Preuve: Posons $P(n) : F_n^{(p)} = \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \binom{n-pk}{k}$

Initialisation : Pour $n = 0$, on a

$$\sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \binom{n-pk}{k} = \sum_{k=0}^{0} \binom{0-pk}{k} = \binom{0}{0} = 1$$

Hérédité : Soit $n \in \mathbb{N}$ tel que $\forall k \in [\![0, n]\!], P(k)$ soit vraie.

$$\begin{aligned}
F_{n+1}^{(p)} &= F_{n-p}^{(p)} + F_n^{(p)} \\
&= \sum_{k=0}^{\lfloor \frac{n-p}{p} \rfloor} \binom{n-p-pk}{k} + \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \binom{n-pk}{k} \\
&= \sum_{k=1}^{\lfloor \frac{n-p}{p} \rfloor + 1} \binom{n-p-p(k-1)}{k-1} + \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \binom{n-pk}{k} \\
&= \sum_{k=0}^{\lfloor \frac{n-p}{p} \rfloor + 1} \binom{n-pk}{k-1} + \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \binom{n-pk}{k} \\
&= \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \left(\binom{n-pk}{k-1} + \binom{n-pk}{k} \right) \\
&= \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \binom{(n+1)-pk}{k} \\
&= \sum_{k=0}^{\lfloor \frac{n+1}{p} \rfloor} \binom{(n+1)-pk}{k}
\end{aligned}$$

Donc $P(n+1)$ est vraie.

Par le principe de récurrence forte, $P(n) : F_n^{(p)} = \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \binom{n-pk}{k}$ ∎