

Équation des géodésiques

Soit le lagrangiens suivant :

$$\mathcal{L}(x^\alpha, \dot{x}^\alpha) = [g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu]^{\frac{1}{2}}$$

avec $\dot{x}^\alpha = \frac{dx^\alpha}{ds}$

Ainsi on n'a :

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{1}{2}[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}} \frac{\partial}{\partial x^\alpha}(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)$$

car \dot{x}^μ et \dot{x}^ν

sont constant

$$\begin{aligned} \text{face à } x^\alpha \rightarrow &= \frac{1}{2}[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \dot{x}^\mu\dot{x}^\nu \\ &= \frac{1}{2}[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}} g_{\mu\nu,\alpha} \dot{x}^\mu\dot{x}^\nu \end{aligned}$$

où on note : $g_{\mu\nu,\alpha} = \frac{\partial g_{\mu\nu}}{\partial x^\alpha}$

Et

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} &= \frac{1}{2}[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}} \frac{\partial}{\partial \dot{x}^\alpha}(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu) = \frac{1}{2}[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}} g_{\mu\nu} \frac{\partial}{\partial \dot{x}^\alpha}(\dot{x}^\mu\dot{x}^\nu) \\ &= \frac{1}{2}[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}} g_{\mu\nu} \left(\underbrace{\dot{x}^\nu \frac{\partial}{\partial \dot{x}^\alpha}(\dot{x}^\mu)}_{=\delta_\alpha^\mu} + \underbrace{\dot{x}^\mu \frac{\partial}{\partial \dot{x}^\alpha}(\dot{x}^\nu)}_{=\delta_\alpha^\nu} \right) \\ &= \frac{1}{2}[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}} g_{\mu\nu} \left(\dot{x}^\nu \delta_\alpha^\mu + \underbrace{\dot{x}^\mu \delta_\alpha^\nu}_{=\dot{x}^\nu \delta_\alpha^\mu \text{ car } \mu \text{ et } \nu \text{ sont muet}} \right) \\ &= \cancel{\frac{1}{2}}[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}} g_{\mu\nu} \cdot \cancel{2} \dot{x}^\nu \delta_\alpha^\mu \\ &= [g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}} g_{\alpha\nu} \dot{x}^\nu \end{aligned}$$

Ainsi par les équation d'Euler-La Grange, on obtient l'équation suivante :

$$\frac{d}{ds} \left([g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}} g_{\alpha\nu} \dot{x}^\nu \right) = \frac{1}{2}[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}} g_{\mu\nu,\alpha} \dot{x}^\mu \dot{x}^\nu$$

En posant :

$$d\lambda = [g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{\frac{1}{2}} ds$$

$$\text{Donc } \frac{d}{d\lambda} = [g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}} \frac{d}{ds}$$

Ainsi, en réécrivant l'équation précédente :

$$\frac{d}{ds} \left([g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}} g_{\alpha\nu} \frac{d}{ds} x^\nu \right) = \frac{1}{2}[g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{-\frac{1}{2}} g_{\mu\nu,\alpha} \frac{d}{ds} x^\mu \frac{d}{ds} x^\nu$$

$$\text{donc } [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{\frac{1}{2}} \frac{d}{d\lambda} \left(\underbrace{[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]}_{=g_{\mu\nu}}^{-\frac{1}{2}} g_{\alpha\nu} \underbrace{[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]}_{=g_{\mu\nu}}^{\frac{1}{2}} \frac{d}{d\lambda} x^\nu \right) = \frac{1}{2} [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{-\frac{1}{2}} [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu] g_{\mu\nu,\alpha} \frac{d}{d\lambda} x^\mu \frac{d}{d\lambda} x^\nu$$

$$\text{donc } \underbrace{[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]}_{=g_{\mu\nu}}^{\frac{1}{2}} \frac{d}{d\lambda} \left(g_{\alpha\nu} \frac{d}{d\lambda} x^\nu \right) = \frac{1}{2} \underbrace{[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]}_{=g_{\mu\nu}}^{\frac{1}{2}} g_{\mu\nu,\alpha} \frac{d}{d\lambda} x^\mu \frac{d}{d\lambda} x^\nu$$

$$\text{donc } \frac{d}{d\lambda} \left(g_{\alpha\nu} \frac{d}{d\lambda} x^\nu \right) = \frac{1}{2} g_{\mu\nu,\alpha} \frac{d}{d\lambda} x^\mu \frac{d}{d\lambda} x^\nu$$

Si on écrit l'action par rapport à s , on a :

$$S_1 = \int \left[g_{\mu\nu} \frac{d}{ds} x^\mu \frac{d}{ds} x^\nu \right]^{\frac{1}{2}} ds$$

Alors en opérant le changement de variable S_1 devient :

$$\begin{aligned} S_1 &= \int \underbrace{[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]}_{=g_{\mu\nu}}^{\frac{1}{2}} \left[g_{\mu\nu} \frac{d}{d\lambda} x^\mu \frac{d}{d\lambda} x^\nu \right]^{\frac{1}{2}} \underbrace{[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]}_{=g_{\mu\nu}}^{-\frac{1}{2}} d\lambda \\ &= \int \left[g_{\mu\nu} \frac{d}{d\lambda} x^\mu \frac{d}{d\lambda} x^\nu \right]^{\frac{1}{2}} d\lambda \end{aligned}$$

Donc comme S_1 est invariante par la transformation $s \rightarrow \lambda$ et que celle-ci est un diféomorphisme, on a alors :

$$\frac{d}{ds} (g_{\alpha\nu} \dot{x}^\nu) = \frac{1}{2} g_{\mu\nu,\alpha} \dot{x}^\mu \dot{x}^\nu$$

Donc en calculant le terme de gauche :

$$\begin{aligned} \frac{d}{ds} (g_{\alpha\nu} \dot{x}^\nu) &= \dot{x}^\nu \frac{d}{ds} g_{\alpha\nu} + g_{\alpha\nu} \ddot{x}^\nu \\ &= g_{\alpha\nu} \ddot{x}^\nu + \dot{x}^\nu \underbrace{\frac{\partial g_{\alpha\nu}}{\partial \mu}}_{=g_{\alpha\nu,\mu}} \underbrace{\frac{dx^\mu}{ds}}_{=\dot{x}^\mu} \\ &= g_{\alpha\nu} \ddot{x}^\nu + g_{\alpha\nu,\mu} \dot{x}^\nu \dot{x}^\mu \end{aligned}$$

Ainsi :

$$\underbrace{g^{\alpha\beta} g_{\alpha\nu}}_{=\delta_\nu^\beta} \ddot{x}^\nu = \frac{1}{2} g_{\mu\nu,\alpha} \dot{x}^\mu \dot{x}^\nu - g_{\alpha\nu,\mu} \dot{x}^\nu \dot{x}^\mu$$

$$\text{Donc } \ddot{x}^\beta = \frac{1}{2} g^{\alpha\beta} (g_{\mu\nu,\alpha} - 2g_{\alpha\nu,\mu}) \dot{x}^\nu \dot{x}^\mu$$

$$\text{Donc } \ddot{x}^\beta = \frac{1}{2} g^{\alpha\beta} (g_{\mu\nu,\alpha} - g_{\alpha\nu,\mu} - g_{\alpha\mu,\nu}) \dot{x}^\nu \dot{x}^\mu$$

En introduisant les symboles de Cristoffel $\Gamma_{\mu\nu}^\beta = \frac{1}{2} g^{\beta\alpha} (g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha})$ on a :

$$\ddot{x}^\beta = -\Gamma_{\mu\nu}^\beta \dot{x}^\mu \dot{x}^\nu$$

$$\text{Soit } \ddot{x}^\beta + \Gamma_{\mu\nu}^\beta \dot{x}^\mu \dot{x}^\nu = 0$$

Commuteur des Co-dérivé

On définit les dérivé covariante par :

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma_{\mu\rho}^\nu v^\rho$$

Ainsi, soit v^ρ un vecteur :

$$\begin{aligned} [\nabla_\mu; \nabla_\nu] v^\rho &= \nabla_\mu (\nabla_\nu v^\rho) - \nabla_\nu (\nabla_\mu v^\rho) \\ &= \nabla_\mu (\partial_\nu v^\rho + \Gamma_{\nu\alpha}^\rho v^\alpha) - \nabla_\nu (\partial_\mu v^\rho + \Gamma_{\mu\alpha}^\rho v^\alpha) \\ &= \partial_\mu (\partial_\nu v^\rho + \Gamma_{\nu\alpha}^\rho v^\alpha) + \Gamma_{\mu\beta}^\rho (\partial_\nu v^\beta + \Gamma_{\nu\alpha}^\beta v^\alpha) - \partial_\nu (\partial_\mu v^\rho + \Gamma_{\mu\alpha}^\rho v^\alpha) - \Gamma_{\nu\beta}^\rho (\partial_\mu v^\beta + \Gamma_{\mu\alpha}^\beta v^\alpha) \\ &= \cancel{\partial_{\nu\mu} v^\rho} + \partial_\mu \Gamma_{\nu\alpha}^\rho v^\alpha + \Gamma_{\mu\beta}^\rho \partial_\nu v^\beta + \Gamma_{\mu\beta}^\rho \Gamma_{\nu\alpha}^\beta v^\alpha - \cancel{\partial_{\mu\nu} v^\rho} - \partial_\nu \Gamma_{\mu\alpha}^\rho v^\alpha - \Gamma_{\nu\beta}^\rho \partial_\mu v^\beta - \Gamma_{\nu\beta}^\rho \Gamma_{\mu\alpha}^\beta v^\alpha \\ &= (\Gamma_{\mu\beta}^\rho \Gamma_{\nu\alpha}^\beta - \Gamma_{\nu\beta}^\rho \Gamma_{\mu\alpha}^\beta + \partial_\mu \Gamma_{\nu\alpha}^\rho - \partial_\nu \Gamma_{\mu\alpha}^\rho) v^\alpha + \underbrace{\Gamma_{\mu\beta}^\rho \partial_\nu v^\beta - \Gamma_{\nu\beta}^\rho \partial_\mu v^\beta}_{=0} \end{aligned}$$

Ainsi on reconnaît le tenseur de Riemann défini par : $R_{\mu\nu\kappa}^\sigma = \partial_\nu \Gamma_{\mu\kappa}^\sigma - \partial_\kappa \Gamma_{\mu\nu}^\sigma + \Gamma_{\nu\lambda}^\sigma \Gamma_{\mu\kappa}^\lambda - \Gamma_{\kappa\lambda}^\sigma \Gamma_{\mu\nu}^\lambda$

Donc on obtient la relation suivante :

$$[\nabla_\mu; \nabla_\nu] v^\rho = R_{\alpha\mu\nu}^\rho v^\alpha$$

Dérivé covariante

On se place dans un système de coordonnée ξ^α , et on s'intéresse à un changement de coordonée vers x^α

Tout d'abord :

$$\begin{aligned} 0 &= \frac{d^2 \xi^\alpha}{d\tau^2} = \frac{d}{d\tau} \left(\frac{d\xi^\alpha}{d\tau} \right) \\ &= \frac{d}{d\tau} \left(\frac{\partial \xi^\alpha}{\partial x^\mu} \frac{dx^\mu}{d\tau} \right) \\ &= \frac{\partial \xi^\alpha}{\partial x^\rho} \frac{d^2 x^\rho}{d\tau^2} + \frac{dx^\mu}{d\tau} \frac{d}{d\tau} \frac{\partial \xi^\alpha}{\partial x^\mu} \end{aligned}$$

Ainsi comme $\frac{\partial \xi^\alpha}{\partial x^\mu} \times \frac{\partial x^\mu}{\partial \xi^\beta} = \delta_\beta^\alpha$, en multipliant des deux cotés :

$$\begin{aligned} 0 &= \frac{d^2 x^\rho}{d\tau^2} + \frac{\partial x^\rho}{\partial \xi^\alpha} \frac{d}{d\tau} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{dx^\mu}{d\tau} \\ &= \frac{d^2 x^\rho}{d\tau^2} + \frac{\partial x^\rho}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \end{aligned}$$

En posant : $\{_\mu^\lambda\}_\nu = \frac{\partial x^\lambda}{\partial \xi^\beta} \frac{\partial^2 \xi^\beta}{\partial x^\mu \partial x^\nu}$, on obtient finalement :

$$0 = \frac{d^2 x^\rho}{d\tau^2} + \{_\mu^\rho\}_\nu \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

En réécrivant le produit scalaire dans les coordonnées x^α à :

$$\eta_{\mu\nu} d\xi^\mu d\xi^\nu = \underbrace{\eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta}}_{=g_{\alpha\beta}} dx^\alpha dx^\beta$$

Donc:

$$g_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta}$$

alors en dérivant:

$$\begin{aligned} \partial_\lambda g_{\alpha\beta} &= \eta_{\mu\nu} \frac{\partial}{\partial x^\lambda} \left(\frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta} \right) \\ &= \underbrace{\eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial^2 \xi^\nu}{\partial x^\beta \partial x^\lambda}}_{=g_{\alpha\mu} \frac{\partial x^\mu}{\partial \xi^\nu}} + \underbrace{\eta_{\mu\nu} \frac{\partial \xi^\nu}{\partial x^\beta} \frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\lambda}}_{=g_{\alpha\nu} \frac{\partial x^\nu}{\partial \xi^\mu}} \\ &= g_{\alpha\mu} \underbrace{\frac{\partial x^\mu}{\partial \xi^\nu} \frac{\partial^2 \xi^\nu}{\partial x^\beta \partial x^\lambda}}_{=\{\beta^\mu_\lambda\}} + g_{\alpha\nu} \underbrace{\frac{\partial x^\nu}{\partial \xi^\mu} \frac{\partial^2 \xi^\mu}{\partial x^\beta \partial x^\lambda}}_{=\{\beta^\nu_\lambda\}} \\ &= \{\beta^\mu_\lambda\} g_{\mu\alpha} + \{\beta^\nu_\lambda\} g_{\nu\alpha} \end{aligned}$$