Gaspar Daguet

Info : DM2

En logique minimale :

Partie I. Lemme :

- **Lemme Minimaliste**

Lemme 1

$$\frac{\Gamma, \varphi_1 \vdash \varphi_2 \qquad \Gamma, \varphi_1 \vdash \neg\varphi_2}{\Gamma \vdash \neg\varphi_1}\,\neg_{ei}$$

preuve :

$$\dfrac{\dfrac{\dfrac{}{\Gamma, \varphi_1 \vdash \varphi_2}\,\text{hypp} \qquad \dfrac{}{\Gamma, \varphi_1 \vdash \neg\varphi_2}\,\text{hypp}}{\Gamma, \varphi_1 \vdash \bot}\,\neg_e}{\Gamma \vdash \neg\varphi_1}\,\neg_i$$

Lemme 2

$$\frac{}{\Gamma, \varphi \to \psi, \varphi \vdash \psi}\,\to_{\text{ax}}$$

preuve :

$$\dfrac{\dfrac{}{\Gamma, \varphi \to \psi, \varphi \vdash \varphi \to \psi}\,\text{ax} \qquad \dfrac{}{\Gamma, \varphi \to \psi, \varphi \vdash \varphi}\,\text{ax}}{\Gamma, \varphi \to \psi, \varphi \vdash \psi}\,\to_e$$

- **Lemme Classique**

Lemme 1

$$\frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}\,\neg\neg_e$$

preuve :

$$\dfrac{\dfrac{}{\Gamma, \neg\varphi \vdash \neg\varphi}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma \vdash \neg\neg\varphi}\,\text{hypp}}{\Gamma, \neg\varphi \vdash \neg\neg\varphi}\,\text{aff}}{\dfrac{\Gamma, \neg\varphi \vdash \bot}{\Gamma \vdash \varphi}\,\text{raa}}\,\neg_e$$

Partie II. Logique minimale :

N=° 1.

$$\dfrac{\dfrac{}{\Gamma \vdash p \vee (p \wedge q)}\,\text{ax} \qquad \dfrac{}{\Gamma, p \vdash p}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma, p \wedge q \vdash p \wedge q}\,\text{ax}}{\Gamma, p \wedge q \vdash p}\,\wedge_e^g}{p \vee (p \wedge q) \vdash p}\,\vee_e$$

Avec $\Gamma = \{p \vee (p \wedge q)\}$

N=° 2.

$$\dfrac{\dfrac{}{\Gamma \vdash p}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma \vdash p \to \neg p}\,\text{ax} \qquad \dfrac{}{\Gamma \vdash p}\,\text{ax}}{\Gamma \vdash \neg p}\,\to_e}{p \to \neg p \vdash \neg p}\,\neg_{\text{ei}}$$

Avec $\Gamma = \{p \to \neg p, p\}$

N=° 3.

$$\dfrac{\dfrac{}{\Gamma, p \vdash p \to q \vee r}\,\text{ax} \quad \dfrac{}{\Gamma, p \vdash p}\,\text{ax}}{\dfrac{\Gamma, p \vdash q \vee r}{\Gamma, p \vdash s}\,{\to_e} \quad \dfrac{}{\Gamma, p, q \vdash s}\,\overset{\to}{\text{ax}} \quad \dfrac{}{\Gamma, p, r \vdash s}\,\overset{\to}{\text{ax}}}\,{\vee_e}}{p \to (q \vee r), q \to s, r \to s \vdash p \to s}\,{\to_i}$$

Avec $\Gamma = \{p \to (q \vee r), q \to s, r \to s\}$

N=° 4.

$$\dfrac{\dfrac{}{\Gamma \vdash p \to q}\,\text{ax} \quad \dfrac{\dfrac{}{\Gamma \vdash p \wedge r}\,\text{ax}}{\Gamma \vdash p}\,{\wedge_e^g}}{\Gamma \vdash q}\,{\to_e} \quad \dfrac{\dfrac{}{\Gamma \vdash r \to s}\,\text{ax} \quad \dfrac{\dfrac{}{\Gamma \vdash p \wedge r}\,\text{ax}}{\Gamma \vdash r}\,{\wedge_e^d}}{\Gamma \vdash s}\,{\to_e}}{\Gamma \vdash q \wedge s}\,{\wedge_i}$$

Avec $\Gamma = \{p \to q, r \to s, p \wedge r\}$

N=° 5.

$$\dfrac{\dfrac{}{\Gamma \vdash p \vee r}\,\text{ax} \quad \dfrac{\dfrac{}{\Gamma, p \vdash q}\,\overset{\to}{\text{ax}}}{\Gamma, p \vdash q \vee s}\,{\vee_i^g} \quad \dfrac{\dfrac{}{\Gamma, r \vdash s}\,\overset{\to}{\text{ax}}}{\Gamma, r \vdash q \vee s}\,{\vee_i}}{\Gamma \vdash q \vee s}\,{\vee_e}$$

Avec $\Gamma = \{p \to q, r \to s, p \vee r\}$

N=° 6.

$$\dfrac{\dfrac{}{\Gamma_1 \vdash (p \vee q) \to r}\,\text{ax} \quad \dfrac{\dfrac{}{\Gamma_1 \vdash p}\,\text{ax}}{\Gamma_1 \vdash p \vee q}\,{\vee_e^g}}{\Gamma_1 \vdash r}\,{\to_e} \quad \dfrac{\dfrac{}{\Gamma_2 \vdash (p \vee q) \to r}\,\text{ax} \quad \dfrac{\dfrac{}{\Gamma_2 \vdash q}\,\text{ax}}{\Gamma_2 \vdash p \vee q}\,{\vee_e^d}}{\Gamma_2 \vdash r}\,{\to_e}}{(p \vee q) \to r \vdash (p \to r) \wedge (q \to r)}\,{\wedge_i + \to_i}$$

Avec $\Gamma_1 = \{(p \vee q) \to r, p\}$ et $\Gamma_2 = \{(p \vee q) \to r, q\}$

Partie III. Lois de De Morgan

N=° 7.

$$\dfrac{\dfrac{}{\Gamma_1 \vdash \neg(p \vee q)}\,\text{ax} \quad \dfrac{\dfrac{}{\Gamma_1 \vdash p}\,\text{ax}}{\Gamma_1 \vdash p \vee q}\,{\vee_i^g}}{\neg(p \vee q) \vdash \neg p}\,{\neg_{ei}} \quad \dfrac{\dfrac{}{\Gamma_2 \vdash \neg(p \vee q)}\,\text{ax} \quad \dfrac{\dfrac{}{\Gamma_2 \vdash q}\,\text{ax}}{\Gamma_2 \vdash p \vee q}\,{\vee_i^d}}{\neg(p \vee q) \vdash \neg q}\,{\neg_{ei}}}{\neg(p \vee q) \vdash \neg p \wedge \neg q}\,{\wedge_i}$$

Avec $\Gamma_1 = \{\neg(p \vee q), p\}$ et $\Gamma_2 = \{\neg(p \vee q), q\}$

N=° 8.

$$\dfrac{\dfrac{\dfrac{}{\Gamma, q \vdash \neg p \wedge \neg q}\,\text{ax}}{\Gamma, q \vdash \neg q}\,{\wedge_e^d} \quad \dfrac{}{\Gamma, q \vdash q}\,\text{ax}}{\Gamma, q \vdash \bot}\,{\neg_e} \quad \dfrac{\dfrac{}{\Gamma \vdash p \vee q}\,\text{ax} \quad \dfrac{\dfrac{}{\Gamma, p \vdash p}\,\text{ax} \quad \dfrac{\dfrac{}{\Gamma, p \vdash \neg p \wedge \neg q}\,\text{ax}}{\Gamma, p \vdash \neg p}\,{\wedge_e^g}}{\Gamma, p \vdash \bot}\,{\neg_e}}{\neg p \wedge \neg q, p \vee q \vdash \bot}\,{\vee_e}}{\neg p \wedge \neg q \vdash (p \vee q)}\,{\neg_i}$$

Avec $\Gamma = \{\neg p \wedge \neg q, p \vee q\}$

N=° 9.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg q \vdash p \wedge q}\text{ax}}{\Gamma, \neg q \vdash q}\wedge_e^d \quad \dfrac{}{\Gamma, \neg q \vdash \neg q}\text{ax}}{\Gamma, \neg q \vdash \bot}\neg_e \quad \dfrac{}{\Gamma \vdash \neg p \vee \neg q}\text{ax} \quad \dfrac{\dfrac{}{\Gamma, \neg p \vdash \neg p}\text{ax} \quad \dfrac{\dfrac{}{\Gamma, \neg p \vdash p \wedge q}\text{ax}}{\Gamma, \neg p \vdash p}\wedge_e^g}{\Gamma, \neg p \vdash \bot}\neg_e}{\neg p \vee \neg q, p \wedge q \vdash \bot}\vee_e}{\neg p \vee \neg q \vdash \neg(p \wedge q)}\neg_i$$

Avec $\Gamma = \{\neg p \vee \neg q, p \wedge q\}$

N=° 10.

$$\dfrac{\dfrac{}{\Gamma \vdash \neg(p \wedge q)}\text{ax} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg\neg p \wedge \neg\neg q}\text{q.7}}{\Gamma \vdash \neg\neg p}\wedge_e^g}{\Gamma \vdash p}\neg\neg_e \quad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg\neg p \wedge \neg\neg q}\text{q.7}}{\Gamma \vdash \neg\neg q}\wedge_e^d}{\Gamma \vdash q}\neg\neg_e}{\Gamma \vdash p \wedge q}\wedge_i}{\Gamma \vdash \bot}\neg_e}{\neg(p \wedge q) \vdash \neg p \vee \neg q}\text{raa}$$

Avec $\Gamma = \{\neg(p \wedge q), \neg(\neg p \vee \neg q)\}$

Partie IV. Logique intuitionniste

N=° 11.

$$\dfrac{\dfrac{\dfrac{}{\neg p, p \vdash p}\text{ax} \quad \dfrac{}{\neg p, p \vdash \neg p}\text{ax}}{\neg p, p \vdash \bot}\neg_e}{\neg p, p \vdash q}\bot_e}{\neg p \vdash p \to q}\to_i$$

N=° 12.

$$\dfrac{\dfrac{}{\Gamma \vdash p \vee q}\text{ax} \quad \dfrac{\dfrac{}{\Gamma, p \vdash p}\text{ax} \quad \dfrac{\dfrac{\dfrac{}{\Gamma, q \vdash q}\text{ax} \quad \dfrac{}{\Gamma, q \vdash \neg q}\text{ax}}{\Gamma, q \vdash \bot}\neg_e}{\Gamma, q \vdash p}\bot_e}{p \vee q, \neg q \vdash p}\vee_e$$

Avec $\Gamma = \{p \vee q, \neg q\}$

N=° 13.

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg(p \to q)} \quad \dfrac{\dfrac{}{\Gamma, p \vdash q}\text{ax}}{\Gamma \vdash p \to q}\to_i}{\Gamma \vdash \bot}\neg_e}{\Gamma \vdash p}\bot_e}{\neg(p \to q) \vdash q \to p}\to_i$$

Avec $\Gamma = \{\neg(p \to q), q\}$

Partie V. logique classique

N=° 14.

$$\dfrac{\dfrac{}{\Gamma \vdash q \vee \neg q}\,\text{t.e.} \qquad \dfrac{\dfrac{}{\Gamma, q, p \vdash q}\,\text{ax}}{\Gamma, q \vdash (p \to q) \vee (p \to r)}\,\vee_i^g + \to_i \qquad \dfrac{\dfrac{\text{Voir ci-dessous}}{\Gamma, \neg q, p \vdash r}\,\text{ax}}{\Gamma, \neg q \vdash (p \to q) \vee (p \to r)}\,\vee_i^d + \to_i}{p \to (q \vee r) \vdash (p \to q) \vee (p \to r)}\,\vee_e$$

Le ci-dessous en question :

$$\dfrac{\dfrac{\dfrac{}{\Gamma_2 \vdash p \to (q \vee r)}\,\text{ax} \qquad \dfrac{}{\Gamma_2 \vdash p}\,\text{ax}}{\Gamma_2 \vdash q \vee r}\,\to_e \qquad \dfrac{}{\Gamma, r \vdash r}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma_2, q \vdash q}\,\text{ax} \qquad \dfrac{}{\Gamma_2, q \vdash \neg q}\,\text{ax}}{\Gamma_2, q \vdash r}\,\neg_e}{\Gamma, \neg q, p \vdash r}\,\vee_e$$

Avec $\Gamma = \{p \to (q \vee r)\}$ et $\Gamma_2 = \Gamma \cup \{\neg q, p\}$