46.

$\forall a,b \in \cancel{\langle H_1 \cap H_2, * \rangle}\ H_1 \cap H_2$

$\Rightarrow a \in H_1 \wedge a \in H_2 \wedge b \in H_1 \wedge b \in H_2$

$\Rightarrow a*b \in H_1 \wedge a*b \in H_2$ （$\langle H_1, * \rangle, \langle H_2, * \rangle$是子群）

$\Rightarrow a*b \in H_1 \cap H_2$

$\forall a \in H_1 \cap H_2$

$\Rightarrow a \in H_1 \wedge a \in H_2$

$\Rightarrow \exists a^{-1} \in H_1, a*a^{-1}=e \wedge \exists a^{-1} \in H_2, a*a^{-1}=e$

$\Rightarrow \exists a \in H_1 \cap H_2, a*a^{-1}=e$

$H_1 \subseteq G, H_2 \subseteq G, H_1 \cap H_2 \subseteq G$

$\therefore \langle H_1 \cap H_2, * \rangle$是$\langle G, * \rangle$的子群

50. (1) $\forall f_1, f_2 \in G$

$f_2 \circ f_1(x) = a_2(a_1 x + b_1) + b_2$

$= a_1 a_2 x + a_2 b_1 + b_2$

$a_1 a_2 \in R \wedge a_1 a_2 \neq 0$, $a_2 b_1 + b_2 \in R$

$\therefore f_2 \circ f_1 \in G$

$\therefore$ 具有封闭性.

$\forall f_1, f_2, f_3 \in G$

$f_3 \circ (f_2 \circ f_1)(x) = a_3(a_1 a_2 x + a_2 b_1 + b_2) + b_3$

$= a_1 a_2 a_3 x + a_2 a_3 b_1 + a_3 b_2 + b_3$

$(f_3 \circ f_2) f_1(x) = a_2 a_3 (a_1 x + b_1) + a_3 b_2 + b_3$

$= a_1 a_2 a_3 x + a_2 a_3 b_1 + a_3 b_2 + b_3$

$f_3\circ(f_2\circ f_1)=(f_3\circ f_2)\circ f_1$

$\therefore$ 满足结合律

易知 $\exists f_e(x)=x$

$\forall f\in G,\ f\circ f_e=f$

$\therefore$ 存在幺元

$\forall f(x)=ax+b\in G$

$\exists f^{-1}(x)=\frac{1}{a}x-\frac{b}{a}$

$f\circ f^{-1}=a(\frac{1}{a}x-\frac{b}{a})+b$

$=x$

$f^{-1}\circ f=\frac{1}{a}(ax+b)-\frac{b}{a}$

$=x+\frac{b}{a}-\frac{b}{a}$

$=x$

$\therefore$ 存在逆元

综上：$\langle G,\circ\rangle$ 是群

(2). 易知 $S_1\in G$ 且 $S_1\neq\varnothing$

$\forall f_1,f_2\in S_1$

$f_2\circ f_1=x+b_1+b_2$

$b_1+b_2\in R$

$\therefore f_2\circ f_1\in S_1$

$\forall f(x)=x+b$

$\exists f^{-1}(x)=x-b$ $f\circ f^{-1}=f^{-1}\circ f_1=f_e=x$

$\therefore$ $\langle S_1,\circ\rangle$ 是 $\langle G,\circ\rangle$ 的子群

$f \in S_2$

$\forall f_1, f_2 \in S_2$

$f_1 \circ f_2^{-1}$

$= \frac{a_1}{a_2} x$

$a_2 \neq 0 \wedge \frac{a_1}{a_2} \in R$

$\therefore f_1 \circ f_2^{-1} \in S_2$

$\therefore \langle S_2, \circ \rangle$是$\langle G, \circ \rangle$的子群

综上所述 $\langle S_1, \circ \rangle$和$\langle S_2, \circ \rangle$都是$\langle G, \circ \rangle$的子群

57.

$\forall x \in G$

$x = e * x * e^{-1}$

$\therefore (x, x) \in R$

$\therefore R$具有自反性

若$(x, y) \in R$

即$\exists z \in G$, $y = z * x * z^{-1}$

$\therefore z^{-1} * y * z = x$

$\therefore x = z^{-1} * y * (z^{-1})^{-1}$

$\therefore (y, x) \in G$

$\therefore R$具有对称性

若$(x, y), (y, z)$

即$\exists z_1 \in G$, $y = z_1 * x * z_1^{-1}$ ①

$\exists z_2 \in G$, $z = z_2 * y * z_2^{-1}$ ②

$\therefore \exists z_3 \in G, z_3 = z_2 * z_1, z_3^{-1} = z_1^{-1} * z_2^{-1}$ （群的封闭性）

$\therefore$ ~~$\exists z_3$~~ ①代入②

$z = (z_2 * z_1) * x * (z_1^{-1} * z_2^{-1}) = z_3 * x * z_3^{-1}$

$\therefore \exists z_3 \in G, z = z_3 * x * z_3^{-1}$

$\therefore (x, z) \in R$

$\therefore R$具有传递性

综上：R

56. $\forall x_1, x_2 \in H$

$f(x_1) = g(x_1)$

$f(x_2) = g(x_2)$

易知 $x_1, x_2 \in X$, $f(x_1), g(x_1) \in Y$, $f(x_2), g(x_2) \in Y$

$\therefore f(x_1) + f(x_2) = g(x_1) + g(x_2) = y'$

$\because f, g$为同态函数

$\therefore f(x_1) + f(x_2) = f(x_1 * x_2) = y'$

$g(x_1) + g(x_2) = g(x_1 * x_2) = y'$

$f(x_1 * x_2) = g(x_1 * x_2) = y'$

$\therefore$ 且 $x_1 * x_2 \in X$ （封闭性）

$\therefore \forall x_1, x_2 \in H, x_1 * x_2 \in H.$

$\forall x \in H$

$f(x * x^{-1}) = f(x) + f(x^{-1}) = f(e)$

$g(x * x^{-1}) = g(x) + g(x^{-1}) = g(e)$

$\because g(e) = f(e)$

$g(x) = f(x)$

$\therefore g(x^{-1}) = f(x^{-1})$

又$\because x^{-1} \in X \Rightarrow x^{-1} H$

$\therefore \forall x \in H, x^{-1} \in H$

$\therefore \langle H, * \rangle$ 是 $\langle X, * \rangle$ 的子群