Équation des géodésiques

Soit le lagrangiens suivant :

$$\mathcal{L}(x^\alpha, \dot{x}^\alpha) = \left[g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu \right]^{\frac{1}{2}}$$

avec $\dot{x}^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}s}$

Ainsi on n'a :

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{1}{2} \left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{-\frac{1}{2}} \frac{\partial}{\partial x^\alpha} \left(g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right)$$

$$\text{car } \dot{x}^\mu \text{ et } \dot{x}^\nu$$

$$\text{sont constant}$$

$$\text{face à } x^\alpha \rightarrow = \frac{1}{2} \left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{-\frac{1}{2}} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \dot{x}^\mu \dot{x}^\nu$$

$$= \frac{1}{2} \left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{-\frac{1}{2}} g_{\mu\nu,\alpha} \dot{x}^\mu \dot{x}^\nu$$

où on note : $g_{\mu\nu,\alpha} = \frac{\partial g_{\mu\nu}}{\partial x^\alpha}$

Et

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = \frac{1}{2} \left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{-\frac{1}{2}} \frac{\partial}{\partial \dot{x}^\alpha} \left(g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right) = \frac{1}{2} \left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{-\frac{1}{2}} g_{\mu\nu} \frac{\partial}{\partial \dot{x}^\alpha} \left(\dot{x}^\mu \dot{x}^\nu \right)$$

$$= \frac{1}{2} \left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{-\frac{1}{2}} g_{\mu\nu} \left(\dot{x}^\nu \underbrace{\frac{\partial}{\partial \dot{x}^\alpha} (\dot{x}^\mu)}_{=\delta^\mu_\alpha} + \dot{x}^\mu \underbrace{\frac{\partial}{\partial \dot{x}^\alpha} (\dot{x}^\nu)}_{=\delta^\nu_\alpha} \right)$$

$$= \frac{1}{2} \left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{-\frac{1}{2}} g_{\mu\nu} \left(\dot{x}^\nu \delta^\mu_\alpha + \underbrace{\dot{x}^\mu \delta^\nu_\alpha}_{=\dot{x}^\nu \delta^\mu_\alpha \text{ car } \mu \text{ et } \nu \text{ sont muet}} \right)$$

$$= \frac{1}{\cancel{2}} \left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{-\frac{1}{2}} g_{\mu\nu} \cdot \cancel{2} \dot{x}^\nu \delta^\mu_\alpha$$

$$= \left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{-\frac{1}{2}} g_{\alpha\nu} \dot{x}^\nu$$

Ainsi par les équation d'Euleur-La Grange, on obtient l'équation suivante :

$$\frac{\mathrm{d}}{\mathrm{d}s} \left(\left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{-\frac{1}{2}} g_{\alpha\nu} \dot{x}^\nu \right) = \frac{1}{2} \left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{-\frac{1}{2}} g_{\mu\nu,\alpha} \dot{x}^\mu \dot{x}^\nu$$

En posant :

$$\mathrm{d}\lambda = \left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{\frac{1}{2}} \mathrm{d}s$$

$$\text{Donc } \frac{\mathrm{d}}{\mathrm{d}\lambda} = \left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{-\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}s}$$

Ainsi, en réécrivant l'équation précédente :

$$\frac{\mathrm{d}}{\mathrm{d}s} \left(\left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{-\frac{1}{2}} g_{\alpha\nu} \frac{\mathrm{d}}{\mathrm{d}s} x^\nu \right) = \frac{1}{2} \left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{-\frac{1}{2}} g_{\mu\nu,\alpha} \frac{\mathrm{d}}{\mathrm{d}s} x^\mu \frac{\mathrm{d}}{\mathrm{d}s} x^\nu$$

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donc $[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{\frac{1}{2}}\dfrac{\mathrm{d}}{\mathrm{d}\lambda}\left(\cancel{[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}}}g_{\alpha\nu}\cancel{[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{\frac{1}{2}}}\dfrac{\mathrm{d}}{\mathrm{d}\lambda}x^\nu\right) = \dfrac{1}{2}[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}}[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]g_{\mu\nu,\alpha}\dfrac{\mathrm{d}}{\mathrm{d}\lambda}x^\mu\dfrac{\mathrm{d}}{\mathrm{d}\lambda}x^\nu$

donc $\cancel{[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{\frac{1}{2}}}\dfrac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\alpha\nu}\dfrac{\mathrm{d}}{\mathrm{d}\lambda}x^\nu\right) = \dfrac{1}{2}\cancel{[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{\frac{1}{2}}}g_{\mu\nu,\alpha}\dfrac{\mathrm{d}}{\mathrm{d}\lambda}x^\mu\dfrac{\mathrm{d}}{\mathrm{d}\lambda}x^\nu$

donc $\dfrac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\alpha\nu}\dfrac{\mathrm{d}}{\mathrm{d}\lambda}x^\nu\right) = \dfrac{1}{2}g_{\mu\nu,\alpha}\dfrac{\mathrm{d}}{\mathrm{d}\lambda}x^\mu\dfrac{\mathrm{d}}{\mathrm{d}\lambda}x^\nu$

Si on écrit l'action par rapport à s, on a :

$$S_1 = \int\left[g_{\mu\nu}\dfrac{\mathrm{d}}{\mathrm{d}s}x^\mu\dfrac{\mathrm{d}}{\mathrm{d}s}x^\nu\right]^{\frac{1}{2}}\mathrm{d}s$$

Alors en oprérant le changement de variable S_1 devient :

$$S_1 = \int\cancel{[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{\frac{1}{2}}}\left[g_{\mu\nu}\dfrac{\mathrm{d}}{\mathrm{d}\lambda}x^\mu\dfrac{\mathrm{d}}{\mathrm{d}\lambda}x^\nu\right]^{\frac{1}{2}}\cancel{[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}}}\mathrm{d}\lambda$$

$$= \int\left[g_{\mu\nu}\dfrac{\mathrm{d}}{\mathrm{d}\lambda}x^\mu\dfrac{\mathrm{d}}{\mathrm{d}\lambda}x^\nu\right]^{\frac{1}{2}}\mathrm{d}\lambda$$

Donc comme S_1 est invariante par la transformation $s\longrightarrow\lambda$ et que celle-ci est un diféomorphisme, on a alors :

$$\dfrac{\mathrm{d}}{\mathrm{d}s}(g_{\alpha\nu}\dot{x}^\nu) = \dfrac{1}{2}g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu$$

Donc en calculant le terme de gauche :

$$\dfrac{d}{\mathrm{d}s}(g_{\alpha\nu}\dot{x}^\nu) = \dot{x}^\nu\dfrac{\mathrm{d}}{\mathrm{d}s}g_{\alpha\nu} + g_{\alpha\nu}\ddot{x}^\nu$$

$$= g_{\alpha\nu}\ddot{x}^\nu + \dot{x}^\nu\underbrace{\dfrac{\partial g_{\alpha\nu}}{\partial\mu}}_{=g_{\alpha\nu,\mu}}\underbrace{\dfrac{\mathrm{d}x^\mu}{\mathrm{d}s}}_{=\dot{x}^\mu}$$

$$= g_{\alpha\nu}\ddot{x}^\nu + g_{\alpha\nu,\mu}\dot{x}^\nu\dot{x}^\mu$$

Ainsi :

$$\underbrace{g^{\alpha\beta}g_{\alpha\nu}}_{=\delta^\beta_\nu}\ddot{x}^\nu = \dfrac{1}{2}g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu - g_{\alpha\nu,\mu}\dot{x}^\nu\dot{x}^\mu$$

Donc $\ddot{x}^\beta = \dfrac{1}{2}g^{\alpha\beta}\big(g_{\mu\nu,\alpha} - 2g_{\alpha\nu,\mu}\big)\dot{x}^\nu\dot{x}^\mu$

Donc $\ddot{x}^\beta = \dfrac{1}{2}g^{\alpha\beta}\big(g_{\mu\nu,\alpha} - g_{\alpha\nu,\mu} - g_{\alpha\mu,\nu}\big)\dot{x}^\nu\dot{x}^\mu$

En introduisant les symboles de Cristoffel $\Gamma^\beta_{\mu\nu} = \frac{1}{2}g^{\beta\alpha}\big(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\big)$ on a :

$$\ddot{x}^\beta = -\Gamma^\beta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

Soit $\ddot{x}^\beta + \Gamma^\beta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$

Commuteur des Co-dérivé

On définis les dérivé covariante par :

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\rho} v^\rho$$

Ainsi, soit v^ρ un vecteur :

$$\left[\nabla_\mu ; \nabla_\nu\right] v^\rho = \nabla_\mu(\nabla_\nu v^\rho) - \nabla_\nu(\nabla_\mu v^\rho)$$

$$= \nabla_\mu(\partial_\nu v^\rho + \Gamma^\rho_{\nu\alpha} v^\alpha) - \nabla_\nu(\partial_\mu v^\rho + \Gamma^\rho_{\mu\alpha} v^\alpha)$$

$$= \partial_\mu(\partial_\nu v^\rho + \Gamma^\rho_{\nu\alpha} v^\alpha) + \Gamma^\rho_{\mu\beta}(\partial_\nu v^\beta + \Gamma^\beta_{\nu\alpha} v^\alpha) - \partial_\nu(\partial_\mu v^\rho + \Gamma^\rho_{\mu\alpha} v^\alpha) - \Gamma^\rho_{\nu\beta}(\partial_\mu v^\beta + \Gamma^\beta_{\mu\alpha} v^\alpha)$$

$$= \cancel{\partial_{\nu\mu} v^\rho} + \partial_\mu \Gamma^\rho_{\nu\alpha} v^\alpha + \Gamma^\rho_{\mu\beta} \partial_\nu v^\beta + \Gamma^\rho_{\mu\beta} \Gamma^\beta_{\nu\alpha} v^\alpha - \cancel{\partial_{\mu\nu} v^\rho} - \partial_\nu \Gamma^\rho_{\mu\alpha} v^\alpha - \Gamma^\rho_{\nu\beta} \partial_\mu v^\beta - \Gamma^\rho_{\nu\beta} \Gamma^\beta_{\mu\alpha} v^\alpha$$

$$= \left(\Gamma^\rho_{\mu\beta} \Gamma^\beta_{\nu\alpha} - \Gamma^\rho_{\nu\beta} \Gamma^\beta_{\mu\alpha} + \partial_\mu \Gamma^\rho_{\nu\alpha} - \partial_\nu \Gamma^\rho_{\mu\alpha}\right) v^\alpha + \underbrace{\Gamma^\rho_{\mu\beta} \partial_\nu v^\beta - \Gamma^\rho_{\nu\beta} \partial_\mu v^\beta}_{=0}$$

Ainsi on reconnais le tenseur de Riemann définis par : $R^\sigma_{\mu\nu\kappa} = \partial_\nu \Gamma^\sigma_{\mu\kappa} - \partial_\kappa \Gamma^\sigma_{\mu\nu} + \Gamma^\sigma_{\nu\lambda} \Gamma^\lambda_{\mu\kappa} - \Gamma^\sigma_{\kappa\lambda} \Gamma^\lambda_{\mu\nu}$

Donc on obtient la relation suivante :

$$\left[\nabla_\mu ; \nabla_\nu\right] v^\rho = R^\rho_{\alpha\mu\nu} v^\alpha$$

Dérivé covariante

On se place dans un sytème de coordonnée ξ^α, et on s'intérèse à un changement de coordonée vers x^α

Tout d'abord :

$$0 = \frac{\mathrm{d}^2 \xi^\alpha}{\mathrm{d}\tau^2} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\xi^\alpha}{\mathrm{d}\tau}\right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right)$$

$$= \frac{\partial\xi^\alpha}{\partial x^\rho} \frac{\mathrm{d}^2 x^\rho}{\mathrm{d}\tau^2} + \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\partial\xi^\alpha}{\partial x^\mu}$$

Ainsi comme $\frac{\partial\xi^\alpha}{\partial x^\mu} \times \frac{\partial x^\mu}{\partial\xi^\beta} = \delta^\alpha_\beta$, en multipliant des deux cotés :

$$0 = \frac{\mathrm{d}^2 x^\rho}{\mathrm{d}\tau^2} + \frac{\partial x^\rho}{\partial\xi^\alpha} \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\partial\xi^\alpha}{\partial x^\mu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$

$$= \frac{\mathrm{d}^2 x^\rho}{\mathrm{d}\tau^2} + \frac{\partial x^\rho}{\partial\xi^\alpha} \frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$

En posant : $\left\{_\mu{}^\lambda{}_\nu\right\} = \frac{\partial x^\lambda}{\partial\xi^\beta} \frac{\partial^2\xi^\beta}{\partial x^\mu \partial x^\nu}$, on obtient finalement :

$$0 = \frac{\mathrm{d}^2 x^\rho}{\mathrm{d}\tau^2} + \left\{_\mu{}^\rho{}_\nu\right\} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$

En réécrivant le produit scalaire dans les coordonnées x^α à:

$$\eta_{\mu\nu}\,\mathrm{d}\xi^{\mu}\,\mathrm{d}\xi^{\nu} = \underbrace{\eta_{\mu\nu}\frac{\partial\xi^{\mu}}{\partial x^{\alpha}}\frac{\partial\xi^{\nu}}{\partial x^{\beta}}}_{=g_{\alpha\beta}}\,\mathrm{d}x^{\alpha}\,\mathrm{d}x^{\beta}$$

Donc :

$$g_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial\xi^{\mu}}{\partial x^{\alpha}}\frac{\partial\xi^{\nu}}{\partial x^{\beta}}$$

alors en dérivant :

$$\partial_{\lambda}g_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial\xi^{\mu}}{\partial x^{\alpha}}\frac{\partial\xi^{\nu}}{\partial x^{\beta}}\right)$$

$$= \underbrace{\eta_{\mu\nu}\frac{\partial\xi^{\mu}}{\partial x^{\alpha}}}_{=g_{\alpha\mu}\frac{\partial x^{\mu}}{\partial\xi^{\nu}}}\frac{\partial^{2}\xi^{\nu}}{\partial x^{\beta}\partial x^{\lambda}} + \underbrace{\eta_{\mu\nu}\frac{\partial\xi^{\nu}}{\partial x^{\beta}}}_{=g_{\alpha\nu}\frac{\partial x^{\nu}}{\partial\xi^{\mu}}}\frac{\partial^{2}\xi^{\mu}}{\partial x^{\alpha}\partial x^{\lambda}}$$

$$= g_{\alpha\mu}\underbrace{\frac{\partial x^{\mu}}{\partial\xi^{\nu}}\frac{\partial^{2}\xi^{\nu}}{\partial x^{\beta}\partial x^{\lambda}}}_{=\{_{\beta}{}^{\mu}{}_{\lambda}\}} + g_{\alpha\nu}\underbrace{\frac{\partial x^{\nu}}{\partial\xi^{\mu}}\frac{\partial^{2}\xi^{\mu}}{\partial x^{\beta}\partial x^{\lambda}}}_{=\{_{\beta}{}^{\nu}{}_{\lambda}\}}$$

$$= \left\{_{\beta}{}^{\mu}{}_{\lambda}\right\}g_{\mu\alpha} + \left\{_{\beta}{}^{\nu}{}_{\lambda}\right\}g_{\nu\alpha}$$