$X\sim Exp(\lambda)$, $E(X)=\frac{1}{\lambda}$

7.1 (1). $E(X)=\frac{1}{\lambda}=\bar{X}$

$\hat{\lambda}=\frac{1}{\bar{X}}$

(2). $E(X)=\int_0^1 x f(x)\,dx$

$=\int_0^1 \theta x^{\theta}\,dx$

$=\frac{\theta}{\theta+1}x^{\theta+1}\Big|_0^1$

$=\frac{\theta}{\theta+1}=\bar{X}$ $\quad\hat{\theta}=(\hat{\theta}+1)\bar{X}$

$\hat{\theta}=\frac{\bar{X}}{1-\bar{X}}$

(3). $E(X)=\int_0^{+\infty}\frac{\beta^k}{(k-1)!}x^k e^{-\beta x}\,dx$

$=\frac{\beta^k}{(k-1)!}\cdot\int_0^{+\infty}x^k e^{-\beta x}\,dx$

$=\frac{\beta^k}{(k-1)!}\cdot\frac{k!}{\beta^{k+1}}$

$=\frac{k}{\beta}=\bar{X}$

$\beta=\frac{k}{\bar{X}}$

(4). $E(X)=\int_a^{+\infty}\frac{x}{\theta}e^{-\frac{x-a}{\theta}}\,dx$

$\xlongequal{t=\frac{x-a}{\theta}}\int_0^{+\infty}(t+\frac{a}{\theta})\,e^{-t}\,\theta\,dt$

$=\int_0^{+\infty}(\theta t+a)\cdot e^{-t}$

$=\theta\int_0^{+\infty}te^{-t}+\int_0^{+\infty}ae^{-t}$

$=a+\theta=\bar{X}$ ①

$E(X^2)=\int_a^{+\infty}x^2\frac{1}{\theta}e^{-\frac{x-a}{\theta}}$

$=\int_0^{+\infty}(a+\theta t)^2e^{-t}\,dt$

$=a^2+2a\theta+2\theta^2=\frac{1}{n}\sum_{i=1}^{n}X_i^2$ ②

联立①②

$$\begin{cases}\hat{\theta}=\sqrt{B_2}\\ \hat{a}=\bar{X}-\sqrt{B_2}\end{cases}$$

(5). $E(X)=mp=\bar{X}$

$\hat{p}=\frac{\bar{X}}{m}$

3. $P\{X=k\}=p(1-p)^{k-1}$ $(k=1,2,\cdots)$

$L(p)=\prod_{i=1}^{n}p(1-p)^{x_i-1}$

$=p^n(1-p)^{\sum_{i=1}^{n}x_i-n}$

$\ln L(p)=n\ln p+(\sum_{i=1}^{n}x_i-n)\ln(1-p)$

$\frac{d(\ln L(p))}{dp}=\frac{n}{p}-\frac{\sum_{i=1}^{n}x_i-n}{1-p}=0$

$\hat{p}=\frac{1}{\bar{X}}$

5. ~~$E(X)$~~ 6. 当$\alpha=1$

$$f(x;\beta)=\begin{cases}\beta x^{-\beta-1}, & x>1\\ 0, & x\le 1\end{cases}$$

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$E(X)=\int_1^{+\infty} x f(x;\beta)\,dx$

$=\int_1^{+\infty} \beta x^{-\beta}\,dx$

$=\frac{\beta}{1-\beta} x^{1-\beta}\Big|_1^{+\infty} \quad (\beta>1)$

$1-\beta<0$

$E(X)=\frac{\beta}{\beta-1}=\bar{X}$

$\hat{\beta}=\frac{\bar{X}}{\bar{X}-1}$

(2) 当 $\beta=2$，$f(x;\alpha,2)=\begin{cases}2\alpha^2 x^{-3}, & x>\alpha\\ 0, & x\le\alpha\end{cases}$

$L(\alpha)=\prod_{i=1}^{n} 2\alpha^2 x_i^{-3}$

$=(2\alpha^2)^n \prod_{i=1}^{n} x_i^{-3}$

~~$\ln L(\alpha)$~~

~~$=n\ln 2\alpha^2-3\ln\prod_{i=1}^{n} x_i$~~

$\ln L(\alpha)=n\ln 2\alpha^2-3\ln\prod_{i=1}^{n} x_i$

$\frac{d\ln L(\alpha)}{d\alpha}=\frac{2n}{\alpha}>0$，单调递增，$x_i>\alpha$

$\therefore \hat{\alpha}=\min\{x_1,x_2\cdots x_n\}$

7. $E(X)=\frac{4-9\theta}{2}=\bar{x}=\frac{7}{8}$

$\theta=\frac{1}{4}$

$L(\theta)=\theta^3\cdot\left(\frac{\theta}{2}\right)^1\cdot\left(\frac{\theta}{2}\right)^5\cdot(1-2\theta)^6$

$=\left(\frac{1}{2}\right)^7\cdot\theta^{10}(1-2\theta)^6$

$\ln L(\theta)=-7\ln 2+10\ln\theta+6\ln(1-2\theta)$

$\frac{d\ln L(\theta)}{d(\theta)}=0 \Rightarrow \hat{\theta}=\frac{5}{16}$.

9. $E\left(C\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2\right)$

$=CE\left(\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2\right)$

$=CE\left(\sum_{i=1}^{n-1}X_{i+1}^2+X_i^2-2X_iX_{i+1}\right)$

$=C\sum_{i=1}^{n-1}\left[2E(X_i^2)-2E^2(X_i)\right]$

$=2C\sum_{i=1}^{n-1}D(X_i)$

$=2C(n-1)\sigma^2=\sigma^2$

$C=\frac{1}{2(n-1)}$

10. $E(\alpha\bar{X}+(1-\alpha)S^2)$

$=\alpha E(\bar{X})+(1-\alpha)E(S^2)$

$\because X\sim P(\lambda)$

原式 $=\alpha\lambda+(1-\alpha)\lambda$

$=\lambda$

$\therefore$ 是无偏估计量.

11. (1) 若存在无偏估计量

$E(\hat{p}^2(X_1))=p^2$

$p\cdot\hat{p}^2(1)+(1-p)\cdot\hat{p}^2(0)=p^2$

$p(\hat{p}^2(1)-\hat{p}^2(0))+\hat{p}^2(0)=p^2$

$\hat{p}^2(x)$ 函数中总含有 p，不存在无偏估计.

(2). $E(\bar{X})=p$, $D(\bar{X})=\frac{p(1-p)}{n}$

$E(\bar{X}^2)=E^2(\bar{X})+D(\bar{X})=\frac{n-1}{n}p^2+\frac{p}{n}$, $E(X^2)=D(X)+E^2(X)=p(1-p)+p^2=p$

$E(S^2)=E(\frac{1}{n-1}\sum_{i=1}^{n}(X_i^2-\bar{X})^2)$

$=\frac{1}{n-1}[\sum_{i=1}^{n}E(X_i^2)-nE(\bar{X}^2)]$

$=\frac{1}{n-1}(np-(n-1)p^2-p]$

$=p-p^2$

$\therefore E(\bar{X}-S^2)=p^2$

$\therefore \bar{X}-S^2$ 为 p^2 的无偏估计量.

13. $\lim\limits_{n\to\infty}E[(\hat{\theta}-\theta)^2]$

$=\lim\limits_{n\to\infty}E(\hat{\theta}^2-2\hat{\theta}\theta+\theta^2)$

已知 $\lim\limits_{n\to\infty}E(\hat{\theta}-\theta)=0$

$\lim\limits_{n\to\infty}E(\hat{\theta})=\theta$

原式 $=\lim\limits_{n\to\infty}E[\hat{\theta}^2-2\hat{\theta}E(\hat{\theta})+E^2(\hat{\theta})]$

$=\lim\limits_{n\to\infty}E[(\hat{\theta}-E(\hat{\theta}))^2]=\lim\limits_{n\to\infty}D(\hat{\theta})=0$ $\therefore\lim\limits_{n\to\infty}E[(\hat{\theta}-\theta)^2]=0$

$\therefore$ 是均方相合估计量，

是相合估计量

14. 由题6.25已知

当 $X \sim Exp(\lambda)$

$2\lambda X \sim \chi(2)$

$\therefore 2\lambda\sum_{i=1}^{12} X_i \sim \chi^2(2\times 12) = \chi^2(24)$

$P\{k_1 < 2\lambda\sum_{i=1}^{12} X_i < k_2\} = 0.9$

$k_1 = \chi^2_{1-\alpha/2}(24) = \chi^2_{0.95}(24) = 13.848$

$k_2 = \chi^2_{\alpha/2}(24) = \chi^2_{0.05}(24) = 36.415$

$P\{\frac{k_1}{2\sum_{i=1}^{12} X_i} < \lambda < \frac{k_2}{2\sum_{i=1}^{12} X_i}\} = 0.9$

其中 $2\sum_{i=1}^{12} X_i = 24820$

置信区间为 $(0.0006, 0.0013)$

$\lambda = \frac{1}{u}$，u 的置信区间 $(681.587, 1792.317)$

$P\{2\lambda\sum_{i=1}^{12} X_i > k_1\} = 0.9$

$\therefore k_1 = \chi^2_{0.9}(24) = 15.659$，$\frac{2\sum_{i=1}^{12} X_i}{k_1} = 1585.031$

$P\{2\lambda\sum_{i=1}^{12} X_i < k_2\} = 0.9$

$k_2 = \chi^2_{0.1}(24) = 33.196$，$\frac{2\sum_{i=1}^{12} X_i}{k_2} = 747.680$

u 的置信下限 747.680，置信上限为 1585.031

15. $X \sim B(1, p)$，

$$U = \frac{(\bar{X} - p)}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$$

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$\therefore -u_{\alpha/2} < \frac{\sqrt{n}(\bar{X}-p)}{\sqrt{p(1-p)}} < u_{\alpha/2}$

$\frac{n(\bar{X}-p)^2}{p(1-p)} < u^2_{\alpha/2}$

$(n+u^2_{\alpha/2})p^2-(2n\bar{X}+u^2_{\alpha/2})p+n\bar{X}^2<0$

解得区间为 $(0.4759, 0.6619)$

16. $\frac{\sum_{i=1}^{n} X_i - n\lambda}{\sqrt{n\lambda}} \sim N(0,1)$

$P\left\{\left|\frac{\sum_{i=1}^{n} X_i - n\lambda}{\sqrt{n\lambda}}\right| < u_{\alpha/2}\right\} = 1-\alpha$

$(\sum_{i=1}^{n} X_i - n\lambda)^2 < u^2_{\alpha/2} n\lambda$

$n\lambda^2 - (2n\bar{X} + u^2_{\alpha/2})\lambda + n\bar{X} < 0$

解得区间 $(\bar{X} + \frac{u_{\alpha/2}}{2n}(u_{\alpha/2} - \sqrt{4n\bar{X}+u^2_{\alpha/2}}),\ \bar{X} + \frac{u_{\alpha/2}}{2n} + \sqrt{4n\bar{X}+u^2_{\alpha/2}})$

17.

$\frac{\sqrt{n}(\bar{X}-u)}{6} \sim N(0,1)$

$P\left\{\left|\frac{\sqrt{n}(\bar{X}-u)}{6}\right| < u_{6/2}\right\} = 1-\alpha$

$(\bar{X} + \frac{6}{\sqrt{n}} u_{\alpha/2}) - (\bar{X} - \frac{6}{\sqrt{n}} u_{\alpha/2}) \le L$

$\frac{26}{\sqrt{n}} u_{\alpha/2} \le L$

$n \geq \frac{46^2 u^2_{\alpha/2}}{L^2}$

18. $P\left\{\left|\frac{\sqrt{n}(\bar{X}-u)}{6}\right| < u_{0.025}\right\} = 0.95$

解得: $\mu \in (\bar{x} - \frac{\sigma}{\sqrt{n}} u_{\alpha/2}, \bar{x} + \frac{\sigma}{\sqrt{n}} u_{\alpha/2})$, $\bar{x} = 21.4$

$= (21.137, 21.663)$

(2). $\cancel{\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)}$ $\cancel{t(n-1)}$

~~n-1~~

$\frac{\sqrt{n}(\bar{X}-\mu)}{S} \sim t(n-1)$

$(\bar{x} - \frac{S}{\sqrt{n}} t_{\alpha/2}, \bar{x} + \frac{S}{\sqrt{n}} t_{\alpha/2}(n-1))$

其中 $S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2}$

区间为 $(20.3355, 22.4645)$

(3). $P\{\frac{\sqrt{n}(\bar{x}-\mu)}{S} > u_{\alpha/2}\} = 0.95$

置信上界: $\cancel{\bar{x} - \sqrt{n}}$ $\bar{x} - \frac{S}{\sqrt{n}} u_{\alpha/2} = 22.2173$

$P\{\frac{\sqrt{n}(\bar{x}-\mu)}{S} < -u_{\alpha/2}\} = 0.95$

置信下界: $\bar{x} + \frac{S}{\sqrt{n}} u_{\alpha/2} = 20.5827$.

19. (1). $\frac{\sum_{i=1}^{6}(X_i-\mu)^2}{\sigma^2} \sim \chi^2(6)$

$\therefore P\{k_1 < \frac{\sum_{i=1}^{6}(X_i-\mu)^2}{\sigma^2} < k_2\} = 0.95$

$k_1 = \chi^2_{0.975}(6)$, $k_2 = \chi^2_{0.025}(6)$

$\frac{\sum_{i=1}^{6}(X_i-\mu)^2}{k_2} < \sigma^2 < \frac{\sum_{i=1}^{6}(X_i-\mu)^2}{k_1}$, $\sum_{i=1}^{6}(X_i-\mu)^2 = 0.35$

区间为 $(0.0242, 0.2829)$

(2) $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) = \chi^2(5)$

$P\{k_1 < \frac{(n-1)S^2}{\sigma^2} < k_2\} = 0.95.$

$k_1 = \chi^2_{0.975}(5)$, $k_2 = \chi^2_{0.025}(5)$

$\frac{(n-1)S^2}{k_2} < \sigma^2 < \frac{(n-1)S^2}{k_1}$, $n=6$

$5S^2 = \sum_{i=1}^{6}(X_i - \bar{X})^2 = 0.35$

解得：区间 $(0.0273, 0.4212)$

20. $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) = \chi^2(9)$

$P\{k_1 < \frac{(n-1)S^2}{\sigma^2} < k_2\} = 0.95$

$k_1 = \chi^2(9)_{0.975}$, $k_2 = \chi^2(9)_{0.025}$.

$\frac{(n-1)S^2}{k_2} < \sigma^2 < \frac{(n-1)S^2}{k_1}$, $n=10$

$(n-1)S^2 = \frac{1}{n-1} \cdot (n-1) \sum_{i=1}^{10}(X_i - \bar{X})^2$

$= 676.4$

区间 $(5.9630, 15.8278)$.

置信下限 $P\{\frac{(n-1)S^2}{\sigma^2} < \chi^2_{0.05}(9)\} = 0.95$.

$P\{\frac{(n-1)S^2}{\chi^2_{0.05}(9)} < \sigma^2\} = 0.95$

$\therefore$ 下限为 $\sqrt{\frac{9S^2}{\chi^2_{0.05}(9)}} = 6.3229$

21. $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) = \chi^2(9)$

$\therefore P\{k_1 < \frac{9S^2}{\sigma^2} < k_2\} = 0.95$

取 $k_1 = \chi^2_{0.975}(9)$, $k_2 = \chi^2_{0.025}(9)$

$\therefore \frac{9S^2}{k_2} < \sigma^2 < \frac{9S^2}{k_1}$

解得区间为 $(0.9462, 6.0067)$

$D(\frac{X^2}{\sigma^2}) = \frac{1}{\sigma^2} D(\frac{X^2}{\sigma^2})$

$\frac{X^2}{\sigma^2} \sim \chi^2(1)$

原式 $= \frac{1}{\sigma^2} \cdot 2 = \frac{2}{\sigma^2}$

$\therefore \frac{2k_1}{9S^2} < \frac{2}{\sigma^2} < \frac{2k_2}{9S^2}$

即 $(0.3000, 2.1137)$

22. $\frac{(\bar{X}-\bar{Y})-(u_1-u_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} \sim N(0,1)$

$P\{\left|\frac{(\bar{X}-\bar{Y})-(u_1-u_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}\right| < u_{\frac{\alpha}{2}}\} = 1-\alpha$

区间 $(\bar{X}-\bar{Y} - u_{\alpha/2} \cdot \sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}, \bar{X}-\bar{Y} + u_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}})$

置信上限 $\bar{X}-\bar{Y} + u_\alpha\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$

置信下限 $\bar{X}-\bar{Y} - u_\alpha\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$

23. $\frac{(\bar{X}-\bar{Y})-(u_1-u_2)}{S_w\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \sim t(n_1+n_2-2)$

$$S_w=\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}$$

$$t_{0.975}(n_1+n_2-2)<\frac{(\bar{X}-\bar{Y})-(u_1-u_2)}{S_w\cdot\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}<t_{0.025}(n_1+n_2-2)$$

区间：$\left(\bar{X}-\bar{Y}-t(n_1+n_2-2)S_w\sqrt{\frac{1}{n_1}+\frac{1}{n_2}},\ \bar{X}-\bar{Y}+t(n_1+n_2-2)S_w\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\right)$

解得：区间 $(-0.0020, 0.0061)$

~~26~~ 24. ∵ 同23. 设 ~~服~~ 放 ~~？~~ 一批为Y

$$\frac{(\bar{X}-\bar{Y})-(u_1-u_2)}{S_w\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}\sim t(100+100-2)$$

$$S_w=\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}$$

~~又~~ 区间代入得：$(0.0299, 0.0501)$

26. $\frac{\sigma_B^2S_A^2}{\sigma_A^2S_B^2}\sim F(9,9)$

$$P\left\{F_{\frac{\alpha}{2}}(9,9)<\frac{\sigma_B^2}{\sigma_A^2}\cdot\frac{S_A^2}{S_B^2}<F_{1-\frac{\alpha}{2}}(9,9)\right\}=0.95.$$

区间：$\cancel{\left(\frac{S_A^2}{S_B^2}\cdot F_{1-\alpha/2}(9,9),\ \frac{S_A^2}{S_B^2}F_{\frac{\alpha}{2}}(9,9)\right)}$ $\left(\frac{S_A^2}{S_B^2F_{\alpha/2}(9,9)},\ \frac{S_A^2}{S_B^2F_{1-\alpha/2}(9,9)}\right)$

即 $(0.2217, 3.6008)$

$$P\left\{\frac{S_A^2\sigma_B^2}{S_B^2\sigma_A^2}>F_{1-\alpha}(9,9)\right\}=0.95.$$

$$P\left\{\frac{S_A^2\sigma_B^2}{S_B^2\sigma_A^2}<F_{\alpha}(9,9)\right\}=0.95.$$

置信下限为 0.2810，置信上限 2.8413