Cosmology Project 1

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Abstract

In this project we explore the evolution of a homogeneous and isotropic Universe, in the standard cosmological model, Λ-CDM, also called the Benchmark model, to do this we need to solve the general Friedmann equation, taking into consideration the model and since it's general, we need to solve this Differential equation numerically. We also compute observable quantities in order to compare with observations. We use the cosmological parameters given by the Planck experiment [1].

1 Introduction

Cosmology is the study of the universe as a whole, which happens at distances beyond hundreds of MegaParsecs were it begins to look "smooth", since this is what we want, we have to come up with ideas that help us study this. Here is where we introduce the cosmological principle, which states that at great distances, the universe is spatially homogeneous and isotropic; by homogeneous we mean that the universe "looks" the same in every point of the universe and by isotropic we mean that it "looks" the same in every direction. The cosmological principle is important because it is where the theory of the Big Bang is based on; the Big Bang is the best description we have of our Universe. [2] And, just to remember, this model explains that the universe started in a state of high density and temperature, then an explosion occurred and the universe began to expand, it is still expanding to this day, we know this thanks to cosmological observations. If we want to see the visible universe to us, observable by the CMB (Cosmic Microwave Background) see figure 1. CMB is the electromagnetic radiation that the universe left from the early stages of it's evolution.

The cosmological and astronomical parameters and values we need are possible to obtain thanks to the implementation of full electromagnetic spectrum observations, by this we mean that we can observe the universe in radio waves, microwaves, infrared light, visible light, ultraviolet light, X-rays and gamma rays which has led us into a wider possibility of getting to know the universe. In cosmology, we can't even possibly takes measurements as we do in earth or even how astronomers do it, so what we have to do is measure parameters like the red shift, which is like the Doppler effect but in light waves. Since everything is receding from us, we can measure red shift from objects. Related to this, we have the Hubble law, which says that the velocity of recession is proportional to the distance of an object were the constant relating both of them is Ho or the Hubble constant that has a value of $68.63\frac{Km}{s \cdot MPc}$. Another important thing we

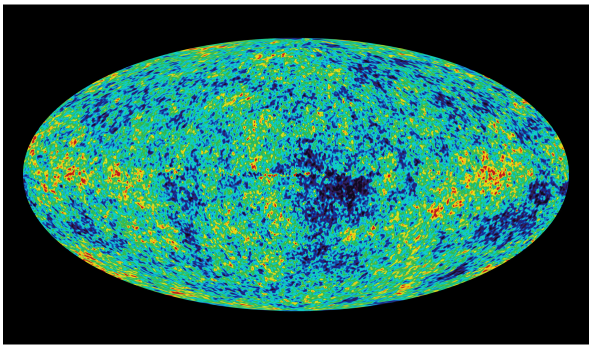

Figure 1: Cosmic Microwave Background. Observable Universe.

can observe from our universe is that when we see it as a whole, we can imagine our universe as a "fluid" and because of that we can measure the scale factor $a(t)$ thanks to the Friedmann equation, which we will discuss later on including how this equation relates to the universe but in a very summarized way, the universe has had principally three eras which include a radiation dominated era, a matter dominated era and a dark energy dominated era which we are currently on. This scale factor can help us identify characteristics or when the three eras occurred. Let´s also remember that the universe is currently $13.7 Gyears$.

From the universe we can get also some observational parameters, right know we will only mention the ones we have used, this includes the Hubble constant, from which we talked earlier. With this constant we can measure the expansion rate of the Universe and as we mentioned, this currently has a value of $68.63 \frac{Km}{s \cdot MPc}$. We can also measure the density parameters which helps us know the density of the universe. We have the density Ω_0 of the universe, we have the ϵ_m which is the energy density of matter in the universe, ϵ_r the energy density of radiation and ϵ_Λ which is the cosmological constant or the dark energy. We will use this densities in our project.

We are only using the Benchmark universe model, but its important that we mention some other of the models we can find of the universe. There is the empty universe, the single component universes (matter, radiation, lambda) and the ones with multiple components these are the matter + radiation model, matter + curvature, matter + lambda, matter + curvature + lambda. From these models, we can see that matter + lambda model universe that is spatially flat is a close approximation to our universe in the present day, so it's important we study it. Some of the characteristics from the Benchmark model is that it is a good fit to the currently available observational data, it is spatially flat, and contains radiation, matter, and a cosmological constant. Some important data from this model is that we have a total radiation of $\Omega_r, 0 = 9x10^{-5}$ to this day, which is the sum of photons and neutrinos density, we also have a $\Omega_{m,0} = 0.31$ of total matter, this is the sum of baryonic matter and nonbaryonic dark matter, $\Omega_{\Lambda,0} = 0.69$, but besides this densities of the components of the universe, we have the 3 important epochs that are the radiation - matter equality with an $a_m r = 2.9x10^-4$ in the year 0.050 Myr, matter - lambda equality whith a

= 0.77 in the year 10.2 Gyr and finally "now" with an a_0 of 1 and in the year 13.7Gyr (actuality). [3]

2 Methods

Lets start with the most important part of the project, the Friedmann equation for the Benchmark model. This equation is:

$$\dot{a}Ho^-1 = (\Omega_m a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2)^{1/2} \tag{1}$$

$$\frac{da}{dt}Ho^-1 = (\Omega_m a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2)^{1/2} \tag{2}$$

$$\frac{da}{\Omega_m a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2)^{1/2}} = Hodt \tag{3}$$

Where:
$\Omega_m = 0.3153$
$\Omega_r = 9x10^-5$
$\Omega_\Lambda = 0.6847$
$Ho = 67.36km/(s * Mpc)$

The Friedmann equation describes the expansion of the Universe, that makes it the most important equation in cosmology. As we can see, this a multiple component equation which can not be resolved analytically, we need to solve it numerically for us to be able to use it, we used the Euler method to solve this equation. Euler's method is the most elementary approximation technique for solving initial-value problems. [4] In this case we can use as a initial condition a(t) = 0 at time 0 (when the universe began) or we can use some of the epochs as conditions. In this case we used the a=0 at time = 0, where a goes from 0 to 1.
Then we have to use the equation:

$$H(t) = \frac{\dot{a}}{a} \tag{4}$$

So that we can obtain how H(t) behaves as the years go by. We can only obtain this when we solve the Friedman equation (1).
There were some problems with our solution, we had to obtain a function that we can use to obtain more parametrs of the universe. So with the values of a that we got, we obtained a Polynomial. To do this we used the Lagrange interpolation Polynomial method (Which I had forgotten we can do) [4]. This method consists of finding a single polynomial that goes through all the data points, giving us a great approximation of the function or graph we need. We need to be careful as to how we use this method and how many points we use for our approximation, in my case when I used 20 points for my approximation, it tends to infinity so I had to use only ten and since we defined a random number generator for our data, we had to be careful as to what numbers we use, we have to compile the code till we get a great approximation.
After we had all of this, we now have to check all of the species we have for the universe (matter, radiation and lambda). First lets start with matter, for this

case we have a w = 0, which varies for every species we use. We also have a time of:

$$t_0 = \frac{2}{3Ho} \tag{5}$$

And a scale factor of:

$$a(t) = (\frac{t}{t_0})^{2/3} \tag{6}$$

Now using our parameter of density (we will use this equation for every case) , we have:

$$\epsilon_i(a) = \epsilon_{i,0} a^{-3(1+\omega_i)} \tag{7}$$

and we now that w=0 for matter, so:

$$\epsilon_m(a) = \epsilon_{m,0} a^{-3} \tag{8}$$

If we want to check how this varies with z (red shift) we remember that:

$$a = \frac{1}{1+z} \tag{9}$$

And substituting this into equation 7 :

$$z = (\frac{\epsilon_m}{\epsilon_{m,0}})^{1/3} - 1 \tag{10}$$

Now with this information we can vary the energy density of matter.
For radiation's case we have a w = 1/3 and a time of:

$$t_0 = \frac{1}{2Ho} \tag{11}$$

with a scale factor of:

$$a(t) = (\frac{t}{t_0})^{1/2} \tag{12}$$

Doing the same things for the case of matter with equation 7, we have:

$$\epsilon_m(a) = \epsilon_{m,0} a^{-4} \tag{13}$$

Finally, to see how it varies along z:

$$z = (\frac{\epsilon_m}{\epsilon_{m,0}})^{1/4} - 1 \tag{14}$$

For the third case of Lambda or dark energy we have to use the Friedmann equation in a lambda only flat universe (flat because we are studying Benchmark universe), so we begin with:

$$\dot{a}^2 = \frac{8\pi G \epsilon_\Lambda}{3c^2} a^2 \tag{15}$$

Were Ho is equal to:

$$Ho = \frac{8\pi G \epsilon_\Lambda}{3c^2} \tag{16}$$

Obtaining after substitution and applying integrals:

$$\int \frac{da}{a} = \int Hodt \tag{17}$$

$$a(t) = e^{H_0(t-t_0)} \tag{18}$$

Now for the energy density of Lambda we have:

$$\epsilon_m(a) = \epsilon_{m,0}a^2 \tag{19}$$

For an w of $-\frac{1}{3}$ and substituting in equation 7:

$$\epsilon_m(a) = \epsilon_{m,0}a^2 \tag{20}$$

Finally for our relation with the red shift, we have:

$$z = (\frac{\epsilon_{m,0}}{\epsilon_m})^{1/2} - 1 \tag{21}$$

After we have done all of this and have found the relations and equations that relate the energy density and the scale factor with redshift and time, we can now obtain the luminosity and angular-diameter distance, related to the proper distance which we will also discuss.
We know that the proper distance can be found as:

$$dp = c\int_{te}^{to} \frac{dt}{a(t)} \tag{22}$$

Where the proper distance is the distance measured from the observer, we know that the proper distance can't be measured, so we can infer this with other things like the red shift.
As we can see, this is an integral we can't possibly solve analytically for the simple fact that we couldn't solve the Friedmann equation analytically, so the way we can solve this equation is with the Simpson composite method [4]. This method consists of approximating the area under a curve over a given interval that involves partitioning the interval by an odd number n + 1 of equally spaced ordinates and adding the areas of the n/2 figures formed by pairs of successive odd-numbered ordinates and the parabolas which they determine with their included even-numbered ordinates.[4].
After we have the proper distance, we can measure stuff like the Luminous distance, (remembering the Benchmark Model with curvature 0) we can define the Luminous distance (dL) as:

$$dL = dp(t_0)(1+z) \tag{23}$$

Also we can measure the angular distance, that is:

$$dA = \frac{dL}{(1+z)^2} \tag{24}$$

The angular distance is the angle between the two sightlines, or between two point objects as viewed from an observer in Earth. This can be viewed more clearly when we see an image of it. See figure 2.

Since my results for the proper distance seems to be wrong for the first model we are going to try to replicate (Benchmark), we will take the proper distance that is mentioned in book [3] with a value of:

$$dp = 3.20c/H0 \tag{25}$$

$$d_A \equiv \frac{\ell}{\delta\theta}.$$

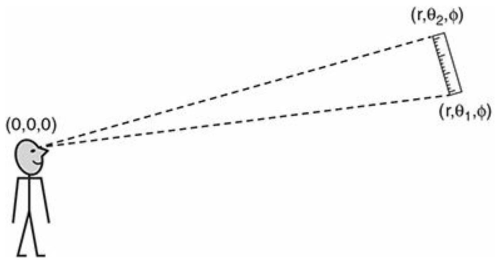

Figure 2: Representation of what the angular distance is. [3]

Where $Ho = 68\frac{km}{s \cdot Mpc}$ and c is the velocity of light. Besides this model, we are replicating the only matter universe model. For this we used equation 22 and the scale factor for this model. Salving the integral we have:

$$dp = 3Cto^{2/3}(to^{1/3} - te^{1/3}) \tag{26}$$

And we solved for equation 23 and equation 24, using $_0$ = to the last value of dp that we get from our program.
Now, for the model of only Lambda, we have the same equations, and solving the integral, we have:

$$dp = -Ce^{-Ho(te-to)} \tag{27}$$

But in our book [3] we have equations 5.74 and 5.75 that state that:

$$dp(to) = \frac{C}{Ho}z \tag{28}$$

and equation:

$$dp(te) = \frac{C}{Ho}\frac{z}{1+z} \tag{29}$$

Were te, means a time before our actual time or the age of the universe, a time between the 0 and 1 scale factor of the universe. Lastly we are using an equation that relates the luminous distance, this equation is:

$$\mu = 5\log_{10}(\frac{d_L}{1\text{Mpc}} + 25) \tag{30}$$

This is called the distance modulus, which is the difference between the apparent magnitude and absolute magnitude of a celestial object (m – M), it also provides a measure of the distance to the object, r. This last equation is important because we have to compare the values and data we obtained, to the data that is provided from [5].

3 Results and Discussion

All of the information and code can be found in Github in the url: “https://github.com/l-cabal/Cosmolgy-Project-1”

- Find solutions for the scale factor $a(t)$ using the cosmological parameters of the baseline cosmology, use it to express the Hubble factor H as function of t.
- Make plots of how the energy density of the different matter-energy species evolves as function of a, z and t.
- Compute the proper, the luminosity and angular-diameter distance. Make plots of such quantities for the base cosmology, and for variations of it. You can get some inspiration by the plots in chapter 5 and 6 of the Barbara Ryden book[3].
- In particular use the Luminosity distance to compute the distance modulus $\mu = 5\log_{10}(\frac{d_L}{1\mathrm{Mpc}} + 25)$ and compare it against the data provided here: `https://supernova.lbl.gov/Union/figures/SCPUnion_mu_vs_z.txt`. You can find more information about this data in [5]

For the first point what we obtained is 3.

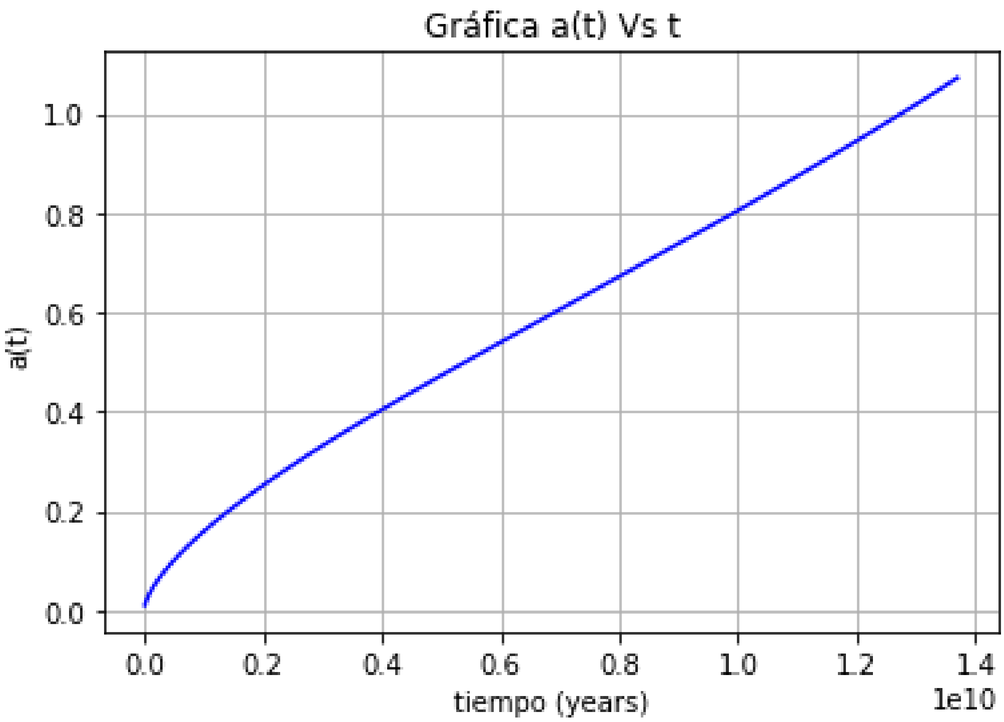

Figure 3: Solution a(t) vs t in Benchmark Model

Due to the baseline model that we are taking to do this problem, at first I did not consider the term for the curvature, since in this model k=0, but we can also find in the book [3], that it is mentioned that despite this, we cannot guarantee that it is totally 0, so we must add to our equation a $(1 - \Omega_0)$ Which, following the book and to compare results, we added $\Omega_0 = 0.96409$ which is the

sum of all our densities.
An attempt was made where Ho was passed to other units, the limits were changed from 0.05 to 13700 Myr, with an initial condition of 2.9e-4, which is the time of equality of matter radiation, and the time was multiplied by 1x10e6 to obtain the graph, but in 13.7x10e9 a(t) does not match at 1, it matches at 0.9 approximately. Also, for the first point, when playing with the curvature term and setting Ω_0 =1.1 which is for positive curvature, a=0.77 occurs in a time 10.001 Gyr When the real is a=0.77 at $t = 10.2x10^9$.
For our graph of H(t) vs t we have figure 4.

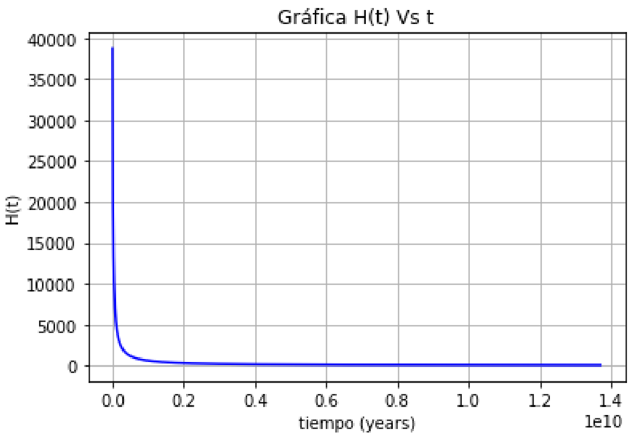

Figure 4: H(t) vs Time for Benchmark model.

In this strange graph we can see how H(t) decreases as time passes. As at the beginning of the universe H(t) was very large, this can refer to the fact that the expansion was very accelerated and is currently decreasing.
For our point 2 first we obtained a polynomial that described the scale factor a(t). This polynomial is figure 5

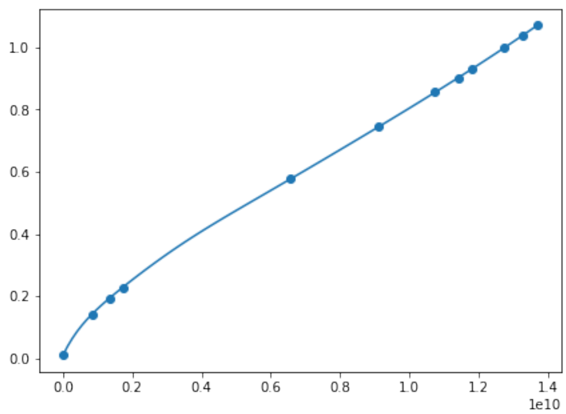

Figure 5: Caption

Which is $5.89577106202795x10^{-110} * x - 5.54363272537958e - 99 * x^{10} +$

$2.29617608373555e-88*x^9-5.49998169983441e-78*x^8+8.40549662958777e-68*x^7-8.52834400734695e-58*x^6+5.794987938611e-48*x^5-2.6023177394379e-38*x^4+7.50628905696984e-29*x^3-1.36497459197923e-19*x^2+2.32041873001947e-10*x+0.01$ So now we can see what we obtain for each case. For the only matter we have the graphs 6, 7, 8 and 9.

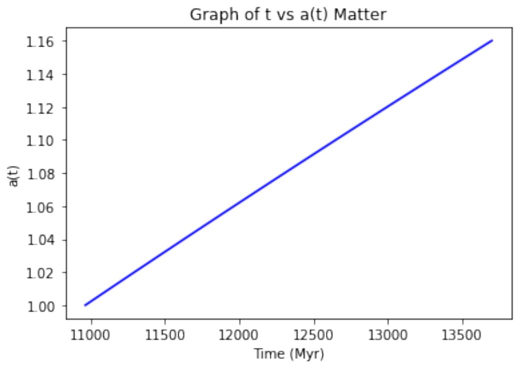

Figure 6: How the scale factor changes through time for Matter

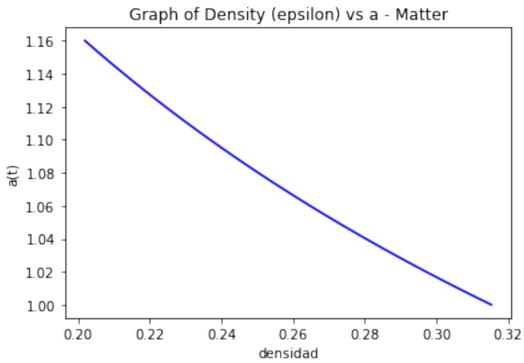

Figure 7: How the energy density changes through the scale factor for Matter

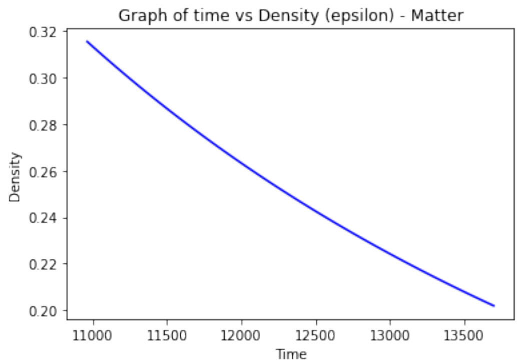

Figure 8: How the scale density changes through time for Matter

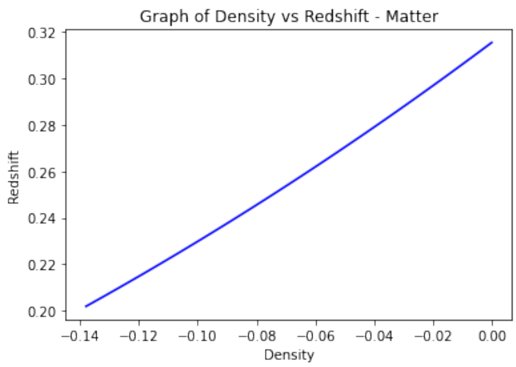

Figure 9: How the density changes through red shift for Matter

While for the case of the radiation we have graphs 10 11 12 13

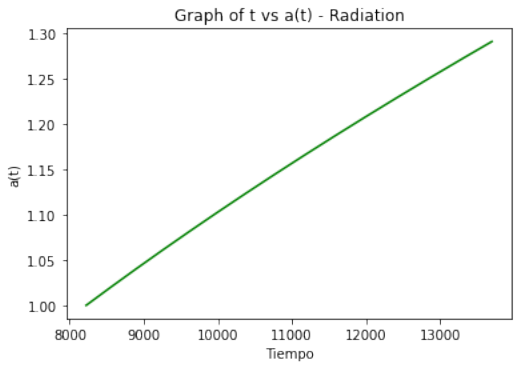

Figure 10: How the scale factor changes through time for radiation

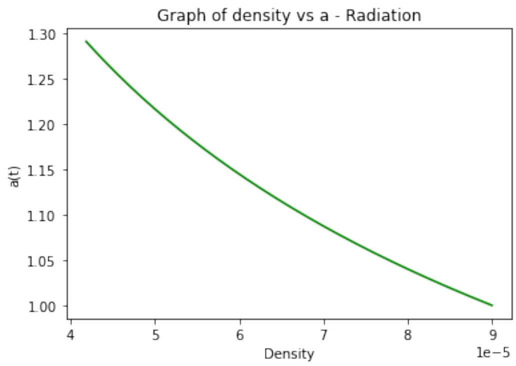

Figure 11: How the energy density changes through the scale factor for radiation

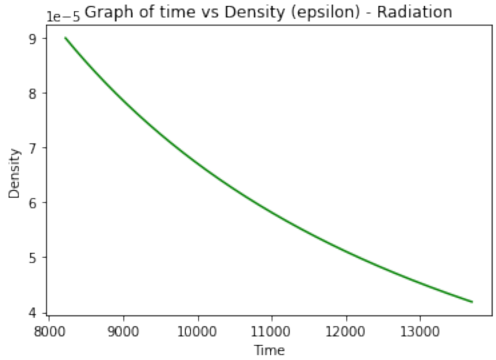

Figure 12: How the scale density changes through time for radiation

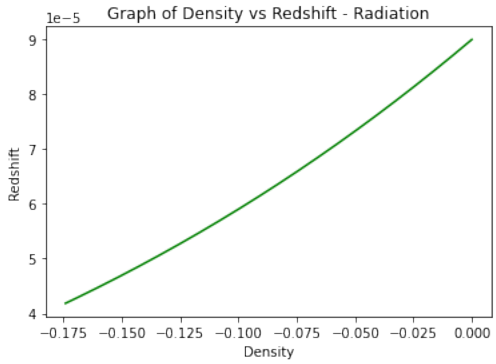

Figure 13: How the density changes through red shift for radiation

Finally for Lambda or dark energy we have graphs 14 15 16 17.

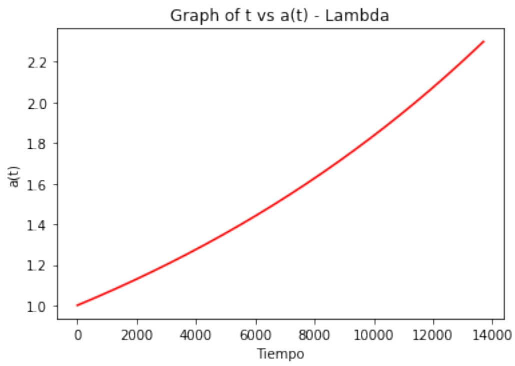

Figure 14: How the scale factor changes through time for Lambda

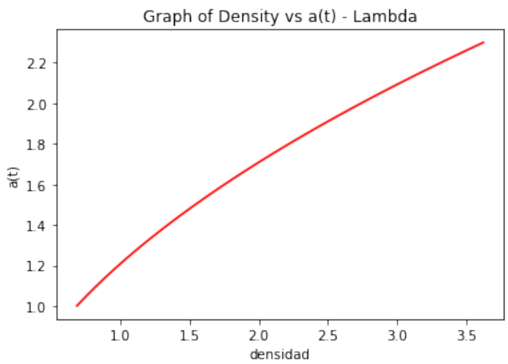

Figure 15: How the energy density changes through the scale factor for lambda

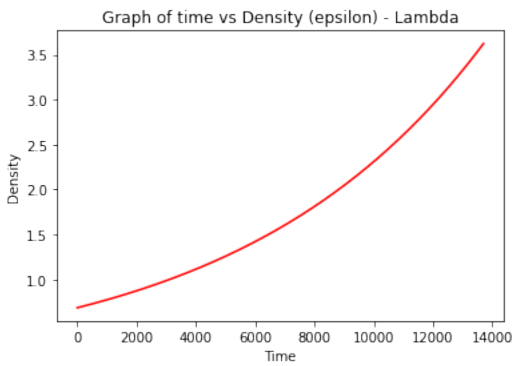

Figure 16: How the scale density changes through time for lambda

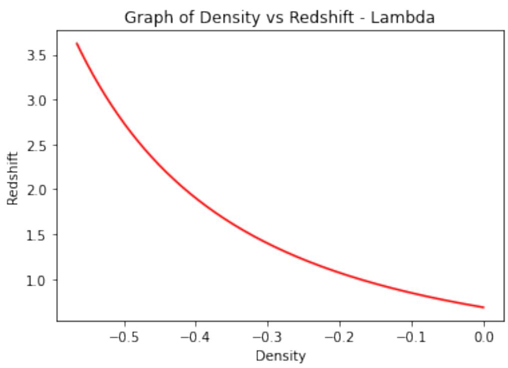

Figure 17: How the density changes through red shift for lambda

For part three we had to compute the proper distance for our Benchmark model, the only matter universe model, and the only lambda universe model, for those and using the equations previously described, we obtained graphs for the Benchmark model 18 19 20

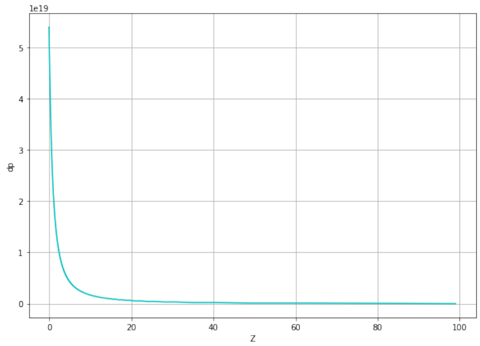

Figure 18: Our proper distance along the redshift for Benchmark

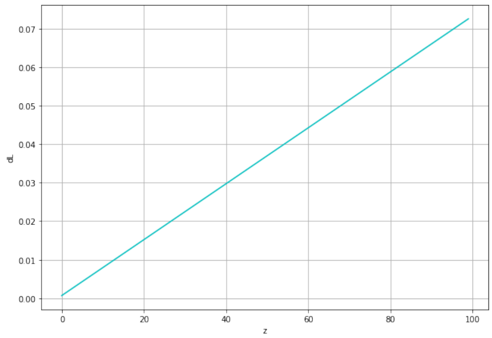

Figure 19: Our luminous distance along the redshift for Benchmark

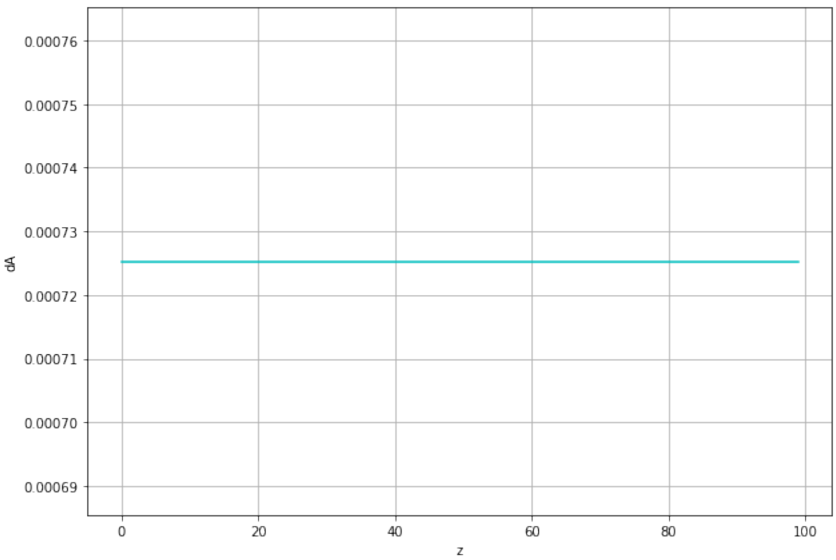

Figure 20: Our angular distance along the redshift for Benchmark

To check the matter and Lambda only universes I recommend to check the project in Github so the results don't end up having a lot of graphs.
Finally for our part 4 we have to compare data from reference [1] and our data from what we obtained. The graph that shows the observed data is figure 21

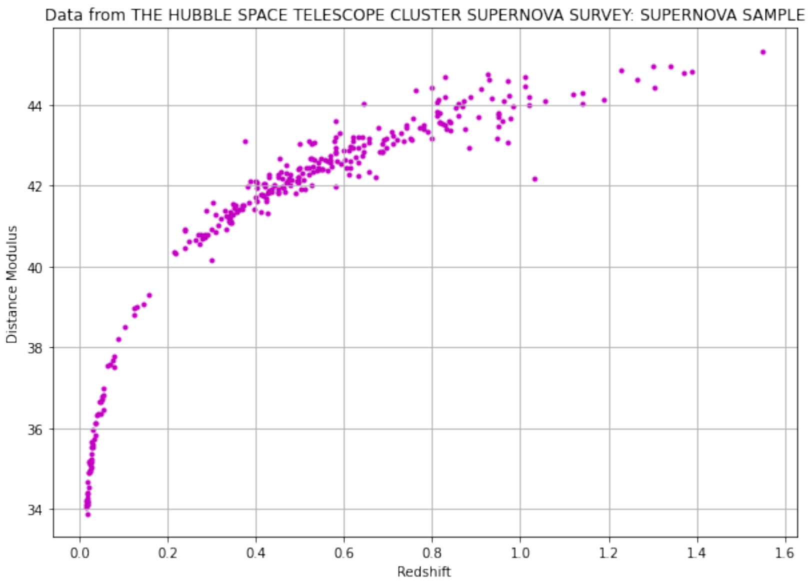

Figure 21: Observed data from article [1]

Our graph for the distance modulus is graph 22

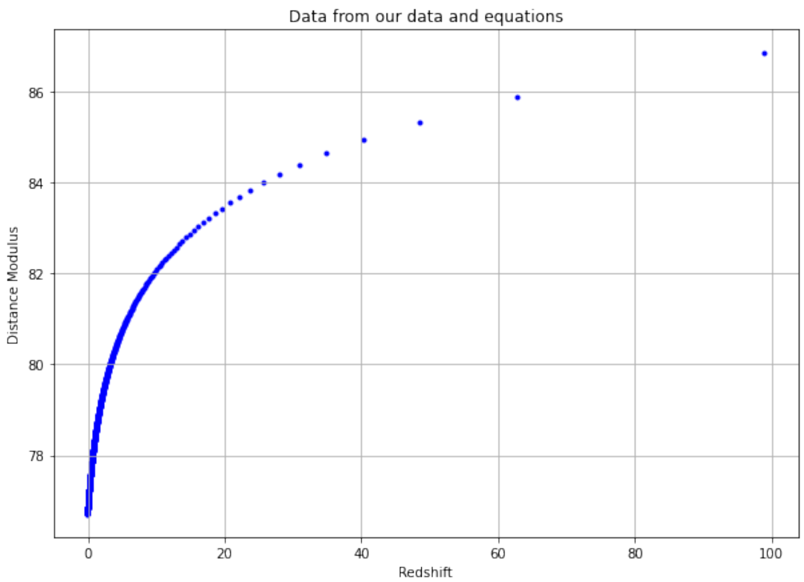

Figure 22: Data obtained from our luminous distance in the Benchmark model

As we can observe, both graphs look like they have the same form, this assures us more that the Benchmark model is the best description of our universe, since the observed data can significantly compare to what the model tries to explain us.

4 Conclusions

In this project, we had to solve the Friedmann equation for our Benchmark model, since this model is the most accurate to the universe and it relates or is based in the Big Bang Theory it is important that we study and understand how this model works. We obtained a lot of cosmological parameters that we used for this model and many others to compare them. Some of the thngs we obtained to compare are the scale factor, the redshift, the energy densities, the proper distance, luminous distance and the angular diameter distance. All of these things helps us better understand and predict what will happen to the Universe.
It's important to mention that there were a lot of mistakes made in this project so some graphs don't look like the ones from book [3]. I have a few possible options as to why this may have occured.

- There may be more exact methods than Euler's and that's why the epochs aren't the same as we know them
- There is a problem in the time interval or in general with time manipulation
- Another step size should have been taken
- I made a mistake either in the equation solution, in an integral, in the method code, etc.

The error I suspect the most is that constants weren't in the right units and the time intervals weren't the correct ones. But besides all that I got to understand a bit better how the universe works.

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