Gaspar Daguet

Info : DM2

En logique minimale :

Partie I. Lemme :

- **Lemme Minimaliste**

Lemme 1

$$\dfrac{\Gamma, \varphi_1 \vdash \varphi_2 \qquad \Gamma, \varphi_1 \vdash \neg\varphi_2}{\Gamma \vdash \neg\varphi_1}\,\neg_{ei}$$

preuve :

$$\dfrac{\dfrac{\dfrac{}{\Gamma, \varphi_1 \vdash \varphi_2}\,\text{hypp} \qquad \dfrac{}{\Gamma, \varphi_1 \vdash \neg\varphi_2}\,\text{hypp}}{\Gamma, \varphi_1 \vdash \bot}\,\neg_e}{\Gamma \vdash \neg\varphi_1}\,\neg_i$$

Lemme 2

$$\dfrac{}{\Gamma, \varphi \to \psi, \varphi \vdash \psi}\,\to_{\text{ax}}$$

preuve :

$$\dfrac{\dfrac{}{\Gamma, \varphi \to \psi, \varphi \vdash \varphi \to \psi}\,\text{ax} \qquad \dfrac{}{\Gamma, \varphi \to \psi, \varphi \vdash \varphi}\,\text{ax}}{\Gamma, \varphi \to \psi, \varphi \vdash \psi}\,\to_e$$

- **Lemme Classique**

Lemme 1

$$\dfrac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}\,\neg\neg_e$$

preuve :

$$\dfrac{\dfrac{\dfrac{}{\Gamma, \neg\varphi \vdash \neg\varphi}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma \vdash \neg\neg\varphi}\,\text{hypp}}{\Gamma, \neg\varphi \vdash \neg\neg\varphi}\,\text{aff}}{\Gamma, \neg\varphi \vdash \bot}\,\neg_e}{\Gamma \vdash \varphi}\,\text{raa}$$

Partie II. Logique minimale :

N=° 1.

$$\dfrac{\dfrac{}{\Gamma \vdash p \vee (p \wedge q)}\,\text{ax} \qquad \dfrac{}{\Gamma, p \vdash p}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma, p \wedge q \vdash p \wedge q}\,\text{ax}}{\Gamma, p \wedge q \vdash p}\,\wedge_e^g}{\underline{p \vee (p \wedge q) \vdash p}}\,\vee_e$$

Avec $\Gamma = \{p \vee (p \wedge q)\}$

N=° 2.

$$\dfrac{\dfrac{}{\Gamma \vdash p}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma \vdash p \to \neg p}\,\text{ax} \qquad \dfrac{}{\Gamma \vdash p}\,\text{ax}}{\Gamma \vdash \neg p}\,\to_e}{\underline{p \to \neg p \vdash \neg p}}\,\neg_{\text{ei}}$$

Avec $\Gamma = \{p \to \neg p, p\}$

N=° 3.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p \vdash p \to q \lor r$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p \vdash p$}
\RightLabel{$\to_e$}
\BinaryInfC{$\Gamma, p \vdash q \lor r$}
\AxiomC{}
\RightLabel{$\to_{\text{ax}}$}
\UnaryInfC{$\Gamma, p, q \vdash s$}
\AxiomC{}
\RightLabel{$\to_{\text{ax}}$}
\UnaryInfC{$\Gamma, p, r \vdash s$}
\RightLabel{$\lor_e$}
\TrinaryInfC{$\Gamma, p \vdash s$}
\RightLabel{$\to_i$}
\UnaryInfC{$p \to (q \lor r), q \to s, r \to s \vdash p \to s$}
\end{prooftree}
$$

Avec $\Gamma = \{p \to (q \lor r), q \to s, r \to s\}$

N=° 4.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash p \to q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash p \land r$}
\RightLabel{$\land_e^g$}
\UnaryInfC{$\Gamma \vdash p$}
\RightLabel{$\to_e$}
\BinaryInfC{$\Gamma \vdash q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash r \to s$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash p \land r$}
\RightLabel{$\land_e^d$}
\UnaryInfC{$\Gamma \vdash r$}
\RightLabel{$\to_e$}
\BinaryInfC{$\Gamma \vdash s$}
\RightLabel{$\land_i$}
\BinaryInfC{$\Gamma \vdash q \land s$}
\end{prooftree}
$$

Avec $\Gamma = \{p \to q, r \to s, p \land r\}$

N=° 5.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash p \lor r$}
\AxiomC{}
\RightLabel{$\to_{\text{ax}}$}
\UnaryInfC{$\Gamma, p \vdash q$}
\RightLabel{$\lor_i^g$}
\UnaryInfC{$\Gamma, p \vdash q \lor s$}
\AxiomC{}
\RightLabel{$\to_{\text{ax}}$}
\UnaryInfC{$\Gamma, r \vdash s$}
\RightLabel{$\lor_i$}
\UnaryInfC{$\Gamma, r \vdash q \lor s$}
\RightLabel{$\lor_e$}
\TrinaryInfC{$\Gamma \vdash q \lor s$}
\end{prooftree}
$$

Avec $\Gamma = \{p \to q, r \to s, p \lor r\}$

N=° 6.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma_1 \vdash (p \lor q) \to r$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma_1 \vdash p$}
\RightLabel{$\lor_e^g$}
\UnaryInfC{$\Gamma_1 \vdash p \lor q$}
\RightLabel{$\to_e$}
\BinaryInfC{$\Gamma_1 \vdash r$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma_2 \vdash (p \lor q) \to r$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma_2 \vdash q$}
\RightLabel{$\lor_e^d$}
\UnaryInfC{$\Gamma_2 \vdash p \lor q$}
\RightLabel{$\to_e$}
\BinaryInfC{$\Gamma_2 \vdash r$}
\RightLabel{$\land_i + \to_i$}
\BinaryInfC{$(p \lor q) \to r \vdash (p \to r) \land (q \to r)$}
\end{prooftree}
$$

Avec $\Gamma_1 = \{(p \lor q) \to r, p\}$ et $\Gamma_2 = \{(p \lor q) \to r, q\}$

Partie III. Lois de De Morgan

N=° 7.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma_1 \vdash \neg(p \lor q)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma_1 \vdash p$}
\RightLabel{$\lor_i^g$}
\UnaryInfC{$\Gamma_1 \vdash p \lor q$}
\RightLabel{$\neg_{ei}$}
\BinaryInfC{$\neg(p \lor q) \vdash \neg p$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma_2 \vdash \neg(p \lor q)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma_2 \vdash q$}
\RightLabel{$\lor_i^d$}
\UnaryInfC{$\Gamma_2 \vdash p \lor q$}
\RightLabel{$\neg_{ei}$}
\BinaryInfC{$\neg(p \lor q) \vdash \neg q$}
\RightLabel{$\land_i$}
\BinaryInfC{$\neg(p \lor q) \vdash \neg p \land \neg q$}
\end{prooftree}
$$

Avec $\Gamma_1 = \{\neg(p \lor q), p\}$ et $\Gamma_2 = \{\neg(p \lor q), q\}$

N=° 8.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, q \vdash \neg p \land \neg q$}
\RightLabel{$\land_e^d$}
\UnaryInfC{$\Gamma, q \vdash \neg q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, q \vdash q$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, q \vdash \bot$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash p \lor q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p \vdash p$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p \vdash \neg p \land \neg q$}
\RightLabel{$\land_e^g$}
\UnaryInfC{$\Gamma, p \vdash \neg p$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, p \vdash \bot$}
\RightLabel{$\lor_e$}
\TrinaryInfC{$\neg p \land \neg q, p \lor q \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$\neg p \land \neg q \vdash \neg(p \lor q)$}
\end{prooftree}
$$

Avec $\Gamma = \{\neg p \land \neg q, p \lor q\}$

N=° 9.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg q \vdash p \wedge q$}
\RightLabel{$\wedge_e^d$}
\UnaryInfC{$\Gamma, \neg q \vdash q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg q \vdash \neg q$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, \neg q \vdash \bot$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg p \vee \neg q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg p \vdash \neg p$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg p \vdash p \wedge q$}
\RightLabel{$\wedge_e^g$}
\UnaryInfC{$\Gamma, \neg p \vdash p$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, \neg p \vdash \bot$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$\neg p \vee \neg q, p \wedge q \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$\neg p \vee \neg q \vdash \neg(p \wedge q)$}
\end{prooftree}
$$

Avec $\Gamma = \{\neg p \vee \neg q, p \wedge q\}$

N=° 10.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg(p \wedge q)$}
\AxiomC{}
\RightLabel{q.7}
\UnaryInfC{$\Gamma \vdash \neg\neg p \wedge \neg\neg q$}
\RightLabel{$\wedge_e^g$}
\UnaryInfC{$\Gamma \vdash \neg\neg p$}
\RightLabel{$\neg\neg_e$}
\UnaryInfC{$\Gamma \vdash p$}
\AxiomC{}
\RightLabel{q.7}
\UnaryInfC{$\Gamma \vdash \neg\neg p \wedge \neg\neg q$}
\RightLabel{$\wedge_e^d$}
\UnaryInfC{$\Gamma \vdash \neg\neg q$}
\RightLabel{$\neg\neg_e$}
\UnaryInfC{$\Gamma \vdash q$}
\RightLabel{$\wedge_i$}
\BinaryInfC{$\Gamma \vdash p \wedge q$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma \vdash \bot$}
\RightLabel{raa}
\UnaryInfC{$\neg(p \wedge q) \vdash \neg p \vee \neg q$}
\end{prooftree}
$$

Avec $\Gamma = \{\neg(p \wedge q), \neg(\neg p \vee \neg q)\}$

Partie IV. Logique intuitionniste

N=° 11.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\neg p, p \vdash p$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\neg p, p \vdash \neg p$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\neg p, p \vdash \bot$}
\RightLabel{$\bot_e$}
\UnaryInfC{$\neg p, p \vdash q$}
\RightLabel{$\to_i$}
\UnaryInfC{$\neg p \vdash p \to q$}
\end{prooftree}
$$

N=° 12.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash p \vee q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p \vdash p$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, q \vdash q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, q \vdash \neg q$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, q \vdash \bot$}
\RightLabel{$\bot_e$}
\UnaryInfC{$\Gamma, q \vdash p$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$p \vee q, \neg q \vdash p$}
\end{prooftree}
$$

Avec $\Gamma = \{p \vee q, \neg q\}$

N=° 13.

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\Gamma \vdash \neg(p \to q)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p \vdash q$}
\RightLabel{$\to_i$}
\UnaryInfC{$\Gamma \vdash p \to q$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma \vdash \bot$}
\RightLabel{$\bot_e$}
\UnaryInfC{$\Gamma \vdash p$}
\RightLabel{$\to_i$}
\UnaryInfC{$\neg(p \to q) \vdash q \to p$}
\end{prooftree}
$$

Avec $\Gamma = \{\neg(p \to q), q\}$

Partie V. logique classique

N=° 14.

$$\dfrac{\dfrac{}{\Gamma \vdash q \vee \neg q}\,\text{t.e.} \qquad \dfrac{\dfrac{}{\Gamma, q, p \vdash q}\,\text{ax}}{\Gamma, q \vdash (p \to q) \vee (p \to r)}\,\vee_i^g + \to_i \qquad \dfrac{\dfrac{\text{Voir ci-dessous}}{\Gamma, \neg q, p \vdash r}\,\text{ax}}{\Gamma, \neg q \vdash (p \to q) \vee (p \to r)}\,\vee_i^d + \to_i}{\underline{p \to (q \vee r) \vdash (p \to q) \vee (p \to r)}}\,\vee_e$$

Le ci-dessous en question :

$$\dfrac{\dfrac{\dfrac{}{\Gamma_2 \vdash p \to (q \vee r)}\,\text{ax} \qquad \dfrac{}{\Gamma_2 \vdash p}\,\text{ax}}{\Gamma_2 \vdash q \vee r}\,\to_e \qquad \dfrac{}{\Gamma, r \vdash r}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma_2, q \vdash q}\,\text{ax} \qquad \dfrac{}{\Gamma_2, q \vdash \neg q}\,\text{ax}}{\Gamma_2, q \vdash r}\,\neg_e}{\Gamma, \neg q, p \vdash r}\,\vee_e$$

Avec $\Gamma = \{p \to (q \vee r)\}$ et $\Gamma_2 = \Gamma \cup \{\neg q, p\}$

N=° 15.

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash q \to r}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \neg q \vdash p}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma, \neg q \vdash \neg q \to \neg p}\,\text{ax} \qquad \dfrac{}{\Gamma, \neg q \vdash \neg q}\,\text{ax}}{\Gamma, \neg q \vdash \neg p}\,\to_e}{\Gamma, \neg q \vdash \bot}\,\neg_e}{\Gamma \vdash q}\,\text{raa}}{\Gamma \vdash r}\,\to_e}{\underline{q \to r, \neg q \to \neg p \vdash p \to r}}\,\to_i$$