❺ Exerciţii rezolvate

Soluţie (Exerciţiul 4.4)

- Pentru a determina relaţia valorilor adevărate ale parametrilor, se înmulţeşte ecuaţia modelului cu $\tilde{z}[n]$ şi se mediază:

$$\theta = E\{\tilde{z}[n]\varphi^T[n]\}^{-1}\left(E\{\tilde{z}[n]y[n]\} - E\{\tilde{z}[n]e[n]\}\right) =$$

$$= \frac{1}{\Delta}\begin{bmatrix} r_u[0] & -r_{uu_f}[0] \\ r_{yu}[0] & -r_{yu_f}[0] \end{bmatrix}\left(\begin{bmatrix} r_{yu_f}[1] \\ r_{yu}[1] \end{bmatrix} - \begin{bmatrix} r_{eu_f}[1] \\ r_{eu}[1] \end{bmatrix}\right),$$

cu $\Delta = r_{yu}[0]\, r_{uu_f}[0] - r_u[0]\, r_{yu_f}[0]$

- Condiţii generale de consistenţă:

$\lim_{N\to\infty} \tilde{\theta}_N = \theta \implies$ 1. $\Delta \neq 0$, $|a| < 1$ (stabilitate)

2. $r_{eu_f}[1] = 0$
3. $r_{eu}[1] = 0$ $\Big\}$ → practic $r_{eu}[k] = 0$, $\forall k \in \mathbb{Z}$

– Se evaluează secvenţele de covarianţă din definiţia discriminantului Δ (pentru $r_{eu}[k] = 0$, $\forall k \in \mathbb{Z}$).

▸ $r_{yu}[k] + a\, r_{yu}[k-1] = b\, r_u[k-1]$ (vezi def. lui u_f)

▸ $r_{yu}[k] \triangleq E\{y[n]u[n-k]\} = b\sum_{m\geq 0}(-a)^m E\{u[n-m-1]u[n-k]\} =$

$= b\sum_{m\geq 0}(-a)^m r_u[k-m-1]$, $\forall k \in \mathbb{Z}$

$\Downarrow$

$r_{yu}[0] = b\sum_{m\geq 0}(-a)^m r_u[m+1]$ (r_u simetrică)

$\left(r_{yu}[1] = b\, r_u[0] - a\, r_{yu}[0]\right)$ (în subsidiar)

❺ Exerciții rezolvate

Soluție (Exercițiul 4.4)

- $$r_{uu_f}[k] \triangleq E\{u[n]\,u_f[n-k]\} = b\sum_{m\geq 0}(-a)^m\, r_u[k+m+1] \quad \forall k\in\mathbb{Z}$$

$$\boxed{u_f[n-k] = \frac{b}{1+aq^{-1}}\,u[n-k-1]}$$

$$\Downarrow$$

$$r_{uu_f}[0] = b\sum_{m\geq 0}(-a)^m\, r_u[m+1] = r_{yu}[0] \quad !$$

- $$r_{yu_f}[k] \triangleq E\{y[n]\,u_f[n-k]\} = b\sum_{m\geq 0}(-a)^m\, r_{yu}[k+m+1] \quad \forall k\in\mathbb{Z}.$$

$$\Downarrow$$

$$r_{yu_f}[0] = b\sum_{m\geq 0}(-a)^m\, r_{yu}[m+1] =$$

$$= b^2\sum_{m\geq 0}\sum_{\ell\geq 0}(-a)^{m+\ell}\, r_u[m-\ell]$$

Soluție (Exercițiul 4.4)

– Rezultă:

$$\Delta = \left(r_{yu}[0]\right)^2 - r_u[0]\, b^2 \sum_{m\geq 0}\sum_{l\geq 0} (-a)^{m+l} r_u[m-l] =$$

$$= b^2 \sum_{m\geq 0}\sum_{l\geq 0} (-a)^{m+l} \Big[r_u[m+1]\, r_u[l+1] - r_u[0]\, r_u[m-l] \Big]$$

– Condiție suficientă: $u = z.a.(0, \sigma_u^2)$

$\Downarrow$

$$\Delta = b^2 \sum_{m\geq 0}\sum_{l\geq 0} (-a)^{m+l} \Big[\underbrace{-r_u[0]}_{-\sigma_u^2}\, \sigma_u^2\, \delta_0[m-l] \Big] =$$

$$= -b^2 \sigma_u^4 \sum_{m\geq 0} a^{2m} =$$

$$= \frac{\sigma_u^4 b^2}{a^2-1} < 0$$

Soluţie (Exerciţiul 4.4)

b) $$\tilde{z}[n] = \left[\frac{bq^{-1}}{1+aq^{-1}}u[n-1] \quad u[n-1]\right]^T = \left[bq^{-1} \quad 1+aq^{-1}\right]^T \frac{u[n-1]}{1+aq^{-1}}$$

$$z_f[n] = \left[\frac{bq^{-1}}{1+aq^{-1}}u[n] \quad \frac{bq^{-1}}{1+aq^{-1}}u[n-1]\right]^T = \left[b \quad bq^{-1}\right]^T \frac{u[n-1]}{1+aq^{-1}}$$

- Se observă că $\exists$ o matrice $P = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ a.î.: $\tilde{z}[n] \equiv P z_f[n]$. Matricea rezultă simplu astfel:

$$\begin{cases} bq^{-1} \equiv \alpha b + \beta b q^{-1} & \Rightarrow \alpha = 0, \ \beta = 1 \\ 1+aq^{-1} \equiv \gamma b + \delta b q^{-1} & \Rightarrow \gamma = \frac{1}{b}, \ \delta = \frac{a}{b} \end{cases}$$

$$P = \begin{bmatrix} 0 & 1 \\ \frac{1}{b} & \frac{a}{b} \end{bmatrix} \quad \text{(inversabilă)}$$

- Atunci: $$\tilde{\theta} = \left[P \sum_{n=1}^{N} z_f[n]\varphi^T[n]\right]^{-1} \left[P \sum_{n=1}^{N} z_f[n]y[n]\right] =$$
$$= \left[\sum_{n=1}^{N} z_f[n]\varphi^T[n]\right]^{-1} \underbrace{P^{-1}P}_{I_2} \left[\sum_{n=1}^{N} z_f[n]y[n]\right] = \theta_f$$