Exercise 10

Confidence intervals

Given is the likelihood function for a measured value x at a given parameter a

$L(X; a) = \frac{1}{\pi} \frac{1}{1+(x-a)^2}$ mit $a > 0$.

> **(a) Using the Neyman construction, determine the central frequentist 90% confidence interval for a when a value $x = 10$ was measured.**

In [10]:

```
import numpy as np
import matplotlib.pyplot as plt
```

The integral of the probability function is analytically determined and equals to:

$P(x; a) = \frac{1}{\pi} arctan(a - x)$

The full area under the function for $x = 10$ is given by:

In [11]:

```
p = (1/ np.pi) * ( np.arctan(np.Inf) - np.arctan(-10) )
print(np.absolute(p))
print(0.05*np.absolute(p))
```

```
0.9682744825694465
0.048413724128472333
```

$\frac{1}{\pi}(\arctan(\infty) - \arctan(-10)) = \frac{1}{\pi}\frac{\pi}{2} + \frac{1}{\pi}1.4711276743037 = 0.9682744825694465$

Since we want the central 90% of the area under the curve, we need to determine for which interval the left and right edges of the area equal to 5% of the total area. This equals to 0.048413724128472333.

For the lower limit we get this equation:

$\frac{1}{\pi}\arctan(I_{unten} - 10) - \frac{1}{\pi}\arctan(-10) = 0.048413724128472333$

$\Leftrightarrow \arctan(I_{unten} - 10) = 0.048413724128472333 \cdot \pi + \arctan(-10)$

$\Leftrightarrow I_{unten} - 10 = \tan(0.048413724128472333 \cdot \pi + \arctan(-10))$

$\Leftrightarrow I_{unten} = \tan(0.048413724128472333 \cdot \pi + \arctan(-10)) + 10 = 6.112318096783665$

In [12]:

```
l_unten = np.tan(0.048413724128847233 * np.pi + np.arctan(-10)) +10
l_unten
```

Out[12]:

```
6.112318096783665
```

For the upper limit we get this equation: $\frac{1}{\pi}\arctan(\infty) - \frac{1}{\pi}\arctan(I_{oben} - 10) = 0.048413724128847233$

$\Leftrightarrow -\arctan(I_{oben} - 10) = 0.048413724128847233 \cdot \pi - \arctan(\infty)$

$\Leftrightarrow \arctan(I_{oben} - 10) = \arctan(\infty) - 0.048413724128847233 \cdot \pi$

$\Leftrightarrow I_{oben} - 10 = \tan(\arctan(\infty) - 0.048413724128847233 \cdot \pi)$

$\Leftrightarrow I_{oben} = \tan(\arctan(\infty) - 0.048413724128847233 \cdot \pi) + 10 = 16.52400912399286$

In [13]:

```
l_oben = np.tan(np.arctan(np.Inf) - 0.048413724128847233 * np.pi) +10
l_oben
```

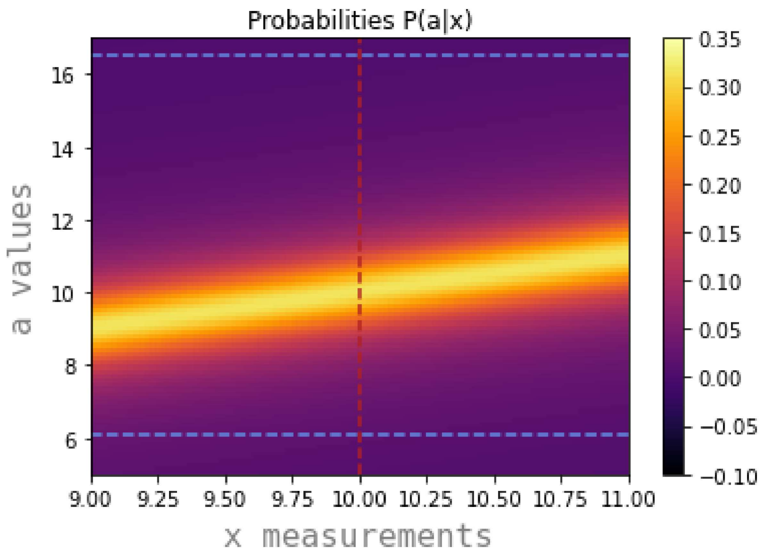
Out[13]:

```
16.52400912399286
```

The interval for which the parameter a lies in with 90% confidence for the measured x value of 10 is $a \in [6.112318, 16.524009]$.

In [16]:

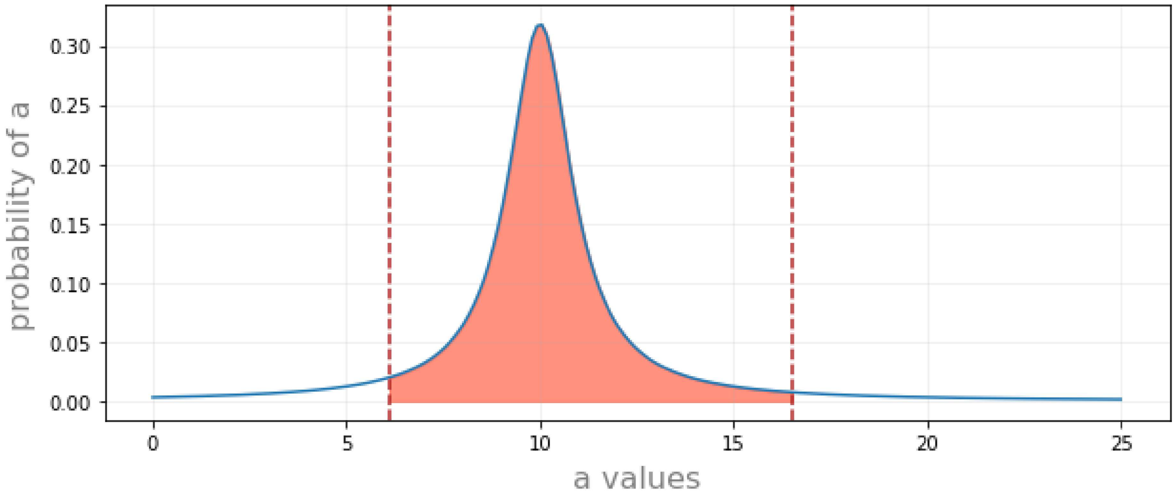
```
#heatmap configurations

x, a = np.meshgrid(np.linspace(9, 11, 100), np.linspace(5, 17, 100))
z = 1/(np.pi *(1+(x-a)**2)) #probability density

z = z[:-1, :-1]

fig, ax = plt.subplots()

c = ax.pcolormesh(x, a, z, cmap='inferno', vmin=-0.1, vmax=0.35)
ax.set_title('Probabilities P(a|x)')

ax.axis([x.min(), x.max(), a.min(), a.max()])

fig.colorbar(c, ax=ax)
plt.xlabel("x measurements", family='monospace', color='grey', weight='normal', size = 16,
plt.ylabel("a values", family='monospace', color='grey', weight='normal', size = 16, labelp
plt.axhline(y=l_unten, color='cornflowerblue', linestyle='dashed')
plt.axhline(y=l_oben, color='cornflowerblue', linestyle='dashed')
plt.axvline(x = 10, color='firebrick', linestyle='dashed')

plt.show()
```



The heatmap plot shows the probabilities of a for certain measurements of x. The blue dashed lines are the confidence interval of a for $x = 10$. The red dashed line is for $x = 10$.

In [18]:

```
def f(n, x=10):
    return 1/(np.pi *(1+(x-n)**2))

#plot

plt.figure(figsize=(10, 4))

b = np.linspace(0, 25, 200)
plt.plot(b,f(b))
plt.xlabel("a values", color='grey', weight='normal', size = 16, labelpad = 6)
plt.ylabel("probability of a", color='grey', weight='normal', size = 16, labelpad = 6)

k = np.arange(l_unten, l_oben, 0.1)

plt.fill_between(k, f(k), color='tomato', alpha=.7)
plt.grid( alpha=.2)

plt.axvline(x=l_unten, color='firebrick', linestyle='dashed')
plt.axvline(x=l_oben, color='firebrick', linestyle='dashed')

plt.show()
```



(b) Assuming a uniform prior distribution in a, determine the central Bayesian credibility interval. (Both sides outside the central confidence interval have the same probability

content).