Assignment 4:

1. Conditional independence:

~~a) Consider:~~

~~this is dependence~~

$\not\perp$ this is not independence

a) Consider:

$B \rightarrow A \Rightarrow A \not\perp B$

$C \rightarrow A \Rightarrow A \not\perp C$

$E \rightarrow C \rightarrow A \Rightarrow A \not\perp E$

$F \rightarrow C \rightarrow A \Rightarrow A \not\perp F$

$G \rightarrow E \rightarrow C \Rightarrow C \not\perp G \Rightarrow A \not\perp G$

$H \rightarrow F \rightarrow C \Rightarrow C \not\perp H \Rightarrow A \not\perp H$

$E \leftarrow G \rightarrow I \Rightarrow E \not\perp I \Rightarrow A \not\perp I$

$D \leftarrow G \rightarrow E \Rightarrow E \not\perp D \Rightarrow A \not\perp D$

$\Rightarrow$ There is no variable independent of A given B

b) Consider:

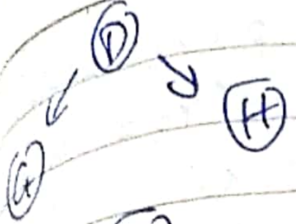
$A \rightarrow G \Rightarrow A \not\perp G$

$A \rightarrow G \rightarrow J \Rightarrow A \not\perp J$

$A \rightarrow G \leftarrow D$, $G \rightarrow J \Rightarrow A \not\perp D$

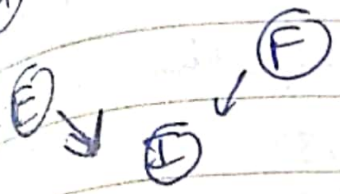
$B \rightarrow D \rightarrow G \Rightarrow B \not\perp D \Rightarrow A \not\perp B$

$D \leftarrow B \rightarrow E \Rightarrow D \not\perp E \Rightarrow A \not\perp E$



$\Rightarrow G \not\perp H \Rightarrow A \not\perp H$

$\Rightarrow H \not\perp I \Rightarrow A \not\perp I$



$\Rightarrow E \perp F \Rightarrow A \perp F$

$C \rightarrow F \quad \Rightarrow C \not\perp F \Rightarrow A \perp C$

So C and F are independent of A given J

2. Bayes Net

a) we have:

$$\begin{aligned} p(x_2 = salmon) &= p(x_2 = salmon \mid x_1 = winter).p(x_1 = winter) \\ &+ p(x_2 = salmon \mid x_1 = spring).p(x_1 = spring) \\ &+ p(x_2 = salmon \mid x_1 = summer).p(x_1 = summer) \\ &+ p(x_2 = salmon \mid x_1 = autumn).p(x_1 = autumn) \\ &= 0,9 . 0,5 + 0 + 0 + 0,8 . 0,5 \\ &= 0,85 \end{aligned}$$

$$\begin{aligned} p(x_2 = sea\ bass) &= p(x_2 = sea\ bass \mid x_1 = winter).p(x_1 = winter) \\ &+ p(x_2 = seabass \mid x_1 = spring).p(x_1 = spring) \\ &+ p(x_2 = sea\ bass \mid x_1 = summer).p(x_1 = summer) \\ &+ p(x_2 = sea\ bass \mid x_1 = autumn).p(x_1 = autumn) \\ &+ p(x_2 = sea\ bass \\ &= 0,1 . 0,5 + 0 + 0 + 0,2 . 0,5 \\ &= 0,15 \end{aligned}$$

Or just $p(x_2 = sea\ bass) = 1 - p(x_2 = salmon) = 0,15$:))

$$p(x_4 = \text{thin}) = p(x_4 = \text{thin} \mid x_2 = \text{salmon}) \cdot p(x_2 = \text{salmon})$$
$$+\ p(x_4 = \text{thin} \mid x_2 = \text{sea bass}) \cdot p(x_2 = \text{sea bass})$$
$$= 0{,}6 \cdot 0{,}85 + 0{,}05 \cdot 0{,}15$$
$$= 0{,}5175$$

And we have

$$p(x_2 = \text{salmon} \mid x_4 = \text{thin}) = \frac{p(x_4 = \text{thin} \mid x_2 = \text{salmon}) \cdot p(x_2 = \text{salmon})}{p(x_4 = \text{thin})}$$
$$= \frac{0{,}6 \cdot 0{,}85}{0{,}5175}$$
$$\approx 0{,}986$$

$$p(x_2 = \text{sea bass} \mid x_4 = \text{thin}) = 1 - 0{,}986$$
$$= 0{,}014$$

=> Most likely to be salmon

~~b) We have:~~

$$\cancel{p(x_3 = \text{light} \mid x_1 = \text{winter}) = p(x_3 = \text{light} \mid x_2 = \text{salmon})}$$
$$\cancel{\cdot\, p(x_2 = \text{salmon} \mid x_1 = \text{winter})}$$
$$\cancel{+\, p(x_3 = \text{light} \mid x_2 = \text{sea bass})}$$
$$\cancel{\cdot\, p(x_2 = \text{sea bass} \mid x_1 = \text{winter})}$$
$$\cancel{= 0{,}33 \cdot 0{,}9 + 0{,}8}$$

b) we have

$$p(x_3 = \text{light} \mid x_1 = \text{winter}) = p(x_3 = \text{light} \mid x_2 = \text{salmon})$$
$$\cdot\, p(x_2 = \text{salmon} \mid x_1 = \text{winter})$$
$$+\, p(x_3 = \text{light} \mid x_2 = \text{sea bass})$$
$$\cdot\, p(x_2 = \text{sea bass} \mid x_1 = \text{winter})$$
$$= 0{,}33 \cdot 0{,}9 + 0{,}8 \cdot 0{,}1$$
$$= 0{,}377$$

b) $p(x_1 | x_3 = medium, x_4 = thin) \propto p(x_3 = medium, x_4 = thin | x_1) . p(x_1)$

$= p(x_3 = medium | x_1) . p(x_4 = thin | x_1) . p(x_1)$

We have:

$p(x_3 = medium | x_1 = winter) = 0,33 . 0,9 + 0,1 . 0,1 = 0,307$

$p(x_3 = medium | x_1 = spring) = 0,33 . 0,3 + 0,1 . 0,7 = 0,169$

$p(x_3 = medium | x_1 = summer) = 0,33 . 0,4 + 0,1 . 0,6 = 0,192$

$p(x_3 = medium | x_1 = autumn) = 0,33 . 0,8 + 0,1 . 0,2 = 0,284$

And:

$p(x_4 = thin | x_1 = winter) = 0,6 . 0,9 + 0,05 . 0,1 = 0,545$

$p(x_4 = thin | x_1 = spring) = 0,6 . 0,3 + 0,05 . 0,7 = 0,215$

$p(x_4 = thin | x_1 = summer) = 0,6 . 0,4 + 0,05 . 0,6 = 0,27$

$p(x_4 = thin | x_1 = autumn) = 0,6 . 0,8 + 0,05 . 0,2 = 0,49$

So:

$p(x_1 = winter | x_3 = medium, x_4 = thin) \propto 0,307 . 0,545 . 0,25 = 0,042$

$p(x_1 = spring | x_3 = medium, x_4 = thin) \propto 0,169 . 0,215 . 0,25 = 0,009$

$p(x_1 = summer | x_3 = medium, x_4 = thin) \propto 0,192 . 0,27 . 0,25 = 0,013$

$p(x_1 = autumn | x_3 = medium, x_4 = thin) \propto 0,284 . 0,49 . 0,25 = 0,035$

$p(x_1 = autumn | x_3$

0,042 is the biggest

$\Rightarrow$ It is likely to be the winter

4. PCA

a) $\tilde{S} \triangleq \frac{1}{n} \tilde{X}.\tilde{X}^T = \frac{1}{n}.(I - v_1.v_1^T)X.[(I - v_1.v_1^T).X]^T$

$= \frac{1}{n}.(I - v_1.v_1^T).X.X^T.(I - v_1.v_1^T)^T$

$= \frac{1}{n}(I - v_1.v_1^T).X.X^T.(I - v_1.v_1^T)$

$= \frac{1}{n}(X.X^T - v_1.v_1^T.X.X^T)(I - v_1.v_1^T)$

$= \frac{1}{n}(X.X^T - v_1.n.\lambda_1.v_1^T)(I - v_1.v_1^T)$

$= \frac{1}{n}(X.X^T - n.\lambda_1.v_1.v_1^T - X.X^T.v_1.v_1^T + n\lambda_1.v_1.v_1^T.v_1.v_1^T$

$= \frac{1}{n}(X.X^T - n.\lambda_1.v_1.v_1^T - n.\lambda_1 v_1.v_1^T + n.\lambda_1.v_1.1.v_1^T)$

$= \frac{1}{n}(X.X^T - n\lambda_1.v_1.v_1^T) = \frac{1}{n}.X.X^T - \lambda_1.v_1.v_2^T$

b) $\tilde{S}.u = \lambda_2.u$

We have: $\tilde{S}.u = (S - \lambda_1.v_1.v_1^T).u$

$= S.u - \lambda_1.v_1.v_1^T.u$

$v_1 \perp u \Rightarrow v_1^T.u = 0$

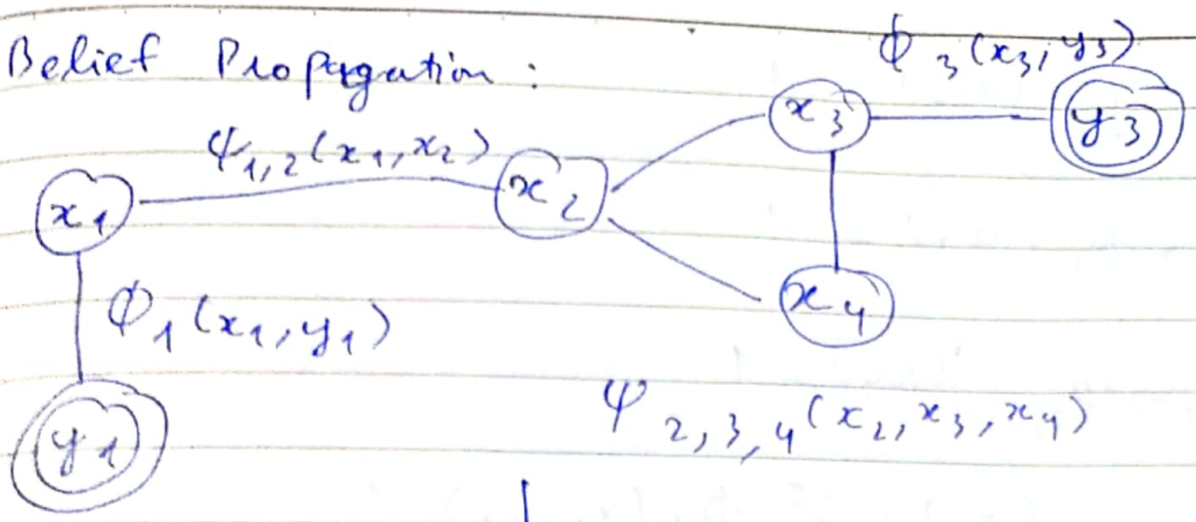
$\Rightarrow \tilde{S}.u = S.u$

So u is an eigenvector of $\tilde{S}$ with the same eigenvalue as S

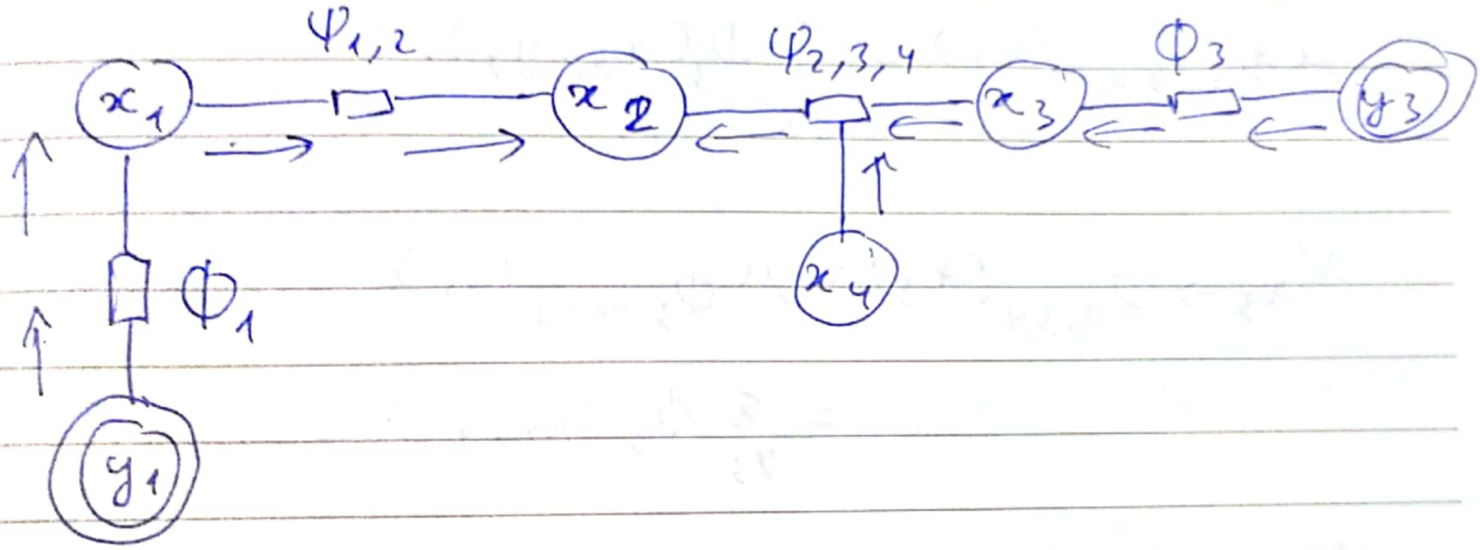
v_1 is the first eigenvector

$\Rightarrow$ u must be the eigenvector corresponding to the second eigenvalue $\lambda_2 \Rightarrow u = v_2$

5. Belief Propagation :



a)



b) to compute the marginal of x_2

we have $p(x_2) = \prod_{v} \mu_{v \to 2}(x_2)$

$$= \mu_{\psi_{1,2} \to 2}(x_2) \cdot \mu_{\psi_{2,3,4} \to 2}(x_2)$$

So we need to compute $\mu_{\psi_{1,2} \to 2}(x_2)$
and $\mu_{\psi_{2,3,4} \to 2}(x_2)$

c) The order in the graph, compute from y_1, y_3 and x_4

d) $\nu_{y_1 \to \phi_1}(y_1) = 1$

$\nu_{y_3 \to \phi_3}(y_3) = 1$

$\nu_{x_4 \to \psi_{2,3,4}}(x_4) = 1$

$\mu_{\phi_1 \to x_1}(x_1) = \sum_{y_1} \phi_1(x_1, y_1) . 1$

$\mu_{\phi_3 \to x_3}(x_3) = \sum_{y_3} \phi_3(x_3, y_3) . 1$

$\nu_{x_3 \to \psi_{2,3,4}}(x_3) = \mu_{\phi_3 \to x_3}(x_3)$

$= \sum_{y_3} \phi_3(x_3, y_3)$

$\nu_{x_1 \to \psi_{1,2}}(x_1) = \mu_{\phi_1 \to x_1}(x_1)$

$= \sum_{y_1} \phi_1(x_1, y_1)$

$\mu_{\psi_{2,3,4} \to x_2}(x_2) = \sum_{x_3, x_4} \psi_{2,3,4}(x_2, x_3, x_4)$

$. \sum_{y_3} \phi_3(x_3, y_3) . 1$

$\mu_{\psi_{1,2} \to x_2}(x_2) = \sum_{x_1} \psi_{1,2}(x_1, x_2) . \sum_{y_1} \phi_1(x_1, y_1)$

So :

$p(x_2) = \sum_{x_3, x_4} \psi_{2,3,4}(x_2, x_3, x_4) . \sum_{y_3=1} \phi_3(x_3, y_3) . \sum_{x_1} \psi_{1,2}(x_1, x_2) . \sum_{y_1=0} \phi_1(x_1, y_1)$

$\Rightarrow \cancel{p(x_1)=}$

$\Rightarrow p(x_2=0) = (0{,}1.0{,}2 + 0{,}1.0{,}2 + 0{,}2.0{,}3 + 0{,}2.0{,}3)$
$\quad . (0{,}1.0{,}3 + 0{,}4.0{,}2)$
$\quad = 0{,}16 . 0{,}11 = 0{,}0176$

$\Rightarrow p(x_2=1) = (0{,}1.0{,}2 + 0{,}1.0{,}2 + 0{,}1.0{,}3 + 0{,}1.0{,}3)$
$\quad . (0{,}4.0{,}3 + 0{,}1.0{,}2) = 0{,}014$

??? Why ???

3. EM for mixtures of Bernoullis:

a) $\cancel{p(x_i|\pi,\mu) = \sum_{i=1}^{N} \pi}$

$$p(x|\pi,\mu) = \sum_{k=1}^{K} \pi_k . p(x|\mu_k)$$

$$p(X|\pi,\mu) = \prod_{i=1}^{N} p(x_i|\pi,\mu)$$

$$\Rightarrow \ln p(X|\mu,\pi) = \ln\left(\prod_{i=1}^{N} p(x_i|\pi,\mu)\right)$$

$$= \sum_{i=1}^{N} \ln\left(\sum_{k=1}^{K} \pi_k . p(x_i|\mu_k)\right)$$

we have: $p(x|z,\mu) = \prod_{k=1}^{K} p(x|\mu_k)^{z_k}$

and $p(z|\pi) = \prod_{k=1}^{K} \pi_k^{z_k}$

So complete data log-likelihood will be:

$$\ln p(X,Z|\mu,\pi) = \sum_{i=1}^{N} \sum_{k=1}^{K} z_{nk} \left(\ln \pi_k + \sum_{j=1}^{D} (x_{ij} \ln \mu_{kj} + (1-x_{ij}) . \ln(1-\mu_{kj})\right)$$

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b)

$$p(z_i \mid x_i, \mu, \pi) = \frac{p(z_i, x_i \mid \mu, \pi)}{p(x_i \mid \mu, \pi)}$$

$$= \frac{p(x_i \mid z_i, \mu) \cdot p(z_i \mid \pi)}{p(x_i \mid \mu, \pi)}$$

$$= \frac{p(x_i \mid \mu_k) \cdot \pi_k}{\sum_{k=1}^{K} \pi_k \cdot p(x_i \mid \mu_k)} = \gamma(z_{ik})$$

c) We have $N_k = \sum_{i=1}^{N} \gamma(z_{ik})$

Derive:

$$\Rightarrow \mu_k = \frac{1}{N_k} \sum_{i=1}^{N} \gamma(z_{ik}) \cdot x_i$$

and $\pi_k = \frac{N_k}{N}$