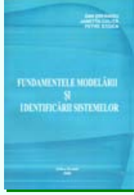
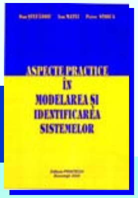


Exerciţiul 3.1



Determinaţi ecuaţiile de estimare a parametrlor necunoscuţi (coeficienţi & dispersie zgomot alb) pentru modelul ARX[1,1], în formă completă, folosind MCMMP. Evaluţi limitele teoretice ale parametrilor pentru o colecţie infinită de date. În ce condiţii parametrii estimaţi sunt consistenţi (adică tind la valorile adevărate)?

Soluţie

- Forma de regresie liniară a modelului ARX[1,1]:

$$y[n] = \varphi^T[n]\theta + e[n]$$

$$\begin{cases} \varphi^T[n] = [\,-y[n-1] \;\vdots\; u[n-1]\,] \\ \theta^T = [\;a \;\vdots\; b\;] \end{cases}$$

- Estimaţia CMMP:

$$\hat{\theta}_N = \underbrace{\left[\frac{1}{N}\sum_{n=1}^{N}\varphi[n]\varphi^T[n]\right]^{-1}}_{R_N^{-1}} \underbrace{\left[\frac{1}{N}\sum_{n=1}^{N}\varphi[n]y[n]\right]}_{r_N}$$

Soluție (Exercițiul 3.1)

$$N R_N = \sum_{n=1}^{N} \begin{bmatrix} -y[n-1] \\ u[n-1] \end{bmatrix} \begin{bmatrix} -y[n-1] & \vdots & u[n-1] \end{bmatrix} = \begin{bmatrix} \sum\limits_{n=1}^{N} y^2[n-1] & -\sum\limits_{n=1}^{N} u[n-1]\times y[n-1] \\ \equiv & \sum\limits_{n=1}^{N} u^2[n-1] \end{bmatrix}$$

$$N r_N = \sum_{n=1}^{N} \begin{bmatrix} -y[n-1] \\ u[n-1] \end{bmatrix} y[n] = \begin{bmatrix} -\sum\limits_{n=1}^{N} y[n]\,y[n-1] \\ \sum\limits_{n=1}^{N} y[n]\,u[n-1] \end{bmatrix}$$

- Notații naturale:

$$r_y^N[k] \triangleq \frac{1}{N}\sum_{n=1}^{N} y[n]\,y[n-k] \qquad r_u^N[k] \triangleq \frac{1}{N}\sum_{n=1}^{N} u[n]\,u[n-k]$$

$$r_{yu}^N \triangleq \frac{1}{N}\sum_{n=1}^{N} y[n]\,u[n-k], \quad \forall k \geq 0$$

- Rezultă:

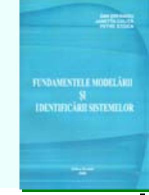
$$R_N \cong \begin{bmatrix} r_y^{N-1}[0] & -r_{yu}^{N-1}[0] \\ -r_{yu}^{N-1}[0] & r_u^{N-1}[0] \end{bmatrix}; \quad r_N \cong \begin{bmatrix} -r_y^{N-1}[1] \\ r_{yu}^{N-1}[1] \end{bmatrix}.$$

(deoarece $y[n] = u[n] = 0$, $\forall n \notin \overline{1,N}$)

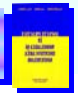
- Proprietate de inversabilitate:

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{\Delta}\begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}, \text{ cu } \Delta = \alpha\delta - \beta\gamma$$

❺ Exerciții rezolvate



Soluție (Exercițiul 3.1)



- Rezultă:

$$\hat{\theta}_N = \frac{1}{\Delta_N}\begin{bmatrix} r_u^{N-1}[0] & r_{yu}^{N-1}[0] \\ r_{yu}^{N-1}[0] & r_y^{N-1}[0] \end{bmatrix}\begin{bmatrix} -r_y^{N-1}[1] \\ r_{yu}^{N-1}[1] \end{bmatrix}$$

$$\Delta_N = r_u^{N-1}[0]\, r_y^{N-1}[0] - \left(r_{yu}^{N-1}[0]\right)^2$$

$$\begin{cases} \hat{\theta}_N = \dfrac{1}{\Delta_N}\begin{bmatrix} r_{yu}^{N-1}[0]\, r_{yu}^{N-1}[1] - r_u^{N-1}[0]\, r_y^{N-1}[1] \\ r_y^{N-1}[0]\, r_{yu}^{N-1}[1] - r_y^{N-1}[1]\, r_{yu}^{N-1}[0] \end{bmatrix} = \begin{bmatrix} \hat{a}_N \\ \hat{b}_N \end{bmatrix} \\ \hat{\lambda}_N^2 = \dfrac{1}{N}\displaystyle\sum_{n=1}^{N}\left(y[n] - \varphi^T[n]\,\hat{\theta}_N\right)^2 \end{cases}$$

- Limite teoretice (în notații naturale):

$$\lim_{N\to\infty} \hat{a}_N = \frac{r_{yu}[0]\, r_{yu}[1] - r_u[0]\, r_y[1]}{r_u[0]\, r_y[0] - \left(r_{yu}[0]\right)^2}$$

$$\lim_{N\to\infty} \hat{b}_N = \frac{r_y[0]\, r_{yu}[1] - r_y[1]\, r_{yu}[0]}{r_u[0]\, r_y[0] - \left(r_{yu}[0]\right)^2}$$

❺ Exerciții rezolvate

Soluție (Exercițiul 3.1)

- Pe de altă parte:

$$\theta = \begin{bmatrix} a \\ b \end{bmatrix} = \left[E\{\varphi[n]\varphi^T[n]\}\right]^{-1}\left[E\{\varphi[n]y[n]\}\right] - \left[E\{\varphi[n]\varphi^T[n]\}\right]^{-1}\left[E\{\varphi[n]e[n]\}\right]$$

$$E\{\varphi[n]\varphi^T[n]\} = \begin{bmatrix} r_y[0] & -r_{yu}[0] \\ -r_{yu}[0] & r_u[0] \end{bmatrix}$$

$$E\{\varphi[n]y[n]\} = \begin{bmatrix} -r_y[1] \\ r_{yu}[1] \end{bmatrix}$$

Vezi Lucrarea #1

$$E\{\varphi[n]e[n]\} = \begin{bmatrix} -r_{ye}[-1] \\ r_{ue}[-1] \end{bmatrix} = \begin{bmatrix} 0 \\ r_{eu}[1] \end{bmatrix}$$

- Deci:

$$a = \frac{r_{yu}[0]\,r_{yu}[1] - r_u[0]\,r_y[1] + r_{yu}[0]\,r_{eu}[1]}{r_u[0]\,r_y[0] - (r_{yu}[0])^2}$$

$$b = \frac{r_y[0]\,r_{yu}[1] - r_y[1]\,r_{yu}[0] + r_y[0]\,r_{eu}[1]}{r_u[0]\,r_y[0] - (r_{yu}[0])^2}$$

- Condiții de consistență ($\lim_{N\to\infty} \hat{\theta}_N = \theta$):

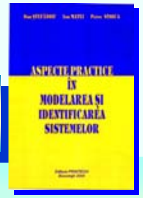
a) $E\{\varphi[n]\varphi^T[n]\}$ = inversabilă ($\underset{\neq}{>} 0$)

b) $E\{e[n]u[n-1]\} = r_{eu}[1] = 0$

c) e = z.a.$(0, \lambda^2)$

❺ Exerciții rezolvate

Exercițiul 3.2



Determinați ecuațiile de estimare a parametrilor necunoscuți (coefcienți & dispersie zgomot alb) pentru modelul ARX[2,2], folosind MCMMP.

Soluție

- Forma de regresie liniară:

$$y[n] = \varphi^T[n]\theta + e[n]\,,$$

$$\begin{cases} \varphi^T[n] = \left[-y[n-1] \;\; -y[n-2] \;\vdots\; u[n-1] \;\; u[n-2] \right] \\ \theta^T = \left[a_1 \;\; a_2 \;\vdots\; b_1 \;\; b_2 \right] \end{cases}$$

Soluție (Exercițiul 3.2)

- Estimația CMMP (cu notațiile din $\boxed{\text{Ex } 3.1}$):

$$\hat{\theta}_N = \underbrace{\left[\begin{array}{cc|cc} r_y^N[0] & r_y^N[1] & -r_{yu}^N[0] & -r_{yu}^N[1] \\ r_y^N[1] & r_y^N[0] & -r_{uy}^N[1] & -r_{uy}^N[0] \\ \hline -r_{yu}^N[0] & -r_{uy}^N[1] & r_u^N[0] & r_u^N[1] \\ -r_{yu}^N[1] & -r_{uy}^N[0] & r_u^N[1] & r_u^N[0] \end{array}\right]^{-1}}_{R_N^{-1}} \underbrace{\left[\begin{array}{c} -r_y^N[1] \\ -r_y^N[2] \\ \hline r_{yu}^N[1] \\ r_{yu}^N[2] \end{array}\right]}_{r_N}$$

$$R_N^{-1} \quad \left(\left[\frac{1}{N}\sum_{n=1}^{N} \varphi[n]\varphi^T[n]\right]^{-1}\right) \qquad r_N \quad \left(\frac{1}{N}\sum_{n=1}^{N} \varphi[n]y[n]\right)$$

$$\hat{\lambda}_N^2 = \frac{1}{N}\sum_{n=1}^{N}\left(y[n] - \varphi^T[n]\hat{\theta}_N\right)^2$$