad \text{“} \quad t = 0 && (2) \\ z = 0 &\quad \text{“} \quad r = a. && (3) \end{aligned}$$

From the symmetry of the supposed initial distortion z must be independent of ϕ, therefore [XI] reduces to

$$D_t^2 z = c^2 \left(D_r^2 z + \frac{1}{r} D_r z \right) \tag{4}$$

and this is the equation for which we wish to find a particular solution.

We shall employ a device not unlike that used in Art. 9.

Assume[8] $z = R.T$ where R is a function of r alone and T is a function of t alone. Substitute this value of z in (4) and we get

$$R D_t^2 T = c^2 T \left(D_r^2 R + \frac{1}{r} D_r R \right)$$

or

$$\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \frac{1}{R} \left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right). \tag{5}$$

The second member of (5) does not involve t, therefore its equal the first member must be independent of t. The first member of (5) does not involve r, and consequently since it contains neither t not r, it must be constant. Let it equal $-\mu^2$, where μ of course is an undetermined constant.

Then (5) breaks up into the two differential equations

$$\frac{d^2 T}{dt^2} + \mu^2 c^2 T = 0 \tag{6}$$

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \mu^2 R = 0 \tag{7}$$

(6) can be solved by familiar methods, and we get $t = \cos \mu ct$ and $t = \sin \mu ct$ as simple particular solutions (v. Int. Cal. p. 319, § 21).

To solve (7) is not so easy. We shall first simplify it by a change of independent variable. Let $r = \frac{x}{\mu}$. (7) becomes

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + R = 0. \tag{8}$$

[8]See note on page 5

Assume, as in Art. 9, that R can be expressed in terms of whole powers of x. Let $R = \sum a_n x^n$ and substitute in (8). We get

$$\sum[n(n-1)a_n x^{n-2} + na_n x^{n-2} + a_n x^n] = 0,$$

an equation which must be true no matter what the value of x. The coefficient of any given power of x, as x^{k-2}, must, then, vanish, and

$$k(k-1)a_k + ka_k + a_{k-2} = 0$$

or

$$k^2 a_k + a_{k-2} = 0$$

whence we obtain

$$a_{k-2} = -k^2 a_k \tag{9}$$

as the only relation that need be satisfied by the coefficients in order that $R = \sum a_k x^k$ shall be a solution of (8).

If

$$k = 0, \quad a_{k-2} = 0, \quad a_{k-4} = 0, \quad \&c.$$

We can then begin with $k = 0$ as our lowest subscript.

From (9)

$$a_k = -\frac{a_k - 2}{k^2}.$$

Then

$$\begin{aligned} a_2 &= -\frac{a_0}{2^2} \\ a_4 &= \frac{a_0}{2^2 \cdot 4^2} \\ a_6 &= -\frac{a_0}{2^2 \cdot 4^2 \cdot 6^2}, \text{ \& c.} \end{aligned}$$

Hence

$$R = a_0 \left[1 - \frac{x^2}{x} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots\right]$$

where a_0 may be taken at pleasure, is a solution of (8), provided the series is convergent.