Exercise 7

Maximum-Likelihood

A random variable x is to be subject to a uniform distribution

$$f(x) = \begin{cases} 1/b & 0 \leq x \leq b \\ 0 & x < 0 \text{ oder } x > b \end{cases}$$

(a) Determine an estimator for the parameter b using the maximum likelihood method from a sample $x_1, x_2, \ldots, x_n$.

Since this is a uniform distribution with f(x) outside of the Parameter 0 and b, the likelihood function is given by:

$L(b, x_1, \ldots, x_n) = \Pi_{i=1}^n f(x_i, b) = \Pi_{i=1}^n \frac{1}{b} = \frac{1}{b^n}$

The log-likelihood function is given by:

$\ln(L(b, x_1, \ldots, x_n)) = -n \ln b = l(b, x_1, \ldots, x_n)$

The log-likelihood function is to be maximised for the parameter b:

$\frac{\partial l(b, x_1, \ldots, x_n)}{\partial b} \stackrel{!}{=} 0$

$-\frac{n}{b} \stackrel{!}{=} 0$

The derivative with respect to b is monotonically decreasing. Thus, the Maximum likelihood estimation for b would be the smallest b possible, which would be:

$\hat{b} = \max(x_1, x_2, \ldots, x_n)$

Therefore b is likely the biggest value that has been measured.

(b) Is this estimate biased? If yes, how can the estimator be corrected in this case?

Obviously $\max(x_1, x_2, \ldots, x_n) < b$ with a probability of one, so the expected value of $\max(x_1, x_2, \ldots, x_n)$ must be smaller than b, so $\max(x_1, x_2, \ldots, x_n)$ is a biased estimator.

The distribution of the max-function is given by: $P(\hat{b} \leq x) = P(x_1 \leq x, x_2 \leq x, \ldots, x_n \leq x) = \left(\frac{x}{b}\right)^n$

$\frac{\partial P}{\partial x} = p(x) = n\left(\frac{x}{b}\right)^{n-1} \frac{1}{b}$

Now the expectation value of the distribution is given by:

$\int_0^b x \cdot n\left(\frac{x}{b}\right)^{n-1} \frac{1}{b} dx = \frac{n}{n+1} b$

The expected value of b is b, therefore $\hat{b} = \frac{n+1}{n} \max(x_1, x_2, \ldots, x_n)$