CSED515 Machine Learning - Assignment 1

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Due date: 23:59pm, March 11th, 2023

Remark

Assignment Submission. All students must submit their homework via PLMS. Submit your answer on PLMS in a single PDF file named with `your_student_id.pdf`. You can scan your hand-written answers or write your answer with a tablet (If you are ambitious enough, try to use LaTeX to write your answers).

Late Homework Policy. We do not allow late submission. If you have any question regarding this, please send an email to the lecturer.

Honor Code. We strongly encourage students to form study groups. Students may discuss and work on homework problems in groups. However, each student must write down their solutions independently, i.e., each student must understand the solution well enough in order to reconstruct it by him/herself. Using code or solutions obtained from the web (GitHub/Google/ etc.) is considered an honor code violation. We check all the submissions for plagiarism. We take the honor code very seriously and expect students to do the same.

Importance Notice Note that the first assignment aims to identify your background on linear algebra, probability and statistics, and vector calculus. Show your work with your answers. You will get zero score if there is no proper explanation. You can use any external resources to solve the problems, however, please make sure that you have understood the details of what you have written. If you think you are not ready to solve these questions, please consider to take preliminary classes before taking this class. The list of preliminary courses are provided during the second lecture.

1. **Variance of sum [10pt].** Show that the variance of a sum is $\mathrm{var}[X+Y] = \mathrm{var}[X] + \mathrm{var}[Y] + 2\,\mathrm{cov}[X,Y]$, where $\mathrm{cov}[X,Y]$ is the covariance between random variable X and Y.

2. **Independence [10pt].** Suppose that two variables X and Y are statistically independent. Show that

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

3. **Conditional independence [10pt].** Is the following property true? Prove or disprove.

$$(A \perp B|D) \wedge (A \perp C|B,D) \Rightarrow (A \perp B,C|D)$$

4. **Mean and variance [10pt].** Let X be a gaussian distributed random variable with mean μ and variance σ^2, i.e., $X \sim \mathcal{N}(\mu, \sigma^2)$. If we transform X to Y with $Y = aX + b$, then what will be the mean and variance of the transformed random variable Y?

5. **Conditional expectation [10pt].** Consider two random variables x, y with joint distribution $p(x, y)$. Show that

$$\mathbb{E}_X[x] = \mathbb{E}_Y[\mathbb{E}_X[x|y]]$$

Here, $\mathbb{E}_X[x|y]$ denotes the expected value of x under the conditional distribution $p(x|y)$, and $\mathbb{E}_X[x]$ and $\mathbb{E}_Y[y]$ denote the expected value of x and y under the marginal distribution $p(x)$ and $p(y)$, respectively.

6. **Derivatives [10pt].** Compute the derivative $f'(x)$ of the function:

$$f(x) = \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

where $\mu, \sigma \in \mathbb{R}$ are constants.

7. **Chain rules [10pt].** Given three functions $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}^D \to \mathbb{R}$, and $h : \mathbb{R}^D \to \mathbb{R}^D$

$$\begin{aligned} f(z) &= \exp\left(-\frac{1}{2}z\right) \\ z = g(\mathbf{y}) &= \mathbf{y}^\top \mathbf{S}^{-1} \mathbf{y} \\ \mathbf{y} = h(\mathbf{x}) &= \mathbf{x} - \boldsymbol{\mu} \end{aligned}$$

where $\mathbf{x}, \boldsymbol{\mu} \in \mathbb{R}^D, \mathbf{S} \in \mathbb{R}^{D \times D}$, answer the following questions.

(a) Compute partial derivative $\frac{\partial f}{\partial z}$ and provide the dimension of the derivative.

(b) Compute partial derivative $\frac{\partial g}{\partial \boldsymbol{y}}$ and provide the dimension of the derivative.

(c) Compute partial derivative $\frac{\partial h}{\partial \boldsymbol{x}}$ and provide the dimension of the derivative.

(d) Compute $\frac{\mathrm{d}f}{\mathrm{d}\boldsymbol{x}}$ using the chain rule.

8. **Inner product space and induced norm [10pt].** Compute the distance between

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

in the inner product space defined with the following inner product:

(a) $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}^\top \boldsymbol{y}$

(b) $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{y}, \quad \boldsymbol{A} := \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

9. **Eigenvalues and eigenvectors [10pt].** Which of the following vectors are eigenvectors of the matrix and what is the corresponding eigenvalue?:

$$\begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 4 \\ 1 & 0 & 3 \end{bmatrix}$$

(a) $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$

10. **Eigenvalues and eigenvectors [10pt].** Find the eigenvalues of $\begin{bmatrix} 1 & \lambda \\ 2 & 1 \end{bmatrix}$ in terms of λ. Can you find an eigenvector corresponding to each of the eigenvalues?