Équation des géodésiques

Soit le lagrangiens suivant :

$$\mathcal{L}(x^\alpha, \dot{x}^\alpha) = \left[g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu\right]^{\frac{1}{2}}$$

avec $\dot{x}^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}s}$
Ainsi on n'a :

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x^\alpha} &= \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right]^{-\frac{1}{2}} \frac{\partial}{\partial x^\alpha}\left(g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right) \\ \text{car } \dot{x}^\mu \text{ et } \dot{x}^\nu \\ \text{sont constant} \\ \text{face à } x^\alpha \rightarrow &= \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right]^{-\frac{1}{2}} \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu \dot{x}^\nu \\ &= \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right]^{-\frac{1}{2}} g_{\mu\nu,\alpha}\dot{x}^\mu \dot{x}^\nu \end{aligned}$$

où on note : $g_{\mu\nu,\alpha} = \frac{\partial g_{\mu\nu}}{\partial x^\alpha}$
Et

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} &= \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right]^{-\frac{1}{2}} \frac{\partial}{\partial \dot{x}^\alpha}\left(g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right) = \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right]^{-\frac{1}{2}} g_{\mu\nu}\frac{\partial}{\partial \dot{x}^\alpha}\left(\dot{x}^\mu \dot{x}^\nu\right) \\ &= \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right]^{-\frac{1}{2}} g_{\mu\nu}\left(\dot{x}^\nu \underbrace{\frac{\partial}{\partial \dot{x}^\alpha}\left(\dot{x}^\mu\right)}_{=\delta^\mu_\alpha} + \dot{x}^\mu \underbrace{\frac{\partial}{\partial \dot{x}^\alpha}\left(\dot{x}^\nu\right)}_{=\delta^\nu_\alpha}\right) \\ &= \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right]^{-\frac{1}{2}} g_{\mu\nu}\left(\dot{x}^\nu \delta^\mu_\alpha + \underbrace{\dot{x}^\mu \delta^\nu_\alpha}_{=\dot{x}^\nu \delta^\mu_\alpha \text{ car } \mu \text{ et } \nu \text{ sont muet}}\right) \\ &= \cancel{\frac{1}{2}}\left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right]^{-\frac{1}{2}} g_{\mu\nu} \cdot \cancel{2}\dot{x}^\nu \delta^\mu_\alpha \\ &= \left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right]^{-\frac{1}{2}} g_{\alpha\nu}\dot{x}^\nu \end{aligned}$$

Ainsi par les équation d'Euleur-La Grange, on obtient l'équation suivante :

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right]^{-\frac{1}{2}} g_{\alpha\nu}\dot{x}^\nu\right) = \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right]^{-\frac{1}{2}} g_{\mu\nu,\alpha}\dot{x}^\mu \dot{x}^\nu$$

En posant :

$$\mathrm{d}\lambda = \left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right]^{\frac{1}{2}} \mathrm{d}s$$

$$\text{Donc } \frac{\mathrm{d}}{\mathrm{d}\lambda} = \left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right]^{-\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}s}$$

Ainsi, en réécrivant l'équation précédente :

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right]^{-\frac{1}{2}} g_{\alpha\nu}\frac{\mathrm{d}}{\mathrm{d}s}x^\nu\right) = \frac{1}{2}\left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right]^{-\frac{1}{2}} g_{\mu\nu,\alpha}\frac{\mathrm{d}}{\mathrm{d}s}x^\mu \frac{\mathrm{d}}{\mathrm{d}s}x^\nu$$

$$\text{donc } [g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\cancel{[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}}}g_{\alpha\nu}\cancel{[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{\frac{1}{2}}}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\nu\right) = \frac{1}{2}[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}}[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]\,g_{\mu\nu,\alpha}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\nu$$

$$\text{donc } \cancel{[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{\frac{1}{2}}}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\alpha\nu}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\nu\right) = \frac{1}{2}\cancel{[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{\frac{1}{2}}}g_{\mu\nu,\alpha}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\nu$$

$$\text{donc } \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\alpha\nu}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\nu\right) = \frac{1}{2}g_{\mu\nu,\alpha}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\nu$$

Si on écrit l'action par rapport à s, on a :

$$S_1 = \int\left[g_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}s}x^\mu\frac{\mathrm{d}}{\mathrm{d}s}x^\nu\right]^{\frac{1}{2}}\mathrm{d}s$$

Alors en opérant le changement de variable S_1 devient :

$$\begin{aligned} S_1 &= \int\cancel{[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{\frac{1}{2}}}\left[g_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\nu\right]^{\frac{1}{2}}\cancel{[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}}}\,\mathrm{d}\lambda \\ &= \int\left[g_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\mu\frac{\mathrm{d}}{\mathrm{d}\lambda}x^\nu\right]^{\frac{1}{2}}\mathrm{d}\lambda \end{aligned}$$

Donc comme S_1 est invariante par la transformation $s \longrightarrow \lambda$ et que celle-ci est un diféomorphisme, on a alors :

$$\frac{\mathrm{d}}{\mathrm{d}s}(g_{\alpha\nu}\dot{x}^\nu) = \frac{1}{2}g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu$$

Donc en calculant le terme de gauche :

$$\begin{aligned} \frac{d}{\mathrm{d}s}(g_{\alpha\nu}\dot{x}^\nu) &= \dot{x}^\nu\frac{\mathrm{d}}{\mathrm{d}s}g_{\alpha\nu} + g_{\alpha\nu}\ddot{x}^\nu \\ &= g_{\alpha\nu}\ddot{x}^\nu + \dot{x}^\nu\underbrace{\frac{\partial g_{\alpha\nu}}{\partial\mu}}_{=g_{\alpha\nu,\mu}}\underbrace{\frac{\mathrm{d}x^\mu}{\mathrm{d}s}}_{=\dot{x}^\mu} \\ &= g_{\alpha\nu}\ddot{x}^\nu + g_{\alpha\nu,\mu}\dot{x}^\nu\dot{x}^\mu \end{aligned}$$

Ainsi :

$$\begin{aligned} \underbrace{g^{\alpha\beta}g_{\alpha\nu}}_{=\delta^\beta_\nu}\ddot{x}^\nu &= \frac{1}{2}g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu - g_{\alpha\nu,\mu}\dot{x}^\nu\dot{x}^\mu \\ \text{Donc } \ddot{x}^\beta &= \frac{1}{2}g^{\alpha\beta}(g_{\mu\nu,\alpha} - 2g_{\alpha\nu,\mu})\dot{x}^\nu\dot{x}^\mu \\ \text{Donc } \ddot{x}^\beta &= \frac{1}{2}g^{\alpha\beta}(g_{\mu\nu,\alpha} - g_{\alpha\nu,\mu} - g_{\alpha\mu,\nu})\dot{x}^\nu\dot{x}^\mu \end{aligned}$$

En introduisant les symboles de Cristoffel $\Gamma^\beta_{\mu\nu} = \frac{1}{2}g^{\beta\alpha}(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha})$ on a :

$$\ddot{x}^\beta = -\Gamma^\beta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

$$\text{Soit } \ddot{x}^\beta + \Gamma^\beta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$$

Commuteur des Co-dérivé

On définis les dérivé covariante par :

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\rho} v^\rho$$

Ainsi, soit v^ρ un vecteur :

$$\begin{aligned}
\left[\nabla_\mu ; \nabla_\nu\right] v^\rho &= \nabla_\mu (\nabla_\nu v^\rho) - \nabla_\nu \left(\nabla_\mu v^\rho\right) \\
&= \nabla_\mu (\partial_\nu v^\rho + \Gamma^\rho_{\nu\alpha} v^\alpha) - \nabla_\nu \left(\partial_\mu v^\rho + \Gamma^\rho_{\mu\alpha} v^\alpha\right) \\
&= \partial_\mu (\partial_\nu v^\rho + \Gamma^\rho_{\nu\alpha} v^\alpha) + \Gamma^\rho_{\mu\beta}\left(\partial_\nu v^\beta + \Gamma^\beta_{\nu\alpha} v^\alpha\right) - \partial_\nu \left(\partial_\mu v^\rho + \Gamma^\rho_{\mu\alpha} v^\alpha\right) - \Gamma^\rho_{\nu\beta}\left(\partial_\mu v^\beta + \Gamma^\beta_{\mu\alpha} v^\alpha\right) \\
&= \cancel{\partial_{\nu\mu} v^\rho} + \partial_\mu \Gamma^\rho_{\nu\alpha} v^\alpha + \Gamma^\rho_{\mu\beta} \partial_\nu v^\beta + \Gamma^\rho_{\mu\beta} \Gamma^\beta_{\nu\alpha} v^\alpha - \cancel{\partial_{\mu\nu} v^\rho} - \partial_\nu \Gamma^\rho_{\mu\alpha} v^\alpha - \Gamma^\rho_{\nu\beta} \partial_\mu v^\beta - \Gamma^\rho_{\nu\beta} \Gamma^\beta_{\mu\alpha} v^\alpha \\
&= \left(\Gamma^\rho_{\mu\beta}\Gamma^\beta_{\nu\alpha} - \Gamma^\rho_{\nu\beta}\Gamma^\beta_{\mu\alpha} + \partial_\mu \Gamma^\rho_{\nu\alpha} - \partial_\nu \Gamma^\rho_{\mu\alpha}\right) v^\alpha + \underbrace{\Gamma^\rho_{\mu\beta} \partial_\nu v^\beta - \Gamma^\rho_{\nu\beta} \partial_\mu v^\beta}_{=0}
\end{aligned}$$

Ainsi on reconnais le tenseur de Riemann définis par : $R^\sigma_{\mu\nu\kappa} = \partial_\nu \Gamma^\sigma_{\mu\kappa} - \partial_\kappa \Gamma^\sigma_{\mu\nu} + \Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\kappa} - \Gamma^\sigma_{\kappa\lambda}\Gamma^\lambda_{\mu\nu}$
Donc on obtient la relation suivante :

$$\left[\nabla_\mu ; \nabla_\nu\right] v^\rho = R^\rho_{\alpha\mu\nu} v^\alpha$$