

Info : DM2

En logique minimale :

Partie I. Lemme :

• Lemme Minimaliste

Lemme 1

$$\frac{\Gamma, \varphi_1 \vdash \varphi_2 \quad \Gamma, \varphi_1 \vdash \neg \varphi_2}{\Gamma \vdash \neg \varphi_1} \neg_{ei}$$

preuve :

$$\frac{\frac{\text{hypp}}{\Gamma, \varphi_1 \vdash \varphi_2} \quad \frac{\text{hypp}}{\Gamma, \varphi_1 \vdash \neg \varphi_2}}{\Gamma, \varphi_1 \vdash \perp} \neg_e \quad \frac{}{\Gamma \vdash \neg \varphi_1} \neg_i$$

Lemme 2

$$\frac{}{\Gamma, \varphi \rightarrow \psi, \varphi \vdash \psi} \rightarrow_{ax}$$

preuve :

$$\frac{\frac{}{\Gamma, \varphi \rightarrow \psi, \varphi \vdash \varphi \rightarrow \psi} ax \quad \frac{}{\Gamma, \varphi \rightarrow \psi, \varphi \vdash \varphi} ax}{\Gamma, \varphi \rightarrow \psi, \varphi \vdash \psi} \rightarrow_e$$

• Lemme Classique

Lemme 1

$$\frac{\Gamma \vdash \neg \neg \varphi}{\Gamma \vdash \varphi} \neg_{\neg e}$$

preuve :

$$\frac{\frac{\frac{}{\Gamma, \neg \varphi \vdash \neg \varphi} ax \quad \frac{\frac{\text{hypp}}{\Gamma \vdash \neg \neg \varphi}}{\Gamma, \neg \varphi \vdash \neg \neg \varphi} aff}{\Gamma, \neg \varphi \vdash \perp} \neg_e}{\Gamma \vdash \varphi} raa$$

Partie II. Logique minimale :

N° 1.

$$\frac{\frac{}{\Gamma \vdash p \vee (p \wedge q)} ax \quad \frac{}{\Gamma, p \vdash p} ax \quad \frac{\frac{}{\Gamma, p \wedge q \vdash p \wedge q} ax}{\Gamma, p \wedge q \vdash p} \wedge_e^g}{\Gamma \vdash p \vee (p \wedge q) \vdash p} \vee_e \quad \text{Avec } \Gamma = \{p \vee (p \wedge q)\}$$

N° 2.

$$\frac{\frac{}{\Gamma \vdash p} ax \quad \frac{\frac{}{\Gamma \vdash p \rightarrow \neg p} ax \quad \frac{}{\Gamma \vdash p} ax}{\Gamma \vdash \neg p} \rightarrow_e}{\Gamma \vdash p \rightarrow \neg p \vdash \neg p} \neg_{ei} \quad \text{Avec } \Gamma = \{p \rightarrow \neg p, p\}$$

N° 3.

$$\begin{array}{c}
\frac{}{\Gamma, p \vdash p \rightarrow q \vee r} \text{ax} \quad \frac{}{\Gamma, p \vdash p} \text{ax} \\
\frac{}{\Gamma, p \vdash q \vee r} \rightarrow_e \quad \frac{}{\Gamma, p, q \vdash s} \rightarrow_{\text{ax}} \quad \frac{}{\Gamma, p, r \vdash s} \rightarrow_{\text{ax}} \\
\frac{}{\Gamma, p \vdash s} \vee_e \\
\frac{}{p \rightarrow (q \vee r), q \rightarrow s, r \rightarrow s \vdash p \rightarrow s} \rightarrow_i
\end{array}$$

Avec $\Gamma = \{p \rightarrow (q \vee r), q \rightarrow s, r \rightarrow s\}$

N° 4.

$$\begin{array}{c}
\frac{}{\Gamma \vdash p \rightarrow q} \text{ax} \quad \frac{}{\Gamma \vdash p \wedge r} \text{ax} \quad \frac{}{\Gamma \vdash p} \wedge_e^g \\
\frac{}{\Gamma \vdash q} \rightarrow_e \quad \frac{}{\Gamma \vdash r \rightarrow s} \text{ax} \quad \frac{}{\Gamma \vdash r} \wedge_e^d \\
\frac{}{\Gamma \vdash s} \rightarrow_e \\
\frac{}{\Gamma \vdash q \wedge s} \wedge_i
\end{array}$$

Avec $\Gamma = \{p \rightarrow q, r \rightarrow s, p \wedge r\}$

N° 5.

$$\begin{array}{c}
\frac{}{\Gamma \vdash p \vee r} \text{ax} \quad \frac{}{\Gamma, p \vdash q} \rightarrow_{\text{ax}} \quad \frac{}{\Gamma, p \vdash q \vee s} \vee_i^g \\
\frac{}{\Gamma \vdash q \vee s} \vee_e \\
\frac{}{\Gamma \vdash q \vee s} \vee_e
\end{array}$$

Avec $\Gamma = \{p \rightarrow q, r \rightarrow s, p \vee r\}$

N° 6.

$$\begin{array}{c}
\frac{}{\Gamma_1 \vdash (p \vee q) \rightarrow r} \text{ax} \quad \frac{}{\Gamma_1 \vdash p} \text{ax} \quad \frac{}{\Gamma_1 \vdash p \vee q} \vee_e^g \\
\frac{}{\Gamma_1 \vdash r} \rightarrow_e \quad \frac{}{\Gamma_2 \vdash (p \vee q) \rightarrow r} \text{ax} \quad \frac{}{\Gamma_2 \vdash q} \text{ax} \quad \frac{}{\Gamma_2 \vdash p \vee q} \vee_e^d \\
\frac{}{\Gamma_2 \vdash r} \rightarrow_e \\
\frac{}{(p \vee q) \rightarrow r \vdash (p \rightarrow r) \wedge (q \rightarrow r)} \wedge_i + \rightarrow_i
\end{array}$$

Avec $\Gamma_1 = \{(p \vee q) \rightarrow r, p\}$ et $\Gamma_2 = \{(p \vee q) \rightarrow r, q\}$

Partie III. Lois de De Morgan

N° 7.

$$\begin{array}{c}
\frac{}{\Gamma_1 \vdash \neg(p \vee q)} \text{ax} \quad \frac{}{\Gamma_1 \vdash p} \text{ax} \quad \frac{}{\Gamma_1 \vdash p \vee q} \vee_i^g \\
\frac{}{\neg(p \vee q) \vdash \neg p} \neg_{ei} \quad \frac{}{\Gamma_2 \vdash \neg(p \vee q)} \text{ax} \quad \frac{}{\Gamma_2 \vdash q} \text{ax} \quad \frac{}{\Gamma_2 \vdash p \vee q} \vee_i^d \\
\frac{}{\neg(p \vee q) \vdash \neg q} \neg_{ei} \\
\frac{}{\neg(p \vee q) \vdash \neg p \wedge \neg q} \wedge_i
\end{array}$$

Avec $\Gamma_1 = \{\neg(p \vee q), p\}$ et $\Gamma_2 = \{\neg(p \vee q), q\}$

N° 8.

$$\begin{array}{c}
\frac{}{\Gamma, q \vdash \neg p \wedge \neg q} \text{ax} \quad \frac{}{\Gamma, q \vdash \neg q} \wedge_e^d \\
\frac{}{\Gamma, q \vdash \perp} \wedge_e \\
\frac{}{\Gamma \vdash p \vee q} \text{ax} \quad \frac{}{\Gamma, p \vdash \neg p \wedge \neg q} \text{ax} \quad \frac{}{\Gamma, p \vdash \neg p} \wedge_e^g \\
\frac{}{\Gamma, p \vdash \perp} \neg_e \\
\frac{}{\neg p \wedge \neg q, p \vee q \vdash \perp} \vee_e \\
\frac{}{\neg p \wedge \neg q \vdash (p \vee q)} \neg_i
\end{array}$$

Avec $\Gamma = \{\neg p \wedge \neg q, p \vee q\}$

N° 9.

$$\begin{array}{c}
\frac{\frac{\frac{}{\Gamma, \neg q \vdash p \wedge q} \text{ax}}{\Gamma, \neg q \vdash q} \wedge_e^d}{\Gamma, \neg q \vdash \perp} \neg_e \quad \frac{\frac{}{\Gamma, \neg q \vdash \neg q} \text{ax}}{\Gamma \vdash \neg p \vee \neg q} \neg_e \quad \frac{\frac{\frac{}{\Gamma, \neg p \vdash \neg p} \text{ax}}{\Gamma, \neg p \vdash p} \neg_e}{\Gamma, \neg p \vdash \perp} \vee_e \\
\frac{\frac{\frac{}{\neg p \vee \neg q, p \wedge q \vdash \perp}}{\neg p \vee \neg q \vdash \neg(p \wedge q)} \neg_i
\end{array}$$

Avec $\Gamma = \{\neg p \vee \neg q, p \wedge q\}$

N° 10.

$$\begin{array}{c}
\frac{\frac{\frac{}{\Gamma \vdash \neg \neg p \wedge \neg \neg q} \text{q.7}}{\Gamma \vdash \neg \neg p} \wedge_e^g}{\Gamma \vdash p} \neg \neg_e \quad \frac{\frac{\frac{}{\Gamma \vdash \neg \neg p \wedge \neg \neg q} \text{q.7}}{\Gamma \vdash \neg \neg q} \wedge_e^d}{\Gamma \vdash q} \neg \neg_e \\
\frac{\frac{\frac{}{\Gamma \vdash \neg(p \wedge q)} \text{ax}}{\Gamma \vdash \neg(p \wedge q)} \text{ax} \quad \frac{\frac{\frac{}{\Gamma \vdash p \wedge q}}{\Gamma \vdash p \wedge q} \wedge_i}{\Gamma \vdash \perp} \neg_e \\
\frac{\Gamma \vdash \perp}{\neg(p \wedge q) \vdash \neg p \vee \neg q} \text{raa}
\end{array}$$

Avec $\Gamma = \{\neg(p \wedge q), \neg(\neg p \vee \neg q)\}$

Partie IV. Logique intuitionniste

N° 11.

$$\begin{array}{c}
\frac{\frac{}{\neg p, p \vdash p} \text{ax}}{\neg p, p \vdash \neg p} \neg_e \\
\frac{\frac{\frac{}{\neg p, p \vdash \perp}}{\neg p, p \vdash q} \perp_e}{\neg p \vdash p \rightarrow q} \rightarrow_i
\end{array}$$

N° 12.

$$\begin{array}{c}
\frac{\frac{\frac{}{\Gamma, q \vdash q} \text{ax}}{\Gamma, q \vdash \neg q} \neg_e}{\Gamma, q \vdash \perp} \perp_e \\
\frac{\frac{\frac{}{\Gamma \vdash p \vee q} \text{ax}}{\Gamma, p \vdash p} \text{ax} \quad \frac{\frac{\frac{}{\Gamma, q \vdash \perp}}{\Gamma, q \vdash p} \perp_e}{p \vee q, \neg q \vdash p} \vee_e
\end{array}$$

Avec $\Gamma = \{p \vee q, \neg q\}$

N° 13.

$$\begin{array}{c}
\frac{\frac{\frac{}{\Gamma, p \vdash q} \text{ax}}{\Gamma \vdash \neg(p \rightarrow q)} \neg_e}{\Gamma \vdash \perp} \perp_e \\
\frac{\frac{\frac{}{\Gamma \vdash \perp}}{\Gamma \vdash p} \perp_e}{\neg(p \rightarrow q) \vdash q \rightarrow p} \rightarrow_i
\end{array}$$

Avec $\Gamma = \{\neg(p \rightarrow q), q\}$

Partie V. logique classique

N° 14.

$$\frac{\frac{\Gamma \vdash q \vee \neg q}{\Gamma \vdash q \vee \neg q} \text{ t.e.} \quad \frac{\frac{\frac{\Gamma, q, p \vdash q}{\Gamma, q \vdash (p \rightarrow q) \vee (p \rightarrow r)} \text{ ax} \quad \vee_i^q + \rightarrow_i}{\Gamma, \neg q \vdash (p \rightarrow q) \vee (p \rightarrow r)} \frac{\text{Voir ci-dessous}}{\Gamma, \neg q, p \vdash r} \text{ ax}}{\frac{p \rightarrow (q \vee r) \vdash (p \rightarrow q) \vee (p \rightarrow r)}{\vee_e} \vee_i^d + \rightarrow_i}$$

Le ci-dessous en question :

$$\frac{\frac{\Gamma_2 \vdash p \rightarrow (q \vee r)}{\Gamma_2 \vdash q \vee r} \text{ ax} \quad \frac{\Gamma_2 \vdash p}{\Gamma, r \vdash r} \text{ ax} \quad \frac{\frac{\Gamma_2, q \vdash q}{\Gamma_2, q \vdash r} \text{ ax} \quad \frac{\Gamma_2, q \vdash \neg q}{\Gamma_2, q \vdash r} \text{ ax}}{\Gamma, \neg q, p \vdash r} \rightarrow_e \vee_e$$

Avec $\Gamma = \{p \rightarrow (q \vee r)\}$ et $\Gamma_2 = \Gamma \cup \{\neg q, p\}$

N° 15.

$$\frac{\frac{\frac{\frac{\Gamma, \neg q \vdash p}{\Gamma, \neg q \vdash \neg q \rightarrow \neg p} \text{ ax} \quad \frac{\Gamma, \neg q \vdash \neg q}{\Gamma, \neg q \vdash \neg p} \text{ ax}}{\Gamma, \neg q \vdash \neg p} \rightarrow_e \quad \frac{\Gamma, \neg q \vdash \perp}{\Gamma \vdash q} \text{ raa}}{\frac{\Gamma \vdash q \rightarrow r}{\Gamma \vdash r} \text{ ax} \quad \rightarrow_e} \rightarrow_i$$