Exercise 14 Balloon Experiment

In an experiment to measure the flux of cosmic rays in the upper atmosphere, protons with an energy between 1 GeV and 100 GeV are counted over a period of one hour from a flying balloon. Over a period of one week, a measurement run of one hour duration is made every day. The measured data are:

a) Assume that the cosmic ray flux is constant over the measurement period. Calculate the most probable count rate using the maximum likelihood method.

We can use a possion-distribution to describe our measurement since this is a counting experiment. So we get our likelihood function $L(\lambda, i) = \prod_i \left(\frac{\lambda^{x_i}}{x_i!} e^{-\lambda} \right)$ We have a $\lambda =$ const ,since the counting rate is constant and λ describes exactly that. To get the most propable count rate we minimize the negativ log likelihood function for λ.

$$\frac{\partial}{\partial \lambda}(-1) \left(\ln \left(\prod_i \left(\frac{\lambda^{x_i}}{x_i!} e^{-\lambda} \right) \right) \right) = 0$$

$$\leftrightarrow \frac{\partial}{\partial \lambda}(-1) \left(\sum_i \ln \left(\frac{\lambda^{x_i}}{x_i!} e^{-\lambda} \right) \right) = 0$$

$$\leftrightarrow \frac{\partial}{\partial \lambda}(-1) \left(\sum_i (x_i \ln(\lambda) - \ln(x_i!) - \lambda) \right) = 0$$

$$\leftrightarrow (-1) \sum_i \left(\frac{x_i}{\lambda} - 1 \right) = (-1) \frac{1}{\lambda} \left(\sum_i x_i \right) + N = 0$$

$$\leftrightarrow \lambda = \frac{1}{N} \sum_i x_i$$

This is the most popable count rate.

b) Your colleague looks at the readings and hypothesizes that the cosmic ray flux is experiencing a dramatic increase. Assume a linearly increasing flux and calculate numerically the most probable flux parameters using the maximum likelihood method.

Now we assume λ has a linear increase. So we have $\lambda_i = a \times i + \lambda_0$ With this we write down the negativ log-likelihood function as:

$$F = -\ln(L) = -\ln \left(\prod_i \left(\frac{(ai + \lambda_0)^{x_i}}{x_i!} e^{-ai - \lambda_0} \right) \right) = \sum_i (-x_i \ln(ai + \lambda_0) + \ln(x_i!) + ai + \lambda_0)$$

In [5]:

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.optimize as op

#numerical calculation of the most propable flux parameters
ii = np.array([1,2,3,4,5,6,7])
x_i = np.array([4135,4202,4203,4218,4227,4231,4310])
def linearlambda(i,params):
    return params[0]*i+params[1]
def F_i(x,i,params):
    return -x*np.log(linearlambda(i,params))+linearlambda(i,params)
def F(params):
    return F_i(x_i[0],ii[0],params)+F_i(x_i[1],ii[1],params)+F_i(x_i[2],ii[2],params)+F_i(x_i[3],ii[3],params)+F_i(x_i[4],ii[4],params)+F_
def lambda_a(x):
    return np.mean(x)
def F_i_new(count,params):
    return F_i(x_new[count],ii_new[count],params)
def F_new(params):
    return F_i_new(1,params)+F_i_new(2,params)+F_i_new(3,params)+F_i_new(4,params)+F_i_new(5,params)+F_i_new(6,params)+F_i_new(7,params)

parameters = op.minimize(F,[x_i[2]-x_i[1],x_i[1]], bounds=[(0,None),(0,None)])
print("The optimal Parameters are:")
print("a =",parameters.x[0])
print("lambda_0 =",parameters.x[1])


#calculation of the most probable lambda in a) for c)
print("Most propable lambda_a =",lambda_a(x_i))
lambda_av = lambda_a(x_i)
```

```
The optimal Parameters are:
a = 20.57199186555189
lambda_0 = 4137.118883371033
Most propable lambda_a = 4218.0
```

c) Calculate the significance of his observation using a likelihood ratio test. Evaluate the significance achieved. Hint: Assume that Wilks' theorem is valid here. Why can you assume this?

With the two likelihood functions from before we can calculate the ratio Γ. $\Gamma(i) = \frac{\prod_i \left(\frac{(ai+\lambda_0)^{x_i}}{x_i!} e^{-ai-\lambda_0} \right)}{\prod_i \left(\frac{\lambda^{x_i}}{x_i!} e^{-\lambda} \right)} = \prod_i \left(\left(\frac{ai+\lambda_0}{\lambda} \right)^{x_i} e^{\lambda-ai-\lambda_0} \right)$ With Wilks Theorem we get:

$$-2\ln(\Gamma) = \sum_i \left(-2 \left(x_i \ln(\frac{ai+\lambda_0}{\lambda}) + \lambda - ai - \lambda_0 \right) \right)$$

In [2]:

```
#using the calculated parameters from before:
params_1 = np.array([lambda_av,parameters.x[0],parameters.x[1]])
def ratio_wilks_i(x,i,params_r):
    return -2*(x*np.log((params_r[1]*i+params_r[2])/params_r[0])+params_r[0]-params_r[1]*i-params_r[2])
ratio = 0
for i in range(len(ii)):
    ratio = ratio + ratio_wilks_i(x_i[i],ii[i],params_1)
print("The Wilks-Ratio(or whatever) is given by Gamma_W =",-ratio)
print("the Ratio is given by Gamma=",np.exp(-2*ratio))
```

```
The Wilks-Ratio(or whatever) is given by Gamma_W = 3.106839501484501
the Ratio is given by Gamma= 499.53566801117654
```

It is valid to use Wilks Theorem, because the Nullhypothesis(λ = linear) is a linear transfrmation of the alternative hypothesis(λ =const) and the sample size is big enough.

d) Your colleague performs another measurement a week later to support his thesis. His measurement results in Day 14 Counts 4402 Calculate (a) to (c) again for this new data set.

In [6]:

```
#initialisation of the new dataset
x_new = np.array([4135,4202,4203,4218,4227,4231,4310,4402])
ii_new = np.array([1,2,3,4,5,6,7,14])

lambda_a_new = lambda_a(x_new)
parameters_new = op.minimize(F_new,[x_new[2]-x_new[1],x_new[1]], bounds=[(0,None),(0,None)])
print("The most propable const lambda =",lambda_a_new)
print("The new optimal Parameters are:")
print("a_new =",parameters_new.x[0])
print("lambda_0_new =",parameters_new.x[1])

#ratio

params_1_new = np.array([lambda_a_new,parameters_new.x[0],parameters_new.x[1]])
ratio_new = 0
for i in range(len(ii_new)):
    ratio_new = ratio_new + ratio_wilks_i(x_new[i],ii_new[i],params_1_new)
print("The Wilks-Ratio(or whatever) is given by Gamma_W =",-ratio_new)
print("the Ratio is given by Gamma=",np.exp(-2*ratio_new))
```

```
The most propable const lambda = 4241.0
The new optimal Parameters are:
a_new = 17.944413991478562
lambda_0_new = 4150.536280945702
The Wilks-Ratio(or whatever) is given by Gamma_W = 9.905151403138007
the Ratio is given by Gamma= 401333297.6592493
```

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e) What is the methodological problem with exercise d)'s approach? Why should you not publish these results, even if the significance is higher then some preset threshold (e.g. 3 or 5σ)?

The new added value has a big time gap between the measurement. So The new data set is not able to describe the phenomenon from day 7 to day 13.

In [ ]: