

## Équation des géodésiques

Soit le lagrangiens suivant :

$$\mathcal{L}(x^\alpha, \dot{x}^\alpha) = [g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu]^{\frac{1}{2}}$$

avec  $\dot{x}^\alpha = \frac{dx^\alpha}{ds}$

Ainsi on n'a :

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{1}{2} [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{-\frac{1}{2}} \frac{\partial}{\partial x^\alpha} (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)$$

car  $\dot{x}^\mu$  et  $\dot{x}^\nu$

sont constant

$$\begin{aligned} \text{face à } x^\alpha \rightarrow &= \frac{1}{2} [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{-\frac{1}{2}} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \dot{x}^\mu \dot{x}^\nu \\ &= \frac{1}{2} [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{-\frac{1}{2}} g_{\mu\nu, \alpha} \dot{x}^\mu \dot{x}^\nu \end{aligned}$$

où on note :  $g_{\mu\nu, \alpha} = \frac{\partial g_{\mu\nu}}{\partial x^\alpha}$

Et

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} &= \frac{1}{2} [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{-\frac{1}{2}} \frac{\partial}{\partial \dot{x}^\alpha} (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu) = \frac{1}{2} [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{-\frac{1}{2}} g_{\mu\nu} \frac{\partial}{\partial \dot{x}^\alpha} (\dot{x}^\mu \dot{x}^\nu) \\ &= \frac{1}{2} [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{-\frac{1}{2}} g_{\mu\nu} \left( \underbrace{\dot{x}^\nu \frac{\partial}{\partial \dot{x}^\alpha} (\dot{x}^\mu)}_{=\delta_\alpha^\mu} + \underbrace{\dot{x}^\mu \frac{\partial}{\partial \dot{x}^\alpha} (\dot{x}^\nu)}_{=\delta_\alpha^\nu} \right) \\ &= \frac{1}{2} [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{-\frac{1}{2}} g_{\mu\nu} \left( \dot{x}^\nu \delta_\alpha^\mu + \underbrace{\dot{x}^\mu \delta_\alpha^\nu}_{=\dot{x}^\nu \delta_\alpha^\mu \text{ car } \mu \text{ et } \nu \text{ sont muet}} \right) \\ &= \cancel{\frac{1}{2}} [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{-\frac{1}{2}} g_{\mu\nu} \cdot \cancel{2} \dot{x}^\nu \delta_\alpha^\mu \\ &= [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{-\frac{1}{2}} g_{\alpha\nu} \dot{x}^\nu \end{aligned}$$

Ainsi par les équation d'Euler-La Grange, on obtient l'équation suivante :

$$\frac{d}{ds} \left( [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{-\frac{1}{2}} g_{\alpha\nu} \dot{x}^\nu \right) = \frac{1}{2} [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{-\frac{1}{2}} g_{\mu\nu, \alpha} \dot{x}^\mu \dot{x}^\nu$$

En posant :

$$d\lambda = [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{\frac{1}{2}} ds$$

$$\text{Donc } \frac{d}{d\lambda} = [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{-\frac{1}{2}} \frac{d}{ds}$$

Ainsi, en réécrivant l'équation précédente :

$$\frac{d}{ds} \left( [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{-\frac{1}{2}} g_{\alpha\nu} \frac{d}{ds} x^\nu \right) = \frac{1}{2} [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{-\frac{1}{2}} g_{\mu\nu, \alpha} \frac{d}{ds} x^\mu \frac{d}{ds} x^\nu$$

$$\begin{aligned}
\text{donc } [g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{\frac{1}{2}} \frac{d}{d\lambda} \left( [\cancel{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}]^{-\frac{1}{2}} g_{\alpha\nu} [\cancel{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}]^{\frac{1}{2}} \frac{d}{d\lambda} x^\nu \right) &= \frac{1}{2} [g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}} [g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu] g_{\mu\nu,\alpha} \frac{d}{d\lambda} x^\mu \frac{d}{d\lambda} x^\nu \\
\text{donc } [\cancel{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}]^{\frac{1}{2}} \frac{d}{d\lambda} \left( g_{\alpha\nu} \frac{d}{d\lambda} x^\nu \right) &= \frac{1}{2} [\cancel{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}]^{\frac{1}{2}} g_{\mu\nu,\alpha} \frac{d}{d\lambda} x^\mu \frac{d}{d\lambda} x^\nu \\
\text{donc } \frac{d}{d\lambda} \left( g_{\alpha\nu} \frac{d}{d\lambda} x^\nu \right) &= \frac{1}{2} g_{\mu\nu,\alpha} \frac{d}{d\lambda} x^\mu \frac{d}{d\lambda} x^\nu
\end{aligned}$$

Si on écrit l'action par rapport à  $s$ , on a :

$$S_1 = \int \left[ g_{\mu\nu} \frac{d}{ds} x^\mu \frac{d}{ds} x^\nu \right]^{\frac{1}{2}} ds$$

Alors en opérant le changement de variable  $S_1$  devient :

$$\begin{aligned}
S_1 &= \int [\cancel{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}]^{\frac{1}{2}} \left[ g_{\mu\nu} \frac{d}{d\lambda} x^\mu \frac{d}{d\lambda} x^\nu \right]^{\frac{1}{2}} [\cancel{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}]^{-\frac{1}{2}} d\lambda \\
&= \int \left[ g_{\mu\nu} \frac{d}{d\lambda} x^\mu \frac{d}{d\lambda} x^\nu \right]^{\frac{1}{2}} d\lambda
\end{aligned}$$

Donc comme  $S_1$  est invariante par la transformation  $s \rightarrow \lambda$  et que celle-ci est un difféomorphisme, on a alors :

$$\frac{d}{ds}(g_{\alpha\nu}\dot{x}^\nu) = \frac{1}{2}g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu$$

Donc en calculant le terme de gauche :

$$\begin{aligned}
\frac{d}{ds}(g_{\alpha\nu}\dot{x}^\nu) &= \dot{x}^\nu \frac{d}{ds} g_{\alpha\nu} + g_{\alpha\nu} \ddot{x}^\nu \\
&= g_{\alpha\nu} \ddot{x}^\nu + \dot{x}^\nu \underbrace{\frac{\partial g_{\alpha\nu}}{\partial \mu} \frac{dx^\mu}{ds}}_{=g_{\alpha\nu,\mu}=\dot{x}^\mu} \\
&= g_{\alpha\nu} \ddot{x}^\nu + g_{\alpha\nu,\mu} \dot{x}^\nu \dot{x}^\mu
\end{aligned}$$

Ainsi :

$$\begin{aligned}
\underbrace{g^{\alpha\beta} g_{\alpha\nu}}_{=\delta_\nu^\beta} \ddot{x}^\nu &= \frac{1}{2} g_{\mu\nu,\alpha} \dot{x}^\mu \dot{x}^\nu - g_{\alpha\nu,\mu} \dot{x}^\nu \dot{x}^\mu \\
\text{Donc } \ddot{x}^\beta &= \frac{1}{2} g^{\alpha\beta} (g_{\mu\nu,\alpha} - 2g_{\alpha\nu,\mu}) \dot{x}^\nu \dot{x}^\mu \\
\text{Donc } \ddot{x}^\beta &= \frac{1}{2} g^{\alpha\beta} (g_{\mu\nu,\alpha} - g_{\alpha\nu,\mu} - g_{\alpha\mu,\nu}) \dot{x}^\nu \dot{x}^\mu
\end{aligned}$$

En introduisant les symboles de Cristoffel  $\Gamma_{\mu\nu}^\beta = \frac{1}{2}g^{\beta\alpha}(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha})$  on a :

$$\ddot{x}^\beta = -\Gamma_{\mu\nu}^\beta \dot{x}^\mu \dot{x}^\nu$$

$$\text{Soit } \ddot{x}^\beta + \Gamma_{\mu\nu}^\beta \dot{x}^\mu \dot{x}^\nu = 0$$

## Commuteur des Co-dérivé

On définit les dérivé covariante par :

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma_{\mu\rho}^\nu v^\rho$$

Ainsi, soit  $v^\rho$  un vecteur :

$$\begin{aligned} [\nabla_\mu; \nabla_\nu] v^\rho &= \nabla_\mu (\nabla_\nu v^\rho) - \nabla_\nu (\nabla_\mu v^\rho) \\ &= \nabla_\mu (\partial_\nu v^\rho + \Gamma_{\nu\alpha}^\rho v^\alpha) - \nabla_\nu (\partial_\mu v^\rho + \Gamma_{\mu\alpha}^\rho v^\alpha) \\ &= \partial_\mu (\partial_\nu v^\rho + \Gamma_{\nu\alpha}^\rho v^\alpha) + \Gamma_{\mu\beta}^\rho (\partial_\nu v^\beta + \Gamma_{\nu\alpha}^\beta v^\alpha) - \partial_\nu (\partial_\mu v^\rho + \Gamma_{\mu\alpha}^\rho v^\alpha) - \Gamma_{\nu\beta}^\rho (\partial_\mu v^\beta + \Gamma_{\mu\alpha}^\beta v^\alpha) \\ &= \cancel{\partial_\nu \partial_\mu v^\rho} + \partial_\mu \Gamma_{\nu\alpha}^\rho v^\alpha + \Gamma_{\mu\beta}^\rho \partial_\nu v^\beta + \Gamma_{\mu\beta}^\rho \Gamma_{\nu\alpha}^\beta v^\alpha - \cancel{\partial_\mu \partial_\nu v^\rho} - \partial_\nu \Gamma_{\mu\alpha}^\rho v^\alpha - \Gamma_{\nu\beta}^\rho \partial_\mu v^\beta - \Gamma_{\nu\beta}^\rho \Gamma_{\mu\alpha}^\beta v^\alpha \\ &= (\Gamma_{\mu\beta}^\rho \Gamma_{\nu\alpha}^\beta - \Gamma_{\nu\beta}^\rho \Gamma_{\mu\alpha}^\beta + \partial_\mu \Gamma_{\nu\alpha}^\rho - \partial_\nu \Gamma_{\mu\alpha}^\rho) v^\alpha + \underbrace{\Gamma_{\mu\beta}^\rho \partial_\nu v^\beta - \Gamma_{\nu\beta}^\rho \partial_\mu v^\beta}_{=0} \end{aligned}$$

Ainsi on reconnait le tenseur de Riemann définis par :  $R_{\mu\nu\kappa}^\sigma = \partial_\nu \Gamma_{\mu\kappa}^\sigma - \partial_\kappa \Gamma_{\mu\nu}^\sigma + \Gamma_{\nu\lambda}^\sigma \Gamma_{\mu\kappa}^\lambda - \Gamma_{\kappa\lambda}^\sigma \Gamma_{\mu\nu}^\lambda$

Donc on obtient la relation suivante :

$$[\nabla_\mu; \nabla_\nu] v^\rho = R_{\alpha\mu\nu}^\rho v^\alpha$$

## Dérivé covariante

On se place dans un système de coordonnées  $\xi^\alpha$ , et on s'intéresse à un changement de coordonnées vers  $x^\alpha$

Tout d'abord :

$$\begin{aligned} 0 &= \frac{d^2 \xi^\alpha}{d\tau^2} = \frac{d}{d\tau} \left( \frac{d\xi^\alpha}{d\tau} \right) \\ &= \frac{d}{d\tau} \left( \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{dx^\mu}{d\tau} \right) \\ &= \frac{\partial \xi^\alpha}{\partial x^\rho} \frac{d^2 x^\rho}{d\tau^2} + \frac{dx^\mu}{d\tau} \frac{d}{d\tau} \frac{\partial \xi^\alpha}{\partial x^\mu} \end{aligned}$$

Ainsi comme  $\frac{\partial \xi^\alpha}{\partial x^\mu} \times \frac{\partial x^\mu}{\partial \xi^\beta} = \delta_\beta^\alpha$ , en multipliant des deux cotés :

$$\begin{aligned} 0 &= \frac{d^2 x^\rho}{d\tau^2} + \frac{\partial x^\rho}{\partial \xi^\alpha} \frac{d}{d\tau} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{dx^\mu}{d\tau} \\ &= \frac{d^2 x^\rho}{d\tau^2} + \frac{\partial x^\rho}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \end{aligned}$$

En posant :  $\left\{ \begin{smallmatrix} \lambda \\ \mu \quad \nu \end{smallmatrix} \right\} = \frac{\partial x^\lambda}{\partial \xi^\beta} \frac{\partial^2 \xi^\beta}{\partial x^\mu \partial x^\nu}$ , on obtient finalement :

$$0 = \frac{d^2 x^\rho}{d\tau^2} + \left\{ \begin{smallmatrix} \rho \\ \mu \quad \nu \end{smallmatrix} \right\} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

En réécrivant le produit scalaire dans les coordonnées  $x^\alpha$  à :

$$\eta_{\mu\nu} d\xi^\mu d\xi^\nu = \underbrace{\eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta}}_{=g_{\alpha\beta}} dx^\alpha dx^\beta$$

Donc :

$$g_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta}$$

alors en dérivant :

$$\begin{aligned} \partial_\lambda g_{\alpha\beta} &= \eta_{\mu\nu} \frac{\partial}{\partial x^\lambda} \left( \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta} \right) \\ &= \underbrace{\eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial^2 \xi^\nu}{\partial x^\beta \partial x^\lambda}}_{=g_{\alpha\mu} \frac{\partial x^\mu}{\partial \xi^\nu}} + \underbrace{\eta_{\mu\nu} \frac{\partial \xi^\nu}{\partial x^\beta} \frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\lambda}}_{=g_{\alpha\nu} \frac{\partial x^\nu}{\partial \xi^\mu}} \\ &= g_{\alpha\mu} \underbrace{\frac{\partial x^\mu}{\partial \xi^\nu} \frac{\partial^2 \xi^\nu}{\partial x^\beta \partial x^\lambda}}_{=\{\beta^\mu{}_\lambda\}} + g_{\alpha\nu} \underbrace{\frac{\partial x^\nu}{\partial \xi^\mu} \frac{\partial^2 \xi^\mu}{\partial x^\beta \partial x^\lambda}}_{=\{\beta^\nu{}_\lambda\}} \\ &= \{\beta^\mu{}_\lambda\} g_{\mu\alpha} + \{\beta^\nu{}_\lambda\} g_{\nu\alpha} \end{aligned}$$