Econometrics I

Law of Large Numbers

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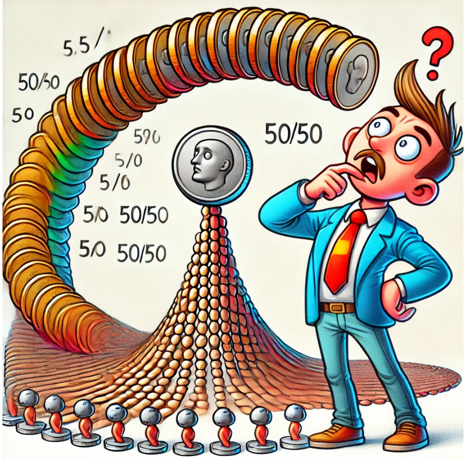
2024

ISET

Why Do Averages Become Reliable with Large Samples?

- Ever wondered why flipping a coin many times results in a nearly perfect 50/50 split?
- In this lecture, we'll explore the Weak Law of Large Numbers (WLLN) and see how larger samples yield more reliable averages.
- To get a deeper understanding, we will use R for demonstrations. The Strong Law of Large Numbers (SLLN) will be covered in the next lecture.



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Note: When observations are both independent and identically distributed, we refer to them as **iid random variables** or a **random sample**.

Introduction

- Let $\{X_n\}$ be a sequence of random variables and $\overline{X}_n$ be the sample mean of the first n terms of the sequence:

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

- A **Law of Large Numbers** is a proposition stating a set of conditions that are sufficient to guarantee the convergence of the **sample mean** to the **population mean**, as the sample size n increases. It is called:
  - a **Weak** Law of Large Numbers (WLLN) if the sequence $\{\overline{X}_n\}$ converges in probability;
  - a **Strong** Law of Large Numbers (SLLN) if the sequence $\{\overline{X}_n\}$ converges almost surely.

Introduction (cont.)

- The Weak Law of Large Numbers (WLLN) involves convergence in probability, while the Strong Law (SLLN) requires almost sure convergence.
- These concepts were introduced in your statistics course.
- But a quick review can be beneficial.
- Today we will only cover convergence in probability.

Convergence in Probability - Intuition

- Two random variables are "close to each other" if there is a high probability that their difference is very small.
- Let $\{X_n\}$ be a sequence of random variables defined on a sample space. Let X be a random variable and ϵ a strictly positive number. Consider the probability:

$$P(|X_n - X| > \epsilon)$$

- X_n is considered far from X when $|X_n - X| > \epsilon$, so this probability measures how likely it is that X_n is far from X.
- If $\{X_n\}$ converges to X, then $P(|X_n - X| > \epsilon)$ should decrease as n increases.

Convergence in Probability - Formal Definition

- A sequence of random variables, $X_1, X_2, \ldots$, converges in probability to a random variable X if and only if:

$$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0 \quad \text{for any } \epsilon > 0.$$

- Here, X is called the probability limit of the sequence, and convergence is indicated by:

$$X_n \xrightarrow{P} X \quad \text{or by} \quad \operatorname{plim} X_n = X \quad \text{as } n \to \infty.$$

Convergence in Probability - Example

- Let X be a discrete random variable with support $R_X = \{0, 1\}$ and probability mass function:

$$p_X(x) = \begin{cases} \frac{1}{3}, & \text{if } x = 1, \\ \frac{2}{3}, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

- Consider a sequence of random variables $\{X_n\}$ whose generic term is:

$$X_n = \left(1 + \frac{1}{n}\right) X$$

- Does $\{X_n\}$ converge in probability to X?

Instruction: Work with the classmate on your left for 3 minutes to figure out the solution.

Convergence in Probability - Example (cont.)

- Take any $\epsilon > 0$. Note that:

$$|X_n - X| = \left(1 + \frac{1}{n}\right) X - X = \frac{1}{n} X$$

- Consider the following cases:
  - **Case 1:** When $X = 0$, which happens with $\frac{2}{3}$ probability:

$$|X_n - X| = \frac{1}{n} \times 0 = 0 \quad \text{so} \quad |X_n - X| \leq \epsilon$$

  - **Case 2:** When $X = 1$, which happens with $\frac{1}{3}$ probability:

$$|X_n - X| = \frac{1}{n} \times 1 = \frac{1}{n}$$

- $|X_n - X| \leq \epsilon$ if and only if $\frac{1}{n} \leq \epsilon$ (i.e., $n \geq \frac{1}{\epsilon}$).

Convergence in Probability - Example (cont.)

- Therefore:

$$P(|X_n - X| \leq \epsilon) = \begin{cases} \frac{2}{3}, & \text{if } n < \frac{1}{\epsilon} \\ 1, & \text{if } n \geq \frac{1}{\epsilon} \end{cases}$$

- And:

$$P(|X_n - X| > \epsilon) = \begin{cases} \frac{1}{3}, & \text{if } n < \frac{1}{\epsilon} \\ 0, & \text{if } n \geq \frac{1}{\epsilon} \end{cases}$$

- Thus, $P(|X_n - X| > \epsilon)$ converges to 0 as n increases.

$$\lim_{n \to \infty} P(|X_n - X| > \epsilon) = 0 \quad \text{for any } \epsilon > 0.$$

WLLN - Theorem

Let $X_1, X_2, \ldots$ be iid random variables with $\mathbb{E}[X_i] = \mu$ and Var$(X_i) = \sigma^2 < \infty$.

Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(|\bar{X}_n - \mu| < \epsilon\right) = 1,$$

that is, $\bar{X}_n$ converges in probability to μ.

- The WLLN asserts that, under general conditions, the sample mean converges to the population mean as $n \to \infty$.
- More general versions of the WLLN require only that the mean is finite.

WLLN - Proof

- The proof is straightforward and uses Chebyshev's Inequality. For every $\epsilon > 0$:

$$P\left(|\bar{X}_n - \mu| \geq \epsilon\right) = P\left((\bar{X}_n - \mu)^2 \geq \epsilon^2\right) \leq \frac{\mathrm{Var}(\bar{X}_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$$

Thus:

$$P\left(|\bar{X}_n - \mu| \geq \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2}$$

- Hence, the probability that $\bar{X}_n$ deviates from μ by more than ϵ approaches 0 as $n \to \infty$:

$$P\left(|\bar{X}_n - \mu| < \epsilon\right) = 1 - P\left(|\bar{X}_n - \mu| \geq \epsilon\right) \geq 1 - \frac{\sigma^2}{n\epsilon^2} \to 1 \text{ as } n \to \infty.$$

Demonstration of the Law of Large Numbers

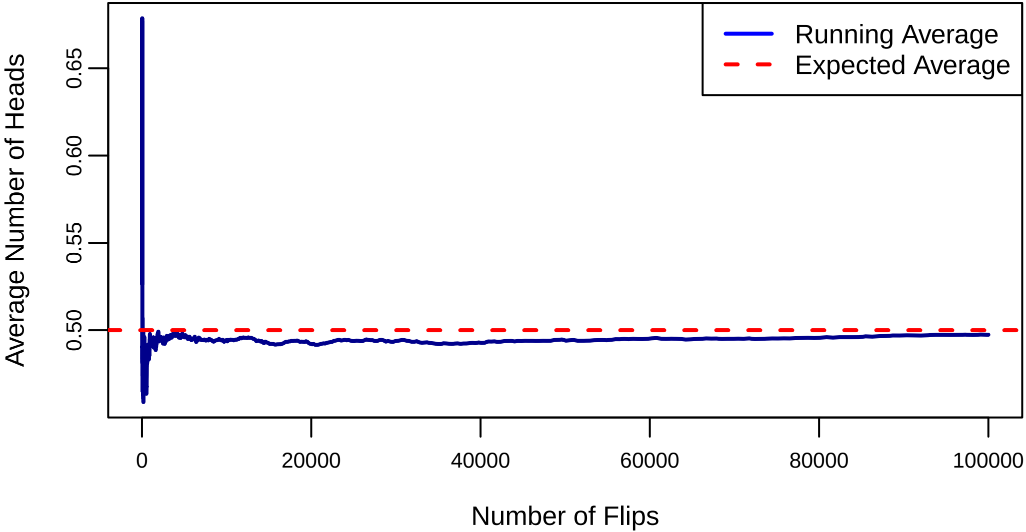
- Ever wondered why flipping a coin many times results in a nearly perfect 50/50 split?

- **Warning:** Do not attempt the strategy shown in the picture—unless you're really, really into flipping coins!

R Code for Coin Flip Simulation

```r
# Set the seed for reproducibility
set.seed(123)
# Set the maximum number of flips
max_flips <- 100000
# Generate a sequence of flip sizes
flip_sizes <- unique(round(exp(seq(log(10),
log(max_flips), length.out = 1000))))
# Simulate the coin flips
flips <- rbinom(max_flips, 1, 0.5)
# Initialize a vector to store the running averages
running_avg <- numeric(length(flip_sizes))
# Calculate the running averages for each flip size
for (i in seq_along(flip_sizes)) {n <- flip_sizes[i]
  running_avg[i] <- mean(flips[1:n])
}
```

Results of the Coin Flip Simulation



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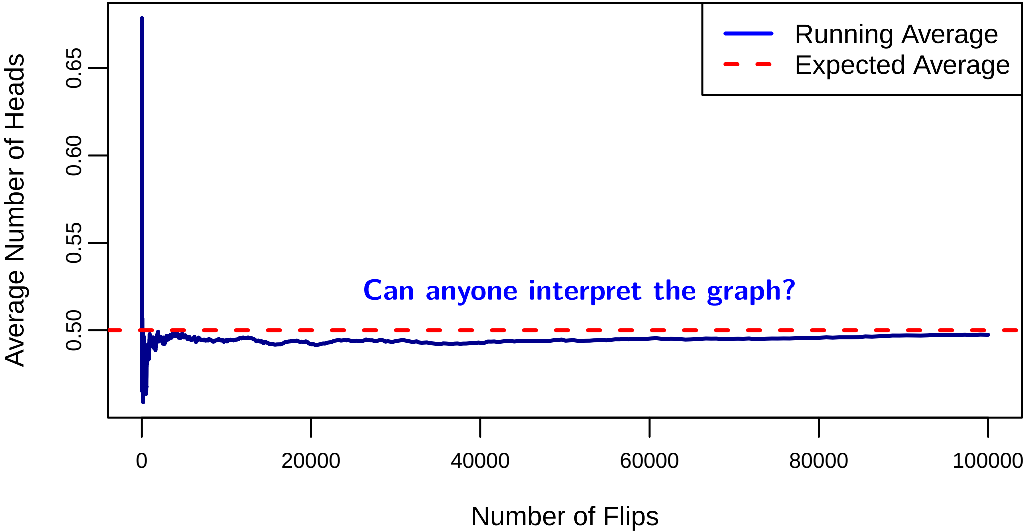

Homework: Law of Large Numbers Self-Exploration

- Download the Jupyter notebook **"HW_law_of_large_numbers.ipynb"** from our course's GitHub repository.
- Work through the notebook, answer the questions, and submit your completed assignment by the start of next week's class.

Takeaway

- The **Law of Large Numbers (LLN)** explains why averages tend to stabilize as the sample size increases.
- **Weak Law of Large Numbers (WLLN)** ensures that the sample mean converges to the population mean in probability.
- Practical demonstration: As the number of coin flips increases, the proportion of heads approaches 50%.
- Key takeaway: Larger samples provide more reliable estimates of population parameters, highlighting the importance of sample size in statistical analysis.

Note: Next Week - The Strong Law of Large Numbers.