x

Équation des géodésiques

Soit le lagrangiens suivant :

$$\mathcal{L}(x^\alpha, \dot{x}^\alpha) = \left[g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu\right]^{\frac{1}{2}}$$

avec $\dot{x}^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}s}$
Ainsi on n'a :

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial x^\alpha} &= \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{-\frac{1}{2}}\frac{\partial}{\partial x^\alpha}\left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right) \\
\begin{matrix}\text{car } \dot{x}^\mu \text{ et } \dot{x}^\nu \\ \text{sont constant} \\ \text{face à } x^\alpha \to\end{matrix} &= \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{-\frac{1}{2}}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu\dot{x}^\nu \\
&= \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{-\frac{1}{2}}g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu
\end{aligned}$$

où on note : $g_{\mu\nu,\alpha} = \frac{\partial g_{\mu\nu}}{\partial x^\alpha}$
Et

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial \dot{x}^\alpha} &= \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{-\frac{1}{2}}\frac{\partial}{\partial \dot{x}^\alpha}\left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right) = \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{-\frac{1}{2}}g_{\mu\nu}\frac{\partial}{\partial \dot{x}^\alpha}\left(\dot{x}^\mu\dot{x}^\nu\right) \\
&= \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{-\frac{1}{2}}g_{\mu\nu}\left(\dot{x}^\nu\underbrace{\frac{\partial}{\partial \dot{x}^\alpha}\left(\dot{x}^\mu\right)}_{=\delta^\mu_\alpha} + \dot{x}^\mu\underbrace{\frac{\partial}{\partial \dot{x}^\alpha}\left(\dot{x}^\nu\right)}_{=\delta^\nu_\alpha}\right) \\
&= \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{-\frac{1}{2}}g_{\mu\nu}\left(\dot{x}^\nu\delta^\mu_\alpha + \underbrace{\dot{x}^\mu\delta^\nu_\alpha}_{=\dot{x}^\nu\delta^\mu_\alpha \text{ car } \mu \text{ et } \nu \text{ sont muets}}\right) \\
&= \cancel{\frac{1}{2}}\left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{-\frac{1}{2}}g_{\mu\nu}\cdot\cancel{2}\dot{x}^\nu\delta^\mu_\alpha \\
&= \left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{-\frac{1}{2}}g_{\alpha\nu}\dot{x}^\nu
\end{aligned}$$

Ainsi par les équations d'Euleur-La Grange, on obtient l'équation suivante :

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{-\frac{1}{2}}g_{\alpha\nu}\dot{x}^\nu\right) = \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{-\frac{1}{2}}g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu$$

En posant :

$$\mathrm{d}\lambda = \left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{\frac{1}{2}}\mathrm{d}s$$

$$\text{Donc } \frac{\mathrm{d}}{\mathrm{d}\lambda} = \left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}s}$$

Ainsi, en réécrivant l'équation précédente :

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{-\frac{1}{2}}g_{\alpha\nu}\frac{\mathrm{d}}{\mathrm{d}s}x^\nu\right) = \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right]^{-\frac{1}{2}}g_{\mu\nu,\alpha}\frac{\mathrm{d}}{\mathrm{d}s}x^\mu\frac{\mathrm{d}}{\mathrm{d}s}x^\nu$$

$$\text{donc } \left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\cancel{\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{-\frac{1}{2}}} g_{\alpha\nu} \cancel{\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{\frac{1}{2}}} \frac{\mathrm{d}}{\mathrm{d}\lambda}x^{\nu}\right) = \frac{1}{2}\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{-\frac{1}{2}}\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right] g_{\mu\nu,\alpha}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^{\mu}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^{\nu}$$

$$\text{donc } \cancel{\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{\frac{1}{2}}} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\alpha\nu}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^{\nu}\right) = \frac{1}{2}\cancel{\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{\frac{1}{2}}} g_{\mu\nu,\alpha}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^{\mu}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^{\nu}$$

$$\text{donc } \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\alpha\nu}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^{\nu}\right) = \frac{1}{2}g_{\mu\nu,\alpha}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^{\mu}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^{\nu}$$

Si on écrit l'action par rapport à s, on a :

$$S_1 = \int \left[g_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}s}x^{\mu}\frac{\mathrm{d}}{\mathrm{d}s}x^{\nu}\right]^{\frac{1}{2}} \mathrm{d}s$$

Alors en opérant le changement de variable S_1 devient :

$$\begin{aligned} S_1 &= \int \cancel{\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{\frac{1}{2}}}\left[g_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^{\mu}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^{\nu}\right]^{\frac{1}{2}} \cancel{\left[g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right]^{-\frac{1}{2}}} \mathrm{d}\lambda \\ &= \int \left[g_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^{\mu}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^{\nu}\right]^{\frac{1}{2}} \mathrm{d}\lambda \end{aligned}$$

Donc comme S_1 est invariante par la transformation $s \longrightarrow \lambda$ et que celle-ci est un diféomorphisme, on a alors :

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(g_{\alpha\nu}\dot{x}^{\nu}\right) = \frac{1}{2}g_{\mu\nu,\alpha}\dot{x}^{\mu}\dot{x}^{\nu}$$

Donc en calculant le terme de gauche :

$$\begin{aligned} \frac{d}{\mathrm{d}s}\left(g_{\alpha\nu}\dot{x}^{\nu}\right) &= \dot{x}^{\nu}\frac{\mathrm{d}}{\mathrm{d}s}g_{\alpha\nu} + g_{\alpha\nu}\ddot{x}^{\nu} \\ &= g_{\alpha\nu}\ddot{x}^{\nu} + \dot{x}^{\nu}\underbrace{\frac{\partial g_{\alpha\nu}}{\partial\mu}}_{=g_{\alpha\nu,\mu}}\underbrace{\frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}}_{=\dot{x}^{\mu}} \\ &= g_{\alpha\nu}\ddot{x}^{\nu} + g_{\alpha\nu,\mu}\dot{x}^{\nu}\dot{x}^{\mu} \end{aligned}$$

Ainsi :

$$\begin{aligned} \underbrace{g^{\alpha\beta}g_{\alpha\nu}}_{=\delta^{\beta}_{\nu}}\ddot{x}^{\nu} &= \frac{1}{2}g_{\mu\nu,\alpha}\dot{x}^{\mu}\dot{x}^{\nu} - g_{\alpha\nu,\mu}\dot{x}^{\nu}\dot{x}^{\mu} \\ \text{Donc } \ddot{x}^{\beta} &= \frac{1}{2}g^{\alpha\beta}\left(g_{\mu\nu,\alpha} - 2g_{\alpha\nu,\mu}\right)\dot{x}^{\nu}\dot{x}^{\mu} \\ \text{Donc } \ddot{x}^{\beta} &= \frac{1}{2}g^{\alpha\beta}\left(g_{\mu\nu,\alpha} - g_{\alpha\nu,\mu} - g_{\alpha\mu,\nu}\right)\dot{x}^{\nu}\dot{x}^{\mu} \end{aligned}$$

En introduisant les symboles de Christoffel $\Gamma^{\beta}_{\mu\nu} = \frac{1}{2}g^{\beta\alpha}\left(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right)$ on a :

$$\ddot{x}^{\beta} = -\Gamma^{\beta}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$$

$$\text{Soit } \ddot{x}^{\beta} + \Gamma^{\beta}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0$$

Commuteur des Co-dérivés covariante

On définit les dérivés covariantes par :

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\rho} v^\rho$$

Ainsi, soit v^ρ un vecteur :

$$\begin{aligned}
[\nabla_\mu ; \nabla_\nu] v^\rho &= \nabla_\mu(\nabla_\nu v^\rho) - \nabla_\nu(\nabla_\mu v^\rho) \\
&= \nabla_\mu(\partial_\nu v^\rho + \Gamma^\rho_{\nu\alpha} v^\alpha) - \nabla_\nu(\partial_\mu v^\rho + \Gamma^\rho_{\mu\alpha} v^\alpha) \\
&= \partial_\mu(\partial_\nu v^\rho + \Gamma^\rho_{\nu\alpha} v^\alpha) + \Gamma^\rho_{\mu\beta}(\partial_\nu v^\beta + \Gamma^\beta_{\nu\alpha} v^\alpha) - \partial_\nu(\partial_\mu v^\rho + \Gamma^\rho_{\mu\alpha} v^\alpha) - \Gamma^\rho_{\nu\beta}(\partial_\mu v^\beta + \Gamma^\beta_{\mu\alpha} v^\alpha) \\
&= \cancel{\partial_{\nu\mu} v^\rho} + \partial_\mu \Gamma^\rho_{\nu\alpha} v^\alpha + \Gamma^\rho_{\mu\beta}\partial_\nu v^\beta + \Gamma^\rho_{\mu\beta}\Gamma^\beta_{\nu\alpha} v^\alpha - \cancel{\partial_{\mu\nu} v^\rho} - \partial_\nu \Gamma^\rho_{\mu\alpha} v^\alpha - \Gamma^\rho_{\nu\beta}\partial_\mu v^\beta - \Gamma^\rho_{\nu\beta}\Gamma^\beta_{\mu\alpha} v^\alpha \\
&= \left(\Gamma^\rho_{\mu\beta}\Gamma^\beta_{\nu\alpha} - \Gamma^\rho_{\nu\beta}\Gamma^\beta_{\mu\alpha} + \partial_\mu \Gamma^\rho_{\nu\alpha} - \partial_\nu \Gamma^\rho_{\mu\alpha}\right) v^\alpha + \underbrace{\Gamma^\rho_{\mu\beta}\partial_\nu v^\beta - \Gamma^\rho_{\nu\beta}\partial_\mu v^\beta}_{=0}
\end{aligned}$$

Ainsi, on reconnait le tenseur de Riemann définis par : $R^\sigma_{\mu\nu\kappa} = \partial_\nu \Gamma^\sigma_{\mu\kappa} - \partial_\kappa \Gamma^\sigma_{\mu\nu} + \Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\kappa} - \Gamma^\sigma_{\kappa\lambda}\Gamma^\lambda_{\mu\nu}$
Donc on obtient la relation suivante :

$$[\nabla_\mu ; \nabla_\nu] v^\rho = R^\rho_{\alpha\mu\nu} v^\alpha$$

Dérivé covariante

On se place dans un système de coordonnée ξ^α, et on s'intérèse à un changement de coordonée vers x^α
Tout d'abord :

$$\begin{aligned}
0 = \frac{\mathrm{d}^2\xi^\alpha}{\mathrm{d}\tau^2} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\xi^\alpha}{\mathrm{d}\tau}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right) \\
&= \frac{\partial\xi^\alpha}{\partial x^\rho}\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}\tau^2} + \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\xi^\alpha}{\partial x^\mu}
\end{aligned}$$

Ainsi comme $\frac{\partial\xi^\alpha}{\partial x^\mu} \times \frac{\partial x^\mu}{\partial\xi^\beta} = \delta^\alpha_\beta$, en multipliant des deux cotés :

$$\begin{aligned}
0 &= \frac{\mathrm{d}^2 x^\rho}{\mathrm{d}\tau^2} + \frac{\partial x^\rho}{\partial\xi^\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \\
&= \frac{\mathrm{d}^2 x^\rho}{\mathrm{d}\tau^2} + \frac{\partial x^\rho}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}
\end{aligned}$$

En posant : $\left\{{}_\mu{}^\lambda{}_\nu\right\} = \frac{\partial x^\lambda}{\partial\xi^\beta}\frac{\partial^2\xi^\beta}{\partial x^\mu \partial x^\nu}$, on obtient finalement :

$$0 = \frac{\mathrm{d}^2 x^\rho}{\mathrm{d}\tau^2} + \left\{{}_\mu{}^\rho{}_\nu\right\}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$

En réécrivant le produit scalaire dans les coordonnées x^α à:

$$\eta_{\mu\nu}\,\mathrm{d}\xi^{\mu}\,\mathrm{d}\xi^{\nu} = \underbrace{\eta_{\mu\nu}\frac{\partial\xi^{\mu}}{\partial x^{\alpha}}\frac{\partial\xi^{\nu}}{\partial x^{\beta}}}_{=g_{\alpha\beta}}\mathrm{d}x^{\alpha}\,\mathrm{d}x^{\beta}$$

Donc :

$$g_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial\xi^{\mu}}{\partial x^{\alpha}}\frac{\partial\xi^{\nu}}{\partial x^{\beta}}$$

alors en dérivant :

$$\begin{aligned}
g_{\alpha\beta,\lambda} = \partial_{\lambda}g_{\alpha\beta} &= \eta_{\mu\nu}\frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial\xi^{\mu}}{\partial x^{\alpha}}\frac{\partial\xi^{\nu}}{\partial x^{\beta}}\right)\\
&= \underbrace{\eta_{\mu\nu}\frac{\partial\xi^{\mu}}{\partial x^{\alpha}}}_{=g_{\alpha\mu}\frac{\partial x^{\mu}}{\partial\xi^{\nu}}}\frac{\partial^{2}\xi^{\nu}}{\partial x^{\beta}\partial x^{\lambda}} + \underbrace{\eta_{\mu\nu}\frac{\partial\xi^{\nu}}{\partial x^{\beta}}}_{=g_{\beta\nu}\frac{\partial x^{\nu}}{\partial\xi^{\mu}}}\frac{\partial^{2}\xi^{\mu}}{\partial x^{\alpha}\partial x^{\lambda}}\\
&= g_{\alpha\mu}\underbrace{\frac{\partial x^{\mu}}{\partial\xi^{\nu}}\frac{\partial^{2}\xi^{\nu}}{\partial x^{\beta}\partial x^{\lambda}}}_{=\{{}_{\beta}{}^{\mu}{}_{\lambda}\}} + g_{\beta\nu}\underbrace{\frac{\partial x^{\nu}}{\partial\xi^{\mu}}\frac{\partial^{2}\xi^{\mu}}{\partial x^{\alpha}\partial x^{\lambda}}}_{=\{{}_{\alpha}{}^{\nu}{}_{\lambda}\}}\\
&= \{{}_{\beta}{}^{\mu}{}_{\lambda}\}g_{\mu\alpha} + \{{}_{\alpha}{}^{\nu}{}_{\lambda}\}g_{\nu\beta}
\end{aligned}$$

Donc en sommant pour difféante dérivé :

$$\begin{aligned}
g_{\alpha\beta,\lambda} + g_{\lambda\beta,\alpha} - g_{\alpha\lambda,\beta} = {} & \{{}_{\alpha}{}^{\mu}{}_{\lambda}\}g_{\mu\beta} + \cancel{\{{}_{\beta}{}^{\nu}{}_{\lambda}\}g_{\nu\alpha}}\\
& + \cancel{\{{}_{\beta}{}^{\mu}{}_{\alpha}\}g_{\mu\lambda}} + \{{}_{\lambda}{}^{\nu}{}_{\alpha}\}g_{\nu\beta}\\
& - \cancel{\{{}_{\alpha}{}^{\mu}{}_{\beta}\}g_{\mu\lambda}} - \cancel{\{{}_{\lambda}{}^{\nu}{}_{\beta}\}g_{\nu\alpha}}\\
& = \{{}_{\alpha}{}^{\mu}{}_{\lambda}\}g_{\mu\beta} + \{{}_{\lambda}{}^{\nu}{}_{\alpha}\}g_{\nu\beta} = 2\{{}_{\alpha}{}^{\mu}{}_{\lambda}\}g_{\mu\beta}
\end{aligned}$$

Ainsi :

$$\begin{aligned}
\Gamma^{\gamma}_{\alpha\lambda} &= \frac{1}{2}g^{\gamma\beta}\left(g_{\alpha\beta,\lambda} + g_{\lambda\beta,\alpha} - g_{\alpha\lambda,\beta}\right) = \frac{1}{2}g^{\gamma\beta}\times 2\{{}_{\alpha}{}^{\mu}{}_{\lambda}\}g_{\mu\beta}\\
&= \{{}_{\alpha}{}^{\mu}{}_{\lambda}\}\delta^{\gamma}_{\mu} = \{{}_{\alpha}{}^{\gamma}{}_{\lambda}\}
\end{aligned}$$

On trouve que : $\Gamma^{\mu}_{\alpha\beta} = \{{}_{\alpha}{}^{\mu}{}_{\beta}\}$, et pas concéquant : $\Gamma^{\mu}_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial\xi^{\nu}}\frac{\partial^{2}\xi^{\nu}}{\partial x^{\alpha}\partial x^{\beta}}$

Transformation de $\Gamma^{\lambda}_{\mu\nu}$

$$\begin{aligned}
\Gamma^{\lambda}_{\mu\nu}{}' = \{{}_{\mu}{}^{\lambda}{}_{\nu}\}' &= \frac{\partial x'^{\lambda}}{\partial\xi^{\beta}}\frac{\partial^{2}\xi^{\beta}}{\partial x'^{\mu}\partial x'^{\nu}}\\
&= \frac{\partial x^{k}}{\partial\xi^{\beta}}\frac{\partial x'^{\lambda}}{\partial x^{k}}\frac{\partial^{2}\xi^{\beta}}{\partial x^{\gamma}\partial x^{\alpha}}\frac{\partial x^{\gamma}}{\partial x'^{\mu}}\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\\
&= \frac{\partial x'^{\lambda}}{\partial x^{k}}\frac{\partial x^{\gamma}}{\partial x'^{\mu}}\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\underbrace{\frac{\partial x^{k}}{\partial\xi^{\beta}}\frac{\partial^{2}\xi^{\beta}}{\partial x^{\gamma}\partial x^{\alpha}}}_{=\{{}_{\gamma}{}^{k}{}_{\alpha}\}}\\
&= \frac{\partial x'^{\lambda}}{\partial x^{k}}\frac{\partial x^{\gamma}}{\partial x'^{\mu}}\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\{{}_{\gamma}{}^{k}{}_{\alpha}\}?
\end{aligned}$$

A.F.L.

Solution de Schwwarzschild

Équation sur $R_{\mu\nu}$

On se place dans un espace de dimension n
D'après l'équation de Einstein :

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$$

On recherche une solution pour un espace vide, statique et à symétrie sphérique
donc $T_{\mu\nu} = 0$, ainsi l'équation devient :

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= 0 \\
\text{donc } g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) &= 0 \\
\text{donc } \underbrace{g^{\mu\nu} R_{\mu\nu}}_{=R} - \frac{1}{2} \underbrace{g^{\mu\nu} g_{\mu\nu}}_{=\delta^{\mu}_{\mu}=n} &= 0 \\
\text{donc } R \left(1 - \frac{1}{2} n \right) &= 0 \\
\text{donc } R(2 - n) &= 0
\end{aligned}$$

Ainsi on a deux cas particulier :

- Soit $n = 2$, dans ce cas la gravité est topologique
- Soit $R = 0$

Or on recherche une solution pour un espace à 4 dimension (espace-temps), ainsi $R = 0$
En injectant dans l'équation d'Einstein :

$$\begin{gathered}
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \underbrace{R}_{=0} = 0 \\
\text{Donc } R_{\mu\nu} = 0
\end{gathered}$$

Écriture de la métrique

On sait que dans un espace isotropique, la métrique doit ressemblé à :

$$\mathrm{d}s^2 = A(r,t)\,\mathrm{d}t^2 - B(r,t)\,\mathrm{d}t(\vec{x} \cdot \mathrm{d}\vec{x}) - C(r,t)(\vec{x} \cdot \mathrm{d}\vec{x})^2 - D(r,t)\,\mathrm{d}\vec{x}^2$$

En opérant le changement de variables en coordonnées sphérique :

$$\begin{aligned}
x^1 &= r \sin(\theta) \cos(\varphi) \\
x^2 &= r \sin(\theta) \sin(\varphi) \\
x^3 &= r \cos(\theta)
\end{aligned}$$

Dans ce cas on a :

$$\vec{x} \cdot \vec{x} = r^2, \qquad \vec{x} \cdot \vec{x} = r\,\mathrm{d}r, \qquad \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x} = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2 \sin(\theta)^2\,\mathrm{d}\varphi^2$$

Ainsi la métrique devient :

$$\begin{aligned} \mathrm{d}s^2 &= A(t,r)\mathrm{d}t^2 - B(t,r)r\,\mathrm{d}t\,\mathrm{d}r - C(t,r)r^2\mathrm{d}r^2 - D(t,r)\mathrm{d}r^2 - D(t,r)r^2\mathrm{d}^2\Omega \\ &= A(t,r)\mathrm{d}t^2 - B'(t,r)\,\mathrm{d}t\,\mathrm{d}r - C'(r,t)\mathrm{d}r^2 - D'\mathrm{d}^2\Omega \end{aligned}$$

avec $\mathrm{d}^2\Omega = \mathrm{d}\theta^2 + \sin(\theta)^2\,\mathrm{d}\varphi^2$ l'angle solide et $B' = Br; \quad C' = Cr^2 + D$ et $D' = D$
On pose $\bar{r}^2 = D'$, alors :

$$\mathrm{d}s^2 = A(t,\bar{r})\mathrm{d}t^2 - B'(t,\bar{r})\,\mathrm{d}t\,\mathrm{d}\bar{r} - C'(t,\bar{r})\,\mathrm{d}\bar{r}^2 - \bar{r}^2\mathrm{d}^2\Omega$$

On pose également $\mathrm{d}\bar{t} = \Phi(t,\bar{r})\left[A(t,\bar{r})\,\mathrm{d}t - \frac{1}{2}B'(t,\bar{r})\,\mathrm{d}\bar{r}\right]$, donc

$$\mathrm{d}\bar{r}^2 = \Phi(t,\bar{r})\left[A^2\mathrm{d}t^2 + \frac{1}{4}B'^2\,\mathrm{d}\bar{r}^2 - AB'\,\mathrm{d}t\,\mathrm{d}\bar{r}\right]$$

$$\text{Donc } A\mathrm{d}t^2 - B'\,\mathrm{d}t\,\mathrm{d}\bar{r} = \frac{1}{A\Phi}\,\mathrm{d}\bar{t}^2 - \frac{B}{4A}\,\mathrm{d}\bar{r}^2$$