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1 Exercise 33 - Maximum-Likelihood

A random variable x is to be subject to a uniform distribution

$$f(x) = \begin{cases} 1/b & 0 \leq x \leq b \\ 0 & x < 0 \quad \text{oder} \quad x > b \end{cases}$$

a) Determine an estimator for the parameter b using the maximum likelihood method from a sample $x_1, x_2, \dots, x_n = \vec{X}$

Likelihood:

$$L\left(\theta \mid x_1, \dots, x_n\right) = f\left(x_1, \dots, x_n; \theta\right) = \prod f\left(x_i; \theta\right) = \prod L\left(\theta \mid x_i\right)$$

Log-Likelihood:

$$l\left(\theta \mid x_1, \dots, x_n\right) = \log L\left(\theta \mid x_1, \dots, x_n\right) = f\left(x_1, \dots, x_n; \theta\right) = \sum \log f\left(x_i; \theta\right) = \sum l\left(\theta \mid x_i\right)$$

Here,

$$\theta = b$$

So

$$L\left(\theta \mid x_1, \dots, x_n\right) = \prod^n \frac{1}{b} = \frac{1}{b^n}$$
$$\rightarrow l\left(b \mid x_i, \dots, x_n\right) = -n \ln(b)$$

$$\text{Maximum:} \quad \frac{\partial l}{\partial b} = \frac{-n}{b} \overset{!}{=} 0 \quad \rightarrow \text{ not reachable.}$$

$$\text{Thus} \quad \hat{b} = \operatorname{argmax} L(b \mid \vec{X}) = \max \vec{X}$$

(b) Is this estimate biased? If yes, how can the estimator be corrected in this case?

Unbiased:

$$\langle \hat{b} \rangle = b$$

Biased:

$$B(\hat{b}) = \langle \hat{b} \rangle - b$$

$\hat{b} = \max \vec{X} < b$, so $\hat{b}$ is definitaly biased.

The PDF $p(x)$ of the max-function can be derived from the CDF.

The CDF of a uniform distribution is assumed as given: $F_{\vec{X}}(x) = \begin{cases} 0, & , x \leqslant 0 \\ \frac{x}{b} & , x \in (0, b) \\ 1 & , x \geqslant b \end{cases}$

Therefore the CDF of the max-function is $P(\max \vec{X} \leqslant x) = P(x_1 \leqslant x, \dots, x_n \leqslant x) = \prod_{i=1}^{n} P(x_i \leqslant x) = \left(F_{\vec{X}}(x)\right)^n$

Then the PDF of the max-function is the derivative: $p(x) = \frac{d}{dx}\left(F_x(x)\right)^n = \frac{d}{dx}\left(\frac{x}{b}\right)^n = \frac{n}{b}\left(\frac{x}{b}\right)^{n-1}$

$\Rightarrow \langle \hat{b} \rangle = \langle \max \vec{X} \rangle_x = \int_0^b x p(x) dx = \frac{n}{b^n} \int_0^b x^n dx = \frac{n}{b^n}\left(\frac{1}{n+1} x^{n+1}\right)_0^b = \frac{n}{n+1} b$

> The mean value $\langle \max \vec{X} \rangle_x$ is calculated regarding to x, despite the confusing notation of $\langle \hat{b} \rangle$, which suggests an integral over b. A closer look at $\hat{b}$ shows that the mean of $\hat{b}$ is indeed the mean of $\max \vec{X}$, since b is assumed as fixed (for a set of random variables $\vec{X}$).

Thus the bias of the estimator for b is $B(\hat{b}) = \langle \hat{b} \rangle - b = \frac{n}{n+1} b - b = b\left(\frac{n}{n+1} - 1\right) = b\left(\frac{n-n-1}{n+1}\right) = -b\left(\frac{1}{n+1}\right)$.

For an unbiased estimator, the correction term is $c = \frac{n+1}{n} \to B(c\hat{b}) = c\langle \hat{b} \rangle - b = b - b = 0$**.**