Internship report:
On the Regge Poles of the Veneziano amplitude

Student: Baptiste Guilleminot
Supervisor: Piotr Tourkine

August 2025

Abstract

In this report, I present some of the work I carried out during my three-month internship at LAPTh, under the supervision of Piotr Tourkine. After a brief and minimal introduction to Regge theory, I discuss my results concerning the Regge poles of the Veneziano amplitude.

In Chapter 3, I introduce a new closed-form expression for the residues of the poles. Chapter 4 is devoted to asymptotic results for large trajectories at fixed energy. Detailed calculations are provided in the appendix. Additionally, a Mathematica notebook containing a summary of the results and illustrative graphs is available here.

Acknowledgements

I would like to thank Samuel Wallon, Robin Zegers, and Björn Herrmann for helping me find this internship in one way or another. I am especially grateful to Piotr Tourkine for his trust and for supervising my work—it was a real pleasure to collaborate with you. Finally, I want to thank everyone at LAPTh for the warm welcome and the enjoyable time I had during these three months.

Contents

Chapter 1

Introduction

In the 40's and 50's, physicists like Schwinger, Tomonaga, Feynman, and Dyson put together an incredible theory which linked two new concepts of the beginning of the century: special relativity and quantum mechanics. The theory starts from a Lagrangian and, instead of a direct prediction, it follows the steps given in figure 1.1.

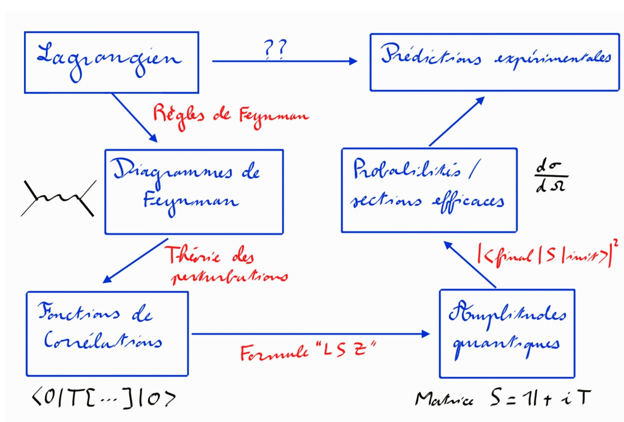


Figure 1.1: Explanation of how perturbative QFT works.

But later, in the 60's, physicists were faced with a problem: the strong force has a coupling constant which prevents a perturbative treatment in some regime. Therefore, the framework used for the electroweak interaction couldn't work. Before chromo-dynamics was born, some people followed this intuition by Heisenberg: "In the absence of knowledge of a microscopic theory, let us try to see what constraints the fundamental principles impose. Principles which we know will remain valid even after the discovery of that theory." This way of building a theory is called a bootstrap.

However, which general considerations should we use for this bootstrap? The first one is causality, which is often imposed through analyticity on certain domains of the function used for the amplitude. I don't have time to go into details here but in order to give a general idea, let's consider a system governed by the Green's function $G(\omega)$ in Fourier space. In order to do the inverse Fourier transform, one needs to compute the following integral.

$$G(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} G(\omega) e^{-i\omega t} \tag{1.1}$$

If $G(\omega)$ is analytic on the upper half of the complex plane and doesn't grow too fast at $i\infty$, the integral is equivalent to a contour integral on this part of the plane for $t \leq 0$ which is null thanks to Cauchy's theorem. Therefore $G(t) = \Theta(t)G(t)$ with Θ the Heaviside function and the system is causal. The same idea can apply for the S-matrix to some degree, for details see [9].

In the following, I will suppose analyticity in the entire complex plane apart from the poles and branch cut imposed by unitarity. This is known as maximal analyticity.

A second property we want for our bootstrap is crossing. To explain this, I will take the example of an interaction with two incident particles and two outgoing particles. Crossing means that the amplitude for $1\ 2 \to 3\ 4$ is the same as for $1\ \bar{3} \to \bar{2}\ 4$ and $1\ \bar{4} \to \bar{2}\ 3$[1]. This property has to be true for any QFT for 2 to 2 scattering as Bros, Epstein and Glaser proved in [1].

Let us define the Mandelstam variables $s = -(p_1 + p_2)^2$, $t = -(p_1 - p_3)^2$, $u = -(p_1 - p_4)^2$ and the amplitude $A(s,t) = \langle p_3p_4|\, T\, |p_1p_2\rangle$ where $S = 1 + iT$ and S is the S-matrix. We can notice that $s + t + u = 4m^2$ where m is the mass of the particles which we suppose was the same for everyone. This relation explains why A doesn't need to depend explicitly on u. In this language, crossing takes the form: $A(s,t) = A(t,s) = A(s,u)$.

If we take into account color ordering, only the crossing between s and t remains, therefore, crossing will translate in the rest of the report by $A(s,t) = A(t,s)$. For more details, see [14].

The last big property we want to impose is unitarity. This property comes from the probabilistic formalism of quantum mechanics. The probability comes from the Hermitian product between states and we want to keep its properties through time. Therefore, a time evolution operator, like the S-matrix, which links states long before interaction to long after, has to be unitary. This translates to the following equation:

$$SS^\dagger = 1 \tag{1.2}$$

Unitarity is a very important property for the bootstrap of the S-matrix. I'll come back to it later in section 3.1.

Of course, there are other things to keep in mind. The first one is that the S-matrix has to respect the Poincaré symmetries. This is the backbone of the proof for the crossing symmetry for example, and I will use it again soon. Also, experimental results like Regge trajectories have to be taken into account.

The bootstrap approach to high energy physics lost some momentum in the 80's. Finding a S-matrix was too ambitious. However, it led Veneziano to propose the following amplitude:

$$A(s,t) = B(-s,-t) = \int_{-1}^{1} du\ (1-u)^{-s-1}u^{-t-1} \tag{1.3}$$

where B is the Euler beta function. This amplitude led to string theory, which is still one of the best contenders for a theory bridging the gap between particle physics and gravity.

It is also at the heart of my internship. The goal of the three months I spent in Annecy was to derive the Regge poles of the Veneziano amplitude in a closed form. This explicit expression could give hints on how the unitarity, which is already known through the no-ghost theorem, appears in the specific case of the Veneziano amplitude. This may prove useful to guide the search within the context of the revival of the bootstrap for the S-matrix. However, before diving into these computations, let me do a brief recap of the tools we will use and investigate.

[1] $\bar{i}$ stands for the antiparticle of i

Chapter 2

A minimal bootstrap toolkit

2.1 The dispersion relations

First, I will give these two classic results which may be find in [2] or [14] :

- Unitarity implies that on the real axis, for $t_0 \in \mathbb{R}$ fixed, $Im[A(s,t_0)] = Disc_s[A(s,t_0)]$, where $Disc_s[A(s,t_0)] = \frac{1}{2i}\lim_{\varepsilon\to 0^+}[A(s+i\varepsilon,t_0) - A(s-i\varepsilon,t_0)]$. As we supposed maximal analyticity, these will be the only discontinuities.
- The optical theorem states that $\sigma_{tot} = \frac{Im[A(s,0)]}{\sqrt{s(s-4m^2)}}$.

Now, we want to know the amplitude at a certain point of the complex plane in s and t. However, we don't have a theory with an exact form for this amplitude. What we can do is use this result of complex analysis :

$$A(s,t) = \frac{1}{2\pi i}\oint_{C_s} ds' \frac{A(s',t)}{s'-s} \tag{2.1}$$

where C_s is a contour around s. If we know that $A(s,t)$ goes to zero when s goes to infinity in any direction [1], then we can send the contour to infinity, picking up just the discontinuities of the function.

Now, using unitarity and the result given higher, we can relate the discontinuities to the imaginary part of the amplitude. If we take the example of only a branch cut going from $4m^2$ to infinity, we have an equation of this form :

$$A(s,t) = \int_{4m^2}^{+\infty} \frac{ds'}{2\pi i}\frac{Im[A(s',t)]}{s'-s} \tag{2.2}$$

This can allow the creation of a test for a self-consistent bootstrap. Also, results like the optical theorem can allow the input from experiments to give us parts of the amplitude and extrapolate from that.

The problem of this approach is that, for high energies, the integration on branch cuts can cause problems. To solve this problem, we need another tool : Regge theory.

2.2 Regge theory

The first idea of Regge theory is to go in the space of angular momentum. Why is that you may ask? The simple reason is that the S matrix has to follow the Poincaré symmetries in which there are rotations. The

[1] For amplitudes that diverges slow enough, it is possible to use substraction, see [14] section 1.3.2

angular momenta are the generators of rotations. Therefore, we know that $[S, J^2] = [S, J_z] = 0$ and we can simultaneously diagonalize these operators.

The second idea is to analytically continue the amplitude $f_J(s)$ in the J complex plane. We will see in section 2.2.3 why it is useful but first we need to go in angular momentum space. While these computations can be generalized to particles with a given helicity, I will limit myself to the case of particles with spin 0.

2.2.1 Going to partial waves

Let's consider the interacting part of the S-matrix T defined in the introduction. We consider the 2 to 2 interaction given the diagram 2.1. With these conventions, we have $|\boldsymbol{p}_i\rangle = |p, 0, 0\rangle$, $|\boldsymbol{p}_f\rangle = |p, \theta, \phi\rangle$ and $A(s,t) = \langle p, \theta, \phi| T |p, 0, 0\rangle$.

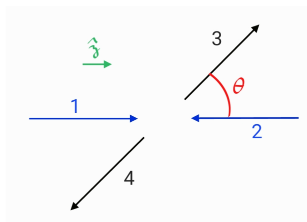


Figure 2.1: Diagram for the 2 to 2 scattering.

Also, for a state of definite angular momentum J, M, because of the rotational invariance explained in the beginning of this section, we have : $\delta_{J,J'}\delta_{M,M'} f_{J,M}(s) \propto \langle p, J, M| T |p, J', M'\rangle$. We can remark that the rotational invariance of T implies that it commutes with $J_\pm$[2], therefore it is independent of M.

Because the harmonic functions $Y_{J,M}(\theta,\phi)$ define an irreducible representation of SO(3) and thanks to the Peter-Weyl theorem, we know that $|p,\theta,\phi\rangle = \sum_{J,M} |p, J, M\rangle \, Y^*_{J,M}(\theta,\phi)$.

Using this change of basis, we can compute the relation between $A(s,t)$ and $f_J(s)$:

$$A(s,t) \propto \sum_{J,M} \sum_{J'} \langle p,\theta,\phi|p,J,M\rangle \, \delta_{J,J'}\delta_{M,0} f_J(s) \, \langle p,J',0|p,0,0\rangle \tag{2.3}$$

$$\propto \sum_J Y_{J,0}(\theta,\phi) Y^*_{J,0}(0,0) f_J(s) \tag{2.4}$$

The spherical harmonics are defined as $Y_{J,0}(\theta,\phi) \propto \sqrt{2J+1} P_J(\cos(\theta))$ where P_J is a Legendre polynomial. We denote $z = \cos(\theta) = 1 + \frac{2t}{s}$[3] and we put a convention factor 16π in front. This gives us :

$$A(s,t) = 16\pi \sum_J (2J+1) f_J(s) P_j(z) \tag{2.5}$$

In order to get more details on all these computations, I recommend chapter 7 of [13]. We can generalize this result for any dimension $d \geq 4$ using $P^{(d)}_J(z) =_2 F_1(-J, J+d-3, \frac{d-2}{2}, \frac{1-z}{2})$ instead of the usual Legendre polynomial (see [3]). Then using the orthogonality relation $\frac{1}{2}\int_{-1}^{1} dz (1-z^2)^{\frac{d-4}{2}} P_J(z) P_{J'}(z) = \frac{\delta_{J,J'}}{\mathcal{N}_d n^{(d)}_J}$ where $n^{(d)}_J = \frac{(4\pi)^{\frac{d}{2}}(d+2J-3)\Gamma(d+J-3)}{\pi\Gamma(\frac{d-2}{2})\Gamma(J+1)}$ one can get :

[2]For detailed definitions of the operators and the basis used, see [13]

[3]I suppose from now on that $m = 0$.

$$f_J(s) = \frac{\mathcal{N}_d}{2}\int_{-1}^{1} dz\ (1-z^2)^{\frac{d-4}{2}} P_J^{(d)}(z) A(s,t(s,z)), \quad \mathcal{N}_d = \frac{(16\pi)^{\frac{2-d}{2}}}{\Gamma(\frac{d-2}{2})} \tag{2.6}$$

The issue with this relation is that it can't be naturally extended to $J \in \mathbb{C}$. In order to do so, we need another technology, the Froissart-Gribov projection.

2.2.2 Froissart-Gribov projection

Before diving in the derivation of the Froissart-Gribov projection, we need to define the Legendre function $Q_J^{(d)}(z) = \frac{c_J^{(d)}}{z^{J+d-3}} {}_2F_1(\frac{J+d-3}{2}, \frac{J+d-2}{2}, J+\frac{d-1}{2}, \frac{1}{z^2})$ where $c_J^{(d)} = \frac{\sqrt{\pi}\Gamma(J+1)\Gamma(\frac{d-2}{2})}{2^{J+1}\Gamma(J+\frac{d-1}{2})}$ and ${}_2F_1$ is an hypergeometric function. We can relate this function to the Legendre polynomial through the equation :

$$\mathrm{Disc}_z\left[(z^2-1)^{\frac{d-4}{2}} Q_J^{(d)}(z)\right] = -\frac{\pi}{2}(1-z^2)^{\frac{d-4}{2}} P_J^{(d)}(z)\ , \qquad z \in [-1,1] \tag{2.7}$$

Writing the integral in 2.6 with the relation 2.7, it becomes an integral along a contour around $[-1,1]$.[4]

$$f_J(s) = \frac{\mathcal{N}_d}{2\pi i}\oint_{[-1,1]} dz\ (z^2-1)^{\frac{d-4}{2}} Q_J^{(d)}(z) A(s,t(s,z)) \tag{2.9}$$

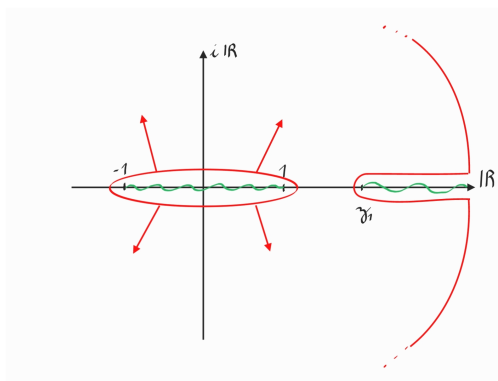


Figure 2.2: Contour for the Froissart-Gribov projection.

We can now send the contour at infinity (see figure 2.2) thanks to the asymptotics of the Legendre $Q_J^{(d)}(z) \approx_{z\to\infty} \textit{constant}\ \times z^{-J-d+3}$ function and the Froissart bounds (see [6]). In doing so, the discontinuities of the amplitude will be caught. Using crossing to put back together all the branch cuts... (see [3] for more details), we arrive at the following expression :

$$f_J(s) = \frac{2\mathcal{N}_d}{\pi}\int_{z_1}^{+\infty} dz\ (z^2-1)^{\frac{d-4}{2}} Q_J^{(d)}(z) Disc_t[A(s,t(s,z))] \tag{2.10}$$

[4]It is easier to see with this expression :

$$(z^2-1)^{\frac{d-4}{2}} Q_J^{(d)}(z) = \frac{1}{2}\int_{-1}^{1} dz'\ (1-z'^2)^{\frac{d-4}{2}}\ \frac{P_J^{(d)}(z')}{z-z'} \tag{2.8}$$

Which is equivalent to 2.7.

All of this may seem pointless, we replaced an integral for another which, at first sight, doesn't seem any easier. That's true for $J \in \mathbb{N}$, something we implicitly suppose until now. However, the Q function has a much better analytic continuation in J. Therefore, the last expression is much more useful for what follows.

We know that 2.6 and 2.10 are the same for $J \in \mathbb{N}$ and that the second one is an analytic continuation. However, they match only on a discrete set of points, so there might be other continuations. It is not the case, thanks to Carlson's theorem (see [8]), we know that it is the only analytic continuation with a "nice" asymptotic.

2.2.3 And back to impulsion space

We now want to go back to impulsion space. One way would be to use 2.5 but this would undo everything we have done and doesn't provide much useful physical information. We are rather going to use the Sommerfeld-Watson transform. The first thing to do is to tweak a little bit equation 2.5 in order to write it first as the sum of integrals around poles of a sinus (red contour) and then as :

$$A(s,t) = -\frac{16\pi}{2i}\int_{\mathcal{C}_0} dJ\ (2J+1) f_J(s) \frac{P_J(-z)}{\sin(\pi J)} \tag{2.11}$$

Where $\mathcal{C}_0$ is the green contour in figure 2.3.

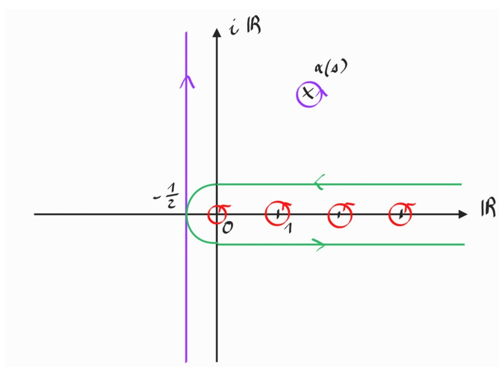


Figure 2.3: Consecutive contours during the derivation of the Sommerfeld-Watson transformation.

We can now "open" the integral, picking the different poles in J of $f_J(s)$. In the case of a meromorphic function, this leaves us with a sum over the contributions of the different Regge poles. We take the notation in order for $f_J(s)$ to take the following form near a pole:

$$f_J(s) \approx \frac{\beta_i(s)}{J - \alpha_i(s)} \tag{2.12}$$

There is also a background integral which has to be place at $Re(J) = \frac{1}{2}$ to minimize its contribution(see [2]). All in all, we get:

$$A(s,t) = -16\pi^2 \sum_i (2\alpha_i(s)+1)\beta_i(s) \frac{P_{\alpha_i(s)}(z)}{\sin(\pi\alpha_i(s))} - \frac{16\pi}{2i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dJ\ (2J+1) f_J(s) \frac{P_J(-z)}{\sin(\pi J)} \tag{2.13}$$

Most of the time, the integral contribution is negligible, leaving mainly the sum over the Regge poles. This is interesting because near a point where $\alpha(s = m_n^2) = n \in \mathbb{N}$, the expression takes the form :

$$A(s,t) \approx 16\pi^2 (n+1)\beta_i(m_n^2) \frac{P_n(z)}{\alpha(s) - n} \tag{2.14}$$

This is literally the exchange of a spin J particle of mass m_n^2. This gives a link between the Regge poles and the coupling constant one may find in a usual QFT.

This is one of the reasons why this travel back and forth between impulsion space and partial waves is so useful, it allows us to extract some perturbative information on the amplitude. Moreover, it is easier to build a consistent bootstrap with $f_J(s)$. For more details, see [14].

Chapter 3

Computing the Regge poles of the Veneziano amplitude

3.1 The path of unitarity

I will start to explain the derivation of the Regge poles in the next section but I first want to emphasize the link with unitarity. In order to do so, we will start from the equation 1.2. Using the definition of T, $S = 1 + iT$, the following equation is straight forward:

$$\frac{1}{i}(T - T^\dagger) = TT^\dagger \tag{3.1}$$

When applying the impulsion braket to this equation, it becomes:

$$2Im[A(s,t)] = \langle p_3, p_4| T\ T^\dagger |p_1, p_2\rangle \tag{3.2}$$

Noting that $TT^\dagger$ is a positive semi-definite matrix, in the forward limit (i.e. when $|p_3, p_4\rangle = |p_1, p_2\rangle$), the equation above implies the positivity of the amplitude:

$$Im[A(s,0)] \geq 0 \tag{3.3}$$

At the poles of the Sommerfeld formula, i.e. in equation 2.14, the imaginary part comes from the $i\pi\delta(\alpha_n(s))\beta_k(m_n^2)$ which appears from $x \to \left(\lim\limits_{\varepsilon\to 0} \frac{1}{x+i\varepsilon}\right) = vp\frac{1}{x} + i\pi\delta$. This means that the positivity implies:

$$\forall k \in \mathbb{N} \ \ \forall n \geq k \ \ \beta_k(m_n^2) \geq 0 \tag{3.4}$$

where $\alpha(m_n^2) = n$.

This property is weaker than unitarity but remains important, in particular for the link with the coupling constant emphasized in section 2.2.3 for example [1]. In the case of the Veneziano amplitude, the no ghost theorem (see [7]) is a proof of this property. The main reason for deriving an explicit expression for the Regge poles was to try to find a new proof for this property and maybe for a stronger one conjectured in section 4.2.2 of [14].

[1] Also, for a meromorphic function, positivity is the best we may get.

3.2 The method to extract Regge poles

In order to study those poles, I will follow the steps of article [5]. The first thing to do is to approximate the integral in 2.10. In order to do so, we need to simplify it by removing the kernel. This can be done using this relation (see DLMF 15.8.12):

$$ {}_2F_1\left(\frac{J+d-3}{2}, \frac{J+d-2}{2}, J+\frac{d-1}{2}, \frac{1}{z^2}\right) = \left(1-\frac{1}{z^2}\right)^{-\frac{d-4}{2}} {}_2F_1\left(\frac{J+1}{2}, \frac{J+2}{2}, J+\frac{d-1}{2}, \frac{1}{z^2}\right) \tag{3.5} $$

The equation 2.10 now has the form:

$$ f_J(s) = \frac{2\mathcal{N}_d\ c_J^{(d)}}{\pi} \int_{z_1}^{+\infty} \frac{dz}{z^{J+1}} {}_2F_1\left(\frac{J+1}{2}, \frac{J+2}{2}, J+\frac{d-1}{2}, \frac{1}{z^2}\right) Disc_t[A(s,z)] \tag{3.6} $$

We are now going to get closer to the Veneziano amplitude by putting $z_1 = 1 + \frac{2}{s}$ and supposing a development of the form $Disc_t[A(s,z)] = z^s \sum_n a_n(s) z^{-n}$. Using the known expression of ${}_2F_1$ (see section 3.4), we arrive at the following form:

$$ f_J(s) \approx \sum_{k=0}^{+\infty} \int_{1+\frac{2}{s}}^{+\infty} dz\ z^{s-J-k-1}\ \tilde{\beta}_k(J,s) \tag{3.7} $$

$$ = \sum_{k=0}^{+\infty} \frac{\tilde{\beta}_k(J,s)}{J-(s-k)} \tag{3.8} $$

We can identify $\alpha_k(s) = s-k$ in this case and $\beta(s) = \tilde{\beta}(s-k,s)$. We want to compute the latter, this means that we are going to need an expression for the asymptotic development of $Disc_t[A(s,z)]$. This is the subject of the next section.

3.3 The Laplace method to expand the amplitude

The expression of the Veneziano amplitude is given in equation 1.3. It is possible to make the discontinuity apparent using the inversion formula of the Euler gamma function. This gives us:

$$ Disc_t[B(-s,-t(s,z))] = \frac{\pi\Gamma(s+t+1)}{\Gamma(s+1)\Gamma(t+1)} \tag{3.9} $$

It is possible to reorganize this to get back a Euler beta function which is defined by an integral. There exist different methods to compute the asymptotic development of a parametric integral. If we change the variable in the integral of equation 2.10, it is possible to use the saddle point method which provides an upper bound for the remainder. However, in order to simplify the following computations, it is better to stay with the current equation and use the Laplace method. This method is based on the following form of integrals:

$$ I(z) = \int_0^{+\infty} du\ e^{-zu} f(u) \tag{3.10} $$

Then, if $f(u)$ is analytic and with the following development : $f(u) = \sum_{n=0}^{+\infty} \frac{a_n}{n!} u^n$. Then the integral has this asymptotic development (see [12] for more details):

$$I(z) = \sum_{n=0}^{+\infty} \frac{a_n}{z^{n+1}} \tag{3.11}$$

After a couple of steps detailed in appendix A.1, I managed to compute the following development:

$$Disc_t[B(-s, -t(s,z))] = \frac{\pi\Gamma(s\frac{z+1}{2})}{\Gamma(s+1)\Gamma(s\frac{z-1}{2})} \tag{3.12}$$

$$\approx \frac{\pi\left(\frac{zs}{2}\right)^s}{\Gamma(s+1)} \sum_{n=0}^{+\infty} \frac{B_n^{(s+1)}\left(\frac{s}{2}\right)}{n!}(-s)_n \left(\frac{zs}{2}\right)^{-n} \tag{3.13}$$

Where $B_n^{(\mu)}(x)$ are the Bernoulli polynomials defined in appendix A.1.

With this development, we can start to compute the Regge poles.

3.4 An explicit form for the Regge poles

In order to compute the development in 3.6, we first need the development of ${}_2F_1$:

$${}_2F_1\left(\frac{J+1}{2}, \frac{J+2}{2}, J + \frac{d-1}{2}, \frac{1}{z^2}\right) = \sum_{n=0}^{+\infty} \frac{\left(\frac{J+1}{2}\right)_n \left(\frac{J+2}{2}\right)_n}{\left(J+\frac{d-1}{2}\right)_n n!} z^{-2n} \tag{3.14}$$

$$= \sum_{n=0}^{+\infty} \frac{(J+1)_{2n}}{\left(J+\frac{d-1}{2}\right)_n 2^{2n} n!} z^{-2n} \tag{3.15}$$

Where $(a)_n = \prod_{l=0}^{n-1}(a+l) = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol. From these two developments, it is obvious that:

$$\tilde{\beta}_k(J,s) = \frac{2\mathcal{N}_d\, c_J^{(d)}}{\pi} \frac{\pi\left(\frac{s}{2}\right)^s}{\Gamma(s+1)} \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(J+1)_{2n}}{\left(J+\frac{d-1}{2}\right)_n 2^{2n} n!} \frac{B_{k-2n}^{(s+1)}\left(\frac{s}{2}\right)}{(k-2n)!} (-s)_{k-2n} \left(\frac{s}{2}\right)^{-(k-2n)} \tag{3.16}$$

It is now time to use $\beta_k(s) = \tilde{\beta}_k(s-k, s)$. Following the steps provided in appendix A.2, one can get:

$$\beta_k(s) = \beta_0(s) \frac{\left(s-k+\frac{d-1}{2}+\lfloor\frac{k}{2}\rfloor\right)_{\lfloor\frac{k+1}{2}\rfloor} p_k(s)}{s^k} \tag{3.17}$$

Where $\beta_0(s) = \frac{\pi^{\frac{3-d}{2}} s^s}{2^{2s+2d-4}\Gamma(s+\frac{d-1}{2})}$ which is clearly positive for $s \geq 0$ and:

$$p_k(s) = (-1)^k 2^{2k} \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} \frac{\left(s-k+\frac{d-1}{2}+n\right)_{\lfloor\frac{k}{2}\rfloor - n}}{2^{4n} n!(k-2n)!} (s)^{2n} B_k^{(s+1)}\left(\frac{s}{2}\right) \tag{3.18}$$

In the Sommerfeld relation, the poles where $J = s - k \leq 0$ are not taken into account, therefore, in order to prove the positivity of the poles, we just need the positivity of $p_k(\tilde{s}+k)$ for $\tilde{s} \in \mathbb{N}$. The stronger property conjectured in [14] is the positivity of the coefficient of these shifted polynomials.

Computing the first few polynomials makes it clear that the positivity fails for $d \geq 10$, which is right at the limit of what the no-ghost theorem allows.

An explicit solution for these poles can't be found in the literature, making it the main result of my internship. Using similar techniques, it is possible to get other explicit expressions. However, this one is nicer for a couple of reasons:

- It is the one I found with the less terms in the sum making the computation more efficient for the computer.
- Even if the degree ($deg[p_k(s)] = 2\lfloor \frac{k}{2} \rfloor$) isn't explicit yet, it is no longer because of the last sum. The reason comes from the explicit expression of the generalized Bernoulli polynomials (see [10] for example) which doesn't make the degree obvious.
- With this expression, it is possible to bound the asymptotic of $\beta_k(s)$ for fixed s making it relatively easy to check the possibility of some kind of Borel transformation for example.

For these reasons, this expression is great for computing large numbers of poles numerically with minimum resources. However, the signs are all hidden in the Bernoulli polynomials, making them complicated to study. Therefore, more work is needed to extract the positivity.

Chapter 4

Study of the poles

4.1 An integral formulation

After looking for a while, it seemed clear that the signs won't come naturally out of the sum and the dependence in the dimension was really hard to take into account. Therefore, we started to look for an asymptotic in hopes of being able to control it and glue the pieces with numerical computations.

We don't know much about the asymptotics of the generalized Bernoulli polynomials when all the indices grow at the same time. Therefore, the first step was removing the sum. In order to do so, we first use an integral around zero of the generating function to express the Bernoulli polynomials. This leaves us with:

$$p_k(s+k) = \frac{(-1)^k 2^{2k}}{2\pi i} \oint \frac{dt}{t^{k+1}} \left(\sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(s+\frac{d-1}{2}+n)_{\lfloor \frac{k}{2} \rfloor - n}}{2^{4n} n!} ((s+k)t)^{2n} \right) e^{t\frac{s+k}{2}} \left(\frac{t}{e^t-1} \right)^{s+k+1} \tag{4.1}$$

The sum in the middle of the integral can be done with Mathematica (code available here) and we get this expression:

$$p_k(s+k) = \frac{(-1)^k 2^{2k}}{2\pi i} (s+\frac{d-1}{2})_{\lfloor \frac{k}{2} \rfloor} \oint \frac{dt}{t^{k+1}} \, {}_0F_1 \left(s+\frac{d-1}{2}; \left(\frac{(s+k)t}{4} \right)^2 \right) e^{t\frac{s+k}{2}} \left(\frac{t}{e^t-1} \right)^{s+k+1} \tag{4.2}$$

There exists an asymptotic development for the hypergeometric function ${}_0F_1$ which is related to the modified Bessel function. Therefore, this expression is convenient to study the large k limit which is the subject of the next section. However, the contour integral makes it inefficient for numerical computation. For this second application, the following expression derived using an integral expression of ${}_0F_1$ (see [4]) is more suited:

$$p_k(s+k) = \frac{(-1)^k 2^{2k}}{\sqrt{\pi} k!} \frac{\Gamma(s+\frac{d-1}{2})}{\Gamma(s+\frac{d-2}{2})} (s+\frac{d-1}{2})_{\lfloor \frac{k}{2} \rfloor} \int_{-1}^{1} du (1-u^2)^{s+\frac{d-4}{2}} B_k^{(s+k+1)} \left(\frac{s+k}{2}(1+u) \right) \tag{4.3}$$

4.2 Study of the asymptotics for large k

Now that everything is set up, it is time to compute the asymptotic for large k. First we need the development of ${}_0F_1$ which can be found at DLMF equation 10.40.5:

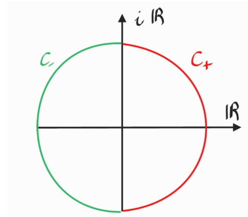


Figure 4.1: Contour used to cut the integral in two.

$$\frac{{}_0F_1\left(s+\frac{d-1}{2};\left(\frac{(s+k)t}{4}\right)^2\right)}{\Gamma(s+\frac{d-1}{2})} = \left(\frac{4}{(s+k)t}\right)^{s+\frac{d-3}{2}} I_{s+\frac{d-3}{2}}\left(\frac{(s+k)t}{2}\right) \tag{4.4}$$

$$\approx \frac{2^{2s+d-3}}{\sqrt{\pi}((s+k)t)^{s+\frac{d-2}{2}}}\begin{cases} e^{t\frac{s+k}{2}} & \Re(t)>0 \\ e^{-i\pi(s+\frac{d-2}{2})}e^{-t\frac{s+k}{2}} & \Re(t)<0 \end{cases} \tag{4.5}$$

This expression can then be used in 4.2 by cutting the contour integral in two in order to use the right approximation on the right part of the plane as described in figure 4.1. We get:

$$p_k(s+k) \approx \frac{(-1)^k 2^{2(s+k)+d-3}}{2\pi i\,\sqrt{\pi}(s+k)^{\frac{d-2}{2}}}\Gamma\left(s+\frac{d-1}{2}+\lfloor\frac{k}{2}\rfloor\right) \tag{4.6}$$

$$\left[\int_{C_+}\frac{dt}{t^{\frac{d-2}{2}}}\frac{e^{t(s+k)}}{(e^t-1)^{s+k+1}} + (-1)^{s+\frac{d-2}{2}}\int_{C_-}\frac{dt}{t^{\frac{d-2}{2}}}\frac{1}{(e^t-1)^{s+k+1}}\right]$$

$$\approx \frac{(-1)^{s+k+\frac{d-2}{2}}2^{2(s+k)+d-3}}{\sqrt{\pi}(s+k)^{\frac{d-2}{2}}}\Gamma\left(s+\frac{d-1}{2}+\lfloor\frac{k}{2}\rfloor\right)\frac{B^{(s+k+1)}_{s+k+\frac{d-2}{2}}(0)}{(s+k+\frac{d-2}{2})!} \tag{4.7}$$

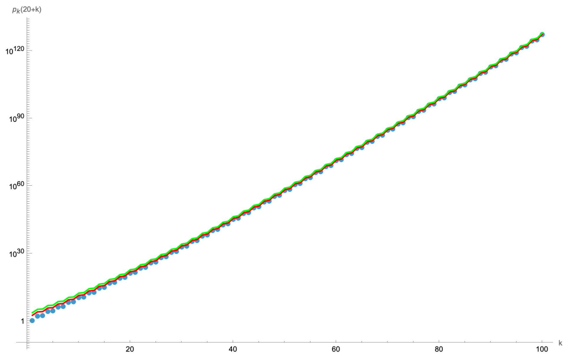
More detail on this derivation can be found in appendix B. This expression is nice and relatively precise, as one can see in figure 4.2. However, the Bernoulli polynomials are back which seems annoying at first. However, the fact that the parameter in parentheses is set at 0 allows us to use the asymptotic given by Temme in [11]. This gives us:

$$p_k(s+k) \approx \frac{2^{2(s+k)+d-3}}{\sqrt{\pi}(s+k)^{\frac{d-2}{2}}}\Gamma\left(s+\frac{d-1}{2}+\lfloor\frac{k}{2}\rfloor\right)\left[\ln\left(s+k+\frac{d-2}{2}\right)\right]^{-\frac{d-2}{2}} \tag{4.8}$$

This development is less precise as one can see in 4.2 (green curve) because the asymptotics of the Bernoulli polynomials is an expansion in logarithms. However, it stays decent compared to the value of the function and makes the sign completely evident.

4.3 Control of the asymptotics and leads to continue

All of this is great. However, it does not prove the positivity. The quality of the approximation is numerically good, but to get a proof, an explicit control of the error is needed. In order to do so, I go back to every step of the derivation available in annex B and evaluate the approximation I have done. To get to 4.7, there are three sources of error: the two parts of the integral I neglected and the original error from the asymptotic of the Bessel function. Taking a relatively large upper bound, I get:



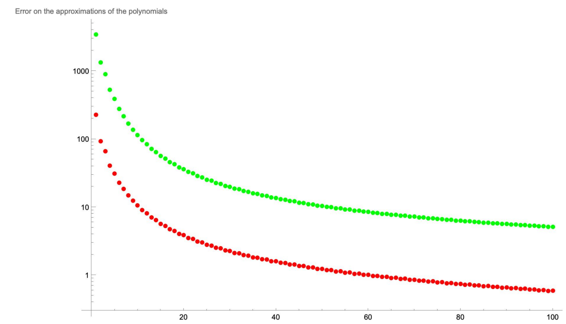


Figure 4.2: On the right the values of $p_k(20+k)$ in function of k with the approximation given in equation 4.7 displayed in red and the asymptotic of 4.8 in green. On the left, the error from the true value is displayed (i.e. $k \to \left|\frac{p_k(20+k)-approximation}{p_k(20+k)}\right|$)

$$\begin{aligned} err_{1+2+3} \leq & \frac{2^{2(s+k)+d-3}}{\sqrt{\pi}(s+k)^{\frac{d-2}{2}}}\Gamma\left(s+\frac{d-1}{2}+\lfloor\frac{k}{2}\rfloor\right) \\ & \left(\frac{4}{\pi^{\frac{d-2}{2}}(s+k-1)}+\frac{64}{\pi^{\frac{d-3}{2}}(s+k)^{\frac{3}{2}}}e^{2\frac{(s+\frac{d-3}{2})^2}{m}}\left|\frac{(s+\frac{d-3}{2})^2-\frac{1}{4}}{2}\right|\right) \end{aligned} \tag{4.9}$$

This expression goes to zero quicker than the expansion (convergence of the last parentheses in $\frac{1}{k}$ against the $(ln(k))^{-\frac{d-2}{2}}$ of the Bernoulli). This means that we would just need to compute a given number of terms. The problem is that the expression 4.7 isn't explicit enough to extract the sign and the monotony.

In order to do so, we would need to use 4.8 and add a bound for the remainder. Unfortunately, I didn't have time to find a good way to control this error. The problem is that the rest of the expansion in [11] is asymptotic and therefore can't be used to get the upper bound. In the same paragraph as the development, they speak about a link to Stirling numbers which may be fruitful.

Chapter 5

Conclusion

5.1 Scientific aspects

From a scientific perspective, this internship has been extremely enriching. The most immediate and significant topic I explored was Regge theory, along with several analytical tools used in high-energy physics. However, my learning extended well beyond that. The various seminars and presentations exposed me to a broad range of subjects, from cosmology to bootstrap methods in quantum field theory.

I also gained practical experience with Mathematica, a powerful tool for both symbolic and numerical computation, and deepened my understanding of several mathematical techniques. Among these, I became familiar with asymptotic expansions using the Laplace method and other saddle point techniques for evaluating integrals. This experience has also revealed to me the true power of complex analysis as a tool in theoretical physics.

Scientifically, the internship led to some interesting results. Using the Laplace method, I derived an explicit expression for the Regge poles of the Veneziano amplitude (see equation 3.18). This closed-form result provides a useful way to compute additional poles, and by combining complex analysis with Mathematica, I was able to investigate the asymptotic behavior of these poles (see equation 4.8).

While I didn't quite reach the goal of super-positivity I had initially set for myself, this work represents a meaningful step toward that objective. It lays the groundwork for achieving positivity through direct analytical computations.

5.2 Personal aspects

From a personal point of view, this internship confirmed my strong interest in theoretical physics. Before starting, I had some doubts about whether a subject so deeply rooted in mathematics would continue to engage me over an extended period. I was concerned it might eventually become monotonous. However, although there were moments of lower motivation, I remained genuinely excited about the topic—even when it felt somewhat abstract or distant from more tangible aspects of physics.

Over the past three months, I also had many opportunities to speak with PhD students and postdoctoral researchers. These discussions were incredibly valuable, offering diverse perspectives on academic careers and the different paths one can take. They helped me better understand what lies ahead and how to prepare for it.

To conclude, this internship has been both educational and inspiring. I've gained a wide range of new concepts in physics and mathematics, while also getting a realistic and exciting preview of what the coming years of study and research may look like. I'm now more motivated than ever to pursue a similar topic next year—and, if possible, during my future PhD.

Appendix A

Derivation of the Regge poles

A.1 Asymptotic expansion of the Veneziano amplitude

As I said earlier, I need to find the asymptotic series for the discontinuity of the Veneziano amplitude ($A(s,t) = B(-s,-t)$ with B the Euler beta function). In order to do so, I first use the relation $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ which makes the discontinuities more clear with the sinus. This leads to A.1 which I rearrange to make the beta function explicit and I use the integral definition of this function.

$$
\begin{aligned}
Disc_t[B(-s,-t)] &= \frac{\pi\Gamma(s+t+1)}{\Gamma(s+1)\Gamma(t+1)} && \text{(A.1)}\\
&= \frac{\pi B(-s, s\frac{z+1}{2})}{\Gamma(s+1)\Gamma(-s)} && \text{(A.2)}\\
&= \frac{\pi}{\Gamma(s+1)\Gamma(-s)} \int_{-1}^{1} dt\ (1-t)^{-1-s} t^{s\frac{z+1}{2}} && \text{(A.3)}\\
&= \frac{2\pi}{s\Gamma(s+1)\Gamma(-s)} \int_{0}^{+\infty} du\ e^{\frac{2u}{s}(\frac{s}{2})} \left(\frac{\frac{2u}{s}}{e^{\frac{2u}{s}}-1}\right)^{s+1} \left(\frac{2u}{s}\right)^{-s-1} e^{-zu} && \text{(A.4)}\\
&\approx \frac{\pi\left(\frac{zs}{2}\right)^{s}}{\Gamma(s+1)} \sum_{n=0}^{+\infty} \frac{B_n^{(s+1)}\left(\frac{s}{2}\right)}{n!} (-s)_n \left(\frac{zs}{2}\right)^{-n} && \text{(A.5)}
\end{aligned}
$$

After the following change of variable: $t = e^{-\frac{2u}{s}}$, and some cleaning, we have A.4 which matches the form required for the Laplace method of equation 3.10. The form of the function we have to expand corresponds to the generating function of the generalized Bernoulli polynomials: $e^{xt}\left(\frac{t}{e^t-1}\right)^{\mu} = \sum_{n=0}^{+\infty} \frac{B_n^{(\mu)}(x)}{n!} t^n$. Replacing the generating function by the sum and computing the integral leads to the result given in the equation 3.13.

A.2 Derivation of the poles

I start from equation 3.16. This expression can easily be transformed to:

$$\tilde{\beta}_k(J,s) = \frac{\pi^{\frac{3-d}{2}}}{2^{J+2d+s-k-4}} \frac{\Gamma(J+1) s^s}{\Gamma(s+1)\Gamma\left(J+\frac{d-1}{2}\right)} \frac{(-1)^k}{s^k} \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(J+1)_{2n}\,(s-k+2n+1)_{k-2n}}{\left(J+\frac{d-1}{2}\right)_n 2^{4n} n!} s^{2n} \frac{B_{k-2n}^{(s+1)}\left(\frac{s}{2}\right)}{(k-2n)!} \tag{A.6}$$

We can now use the fact that $\beta_k(s) = \tilde{\beta}_k(s-k,s)$ and the expression of the Pochhammer with the Gamma function: $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$. This leads to:

$$\beta_k(s) = \frac{\pi^{\frac{3-d}{2}}}{2^{2s+2d-4}} \frac{s^s}{\Gamma\left(s+\frac{d-1}{2}\right)} \frac{(-1)^k 2^{2k}}{s^k} \sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} \frac{\left(s-k+\frac{d-1}{2}\right)_k}{\left(s-k+\frac{d-1}{2}\right)_n 2^{4n} n!} s^{2n} \frac{B_{k-2n}^{(s+1)}\left(\frac{s}{2}\right)}{(k-2n)!} \tag{A.7}$$

Reorganizing this expression and using that $\lfloor \frac{k+1}{2} \rfloor + \lfloor \frac{k}{2} \rfloor = k$, one can get 3.17.

Appendix B

Asymptotic of the Regge poles

B.1 Computation of the asymptotic

In this section, I will comment on the different steps of the computation of section 4.2. First, we need to put 4.5 in equation 4.2. This leaves us with the first equation with the paths given in 4.1:

$$p_k(s+k) \approx \frac{(-1)^k 2^{2(s+k)+d-3}}{2\pi i\ \sqrt{\pi}(s+k)^{\frac{d-2}{2}}}\Gamma\left(s+\frac{d-1}{2}+\lfloor\frac{k}{2}\rfloor\right) \tag{B.1}$$

$$\left[\int_{C_+}\frac{dt}{t^{\frac{d-2}{2}}}\frac{e^{t(s+k)}}{(e^t-1)^{s+k+1}}+(-1)^{s+\frac{d-2}{2}}\int_{C_-}\frac{dt}{t^{\frac{d-2}{2}}}\frac{1}{(e^t-1)^{s+k+1}}\right]$$

$$= \frac{(-1)^k 2^{2(s+k)+d-3}}{2\pi i\ \sqrt{\pi}(s+k)^{\frac{d-2}{2}}}\Gamma\left(s+\frac{d-1}{2}+\lfloor\frac{k}{2}\rfloor\right)\int_{C_+}\frac{dt}{t^{\frac{d-2}{2}}}\frac{e^{t(s+k)}}{(e^t-1)^{s+k+1}}\left(1+(-1)^k e^t\right) \tag{B.2}$$

$$\approx \frac{(-1)^{s+k+\frac{d-2}{2}} 2^{2(s+k)+d-3}}{2\pi i\ \sqrt{\pi}(s+k)^{\frac{d-2}{2}}}\Gamma\left(s+\frac{d-1}{2}+\lfloor\frac{k}{2}\rfloor\right)\oint\frac{dt}{t^{\frac{d-2}{2}}}\frac{e^{t(s+k)}}{(e^t-1)^{s+k+1}}e^t \tag{B.3}$$

$$= \frac{(-1)^{s+k+\frac{d-2}{2}} 2^{2(s+k)+d-3}}{\sqrt{\pi}(s+k)^{\frac{d-2}{2}}}\Gamma\left(s+\frac{d-1}{2}+\lfloor\frac{k}{2}\rfloor\right)\frac{B^{(s+k+1)}_{s+k+\frac{d-2}{2}}(0)}{(s+k+\frac{d-2}{2})!} \tag{B.4}$$

Then, a change of variable is done in the second integral. To get to B.3, we neglect the integral $\int_{C_+}\frac{dt}{t^{\frac{d-2}{2}}}\frac{e^{t(s+k)}}{(e^t-1)^{s+k+1}}$ and we add $(-1)^k\int_{C_-}\frac{dt}{t^{\frac{d-2}{2}}}\frac{e^{t(s+k)}}{(e^t-1)^{s+k+1}}e^t$ which is negligible compared to $\int_{C_+}\frac{dt}{t^{\frac{d-2}{2}}}\frac{e^{t(s+k)}}{(e^t-1)^{s+k+1}}e^t$ (easier to see with the path deformed as in B.1). Using the link between the Bernoulli polynomial and its generating function, one gets 4.7.

B.2 Control of the asymptotic

In the previous section, we did 3 approximations, first with the Bessel function which I will call err_1. Then with the two integrals neglected to get to B.3 which I'll call err_2 and err_3. Let's start with the integrals:

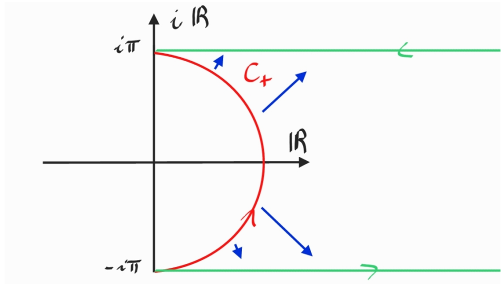


Figure B.1: How to deform the path to make the weight of each part clear.

$$err_2 = \left| \int_{C_+} \frac{dt}{t^{\frac{d-2}{2}}} \frac{e^{t(s+k)}}{(e^t-1)^{s+k+1}} \right| \tag{B.5}$$
$$= \left| \int_{\mathbb{R}+} dt \frac{e^{-t}}{(1+e^{-t})^{s+k+1}} \frac{(t+i\pi)^{\frac{d-2}{2}} - (t+i\pi)^{\frac{d-2}{2}}}{|t+i\pi|^{d-2}} \right| \tag{B.6}$$
$$\leq \frac{2}{\pi^{\frac{d-2}{2}}} \int_{\mathbb{R}+} dt \frac{e^{-t}}{(1+e^{-t})^{s+k+1}} \tag{B.7}$$
$$= \frac{2}{\pi^{\frac{d-2}{2}}} \frac{1-2^{-s-k}}{s+k-1} \tag{B.8}$$
$$\leq \frac{2}{\pi^{\frac{d-2}{2}}} \frac{1}{s+k-1} \tag{B.9}$$

The computation for $err_3 \leq \frac{2}{\pi^{\frac{d-2}{2}}(s+k+1)}$. This leaves us with the error coming from the Bessel function. The bound on 4.5 is:

$$err_{Bessel} \leq \left(\frac{4}{(s+k)t}\right)^{s+\frac{d-3}{2}} 8\frac{(s+\frac{d-3}{2})^2 - \frac{1}{4}}{2} e^{2\pi\frac{(s+\frac{d-3}{2})^2-\frac{1}{4}}{(s+k)|t|}} \tag{B.10}$$

I first replace $|t|$ by π in the exponential. This leaves us with the following integral:

$$\oint \frac{dt}{|t|^{s+k+\frac{d-3}{2}+1}} \left| e^{t\frac{s+k}{2}} \left(\frac{t}{e^t-1}\right)^{s+k+1} \right| \leq \frac{2}{\pi^{\frac{d-3}{2}}} \int_{\mathbb{R}} dt \frac{e^{t\frac{s+k}{2}}}{(e^t+1)^{s+k+1}} \tag{B.11}$$
$$\leq \frac{4}{\pi^{\frac{d-3}{2}}} \int_{\mathbb{R}+} dt\ e^{-t\frac{s+k}{2}} \tag{B.12}$$
$$\leq \frac{8}{\pi^{\frac{d-3}{2}}(s+k)} \tag{B.13}$$

This leaves us with an error of the form:

$$err_3 \leq \frac{2^{2(s+k)+d-3}}{\sqrt{\pi}(s+k)^{\frac{d-2}{2}}} \Gamma\left(s+\frac{d-1}{2}+\lfloor\frac{k}{2}\rfloor\right) \frac{64}{\pi^{\frac{d-3}{2}}(s+k)^{\frac{3}{2}}} e^{2\frac{(s+\frac{d-3}{2})^2}{m}} \left| \frac{(s+\frac{d-3}{2})^2 - \frac{1}{4}}{2} \right| \tag{B.14}$$

In total, the error is bounded as equation 4.9 states.

Bibliography

[1] J. Bros, H. Epstein, and V. Glaser. A proof of the crossing property for two-particle amplitudes in general quantum field theory. *Commun. Math. Phys.*, 1(3):240–264, 1965.

[2] P. D. B. Collins. *An Introduction to Regge Theory and High Energy Physics*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1977.

[3] Miguel Correia, Amit Sever, and Alexander Zhiboedov. An analytical toolkit for the s-matrix bootstrap, 2020.

[4] Jan Dereziński. Hypergeometric type functions and their symmetries. *Annales Henri Poincaré*, 15(8):1569–1653, September 2013.

[5] Christopher Eckner, Felipe Figueroa, and Piotr Tourkine. The regge bootstrap, from linear to non-linear trajectories, 2024.

[6] Marcel Froissart. Asymptotic behavior and subtractions in the mandelstam representation. *Phys. Rev.*, 123:1053–1057, Aug 1961.

[7] P. Goddard and C.B. Thorn. Compatibility of the dual pomeron with unitarity and the absence of ghosts in the dual resonance model. *Physics Letters B*, 40(2):235–238, 1972.

[8] G. H. Hardy. On two theorems of F. Carlson and S. Wigert. *Acta Mathematica*, 42(none):327 – 339, 1920.

[9] Gerald Minerbo. Causality and analyticity in formal scattering theory. *Phys. Rev. D*, 3:928–932, Feb 1971.

[10] Hari Srivastava, Mridula Garg, and Sangeeta Choudhary. A new generalization of the bernoulli and related polynomials. *Russian Journal of Mathematical Physics*, 17:251–261, 06 2010.

[11] Nico Temme. Bernoulli polynomials old and new: Generalizations and asymptotics. 11 1996.

[12] Nico M Temme. *Asymptotic Methods for Integrals*. WORLD SCIENTIFIC, 2014.

[13] Wu-Ki Tung. *Group Theory in Physics*. WORLD SCIENTIFIC, 1985.

[14] Felipe Figueroa Vilar Del Valle. *Dual models revisited : a bootstrap perspective from QCD to string theory*. PhD thesis, Ecole doctorale de physique de Grenoble, 2024.