Gaspar Daguet

Info : DM2

En logique minimale :

Partie I. Lemme :

- **Lemme Minimaliste**

Lemme 1

$$\dfrac{\Gamma, \varphi_1 \vdash \varphi_2 \qquad \Gamma, \varphi_1 \vdash \neg\varphi_2}{\Gamma \vdash \neg\varphi_1}\,\neg_{ei}$$

preuve :

$$\dfrac{\dfrac{\dfrac{}{\Gamma, \varphi_1 \vdash \varphi_2}\,\text{hypp} \qquad \dfrac{}{\Gamma, \varphi_1 \vdash \neg\varphi_2}\,\text{hypp}}{\Gamma, \varphi_1 \vdash \bot}\,\neg_e}{\Gamma \vdash \neg\varphi_1}\,\neg_i$$

Lemme 2

$$\dfrac{}{\Gamma, \varphi \to \psi, \varphi \vdash \psi}\,\to_{\text{ax}}$$

preuve :

$$\dfrac{\dfrac{}{\Gamma, \varphi \to \psi, \varphi \vdash \varphi \to \psi}\,\text{ax} \qquad \dfrac{}{\Gamma, \varphi \to \psi, \varphi \vdash \varphi}\,\text{ax}}{\Gamma, \varphi \to \psi, \varphi \vdash \psi}\,\to_e$$

- **Lemme Classique**

Lemme 1

$$\dfrac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}\,\neg\neg_e$$

preuve :

$$\dfrac{\dfrac{\dfrac{}{\Gamma, \neg\varphi \vdash \neg\varphi}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma \vdash \neg\neg\varphi}\,\text{hypp}}{\Gamma, \neg\varphi \vdash \neg\neg\varphi}\,\text{aff}}{\Gamma, \neg\varphi \vdash \bot}\,\neg_e}{\Gamma \vdash \varphi}\,\text{raa}$$

Partie II. Logique minimale :

N=° 1.

$$\dfrac{\dfrac{}{\Gamma \vdash p \vee (p \wedge q)}\,\text{ax} \qquad \dfrac{}{\Gamma, p \vdash p}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma, p \wedge q \vdash p \wedge q}\,\text{ax}}{\Gamma, p \wedge q \vdash p}\,\wedge_e^g}{\underline{p \vee (p \wedge q) \vdash p}}\,\vee_e$$

Avec $\Gamma = \{p \vee (p \wedge q)\}$

N=° 2.

$$\dfrac{\dfrac{}{\Gamma \vdash p}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma \vdash p \to \neg p}\,\text{ax} \qquad \dfrac{}{\Gamma \vdash p}\,\text{ax}}{\Gamma \vdash \neg p}\,\to_e}{\underline{p \to \neg p \vdash \neg p}}\,\neg_{\text{ei}}$$

Avec $\Gamma = \{p \to \neg p, p\}$

N=° 3.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,p\vdash p\to q\vee r}\text{ax}\qquad \dfrac{}{\Gamma,p\vdash p}\text{ax}}{\Gamma,p\vdash q\vee r}\to_e \qquad \dfrac{}{\Gamma,p,q\vdash s}\to_{\text{ax}} \qquad \dfrac{}{\Gamma,p,r\vdash s}\to_{\text{ax}}}{\Gamma,p\vdash s}\vee_e}{p\to(q\vee r),q\to s,r\to s\vdash p\to s}\to_i$$

Avec $\Gamma=\{p\to(q\vee r),q\to s,r\to s\}$

N=° 4.

$$\dfrac{\dfrac{\dfrac{}{\Gamma\vdash p\to q}\text{ax}\qquad \dfrac{\dfrac{}{\Gamma\vdash p\wedge r}\text{ax}}{\Gamma\vdash p}\wedge_e^g}{\Gamma\vdash q}\to_e \qquad \dfrac{\dfrac{}{\Gamma\vdash r\to s}\text{ax}\qquad \dfrac{\dfrac{}{\Gamma\vdash p\wedge r}\text{ax}}{\Gamma\vdash r}\wedge_e^d}{\Gamma\vdash s}\to_e}{\Gamma\vdash q\wedge s}\wedge_i$$

Avec $\Gamma=\{p\to q,r\to s,p\wedge r\}$

N=° 5.

$$\dfrac{\dfrac{}{\Gamma\vdash p\vee r}\text{ax}\qquad \dfrac{\dfrac{}{\Gamma,p\vdash q}\to_{\text{ax}}}{\Gamma,p\vdash q\vee s}\vee_i^g \qquad \dfrac{\dfrac{}{\Gamma,r\vdash s}\to_{\text{ax}}}{\Gamma,r\vdash q\vee s}\vee_i}{\Gamma\vdash q\vee s}\vee_e$$

Avec $\Gamma=\{p\to q,r\to s,p\vee r\}$

N=° 6.

$$\dfrac{\dfrac{\dfrac{}{\Gamma_1\vdash(p\vee q)\to r}\text{ax}\qquad \dfrac{\dfrac{}{\Gamma_1\vdash p}\text{ax}}{\Gamma_1\vdash p\vee q}\vee_e^g}{\Gamma_1\vdash r}\to_e \qquad \dfrac{\dfrac{}{\Gamma_2\vdash(p\vee q)\to r}\text{ax}\qquad \dfrac{\dfrac{}{\Gamma_2\vdash q}\text{ax}}{\Gamma_2\vdash p\vee q}\vee_e^d}{\Gamma_2\vdash r}\to_e}{(p\vee q)\to r\vdash(p\to r)\wedge(q\to r)}\wedge_i+\to_i$$

Avec $\Gamma_1=\{(p\vee q)\to r,p\}$ et $\Gamma_2=\{(p\vee q)\to r,q\}$

Partie III. Lois de De Morgan

N=° 7.

$$\dfrac{\dfrac{\dfrac{}{\Gamma_1\vdash\neg(p\vee q)}\text{ax}\qquad \dfrac{\dfrac{}{\Gamma_1\vdash p}\text{ax}}{\Gamma_1\vdash p\vee q}\vee_i^g}{\neg(p\vee q)\vdash\neg p}\neg_{ei} \qquad \dfrac{\dfrac{}{\Gamma_2\vdash\neg(p\vee q)}\text{ax}\qquad \dfrac{\dfrac{}{\Gamma_2\vdash q}\text{ax}}{\Gamma_2\vdash p\vee q}\vee_i^d}{\neg(p\vee q)\vdash\neg q}\neg_{ei}}{\neg(p\vee q)\vdash\neg p\wedge\neg q}\wedge_i$$

Avec $\Gamma_1=\{\neg(p\vee q),p\}$ et $\Gamma_2=\{\neg(p\vee q),q\}$

N=° 8.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,q\vdash\neg p\wedge\neg q}\text{ax}}{\Gamma,q\vdash\neg q}\wedge_e^d\qquad \dfrac{}{\Gamma,q\vdash q}\text{ax}}{\Gamma,q\vdash\bot}\neg_e \qquad \dfrac{}{\Gamma\vdash p\vee q}\text{ax} \qquad \dfrac{\dfrac{}{\Gamma,p\vdash p}\text{ax}\qquad \dfrac{\dfrac{}{\Gamma,p\vdash\neg p\wedge\neg q}\text{ax}}{\Gamma,p\vdash\neg p}\wedge_e^g}{\Gamma,p\vdash\bot}\neg_e}{\neg p\wedge\neg q,p\vee q\vdash\bot}\vee_e}{\neg p\wedge\neg q\vdash\neg(p\vee q)}\neg_i$$

Avec $\Gamma=\{\neg p\wedge\neg q,p\vee q\}$

N=° 9.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg q \vdash p \wedge q$}
\RightLabel{$\wedge_e^d$}
\UnaryInfC{$\Gamma, \neg q \vdash q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg q \vdash \neg q$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, \neg q \vdash \bot$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg p \vee \neg q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg p \vdash \neg p$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg p \vdash p \wedge q$}
\RightLabel{$\wedge_e^g$}
\UnaryInfC{$\Gamma, \neg p \vdash p$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, \neg p \vdash \bot$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$\neg p \vee \neg q, p \wedge q \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$\neg p \vee \neg q \vdash \neg(p \wedge q)$}
\end{prooftree}
$$

Avec $\Gamma = \{\neg p \vee \neg q, p \wedge q\}$

N=° 10.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg(p \wedge q)$}
\AxiomC{}
\RightLabel{q.7}
\UnaryInfC{$\Gamma \vdash \neg\neg p \wedge \neg\neg q$}
\RightLabel{$\wedge_e^g$}
\UnaryInfC{$\Gamma \vdash \neg\neg p$}
\RightLabel{$\neg\neg_e$}
\UnaryInfC{$\Gamma \vdash p$}
\AxiomC{}
\RightLabel{q.7}
\UnaryInfC{$\Gamma \vdash \neg\neg p \wedge \neg\neg q$}
\RightLabel{$\wedge_e^d$}
\UnaryInfC{$\Gamma \vdash \neg\neg q$}
\RightLabel{$\neg\neg_e$}
\UnaryInfC{$\Gamma \vdash q$}
\RightLabel{$\wedge_i$}
\BinaryInfC{$\Gamma \vdash p \wedge q$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma \vdash \bot$}
\RightLabel{raa}
\UnaryInfC{$\neg(p \wedge q) \vdash \neg p \vee \neg q$}
\end{prooftree}
$$

Avec $\Gamma = \{\neg(p \wedge q), \neg(\neg p \vee \neg q)\}$

Partie IV. Logique intuitionniste

N=° 11.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\neg p, p \vdash p$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\neg p, p \vdash \neg p$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\neg p, p \vdash \bot$}
\RightLabel{$\bot_e$}
\UnaryInfC{$\neg p, p \vdash q$}
\RightLabel{$\rightarrow_i$}
\UnaryInfC{$\neg p \vdash p \rightarrow q$}
\end{prooftree}
$$

N=° 12.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash p \vee q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p \vdash p$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, q \vdash q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, q \vdash \neg q$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, q \vdash \bot$}
\RightLabel{$\bot_e$}
\UnaryInfC{$\Gamma, q \vdash p$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$p \vee q, \neg q \vdash p$}
\end{prooftree}
$$

Avec $\Gamma = \{p \vee q, \neg q\}$

N=° 13.

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\Gamma \vdash \neg(p \rightarrow q)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p \vdash q$}
\RightLabel{$\rightarrow_i$}
\UnaryInfC{$\Gamma \vdash p \rightarrow q$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma \vdash \bot$}
\RightLabel{$\bot_e$}
\UnaryInfC{$\Gamma \vdash p$}
\RightLabel{$\rightarrow_i$}
\UnaryInfC{$\neg(p \rightarrow q) \vdash q \rightarrow p$}
\end{prooftree}
$$

Avec $\Gamma = \{\neg(p \rightarrow q), q\}$

Partie V. logique classique

N=° 14.

$$\dfrac{\dfrac{}{\Gamma \vdash q \lor \neg q}\,\text{t.e.} \qquad \dfrac{\dfrac{}{\Gamma, q, p \vdash q}\,\text{ax}}{\Gamma, q \vdash (p \to q) \lor (p \to r)}\,\lor_i^g + \to_i \qquad \dfrac{\dfrac{\text{Voir ci-dessous}}{\Gamma, \neg q, p \vdash r}\,\text{ax}}{\Gamma, \neg q \vdash (p \to q) \lor (p \to r)}\,\lor_i^d + \to_i}{\underline{p \to (q \lor r) \vdash (p \to q) \lor (p \to r)}}\,\lor_e$$

Le ci-dessous en question :

$$\dfrac{\dfrac{\dfrac{}{\Gamma_2 \vdash p \to (q \lor r)}\,\text{ax} \qquad \dfrac{}{\Gamma_2 \vdash p}\,\text{ax}}{\Gamma_2 \vdash q \lor r}\,\to_e \qquad \dfrac{}{\Gamma, r \vdash r}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma_2, q \vdash q}\,\text{ax} \qquad \dfrac{}{\Gamma_2, q \vdash \neg q}\,\text{ax}}{\Gamma_2, q \vdash r}\,\neg_e}{\Gamma, \neg q, p \vdash r}\,\lor_e$$

Avec $\Gamma = \{p \to (q \lor r)\}$ et $\Gamma_2 = \Gamma \cup \{\neg q, p\}$

N=° 15.

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash q \to r}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \neg q \vdash p}\,\text{ax} \qquad \dfrac{\dfrac{}{\Gamma, \neg q \vdash \neg q \to \neg p}\,\text{ax} \qquad \dfrac{}{\Gamma, \neg q \vdash \neg q}\,\text{ax}}{\Gamma, \neg q \vdash \neg p}\,\to_e}{\Gamma, \neg q \vdash \bot}\,\neg_e}{\Gamma \vdash q}\,\text{raa}}{\Gamma \vdash r}\,\to_e}{\underline{q \to r, \neg q \to \neg p \vdash p \to r}}\,\to_i$$