59.

$\forall a,b\in Z$

$a\oplus b=a+b-1\in Z$

封闭

$(a\oplus b)\oplus c$

$=(a+b-1)\oplus c$

$=a+b+c-2$

$a\oplus(b\oplus c)$

$=a\oplus(b+c-1)$

$=a+b+c-2$

$(a\oplus b)\oplus c=a\oplus(b\oplus c)$

∴满足结合律

$\forall a\in Z,\exists 1\in Z$

$a\oplus 1=a+1-1=a$

且 $1\oplus a=a+1-1=a$

∴有幺元

$\forall a\in Z,\exists 2-a\in Z$

~~$a\oplus(1-a)=a+1-a-1=$~~

$a\oplus(2-a)=a+2-a-1=1$

∴ $\forall a\in Z$ 有逆元

$a\oplus b=b\oplus a=a+b-1$

满足交换律

$\therefore \langle Z, \oplus \rangle$ 是交换群

~~$a \otimes b = a + b - a$~~

$\forall a, b \in Z$, $a \otimes b = a + b - ab \in Z$ 封闭

$(a \otimes b) \otimes c$

$= (a + b - ab) \otimes c$

$= a + b - ab + c - (a + b - ab)c$

$= a + b + c - ab - ac - bc + abc$

$a \otimes (b \otimes c)$

$= a \otimes (b + c - bc)$

$= a + b + c - bc - a(b + c - bc)$

$= a + b + c - ab - ac - bc + abc$

$(a \otimes b) \otimes c = a \otimes (b \otimes c)$ 满足结合律 $\exists 0 \in Z$, $a \otimes 0 = 0 \otimes a = a$

$\therefore \langle Z, \otimes \rangle$ 为~~半群~~含幺半群 $\therefore$ 含幺

$\forall a, b, c \in Z$

$a \otimes (b \oplus c)$

$= a \otimes (b + c - 1)$

$= a + b + c - 1 - a(b + c - 1)$

$= 2a + b + c - ab - ac - 1$

$(a \otimes b) \oplus (a \otimes c)$

$= (a + b - ab) \oplus (a + c - ac)$

$= 2a + b + c - ab - ac - 1$

$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$

$\therefore \langle Z, \oplus, \otimes \rangle$是有幺元的交换环

62、

$\forall (x_1, y_1), (x_2, y_2) \in X$

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in X$ （封闭）

$\forall (x_1, y_1), (x_2, y_2), (x_3, y_3) \in X$

$((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3) = (x_1 + x_2, y_1 + y_2) \oplus (x_3, y_3)$

$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$

$= (x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3))$

满足结合律

$\exists (0, 0) \in X$

$\forall (x, y) \in X$

$(x, y) \oplus (0, 0) = (x, y) = (0, 0) \oplus (x, y)$

存在幺元.

$\forall (x, y) \in X, \exists (-x, -y) \in X$

$(x, y) \oplus (-x, -y) = (0, 0)$

存在逆元

$\forall (x_1, y_1), (x_2, y_2) \in X$

$(x_1, y_1) \oplus (x_2, y_2) = (x_2, y_2) \oplus (x_1, y_1) = (x_1 + x_2, y_1 + y_2)$

（满足交换律）

$\therefore \langle X, \oplus \rangle$为群

$\forall (x_1, y_1), (x_2, y_2) \in X$

$(x_1, y_1) \otimes (x_2, y_2) = (x_1 x_2, y_1 y_2) \in X$ （封闭性）

$\forall (x_1, y_1) (x_2, y_2) (x_3, y_3) \in X$

$(x_1, y_1) \otimes ((x_2, y_2) \otimes (x_3, y_3)) = ((x_1, y_1) \otimes (x_2, y_2)) \otimes (x_3, y_3) = (x_1 x_2 x_3, y_1 y_2 y_3)$

结合律

$\langle X, \otimes \rangle$ 为群

~~$(x_1, y_1) \otimes (x_2$ 上~~

~~$(x_1, y_1) \otimes (x_2, y_2) = (x_2, y_2) \otimes (x_1, y_1) = (x_1 x_2, y_1 y_2)$~~

~~满足交换率~~

$(x_1, y_1) \otimes ((x_2, y_2) \oplus (x_3, y_3)) = (x_1, y_1) \otimes (x_2 + x_3, y_2 + y_3) = (x_1(x_2+x_3), y_1(y_2+y_3))$

$(x_1, y_1) \otimes (x_2, y_2) \oplus ((x_1, y_1) \otimes (x_3, y_3))$

$= (x_1 x_2, y_1 y_2) \oplus (x_1 x_3, y_1 y_3)$

$= (x_1(x_2+x_3), y_1(y_2+y_3))$

$\therefore (x_1, y_1) \otimes ((x_2, y_2) \oplus (x_3, y_3)) = ((x_1, y_1) \otimes (x_2, y_2)) \oplus ((x_1, y_1) \otimes (x_3, y_3))$

$\oplus$、$\otimes$ 都满足交换率

$\therefore ((x_2, y_2) \oplus (x_3, y_3)) \otimes (x_1, y_1) =$ ~~$((x_1, y_2) \otimes (x_2, y_2$~~ $((x_2, y_2) \otimes (x_1, y_1)) \oplus ((x_3, y_3), (x_1, y_1))$

10. (1).

易知此时不为整环

$\because$ 不存在 $x_0 \in X = \{x \mid x = 2n, n \in Z\}$

$\forall x \in X, x * x_0 = x_0 * x = x$

$\therefore$ 在 $\langle X, \cdot X \rangle$ 不含幺元

$\therefore \langle X, +, \cdot X \rangle$ 不可能是整环

(2). ~~易知 $\langle X, + \rangle$ 封闭~~

$\forall x_1, x_2 \in X, x_1 + x_2 = 2n_1 + 1 + 2n_2 + 1$

$= 2(n_1 + n_2 + 1)$

$n_1 + n_2 + 1 \in Z$, $\therefore \langle X, + \rangle$ 不封闭

$\therefore \langle X, +, \cdot X \rangle$ 不可能是环,不可能是整环

(3). $\forall x_1, x_2 \in X$

$x_1+x_2 \in Z \wedge x_1+x_2 \geq 0$

$\therefore \langle X, + \rangle$ 封闭

~~$\exists 0 \in X$ ∃~~

$\forall x_1, x_2, x_3 \in X$

$(x_1+x_2)+x_3 = x_1+(x_2+x_3) = x_1+x_2+x_3$

$\therefore$ 满足结合律

$\exists 0 \in X, \forall x \in X, x+0=0+x=x$

$\therefore$ 有幺元

当 $x \in X \wedge x \neq 0$

$\forall x \in X, x+x_1 \geq x > 0 \neq 0$

$\therefore$ 不是每个元素都有逆元

$\therefore \langle X, + \rangle$ 无法构成群

$\therefore \langle X, +, \times \rangle$ 不是整环

(4). $\forall x_1, x_2 \in X$

$x_1+x_2 = \cancel{(a_1+b_1)+(a}\ (a_1+a_2)+(b_1+b_2)\sqrt{5}$

$a_1+a_2 \in Q, b_1+b_2 \in Q$

$\therefore x_1+x_2 \in X$

$\therefore \langle X, + \rangle$ 封闭

$(x_1+x_2)+x_3 = x_1+(x_2+x_3) = (a_1+a_2+a_3)+(b_1+b_2+b_3)\sqrt{5}$

$\therefore$ 满足结合律

$\exists a=0, b=0, x=0 \in X$

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$\forall x \in X, x+0=0+x=x$

$\therefore$ 含幺元

~~$\forall x \in X, \exists x$~~

$\forall x = a+b\sqrt{5}$

$\exists -x = -a-b\sqrt{5} \in X$

$x+(-x)=(-x)+x=0$

$\therefore$ 每个元素都有逆元

$\forall x_1, x_2 \in X$

$x_1+x_2=x_2+x_1=(a_1+a_2)+(b_1+b_2)\sqrt{5}$

$\therefore$ 满足交换律

$\therefore \langle X, +\rangle$ 为交换群

对 $\langle X, \times\rangle$

$\forall x_1, x_2 \in X$

$x_1 \times x_2 = (a_1+b_1\sqrt{5})(a_2+b_2\sqrt{5})$

$= a_1a_2+(a_1b_2+a_2b_1)\sqrt{5}+b_1b_2\sqrt{5} \notin X$

$\therefore \langle X, \times\rangle$ 不封闭

$\therefore \langle X, +, \times\rangle$ 不是环，也不是整环.

(5). 由(4)证易知 $\langle X, +\rangle$ 为交换群

$\forall x_1, x_2 \in X$

$x_1 \times x_2 = (a_1+b_1\sqrt{3})(a_2+b_2\sqrt{3})$

$= a_1a_2+(a_1b_2+a_2b_1)\sqrt{3}+3b_1b_2$

$= (a_1a_2+3b_1b_2)+(a_1b_2+a_2b_1)\sqrt{3}$

$a_1a_2+3b_1b_2 \in Q, a_1b_2+a_2b_1 \in Q$

$\therefore \langle X, \times \rangle$ 封闭

由数乘定义可知

~~$\langle X, \times \rangle$~~ $\times$ 对 X 满足结合律

$\therefore \langle X, \times \rangle$ 是半群

~~$\forall x_1, x_2, x_3 \in X$~~ $\forall x_1, x_2, x_3 \in X$

$x_1 \times (x_2 + x_3)$

$= (a_1 + b_1\sqrt{3}) \times ((a_2 + a_3) + (b_2 + b_3)\sqrt{3})$

$= a_1(a_2 + a_3) + a_1(b_2 + b_3)\sqrt{3} + (a_2 + a_3)b_1\sqrt{3} + 3b_1(b_2 + b_3)$

$= (a_1a_2 + 3b_1b_2) + (a_1b_2 + a_2b_1)\sqrt{3} + (a_1a_3 + 3b_1b_3) + (a_1b_3 + a_3b_1)\sqrt{3}$

$= x_1 \times x_2 + x_1 \times x_3$

$\therefore$ $\times$ 对 $+$ 满足分配律 $\therefore \langle X, +, \times \rangle$ 为环

$\exists a_1 = 1, b_1 = 0$

$x_0 = 1 \in X, \forall x \in X, x \times x_0 = x$

$\therefore$ 存在幺元

由数乘性质易知，X 满足交换率

设 $x_1, x_2 \in X \wedge x_1 \neq 0 \wedge x_2 \neq 0$

$x_1 \times x_2 = 0$

$\Rightarrow (a_1 + b_1\sqrt{3}) \times (a_2 + b_2\sqrt{3}) = 0$

$(a_1a_2 + 3b_1b_2) + \sqrt{3}(a_1b_2 + a_2b_1) = 0$

即 $\begin{cases} a_1a_2 + 3b_1b_2 = 0 \\ a_1b_2 + a_2b_1 = 0 \\ a_1, b_1 \text{不同时为零} \\ \text{且 } a_2, b_2 \text{不同时为零} \end{cases}$ 无解

∵不含零因子

∴$\langle X,+,\times\rangle$为整环。

68.(1). 易知 $\forall x_1,x_2\in X$

$x_1+x_2\in Z \wedge x_1+x_2\geq 0$

∴$\langle X,+\rangle$封闭

$\exists 0\in X,\ \forall x\in X,\ x+0=0+x=x$

∴+存在幺元0

$\forall x\in X$, 若$\exists x^{-1}\in X$

$x+x^{-1}=0$

$x^{-1}=-x\leq 0$

∴当$x>0$, x在X中无逆元

∴$\langle X,+\rangle$不是群

∴$\langle X,+,\times\rangle$不是一个域

(2). 由60.(5)已知

$\langle X,+,\times\rangle$是个环

对$X\setminus\{0\}$

$\exists 1\in X\setminus\{0\},\ \forall x\in X\setminus\{0\}$

$x\times 1=1\times x=x$

含幺元

$\forall x\in X\setminus\{0\}$, 若$\exists x_2\in X$

$x_1\times x_2=1$

$x_1x_2=(a_1a_2+3b_1b_2)+\sqrt{3}(a_1b_2+a_2b_1)=1$

$(a_1^2-3b_1^2\neq 0)$

$$\begin{cases} a_1a_2+3b_1b_2=1 \\ a_1b_2+a_2b_1=0 \end{cases} \Rightarrow \begin{cases} a_2=\dfrac{a_1}{a_1^2-3b_1^2} \in Q \\ b_2=\dfrac{-b_1}{a_1^2-3b_1^2} \in Q \end{cases}$$

$\therefore$ ~~$x_2=a_2+b_2\sqrt{3}$~~

~~$=a_1^2-3b_1^2$~~

$\therefore x_2\in X$

$\therefore \langle X,\times\rangle$有逆元

由数乘的性质易知,满足交换律

~~$\therefore\langle X,\times\rangle$是~~

$\therefore\langle X\setminus\{0\},\times\rangle$是一个交换群

又$\because\langle X,+,\times\rangle$是一个环

$\therefore\langle X,+,\times\rangle$是域

(3). 对$\langle X,\times\rangle$

$\forall x_1,x_2\in X$

$x_1\times x_2=(a_1+b_1\sqrt[3]{5})(a_2+b_2\sqrt[3]{5})$

$=a_1a_2+(a_1b_2+a_2b_1)\sqrt[3]{5}+b_1b_2 5^{\frac{2}{3}}\notin X$

$\therefore$不封闭 $\langle X,+,\times\rangle$不是域.

(4). 同(3)可得,$\langle X,+,\times\rangle$是域.

(5) 易知,$1\notin X$,若$1\in X$,则与$a\neq kb$矛盾

则X无幺元1

$\therefore\langle X,+,\times\rangle$不是域.

69、$S_1 \cap S_2 \subseteq S_1$，$S_1 \subseteq F$

$S_1 \cap S_2 \subseteq F$

∃关于$\otimes$的幺元$e \in S_1$

同理 $e \in S_2$

$\therefore S_1 \cap S_2 \neq \emptyset$　易知$\langle S_1 \cap S_2, \oplus \rangle$封闭

$\forall a, b \in S_1 \cap S_2$

$b \in S_1 \wedge b \in S_2$

$-b \in S_1 \wedge -b \in S_2$（$\langle S_1, \oplus \rangle \langle S_2, \oplus \rangle$为群）

$\therefore -b \in S_1 \cap S_2$

$a - b \in S_1 \cap S_2$

$\therefore \langle S_1 \cap S_2, \oplus \rangle$是$S_1$的子群

$\therefore \langle S_1 \cap S_2, \oplus \rangle$为交换群

同理可证

$\langle S_1 \cap S_2 \setminus \{0\}, \otimes \rangle$为交换群且$\langle S_1 \cap S_2 \setminus \{0\}, \otimes \rangle$为$\langle S_1, \otimes \rangle$的子群

$\langle S_1, \oplus, \otimes \rangle$为域

在S_1中$\therefore \otimes$对$\oplus$满足分配律

在$\langle S_1 \cap S_2, \oplus, \otimes \rangle$中$\otimes$对$\oplus$满足分配律（子代数系统的继承性

$\langle S_1 \cap S_2, \oplus, \otimes \rangle$为域

$S_1 \cap S_2 \subseteq F$

$\therefore \langle S_1 \cap S_2, \oplus, \otimes \rangle$是$\langle F, \oplus, \otimes \rangle$的子域

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