EE2211 Tutorial 4

(Systems of Linear Equations)
Question 1:

Given $\mathbf{X}\mathbf{w} = \mathbf{y}$ where $\mathbf{X} = \begin{bmatrix}1 & 1\\ 3 & 4\end{bmatrix}$, $\mathbf{y} = \begin{bmatrix}0\\ 1\end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)
(b) Is $\mathbf{X}$ invertible? Why? → Yes, $\det(X) \neq 0$
(c) Solve for $\mathbf{w}$ if it is solvable.

c) $X = \begin{bmatrix}1 & 1\\ 3 & 4\end{bmatrix}$ $\quad C = \begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$

$\mathrm{adj}(X) = C^T = \begin{bmatrix}4 & -1\\ -3 & 1\end{bmatrix}$

$\det(X) = ad - bc$
$= 1$

$X^{-1} = \dfrac{\mathrm{adj}(X)}{\det(X)} = \begin{bmatrix}4 & -1\\ -3 & 1\end{bmatrix}$

$w = X^{-1}y$

$= \begin{bmatrix}4 & -1\\ -3 & 1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}-1\\ 1\end{bmatrix}$

(Systems of Linear Equations)
Question 2:

Given $\mathbf{X}\mathbf{w} = \mathbf{y}$ where $\mathbf{X} = \begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}$, $\mathbf{y} = \begin{bmatrix}0\\ 1\end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)
(b) Is $\mathbf{X}$ invertible? Why? → no, $\det(X) = 0$
(c) Solve for $\mathbf{w}$ if it is solvable. → cannot solve

(Systems of Linear Equations)
Question 3:

tall matrix (more eqtns than unknowns, hence over-determined)

Given $\mathbf{X}\mathbf{w} = \mathbf{y}$ where $\mathbf{X} = \begin{bmatrix}1 & 2\\ 2 & 4\\ 1 & -1\end{bmatrix}$, $\mathbf{y} = \begin{bmatrix}0\\ 0.1\\ 1\end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)
(b) Is $\mathbf{X}$ invertible? Why? → no, not square (even)
(c) Solve for $\mathbf{w}$ if it is solvable. → left inverse $(X^TX)^{-1}X^Ty$
must be invertible

c) $(X^TX)^{-1}X^T$

$= \left(\begin{bmatrix}1 & 2 & 1\\ 2 & 4 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 4\\ 1 & -1\end{bmatrix}\right)^{-1} X^T$

$= \underbrace{\begin{bmatrix}6 & 9\\ 9 & 21\end{bmatrix}}_{A}^{-1} \begin{bmatrix}1 & 2 & 1\\ 2 & 4 & -1\end{bmatrix}$

$C = \begin{bmatrix}21 & -9\\ -9 & 6\end{bmatrix}$ $\quad \mathrm{adj}(X) = C_T = \begin{bmatrix}21 & -9\\ -9 & 6\end{bmatrix}$

$\det(A) = ad - bc = 45$

$X^{-1} = \dfrac{\mathrm{adj}(A)}{\det(A)} = \begin{bmatrix}21/45 & -9/45\\ -9/45 & 6/45\end{bmatrix}$

$\underset{2\times2}{\begin{bmatrix}21/45 & -9/45\\ -9/45 & 6/45\end{bmatrix}} \underset{2\times3}{\begin{bmatrix}1 & 2 & 1\\ 2 & 4 & -1\end{bmatrix}} \underset{3\times1}{\begin{bmatrix}0\\ 0.1\\ 1\end{bmatrix}}$

$= \begin{bmatrix}0.68\\ -0.32\end{bmatrix}$

(Systems of Linear Equations)
Question 4:

wide matrix (more unknowns than eqtns)

Given $\mathbf{X}\mathbf{w} = \mathbf{y}$ where $\mathbf{X} = \begin{bmatrix}1 & 0 & 1 & 0\\ 1 & -1 & 1 & -1\\ 1 & 1 & 0 & 0\end{bmatrix}$, $\mathbf{y} = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)
(b) Is $\mathbf{X}$ invertible? Why? → no, not square (even)
(c) Solve for $\mathbf{w}$ if it is solvable. → right inverse
$X^T(XX^T)^{-1}y$ must be invertible

$= \begin{pmatrix}0.5\\ 0.5\\ 0.5\\ 0.5\end{pmatrix}$

(Systems of Linear Equations)
Question 5:

$(AB)^T = B^T A^T$

$(w^T X)^T = X^T w \quad (y^T)^T = y \qquad X^T = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

$X^T w = y \qquad \det(X^T) = 0$

Given $\mathbf{w}^T\mathbf{X} = \mathbf{y}^T$ where $\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)
(b) Is $\mathbf{X}$ invertible? Why? → no. $\det(X) = 0$
(c) Solve for $\mathbf{w}$ if it is solvable. → not solvable

(Systems of Linear Equations)
Question 6:

Given $\mathbf{w}^T\mathbf{X} = \mathbf{y}^T$ where

$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$(AB)^T = B^T A^T$

$(w^T X)^T = (y^T)^T$

bring to this form $X^T w = y \Rightarrow X^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix}$ → wide matrix (more unknowns then eqns) (under-determined)

$[Xw = y]$

(a) What kind of system is this? (even-, over- or under-determined?)
(b) Is $\mathbf{X}$ invertible? Why? → no, not square
(c) Solve for $\mathbf{w}$ if it is solvable. → right inverse

$X^T(XX^T)^{-1}y = \begin{bmatrix} 0.067 \\ 0.133 \\ -0.333 \end{bmatrix}$

(Systems of Linear Equations)
Question 7:
This question is related to determination of types of system where an appropriate solution can be found subsequently. The following matrix has a left inverse.

$\mathbf{X} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

→ $(X^T X)^{-1} X^T$
→ invertible, $\det \neq 0$

$\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \overbrace{\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{A}$

a) True
b) False ✓

$\det(A) = 4 \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$

$= 4 \, |0 - 0|$

$= 0$

∴ Since $X^T X$ is not invertible, X does not have a left inverse

(Systems of Linear Equations)
Question 8:
MCQ: Which of the following is/are true about matrix $\mathbf{A}$ below? **There could be more than one answer.**

$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

~~a)~~ $\mathbf{A}$ is invertible
~~b)~~ $\mathbf{A}$ is left invertible → $(A^T A)^{-1} A^T$ → $\det \neq 0$
c) ✓ $\mathbf{A}$ is right invertible → $A^T (AA^T)^{-1}$ → $\det \neq 0$
~~d)~~ $\mathbf{A}$ has no determinant
~~e)~~ None of the above