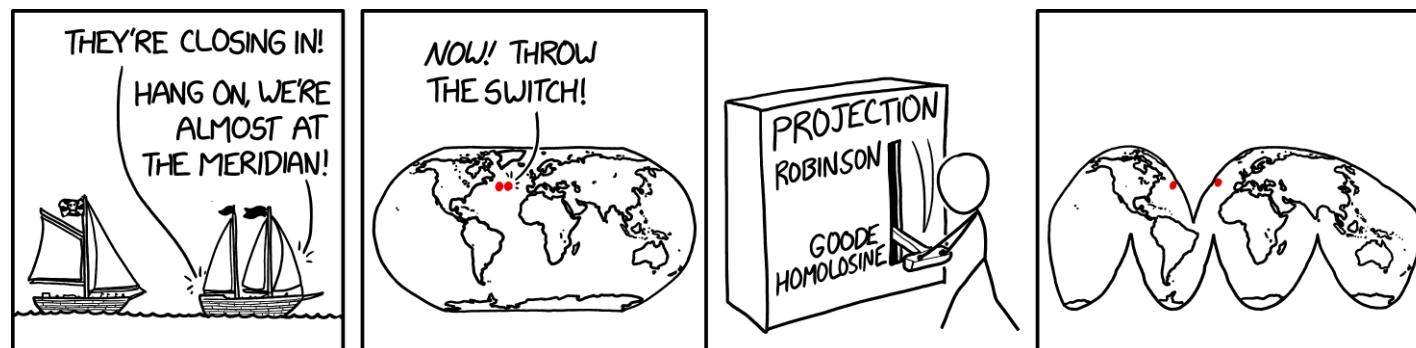
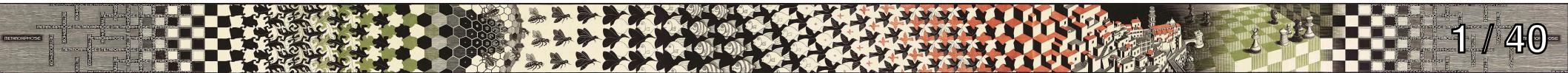


Projection cartographique de la pseudosphère sur le plan

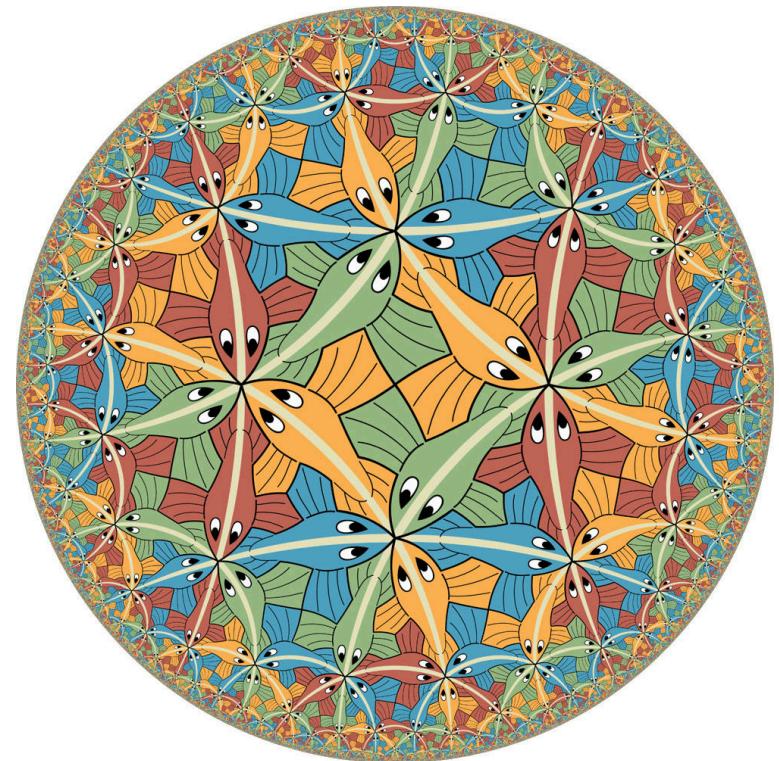
Gaspar Daguet, n°= 21528



Sea Chase - Xkcd

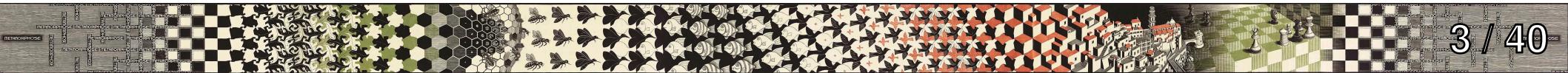


- 1) Introduction et problématique
- 2) La pseudosphère
- 3) La projection
- 4) Projection des droites



Cercle Limite III — M. C. Escher

1) Introduction et problématique

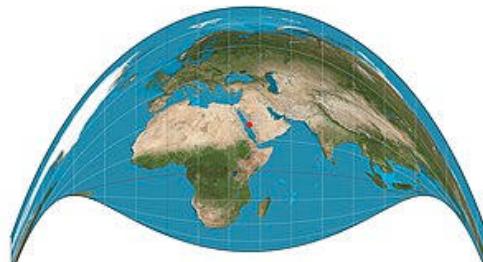


1) Introduction et problématique

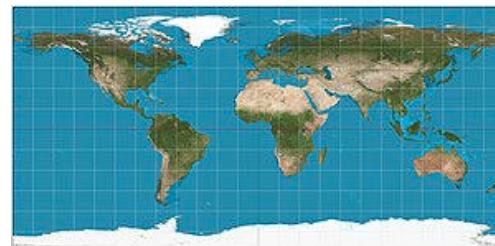
Gaspar Daguet, n°=21528



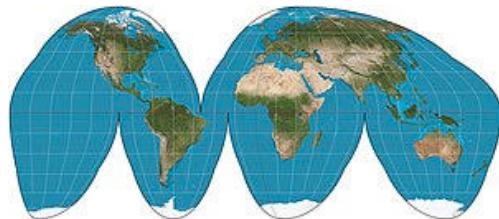
Cahill-Keyes :
1975



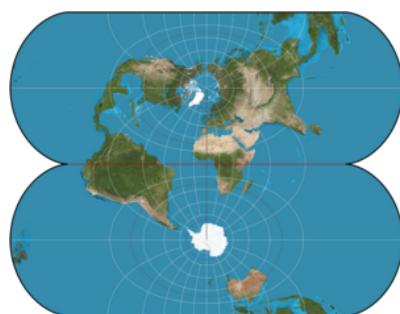
Rétro-azimutale
de Craig : 1909



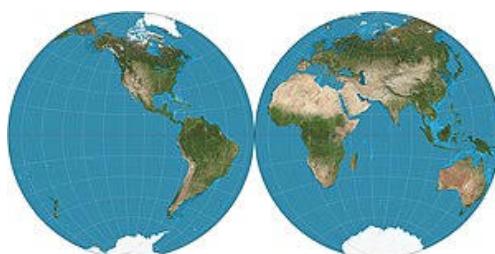
Équirectangulaire :
120



Goode : 1923



Transverse
Universelle de
Mercator : 1822

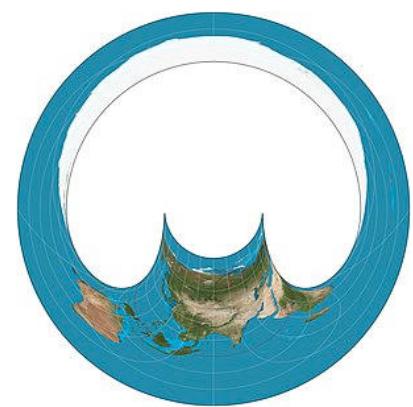


Globulaire de
Nicolosi : 1000



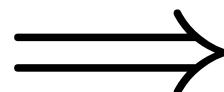
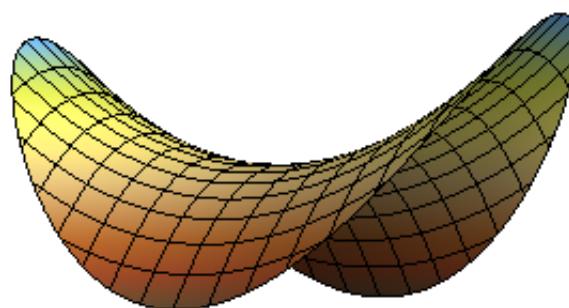
Stéréographique :
200 av J.C.
Rétro-azimutale
de Hammer :

1910



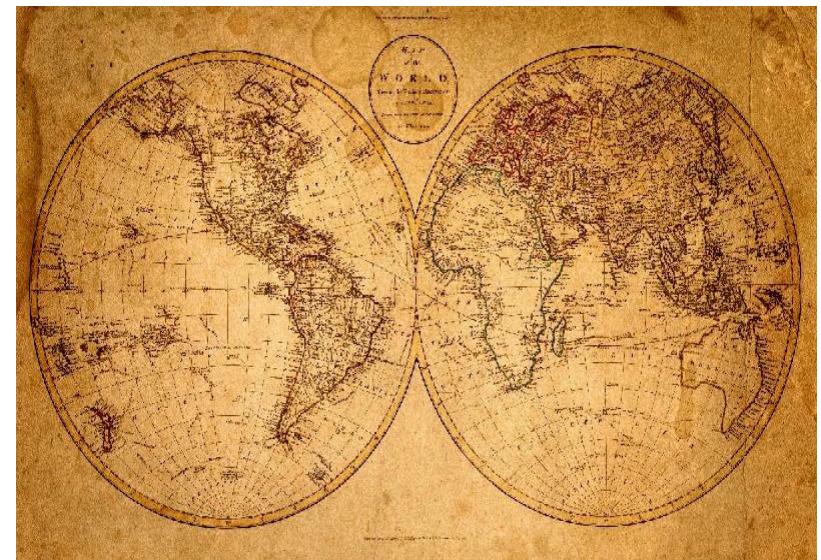
Problématique :

Comment projeter une surface hyperbolique sur le plan ?



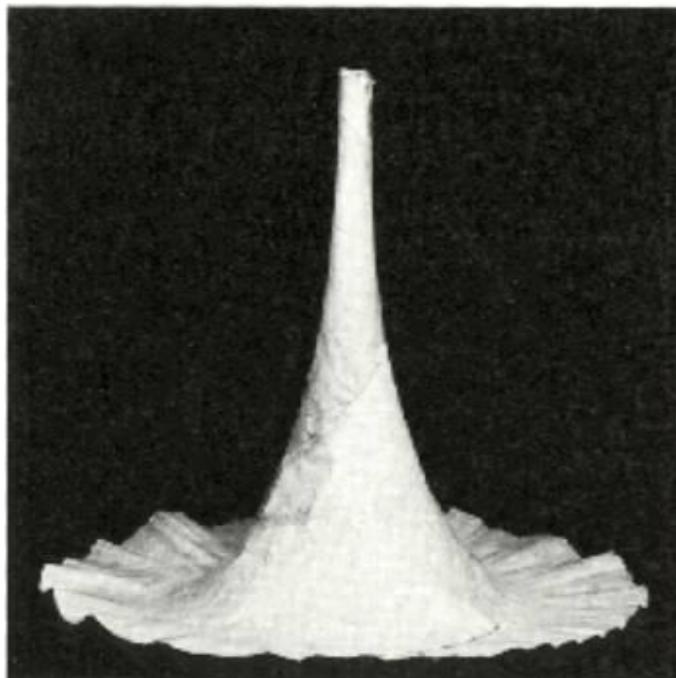
$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$$

une chips

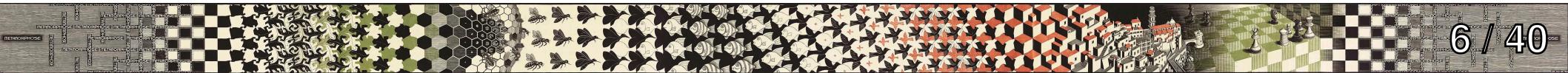


une carte





2) La pseudosphère



2) La pseudosphère

Gaspar Daguet, n°=21528

$$P : \left\{ \begin{array}{ccc} [0; 2\pi] \times \mathbb{R}_+ & \longrightarrow & \mathbb{R}^3 \\ (u, v) & \longmapsto & \left(\begin{array}{c} \frac{\cos(u)}{\operatorname{ch}(v)} \\ \frac{\sin(u)}{\operatorname{ch}(v)} \\ v - \operatorname{th}(v) \end{array} \right) \end{array} \right.$$



2) La pseudosphère

Gaspar Daguet, n°=21528

On note $\frac{\partial P}{\partial u} = P_u$

$$E(p) = \|P_u\|^2$$

$$F(p) = \langle P_u \mid P_v \rangle$$

$$G(p) = \|P_v\|^2$$

$$\mathcal{L}(p) = \langle P_{uu} \mid N \rangle$$

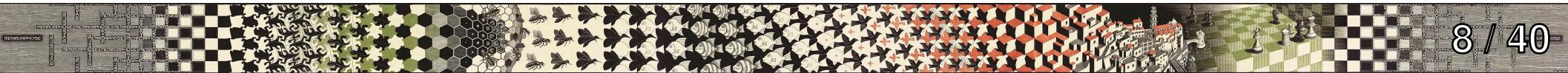
$$\mathcal{M}(p) = \langle P_{uv} \mid N \rangle$$

$$\mathcal{N}(p) = \langle P_{vv} \mid N \rangle$$

L'application Normale :

$$N : [0; 2\pi] \times \mathbb{R}_+ \longrightarrow \mathbb{R}^3$$

$$p \longmapsto \frac{P_u(p) \wedge P_v(p)}{\|P_u(p) \wedge P_v(p)\|}$$

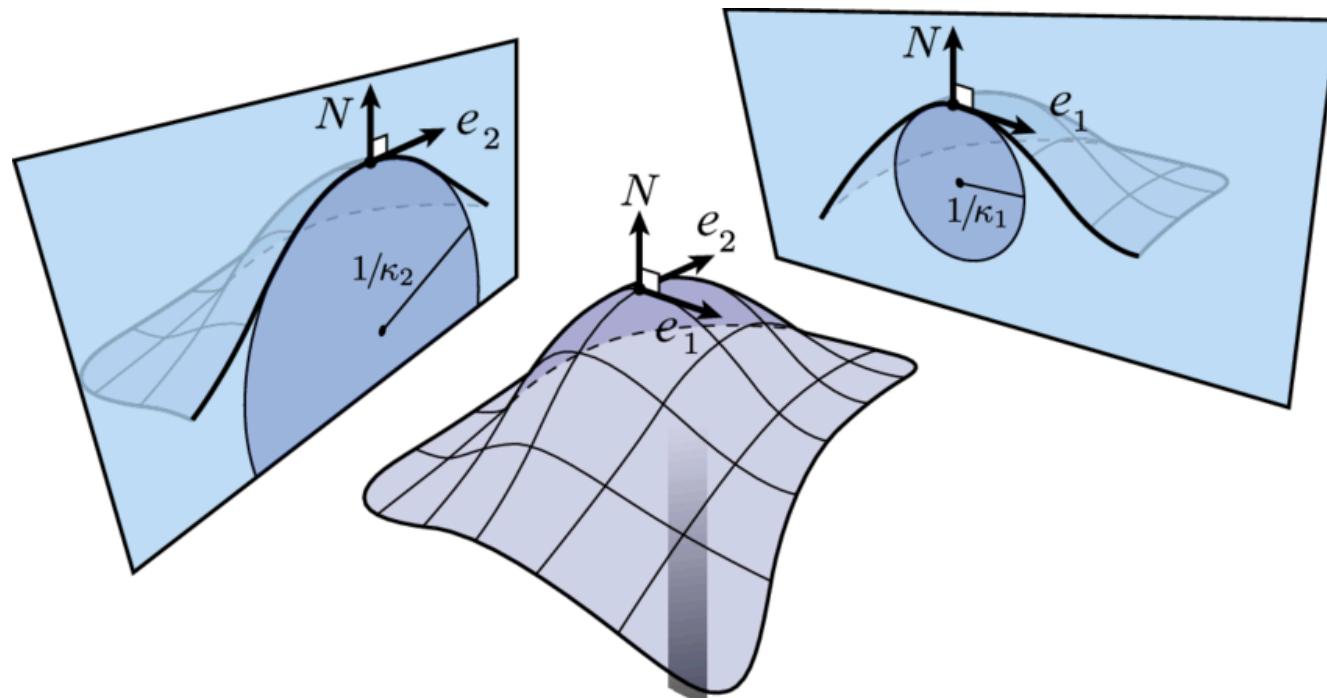


2) La pseudosphère

Gaspar Daguet, n°=21528

La courbure en $p \in [0; 2\pi] \times \mathbb{R}_+$

$$K(p) = \frac{\mathcal{L}(p)\mathcal{N}(p) - \mathcal{M}(p)^2}{E(p)G(p) - F(p)^2}$$



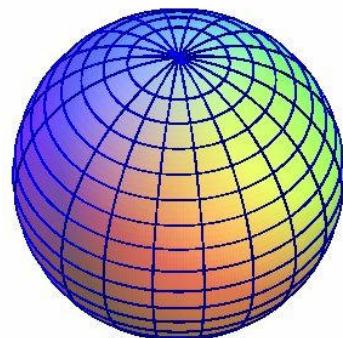
source : Localisation d'objets 3D industriels à l'aide d'un algorithme de SLAM contraint au modèle



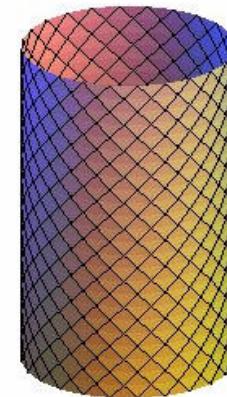
2) La pseudosphère

Gaspar Daguet, n°=21528

Sphérique: $K > 0$

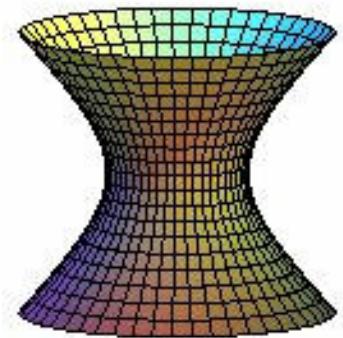


Plan: $K = 0$



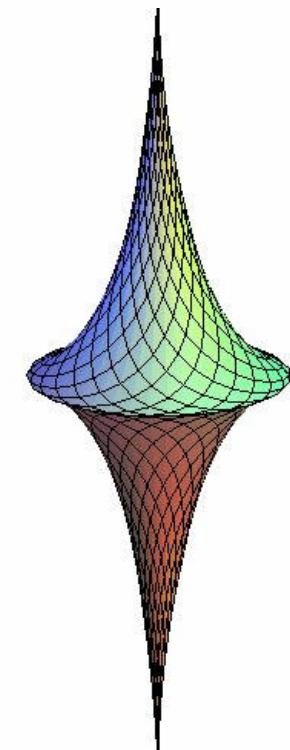
Hyperbolique:

$$K < 0$$



Pseudosphère:

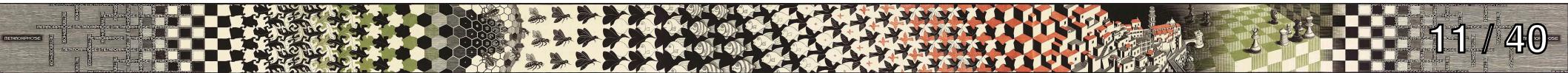
$$K = -1$$



source: <https://mathcurve.com>



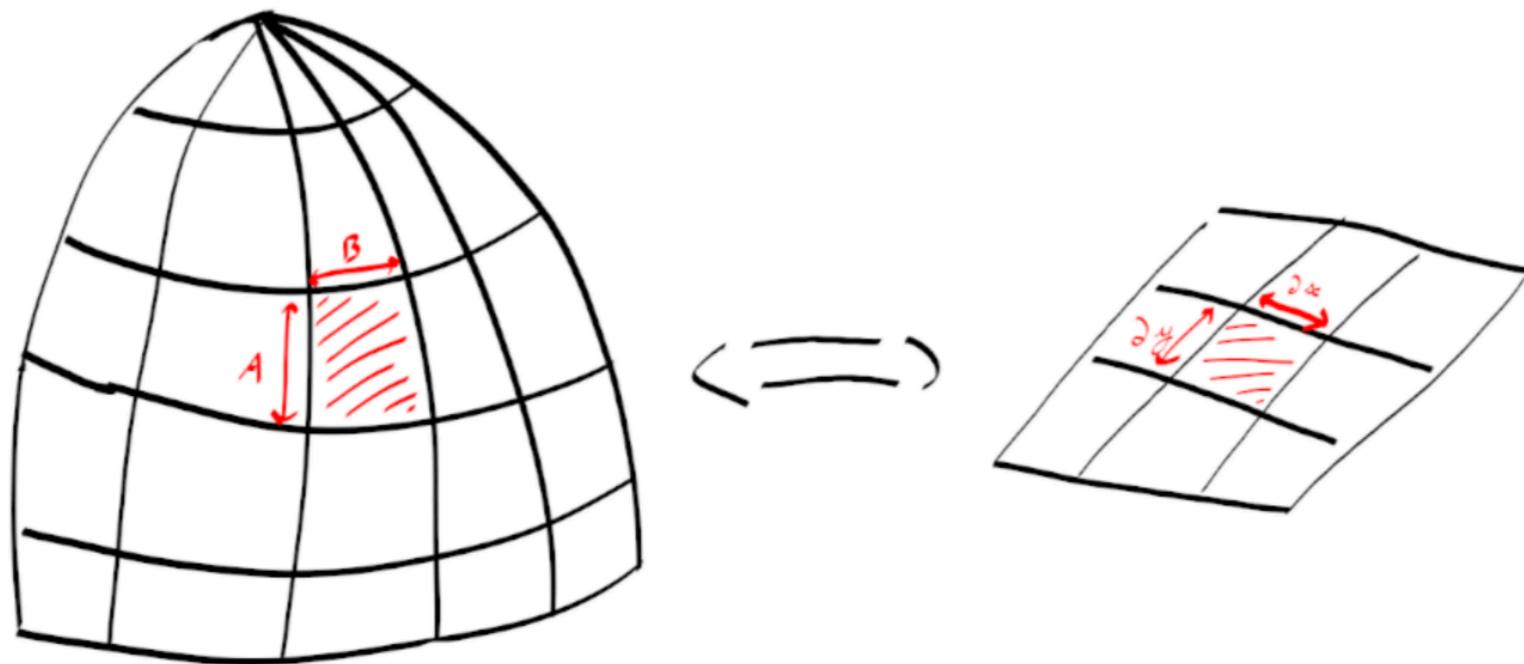
3) La projection



3) La projection

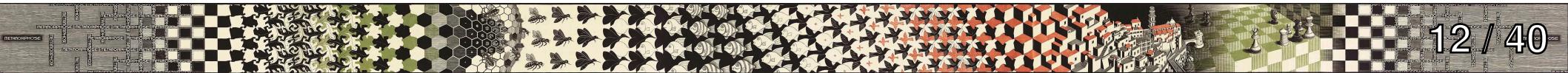
Gaspar Daguet, n°=21528

Idée de la projection de Mercator :



$$\frac{\partial y}{\partial x} = \frac{A}{B}$$

-1

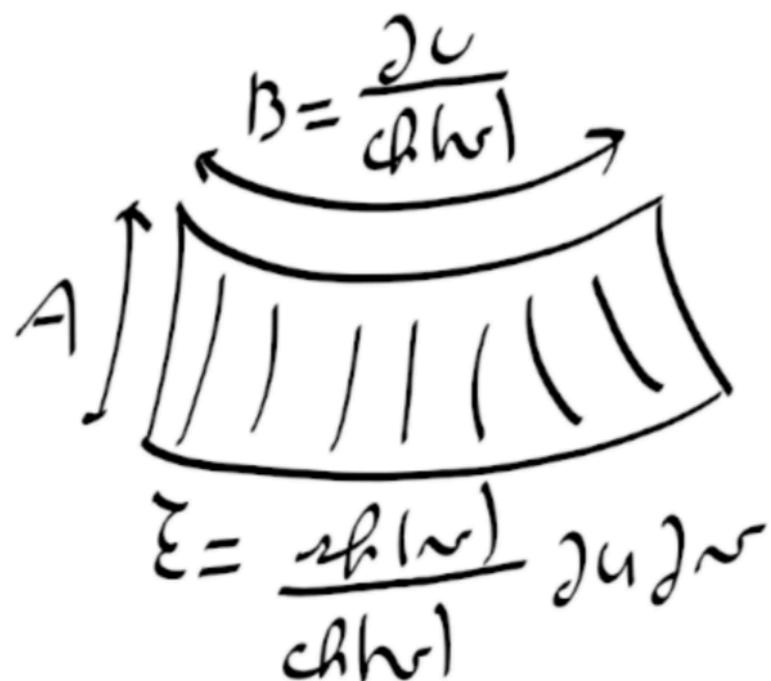


3) La projection

Gaspar Daguet, n°=21528

Surface élémentaire :

$$\mathcal{E} = \|P_u \wedge P_v\| \partial u \partial v$$



Pour la pseudosphère :

$$\mathcal{E} = \frac{\text{sh}(v)}{\text{ch}^2(v)} \partial u \partial v$$

Or :

$$B = \frac{\partial u}{\text{ch}(v)}$$

Donc :

$$\frac{A}{B} = \frac{\mathcal{E}}{B^2} = \frac{\text{sh}(v) \partial v}{\partial u}$$

La projection :

$$C : \begin{cases} [0; 2\pi] \times \mathbb{R}_+ \longrightarrow \mathbb{R}^2 \\ p = (u, v) \longmapsto \begin{pmatrix} u \\ \operatorname{ch}(v) \end{pmatrix} \end{cases}$$

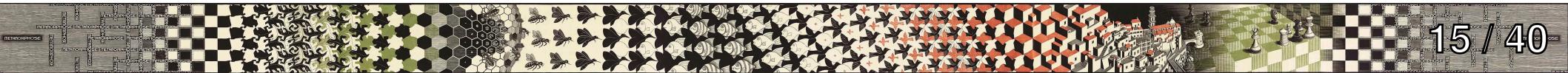
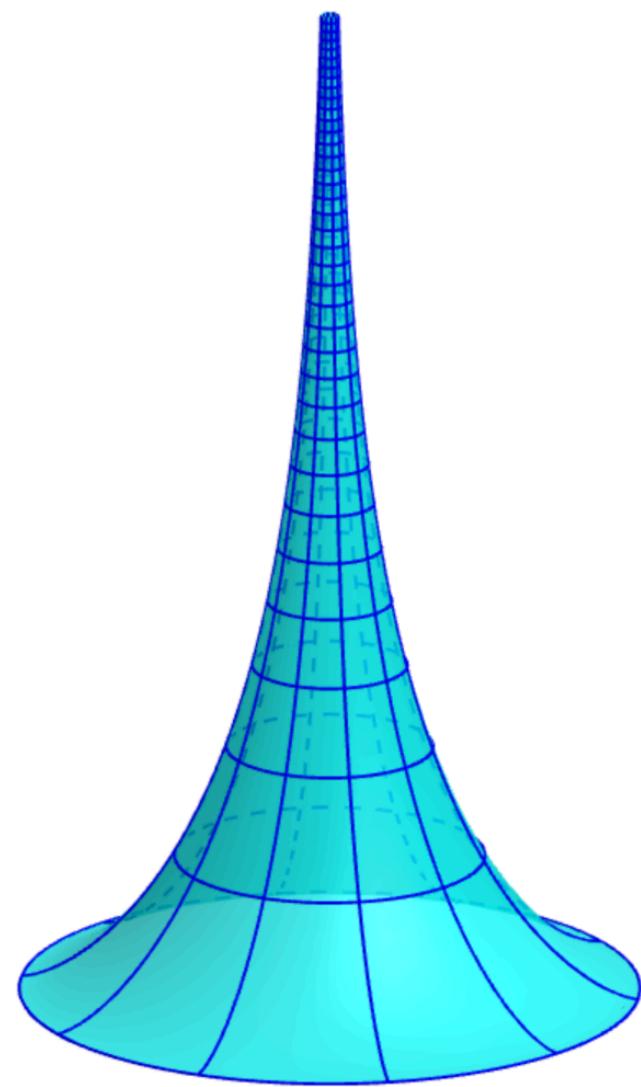
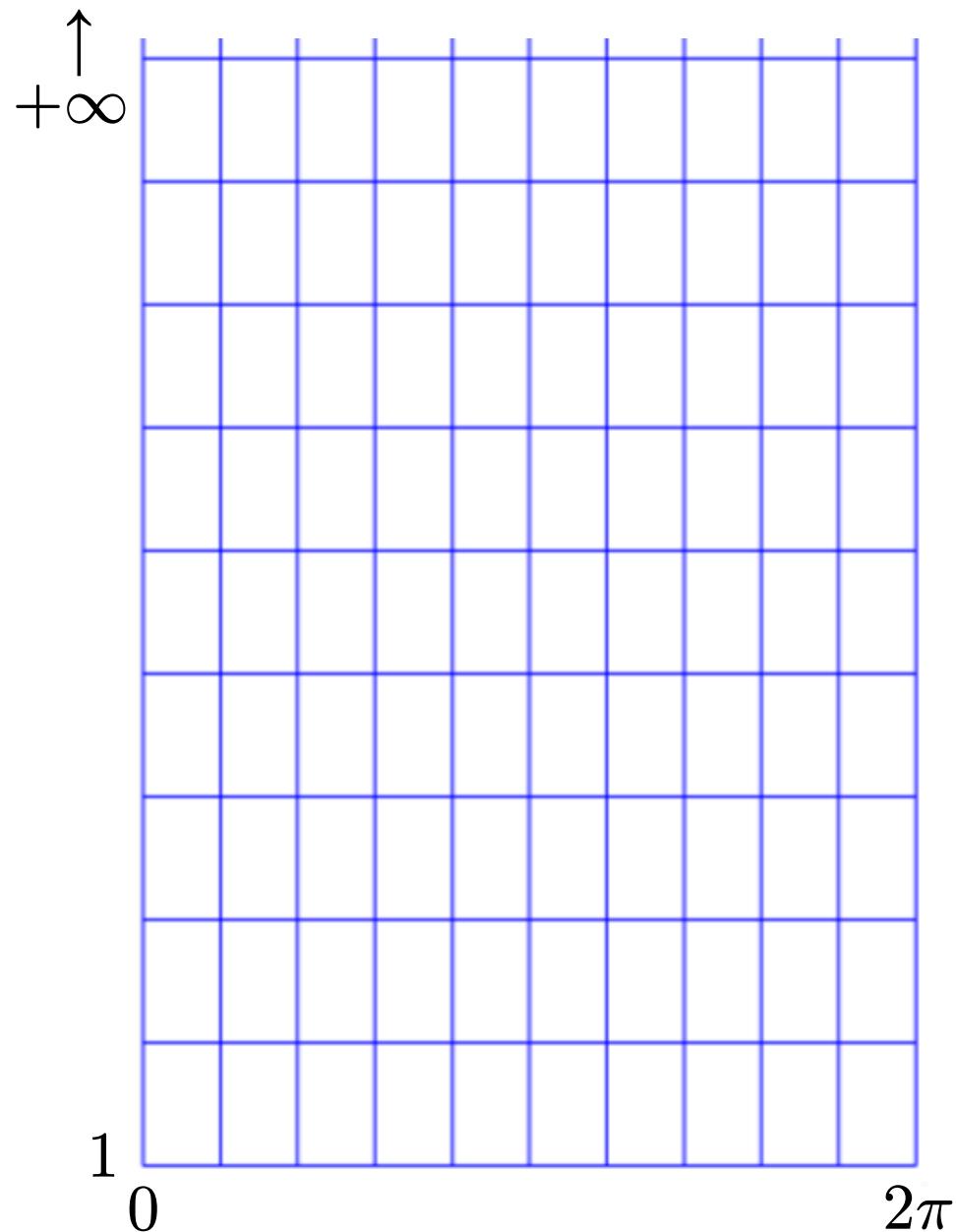
Celle de Mercator:

$$M : \begin{cases} [0; 2\pi] \times [-\pi; \pi] \longrightarrow \mathbb{R}^2 \\ p = (u, v) \longmapsto \begin{pmatrix} u \\ \ln(\tan(\frac{v}{2} + \frac{\pi}{4})) \end{pmatrix} \end{cases}$$



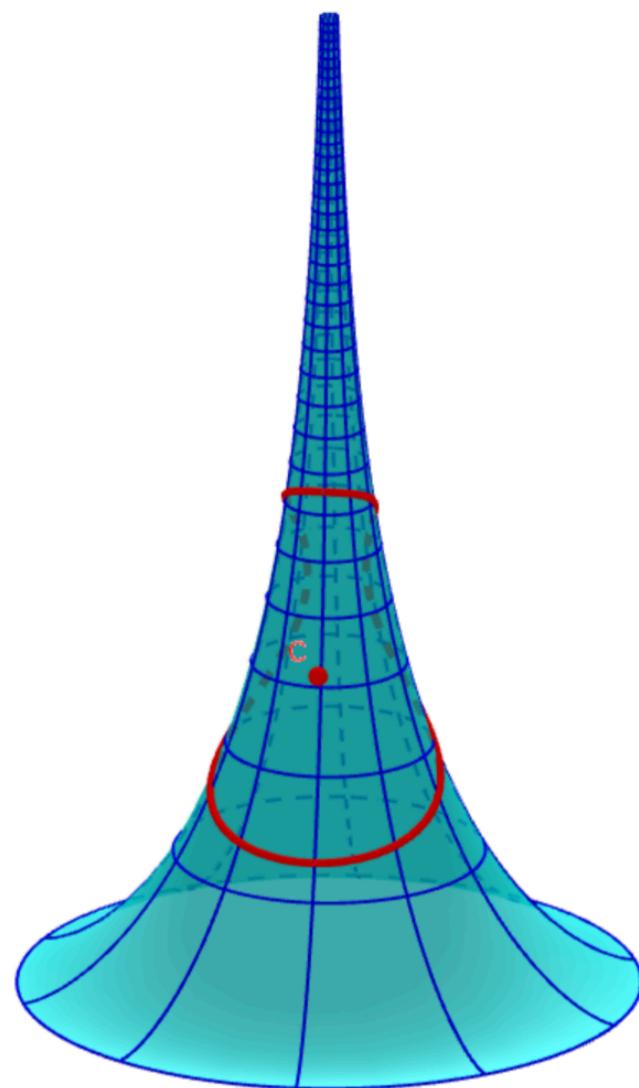
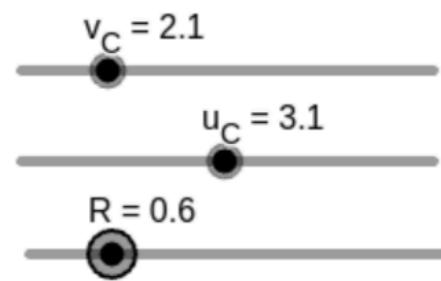
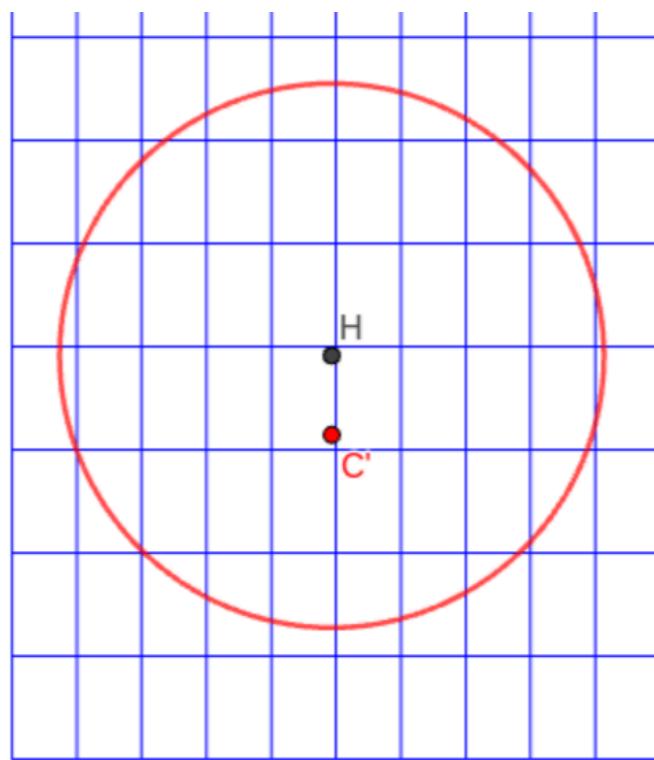
3) La projection

Gaspar Daguet, n°=21528

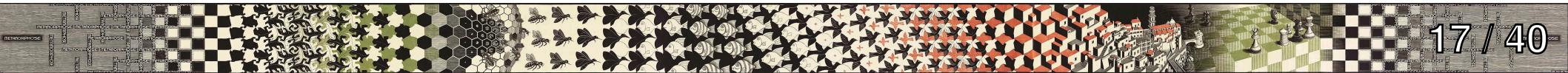


3) La projection

Gaspar Daguet, n°=21528



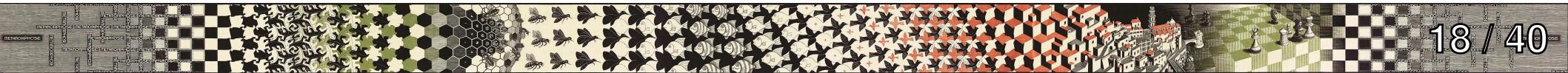
4) Projection des droites



Équation des droites (géodésique):

- Méridiens: $g : t \mapsto P(u, t)$
- Autres droites:

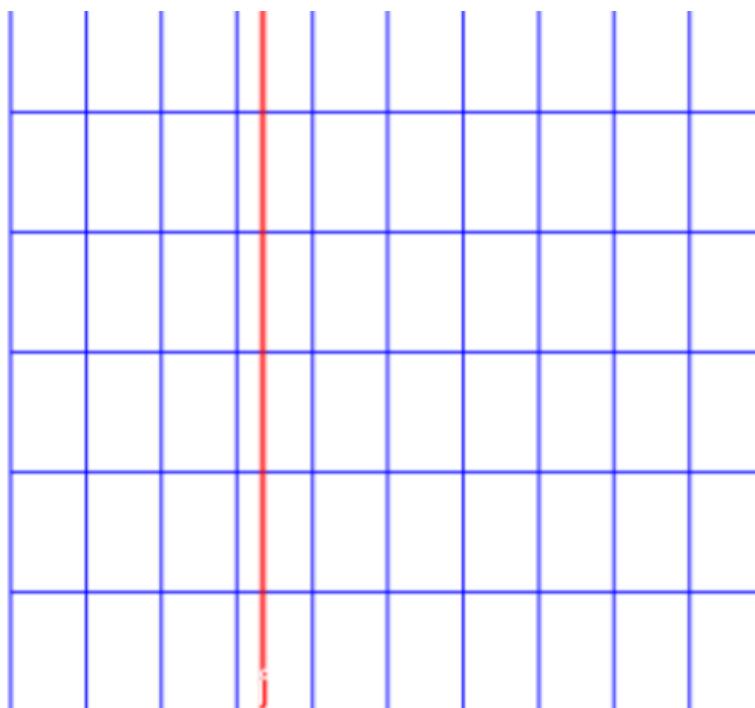
$$g : t \mapsto P\left(t, \operatorname{ch}\left(\sqrt{k^2 - (t + c)^2}\right)\right)$$



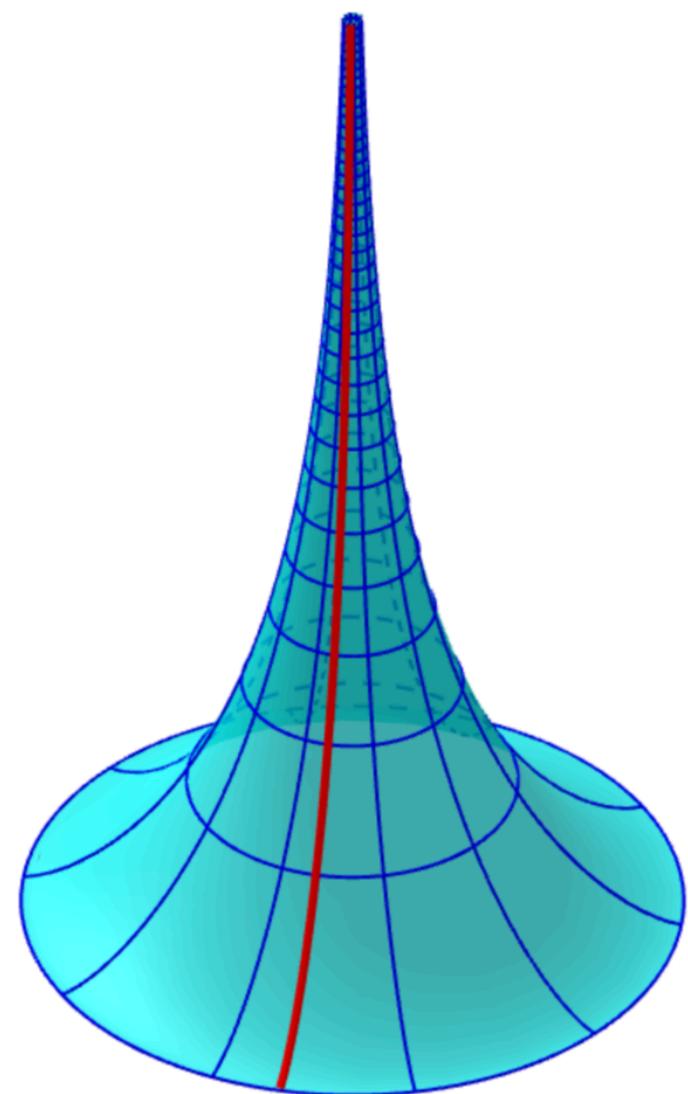
4) Projection des droites

Gaspar Daguet, n°=21528

Méridiens :



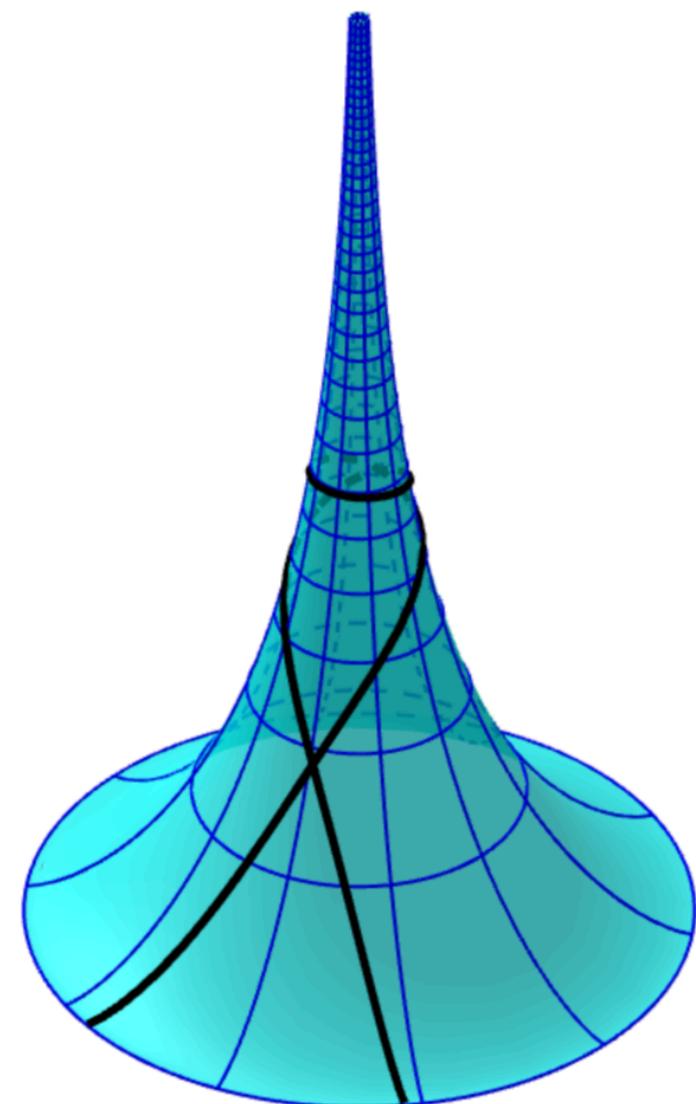
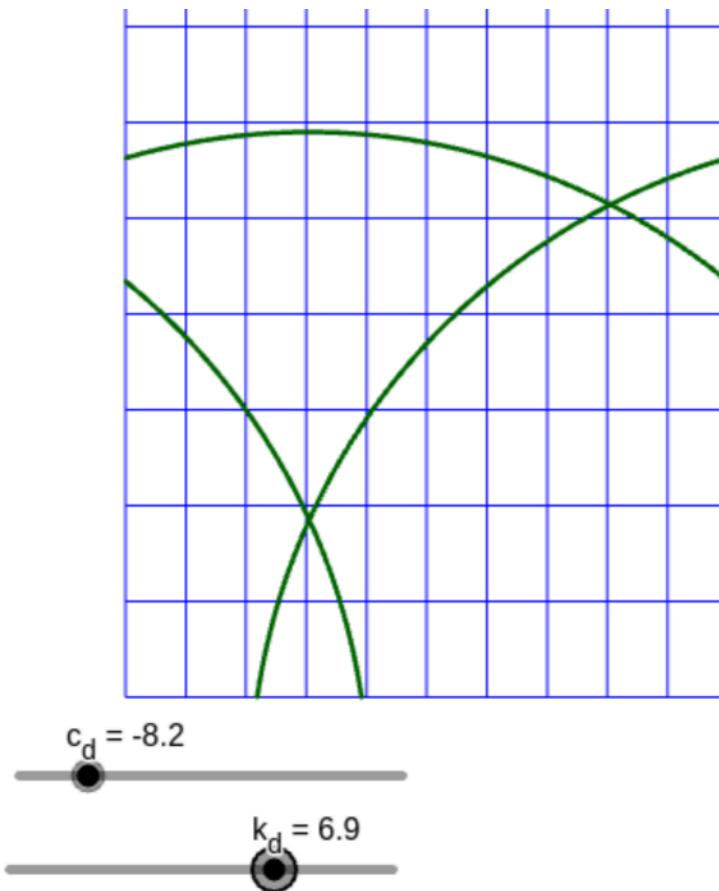
$$u = 2.1$$



4) Projection des droites

Gaspar Daguet, n°=21528

Droites quelconques :



4) Projection des droites

Gaspar Daguet, n°=21528

Sur les méridiens ($v_A = v_B$):

Sur la pseudosphère :

$$d(A, B) = \ln\left(\frac{\operatorname{ch}(u_B)}{\operatorname{ch}(u_A)}\right)$$

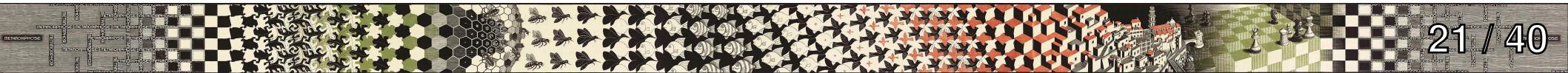
Longueur d'arc :

$$d(A, B) = \int_{t_A}^{t_B} \|g'(t)\| dt \neq$$

Sur la carte :

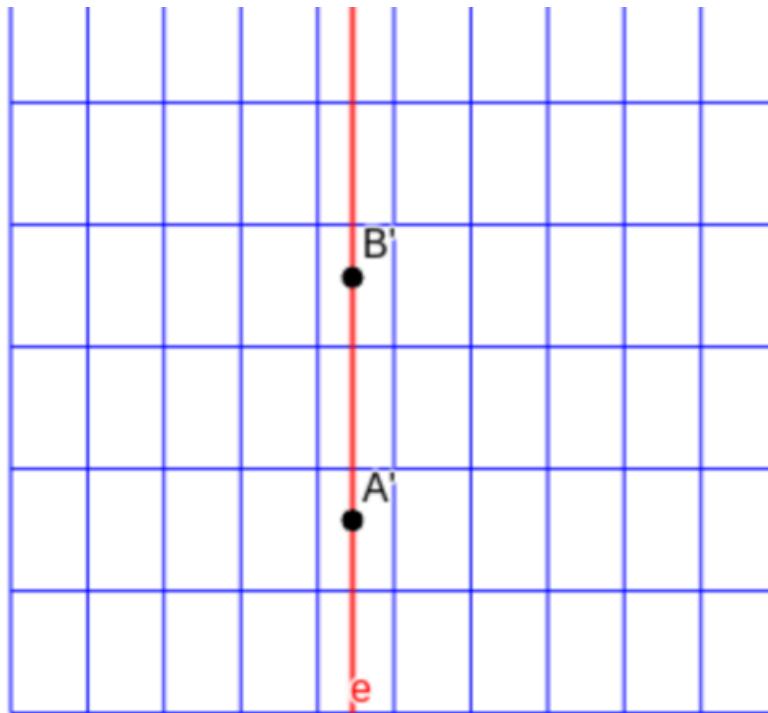
$$d(A, B) = \operatorname{ch}(u_B) - \operatorname{ch}(u_A)$$

Donc la projection n'est pas équivalente

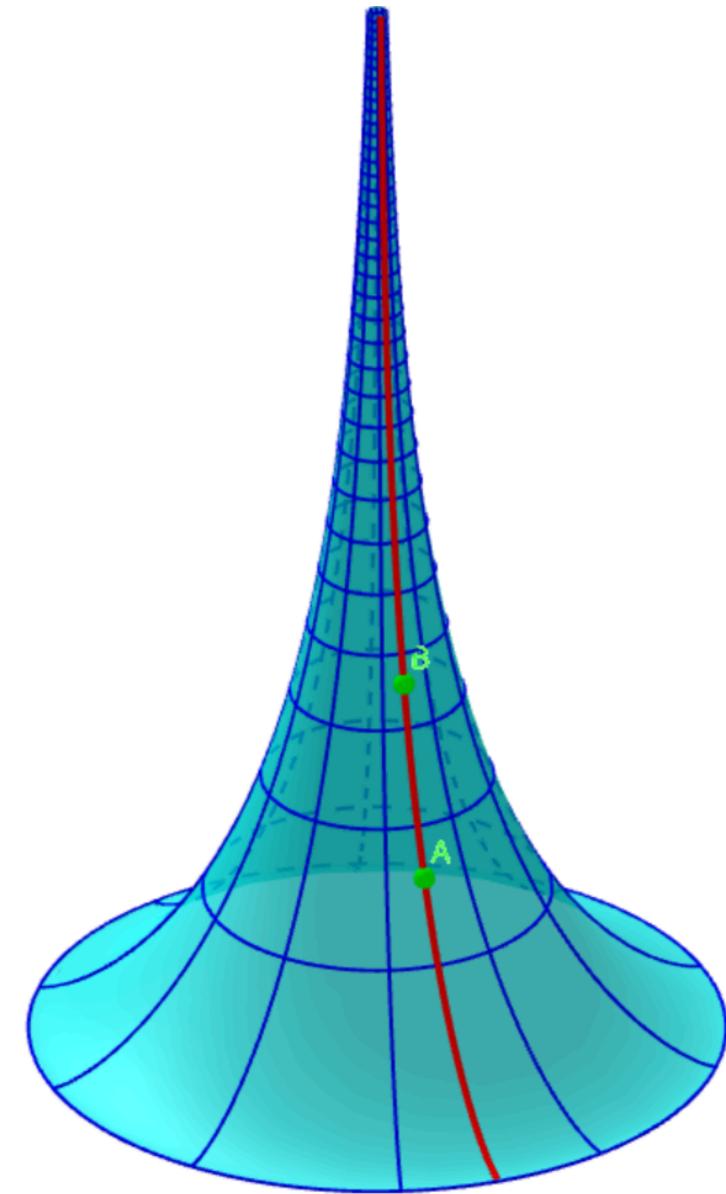


4) Projection des droites

Gaspar Daguet, n°=21528



$$d(A, B) = 1,99$$

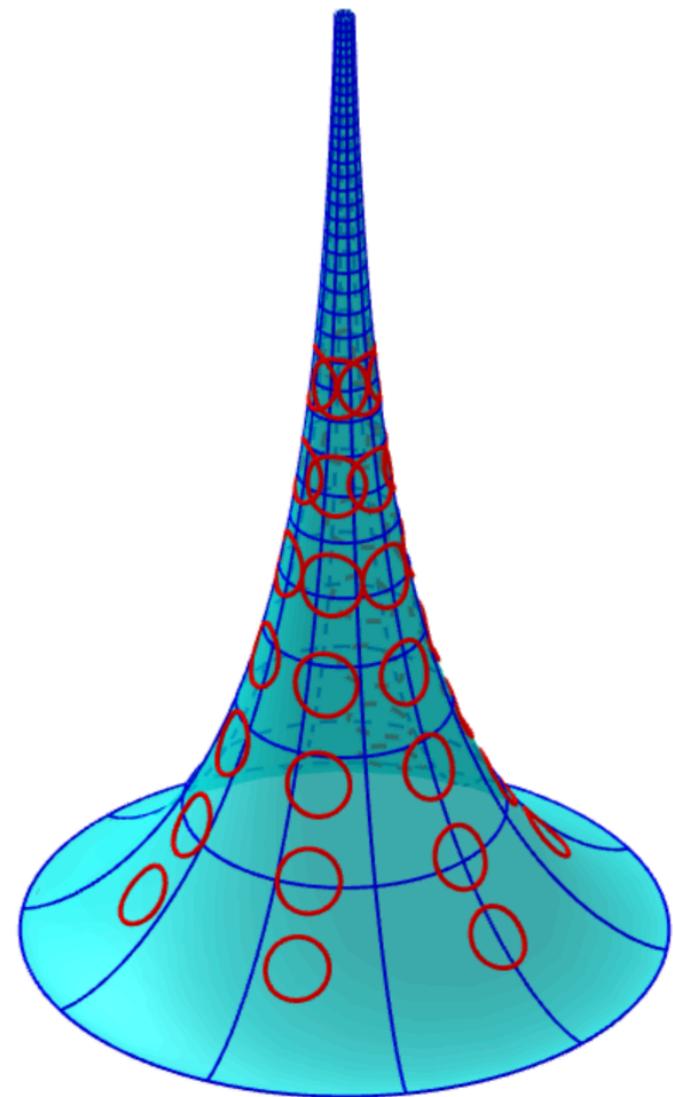
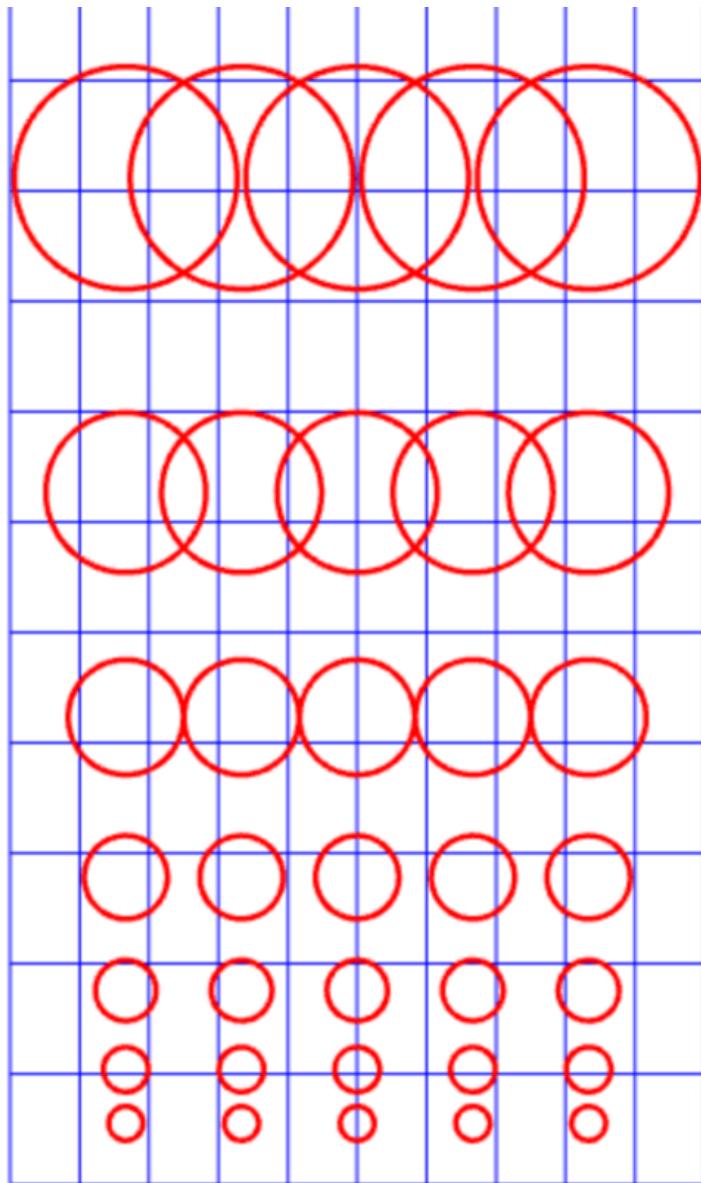


$$d(A, B) = 0,57$$



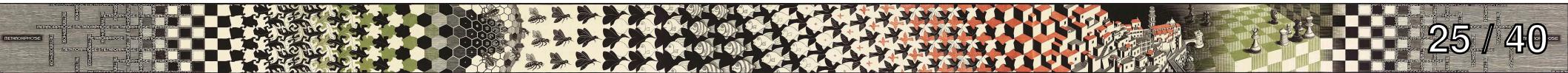
4) Projection des droites

Gaspar Daguet, n°=21528





Annexes



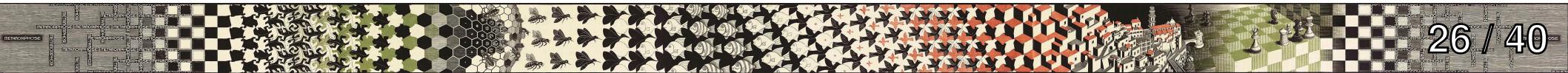
La première forme fondamentale est l'écriture formelle du produit scalaire dans le plan tangent au point p de la surface S

Or le produit scalaire entre deux vecteur du plan tangent peut s'écrire comme la combinaison linéaire entre $P_u(p)$ et $P_v(p)$, donc le produit scalaire peut s'écrire :

$$\begin{aligned} I(aP_u + bP_v, cP_u + dP_v) &= \langle aP_u + bP_v \mid cP_u + dP_v \rangle \\ &= \underbrace{ac(P_u)^2}_{=E} + \underbrace{(ad + bc)\langle P_u \mid P_v \rangle}_{=F} + \underbrace{bd\|P_v\|^2}_{=G} \end{aligned}$$

Ainsi sous forme matricielle pour les vecteur x et y du plan tangent :

$$I(x, y) = x^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} y$$



Dans le repère $(P_{u(p)}, P_{v(p)}, N(p))$, on peut faire un développement limité de la surface de la forme :

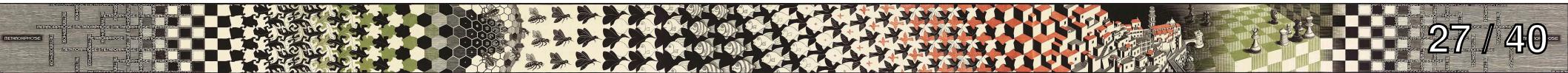
$$z = \underbrace{\langle P_{uu} \mid N \rangle}_{= \mathcal{L}} \frac{x^2}{2} + \underbrace{\langle P_{uv} \mid N \rangle}_{= \mathcal{M}} + \underbrace{\langle P_{vv} \mid N \rangle}_{= \mathcal{N}} \frac{x^2}{2}$$

Ainsi on définit la seconde forme fondamentale par :

$$\text{II} = L \, dx^2 + 2M \, dx \, dy + N \, dy^2$$

Et donc sous forme matricielle :

$$\text{II} = \begin{pmatrix} \mathcal{L} & \mathcal{M} \\ \mathcal{M} & \mathcal{N} \end{pmatrix}$$



L'endomorphisme de Weingarten (ou opérateur de forme) est défini par :

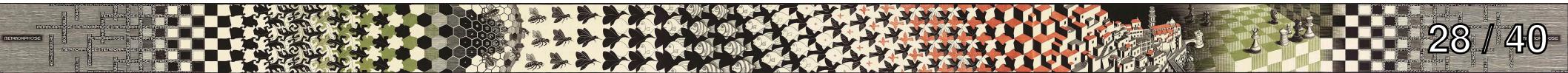
$$W = dN$$

De plus, ses valeurs propres sont les courbures principales au point p , et ses vecteurs propres sont la direction des courbures principales

Comme l'on définit la courbure de Gauss par le produit des valeurs propres, alors le déterminant de l'endomorphisme de Weingarten est la courbure de Gauss

$$K(p) = \det(W)$$

Nous avons également que la seconde forme fondamentale est la forme quadratique associée à l'endomorphisme de Weingarten

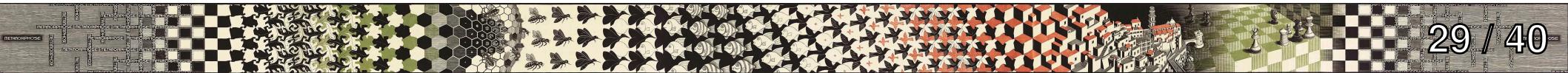


Soit \vec{x}, \vec{y} deux vecteurs du plan tangent au point p sur notre surface P tel que \vec{y} soit un vecteur propre de W associé à la valeur propre λ . Ainsi la première forme fondamentale nous donne le produit scalaire entre ces deux vecteurs :

$$\langle \vec{x} \mid \vec{y} \rangle = X^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} Y$$

Et comme la seconde forme fondamentale est la quadratique associée à l'endomorphisme de Weingarten :

$$\langle \vec{x} \mid W(\vec{y}) \rangle = X^T \begin{pmatrix} \mathcal{L} & \mathcal{M} \\ \mathcal{M} & \mathcal{N} \end{pmatrix} Y$$



Donc comme \vec{y} est vecteur propre:

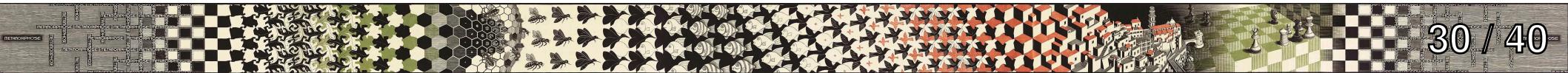
$$X^T \begin{pmatrix} \mathcal{L} & \mathcal{M} \\ \mathcal{M} & \mathcal{N} \end{pmatrix} Y = \langle \vec{x} \mid W(\vec{y}) \rangle = \lambda \langle \vec{x} \mid \vec{y} \rangle = \lambda X^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} Y$$

Comme ceci est vrai pour tout \vec{x} , on a:

$$\begin{pmatrix} \mathcal{L} & \mathcal{M} \\ \mathcal{M} & \mathcal{N} \end{pmatrix} Y = \lambda \begin{pmatrix} E & F \\ F & G \end{pmatrix} Y$$

Donc

$$\left(\begin{pmatrix} \mathcal{L} & \mathcal{M} \\ \mathcal{M} & \mathcal{N} \end{pmatrix} - \lambda \begin{pmatrix} E & F \\ F & G \end{pmatrix} \right) Y = 0$$



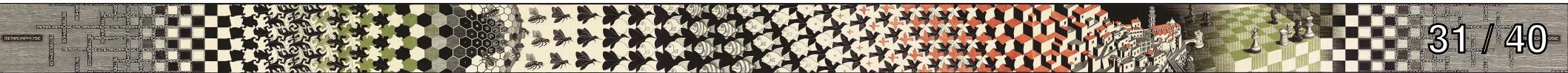
Donc $\begin{pmatrix} \mathcal{L} & \mathcal{M} \\ \mathcal{M} & \mathcal{N} \end{pmatrix} - \lambda \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \mathcal{L}-\lambda E & \mathcal{M}-\lambda F \\ \mathcal{M}-\lambda F & \mathcal{N}-\lambda G \end{pmatrix}$ est non inversible

Et donc $\begin{pmatrix} \mathcal{L}-\lambda E & \mathcal{M}-\lambda F \\ \mathcal{M}-\lambda F & \mathcal{N}-\lambda G \end{pmatrix} = 0$

Soit $(EG - F^2)\lambda^2 + (E \mathcal{N} + G \mathcal{L} - 2 \mathcal{M}F)\lambda + \mathcal{L}\mathcal{N} - \mathcal{M}^2 = 0$

Ainsi le terme $\frac{\mathcal{L}\mathcal{N}-\mathcal{M}^2}{EG-F^2}$ vaut le produit des valeurs propres et donc des courbures principales, donc :

$$K = \frac{\mathcal{L}\mathcal{N}-\mathcal{M}^2}{EG-F^2}$$



Annexes - Géogébra

Gaspar Daguet, n°=21528

$$S = \text{Surface}\left(\frac{\cos(u)}{\cosh(v)}, \frac{\sin(u)}{\cosh(v)}, v - \text{th}(v), u, 0, 2\pi, v, 0, 100\right)$$

$$= \begin{pmatrix} \frac{\cos(u)}{\cosh(v)} \\ \frac{\sin(u)}{\cosh(v)} \\ v - \text{th}(v) \end{pmatrix}$$

$S_x(x, y) = \frac{\cos(x)}{\cosh(y)}$

$S_y(x, y) = \frac{\sin(x)}{\cosh(y)}$

$S_z(x, y) = y - \text{th}(y)$

$\text{Sur} = 1$

-5 5

$c = 1$

0 2

$h_n = 100$

1 100

$$l1 = \text{Séquence}\left(\text{Courbe}\left(\frac{2\pi}{n}k, t, t, 1, c h_n + 1\right), k, 0, n\right)$$

$$= \left\{ \left(\frac{2\pi}{10} \cdot 0, t\right), \left(\frac{2\pi}{10} \cdot 1, t\right), \left(\frac{2\pi}{10} \cdot 2, t\right), \left(\frac{2\pi}{10} \cdot 3, t\right), \left(\frac{2\pi}{10} \cdot 4, t\right), \left(\frac{2\pi}{10} \cdot 5, t\right), \left(\frac{2\pi}{10} \cdot 6, t\right), \left(\frac{2\pi}{10} \cdot 7, t\right), \left(\frac{2\pi}{10} \cdot 8, t\right), \left(\frac{2\pi}{10} \cdot 9, t\right), \left(\frac{2\pi}{10} \cdot 10, t\right) \right\}$$

$$l2 = \text{Séquence}\left(\text{Courbe}(S_x(t, \cosh^{-1}(c k + 1)), S_y(t, \cosh^{-1}(c k + 1)), S_z(t, \cosh^{-1}(c k + 1)), t, 0, 2\pi), k, 0, h_n - 1\right)$$

$$= \left\{ \left(\frac{\cos(t)}{\cosh(\cosh^{-1}(1 \cdot 0 + 1))}, \frac{\sin(t)}{\cosh(\cosh^{-1}(1 \cdot 0 + 1))}, \cosh^{-1}(1 \cdot 0 + 1) - \text{th}(\cosh^{-1}(1 \cdot 0 + 1))\right), \left(\frac{\cos(t)}{\cosh(\cosh^{-1}(1 \cdot 1 + 1))}, \frac{\sin(t)}{\cosh(\cosh^{-1}(1 \cdot 1 + 1))}, \cosh^{-1}(1 \cdot 1 + 1) - \text{th}(\cosh^{-1}(1 \cdot 1 + 1))\right), \left(\frac{\cos(t)}{\cosh(\cosh^{-1}(1 \cdot 2 + 1))}, \frac{\sin(t)}{\cosh(\cosh^{-1}(1 \cdot 2 + 1))}, \cosh^{-1}(1 \cdot 2 + 1) - \text{th}(\cosh^{-1}(1 \cdot 2 + 1))\right) \right\}$$

$$l3 = \text{Séquence}\left(\text{Courbe}\left(S_x\left(\frac{2\pi}{n}k, \cosh^{-1}(t)\right), S_y\left(\frac{2\pi}{n}k, \cosh^{-1}(t)\right), S_z\left(\frac{2\pi}{n}k, \cosh^{-1}(t)\right), t, 1, h_n - 1\right), k, 0, n\right)$$



Donc on a :

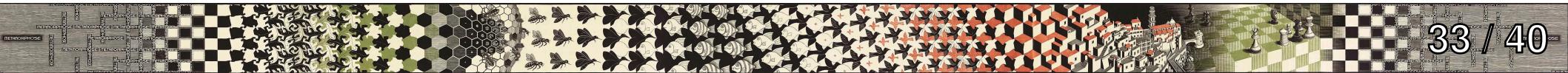
$$\frac{\partial y}{\partial x} = \frac{A}{B} = \frac{\operatorname{sh}(v)\partial v}{\partial u}$$

Ainsi en posant $x(u) = u \Leftrightarrow \frac{\partial x}{\partial u} = 1$,

$$\frac{\partial y}{\partial v} = \operatorname{sh}(v) \underbrace{\frac{\partial x}{\partial u}}_{=1}$$

Donc

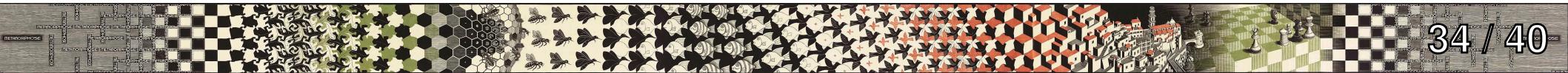
$$y(v) = \operatorname{ch}(v)$$



$$E = \|P_u\|^2 = \frac{1}{\operatorname{ch}(v)^2}$$

$$F = \langle P_u \mid P_v \rangle = \cos(u) \sin(u) \frac{\operatorname{sh}(v)}{\operatorname{ch}(v)^3} - \cos(u) \sin(u) \frac{\operatorname{sh}(v)}{\operatorname{ch}(v)^3} = 0$$

$$G = \|P_v\|^2 = \frac{\operatorname{sh}(v)^2}{\operatorname{ch}(v)^4} + \frac{\operatorname{sh}(v)^4}{\operatorname{ch}(v)^4} = \frac{\operatorname{sh}(v)^2}{\operatorname{ch}(v)^4} \underbrace{\left(1 + \operatorname{sh}(v)^2\right)}_{=\operatorname{ch}(v)^2} = \frac{\operatorname{sh}(v)^2}{\operatorname{ch}(v)^2}$$



Annexes - Calculs des Dérivés

Gaspar Daguet, n°=21528

$$P_u(u, v) = \begin{pmatrix} -\frac{\sin(u)}{\operatorname{ch}(v)} \\ \frac{\cos(u)}{\operatorname{ch}(v)} \\ 0 \end{pmatrix}$$

$$P_v(u, v) = \begin{pmatrix} -\cos(u) \frac{\operatorname{sh}(v)}{\operatorname{ch}(v)^2} \\ -\sin(u) \frac{\operatorname{sh}(v)}{\operatorname{ch}(v)^2} \\ \operatorname{th}(v)^2 \end{pmatrix}$$

$$P_{uu}(u, v) = \begin{pmatrix} -\frac{\cos(u)}{\operatorname{ch}(v)} \\ -\frac{\sin(u)}{\operatorname{ch}(v)} \\ 0 \end{pmatrix}$$

$$P_{vv}(u, v) = \begin{pmatrix} \cos(u) \frac{2\operatorname{sh}(v)^2 - \operatorname{ch}(v)^2}{\operatorname{ch}(v)^3} \\ \sin(u) \frac{2\operatorname{sh}(v)^2 - \operatorname{ch}(v)^2}{\operatorname{ch}(v)^3} \\ 2 \frac{\operatorname{sh}(v)}{\operatorname{ch}(v)^3} \end{pmatrix}$$

$$P_{uv} = \begin{pmatrix} \sin(u) \frac{\operatorname{sh}(v)}{\operatorname{ch}(v)^2} \\ -\cos(u) \frac{\operatorname{sh}(v)}{\operatorname{ch}(v)^2} \\ 0 \end{pmatrix}$$



Tout d'abord:

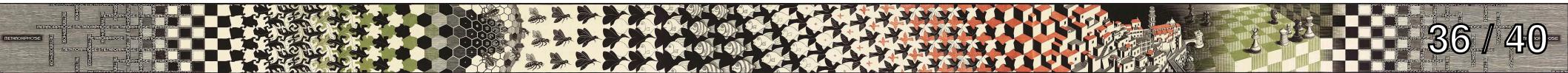
$$P_u \wedge P_v = \begin{pmatrix} \cos(u) \frac{\operatorname{sh}(v)^2}{\operatorname{ch}(v)^3} \\ \sin(u) \frac{\operatorname{sh}(v)^2}{\operatorname{ch}(v)^3} \\ \frac{\operatorname{sh}(v)}{\operatorname{ch}(v)^3} \end{pmatrix}$$

Donc:

$$\begin{aligned} \|P_u \wedge P_v\| &= \sqrt{\frac{\operatorname{sh}(v)^4}{\operatorname{ch}(v)^6} + \frac{\operatorname{sh}(v)^2}{\operatorname{ch}(v)^6}} \\ &= \frac{\operatorname{sh}(v)}{\operatorname{ch}(v)^3} \underbrace{\sqrt{\operatorname{sh}(v)^2 + 1}}_{=\operatorname{ch}(v)} = \frac{\operatorname{sh}(v)}{\operatorname{ch}(v)^2} \end{aligned}$$

Donc:

$$N = \frac{P_u \wedge P_v}{\|P_u \wedge P_v\|} = \begin{pmatrix} \cos(u) \frac{\operatorname{sh}(v)}{\operatorname{ch}(v)} \\ \sin(u) \frac{\operatorname{sh}(v)}{\operatorname{ch}(v)} \\ \frac{1}{\operatorname{ch}(v)} \end{pmatrix}$$



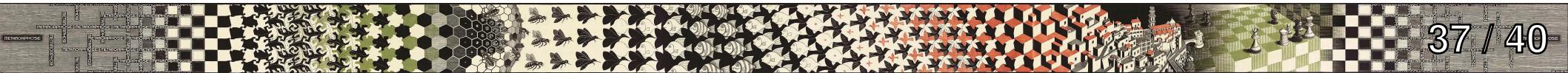
$$\mathcal{L} = \langle P_{uu} \mid N \rangle = -\frac{\operatorname{sh}(v)}{\operatorname{ch}(v)^2}$$

$$\mathcal{M} = \langle P_{uv} \mid N \rangle = \frac{\operatorname{sh}(v)^2}{\operatorname{ch}(v)^3} (\cos(u) \sin(u) - \cos(u) \sin(u)) = 0$$

$$\mathcal{N} = \langle P_{vv} \mid N \rangle = \frac{\operatorname{sh}(v)}{\operatorname{ch}(v)^4} (2\operatorname{sh}(v)^2 - \operatorname{ch}(v)^2 + 2)$$

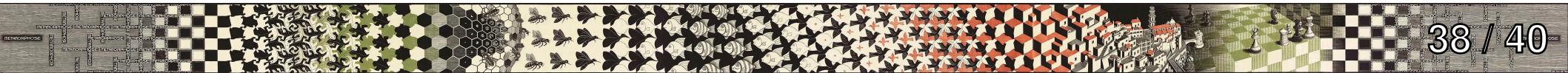
$$= \frac{\operatorname{sh}(v)}{\operatorname{ch}(v)^4} (2(\operatorname{ch}(v)^2 - 1) - \operatorname{ch}(v)^2 + 2)$$

$$= \frac{\operatorname{sh}(v)}{\operatorname{ch}(v)^4} \operatorname{ch}(v)^2 = \frac{\operatorname{sh}(v)}{\operatorname{ch}(v)^2}$$



On a :

$$\begin{aligned} K &= \frac{\mathcal{L}\mathcal{N}-\mathcal{M}^2}{EG-F^2} = -\frac{\operatorname{sh}(v)^2}{\operatorname{ch}(v)^4} \times \frac{1}{\frac{\operatorname{sh}(v)^2}{\operatorname{ch}(v)^4}} \\ &= -\frac{\operatorname{sh}(v)^2}{\operatorname{ch}(v)^4} \times \frac{\operatorname{ch}(v)^4}{\operatorname{sh}(v)^2} = -1 \end{aligned}$$



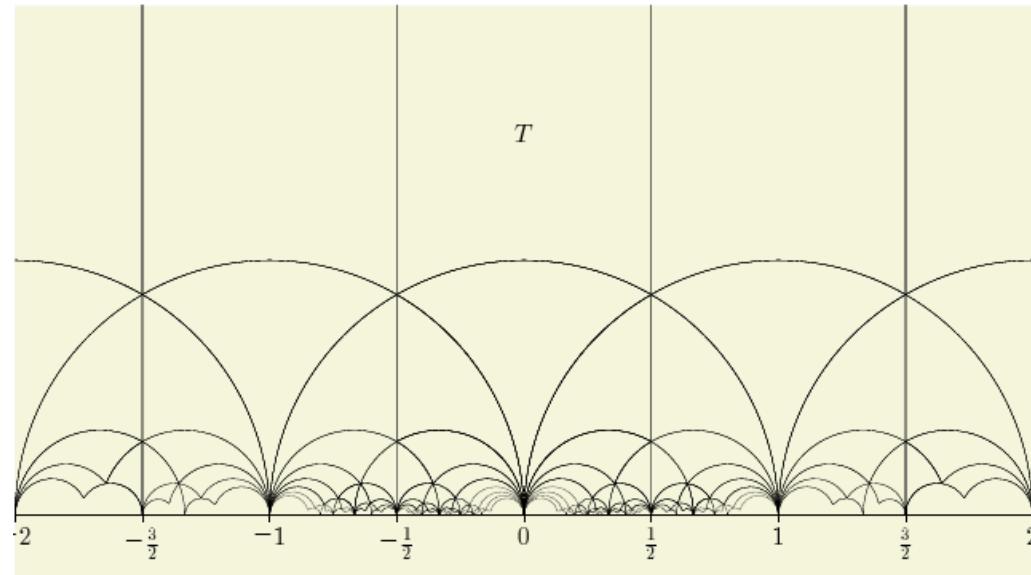
Le demi plan de Poincaré est un modèle de géométrie hyperbolique

Il est défini par :

$$\mathfrak{H}_2 = \{x + iy \in \mathbb{C}, y > 0\}$$

munis de la métrique :

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$



NB: Une métrique est une application g qui permet de définir le produit scalaire entre deux vecteurs, dans notre cas on note $ds^2 = g_{ij} dx_i dx_j$ où g_{ij} sont les composante de la matrice associé et où ds^2 est la longueur d'arc infinitésimal

Tout d'abord on peut définir la métrique sur notre surface par :

$$ds^2 = E dx^2 + 2F dx dy + G dy^2$$

Sur notre surface, la métrique s'écrit :

$$ds^2 = \frac{du^2}{\operatorname{ch}(v)^2} + \frac{\operatorname{sh}(v)^2}{\operatorname{ch}(v)^2} dv^2 = \frac{du^2 + \operatorname{sh}(v)^2 dy^2}{\operatorname{ch}(v)^2}$$

En appliquant notre porjection sur la métrique :

$$\begin{cases} x = u \\ y = \operatorname{ch}(v) \end{cases} \Leftrightarrow \begin{cases} dx = dy \\ dy = \operatorname{sh}(v) dv \end{cases}$$

donc :

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

