Out-of-time-order correlator in the XX chain with a staggered field

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THESIS

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Abstract

While the out-of-time-order correlator (OTOC) is mostly used as an diagnostic for quantum chaos, it can still be studied in an integrable model to analyze the operator spreading and information scrambling aspects. This thesis studies the integrable model of the XX spin chain in a staggered magnetic field, under a quench setting. The OTOC is calculated for both the total and local staggered magnetization. The global OTOC is seen to grow in time (t) with power t^2 or t^4 at first and then reach a plateau at late time with power $t^{-1/2}$. The plateau's height cannot be predicted by the generalized Gibbs ensemble. On the other hand, the local OTOC at all distances decays to zero with power law of t^{-1}, which shows no sign of information scrambling. The local correlation propagates along the spin chain with a constant velocity which equals the maximum quasi-particle velocities.

Contents

Chapter 1

Introduction

Given a system evolving in time with hamiltonian H, the out-of-time-order correlator (OTOC) between two hermitian operators A and B is defined as

$$C_{A,B}(t) = \langle [A_t, B]^\dagger [A_t, B] \rangle = \langle B A_t A_t B \rangle + \langle A_t B B A_t \rangle - 2\,\mathrm{Re}\langle A_t B A_t B \rangle. \tag{1.1}$$

where $\langle \cdots \rangle$ is the expectation value in either a pure state or an ensemble, and $A_t = e^{i\hat{H}t} A e^{-i\hat{H}t}$ is the time-evolved version of A. The name OTOC comes from the term $\mathcal{F}_{A,B}(t) = \langle A_t B A_t B \rangle$, in which the operators are not ordered in time [1] [2]. Being the expectation value of the hermitian operators $i[A_t, B]$ squared, $C_{A,B}(t)$ is therefore a non-negative real number.

The OTOC has received considerable interest and was measured experimentally [1–3]. Firstly, its dynamics was proposed as a diagnostic for quantum chaos. If A^x and B^0 are local operators separated in space, $C_{A,B}(x,t)$ determines how the perturbation by B^0 will affect the measurement of A^x after the time t. For a strongly chaotic system, one expects the OTOC to grow large with time (exponential at early-time) for most choices of A and B [4, 5]. In another perspective, the OTOC is used to study how quantum information spreads and is effectively lost in the process of thermalization [1]. For example, if the system fails to thermalize, as in many-body localization, the information spreads logarithmically slowly in time [6].

The OTOC has been studied in spin chains. These systems usually goes with bounded local observables, and therefore may not provide enough space the exponential growth of the OTOC. *Kukuljan et al,*, instead of the OTOC between local operators, proposed the OTOC density of nonlocal extensive operators as the proper measure of "weak chaos" in these systems [7]. They studied the kicked-Ising-model, in which the OTOC of an extensive quantity grows linearly with time. Later, *Lin and Motrunich* studied the OTOC of local operators, not to quantify chaos, but as a signature for operator growth and information scrambling, in the integrable transverse-field Ising model.

This thesis will calculate the OTOC in the integrable XX spin chain (hard-core bosons) with a staggered magnetic field V, in a out-of equilibrium state. One property of the model is that it turns from a one-band to a two-band free-fermion system when V is switched on. We will let the system undergo a quench of V and study the OTOC dynamics of both the total and local staggered magnetization.

Furthermore, since the generalized Gibbs ensemble (GGE) has been shown to correctly predict the one-body observables and two-spin correlations in spin chains [8], it is worth checking whether the GGE can estimate the stationary value of the OTOC, which is a four-body correlation. The thesis will have this comparison done for the total magnetization's OTOC, which will be shown to have a non-zero stationary value.

[1] Most authors call $\mathcal{F}$ the actual OTOC, instead of C. It is noted that $C_{A,B}(t) = 2 - 2\,\mathrm{Re}\,\mathcal{F}_{A,B}(t)$ if A and B are both unitary and hermitian

[2] The term $\langle B A_t A_t B \rangle$ can be measured as the expectation value of A_t^2 in the state being acted on by B. Therefore it is not really out of time order.

Outline

The structure of the thesis is as follows. Chapter 2 introduces the hamiltonian and its diagonalization. Therein, it describes how the system goes under a quench of the external field, and how the generalized Gibbs ensemble for the post-quench state is constructed. Chapter 3 calculates the OTOC density of the global staggered magnetization, looking at its growth and saturation. The generalized Gibbs ensemble is used to estimate also the stationary value of the OTOC and then compare it with the exact result. As the last major part, Chapter 4 calculates the OTOC of the local staggered magnetization, looking at how it propagates in space and decays over time. The light-cone effect is observed, showing how the information spread through the spin chain.

Convention remarks: the reduced Plank constant $\hbar$ is suppressed in all the equations in this thesis. The letter h is used for the coupling strength between neighbor spins. All the plots are of intensive quantities and visually converged in the thermodynamic limit.

Chapter 2

Model: The XX chain in a staggered field

The XX spin chain that we are going to studied can be mapped into the 1D lattice of hard-core bosons (Tonks–Girardeau gas [9]). It has been realized experimentally and well studied theoretically, for example in Refs. [10, 11].

2.1 The 1D XX model in a staggered field

The XX model's hamiltonian is defined as

$$H = \sum_{n=1}^{L} 2h \left[S_n^x S_{n+1}^x + S_n^y S_{n+1}^y \right] - V_n S_n^z, \tag{2.1}$$

where S_n^i is the spin operators, $2h$ the hopping strength and V_n the magnetic fields at each site. The spin operators at different sites commute and at a single site satisfy $\left[S_n^i, S_n^j\right] = i\varepsilon^{ijk} S_n^k$ and $\left\{S_n^i, S_n^j\right\} = \frac{1}{2}\delta^{ij}\mathbb{1}$. The system is defined to be on a ring geometry, with $S_{L+1}^i \equiv S_1^i$.

By defining $b_n^\dagger \equiv S_n^x + iS_n^y$, the hamiltonian is rewritten as

$$\begin{aligned} H &= \sum_{n=1}^{L} 2h \left[S_n^x S_{n+1}^x + S_n^y S_{n+1}^y \right] + iV_n [S_n^x, S_n^y] \\ &= \sum_{n=1}^{L} 2h \left[\frac{b_n^\dagger + b_n}{2} \frac{b_{n+1}^\dagger + b_{n+1}}{2} + \frac{b_n^\dagger - b_n}{2i} \frac{b_{n+1}^\dagger - b_{n+1}}{2i} \right] + iV_n \left[\frac{b_n^\dagger + b_n}{2}, \frac{b_n^\dagger - b_n}{2i} \right] \\ &= \sum_{n=1}^{L} h \left[b_n^\dagger b_{n+1} + b_{n+1}^\dagger b_n \right] + V_n [b_n^\dagger b_n - \frac{1}{2}]. \end{aligned} \tag{2.2}$$

It is straight forward to show that $b_n^\dagger$ and b_n at different sites also commute, while at a single site they behave as fermionic operators: $\{b_n, b_n^\dagger\} = \mathbb{1}$ and $\{b_n, b_n\} = 0$. They are thus called hardcore bosons' operators.

We will study the system with a staggered field $V_n = (-1)^n V$.

$$H = \sum_{n=1}^{L} h \left[b_n^\dagger b_{n+1} + h.c. \right] + V(-1)^n b_n^\dagger b_n. \tag{2.3}$$

It is at this point that L is assumed to be an even integer, so that the hamiltonian has translational symmetry with the period of 2 lattice points.

2.2 Diagonalization

2.2.1 Jordan-Wigner transformation

In order to diagonalize the hamiltonian in eq.(2.3), the Jordan-Wigner transformation is performed

$$a_i^\dagger \equiv \exp\left\{ i\pi \sum_{j=1}^{i-1} b_j^\dagger b_j \right\} b_i^\dagger = \prod_{j=1}^{i-1} e^{i\pi b_j^\dagger b_j} b_i^\dagger, \quad i \in \{1, 2, ..., L\} \tag{2.4}$$

Using the properties of these operators in appendix A, the hamiltonian is rewritten as

$$H = \sum_{i=1}^{L-1} h \left[a_i^\dagger a_{i+1} + h.c. \right] - he^{i\pi\mathcal{N}} \left[a_L^\dagger a_1 + h.c. \right] + \sum_{i=1}^{L} V(-1)^i a_i^\dagger a_i, \tag{2.5}$$

where $\mathcal{N} \equiv \sum_{i=1}^{L} n_i = \sum_{i=1}^{L} a_i^\dagger a_i = \sum_{i=1}^{L} b_i^\dagger b_i$.

2.2.2 The odd and even sectors of $\mathcal{N}$

Since $[H, \mathcal{N}] = 0$, H and $\mathcal{N}$ can be simultaneously diagonalized. The eigenvectors of $\mathcal{N}$ are constructed by applying products of creation operators $a_i^\dagger$ to the vacuum, and the eigenvalues of $\mathcal{N}$ are thus integers in the range $[0, L]$. Within $\mathcal{N}$'s subspace of odd and even eigenvalues, the hamiltonian has simpler forms

$$H^\pm = \sum_{i=1}^{L-1} h \left[a_i^\dagger a_{i+1} + h.c. \right] \mp h \left[a_L^\dagger a_1 + h.c. \right] + \sum_{i=1}^{L} V(-1)^i a_i^\dagger a_i, \tag{2.6}$$

where (below) sign denotes $\mathcal{N}$'s even (odd) subspace. One way of solving these hamiltonians is sketched in the footnote. [1] The following way use the perspective of an infinite chain hamiltonian.

Although the even and odd hamiltonians $H^\pm$ are contrastingly different at the term $a_L^\dagger a_1$, they can be recast to the same form by the following trick. Considering the periodic infinite chain Hamiltonian

$$\mathcal{H} = \sum_{i\in\mathbb{Z}} h \left[a_i^\dagger a_{i+1} + h.c. \right] + \sum_{i\in\mathbb{Z}} V(-1)^i a_i^\dagger a_i. \tag{2.7}$$

If the boundary condition $a_{n+L} = a_n$ is imposed, $\mathcal{H}$ is equivalent to $H^{(-)}$. On the other hand, if $a_{n+L} = -a_n$, $\mathcal{H}$ is equivalent to $H^{(+)}$. In either case, the translational symmetry group of the system is the same, and partial diagonalization is done via the Fourier transform

$$f_k^\dagger \equiv \frac{1}{\sqrt{L}} \sum_{n=1}^{L} e^{ikn} a_n^\dagger \quad \Rightarrow \quad a_n^\dagger = \frac{1}{\sqrt{L}} \sum_{k\in\mathcal{K}_\varphi} e^{-ikn} f_k^\dagger \tag{2.8}$$

$$\text{with} \quad \sum_{k\in\mathcal{K}_\varphi} e^{ik(n-n')} = \delta_{n,n'}, \qquad \sum_{n=1}^{L} e^{i(k-k')n} = \delta_{k,k'}, \tag{2.9}$$

where $k \in \mathcal{K}_\varphi \equiv \left\{ \frac{2\pi}{L}(m+\varphi) \middle| m \in Z_L \right\}$ and $\varphi \in R$ is arbitrary. It is the boundary condition that fixes the

[1] While $H^{(-)}$ can be solved by the standard Fourier transform $f_k = 1/\sqrt{L}\sum_{n=1}^{L} e^{i2\pi/Lkn} a_n$ with $k \in Z_L$, solving $H^{(+)}$ is more subtle with the minus sign at the end of the chain. One way of resolving the minus sign in $H^{(+)}$ is applying the mapping $a_n^\dagger \mapsto e^{i\frac{2\pi}{L}n} a_n^\dagger$to accumulate a phase factor along the chain, after which the normal Fourier transform can be used. This is equivalent to performing a shifted Fourier transform in k-space, $f_k = 1/\sqrt{L}\sum_{n=1}^{L} e^{i\frac{2\pi}{L}(k+1/2)n} a_n$.

freedom of φ.

$$\begin{cases} a^\dagger_{n+L} = a^\dagger_n \Leftrightarrow \sum_{k\in\mathcal{K}_\varphi}^{L} e^{-ik(n+L)} f^\dagger_k = \sum_{k\in\mathcal{K}_\varphi}^{L} e^{-ikn} f^\dagger_k \Rightarrow e^{ikL} = 1 \Rightarrow k = (m+\frac{1}{2})\frac{2\pi}{L} \in \mathcal{K}_0. \\ a^\dagger_{n+L} = -a^\dagger_n \Leftrightarrow \sum_{k\in\mathcal{K}_\varphi}^{L} e^{-ik(n+L)} f^\dagger_k = -\sum_{k\in\mathcal{K}_\varphi}^{L} e^{-ikn} f^\dagger_k \Rightarrow e^{ikL} = -1 \Rightarrow k = m\frac{2\pi}{L} \in \mathcal{K}_{\frac{1}{2}}. \end{cases} \tag{2.10}$$

The hamiltonians in these two sectors are therefore rewritten as

$$\begin{aligned} H^{\pm} &= \sum_{n=1}^{L} h\left[a^\dagger_n a_{n+1} + h.c.\right] + V(-1)^n a^\dagger_n a_n, \qquad \text{with } a_L = \mp a_1 \\ &= \sum_{n=1}^{L}\sum_{k,k'}^{\mathcal{K}^\pm} h\left[e^{-ikn+ik'(n+1)} f^\dagger_k f_{k'} + h.c.\right] + V e^{i\pi n} e^{-ikn+ik'n} f^\dagger_k f_{k'} \\ &= \sum_{k,k'}^{\mathcal{K}^\pm} h\left[e^{ik'}\delta_{k,k'} f^\dagger_k f_{k'} + h.c.\right] + V\delta_{k,k'+\pi} f^\dagger_k f_{k'} \\ &= \sum_{k}^{\mathcal{K}^\pm} 2h\cos(k) f^\dagger_k f_k + V f^\dagger_k f_{k-\pi}, \end{aligned} \tag{2.11}$$

where $\mathcal{K}^+ = \left\{\left(m+\frac{1}{2}\right)\frac{2\pi}{L}\right\}, \mathcal{K}^- = \left\{m\frac{2\pi}{L}\right\}$, and $k\pm\pi \equiv k\pm\pi \mod 2\pi$. Because L is an even integer, if $k \in \mathcal{K}^\pm$ then $k\pm\pi \in \mathcal{K}^\pm$ as well, which gives the second Kronecker delta. The new operators f_k are still fermionic as the a_n operators.

Since the period of the term $\sum_n V(-1)^n a^\dagger_n a_n$, and hence for the full hamiltonian, is two lattice points, but the Fourier transform is for a period of one, the sum in eq.(2.11) has double redundancy that is reduced below.

$$\begin{aligned} H^{\pm} &= \sum_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} 2h\cos(k)\left[f^\dagger_k f_k - f^\dagger_{k-\pi} f_{k-\pi}\right] + V\left[f^\dagger_k f_{k-\pi} + f^\dagger_{k-\pi} f_k\right] \\ &= \sum_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \begin{pmatrix} f^\dagger_k \\ f^\dagger_{k-\pi} \end{pmatrix}^T \begin{pmatrix} 2h\cos(k) & V \\ V & -2h\cos(k) \end{pmatrix} \begin{pmatrix} f_k \\ f_{k-\pi} \end{pmatrix}. \end{aligned} \tag{2.12}$$

The hamiltonian is already block-diagonal in k-space, and the job left is only diagonalizing a 2×2 matrix.

$$\begin{aligned} H^{\pm} &= \sum_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \begin{pmatrix} f^\dagger_k \\ f^\dagger_{k-\pi} \end{pmatrix}^T U_k \begin{pmatrix} \varepsilon_k & 0 \\ 0 & -\varepsilon_k \end{pmatrix} U^\dagger_k \begin{pmatrix} f_k \\ f_{k-\pi} \end{pmatrix} \\ &\equiv \sum_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \begin{pmatrix} \gamma^\dagger_k \\ \gamma^\dagger_{k-\pi} \end{pmatrix}^T \begin{pmatrix} \varepsilon_k & 0 \\ 0 & -\varepsilon_k \end{pmatrix} \begin{pmatrix} \gamma_k \\ \gamma_{k-\pi} \end{pmatrix} \\ &= \sum_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \varepsilon_k\left[\gamma^\dagger_k\gamma_k - \gamma^\dagger_{k-\pi}\gamma_{k-\pi}\right] \equiv \sum_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \varepsilon_k\left[\gamma^\dagger_k\gamma_k - \eta_k\eta^\dagger_k\right] \\ &= \sum_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \varepsilon_k\left[\gamma^\dagger_k\gamma_k + \eta^\dagger_k\eta_k - 1\right], \end{aligned} \tag{2.13}$$

where the new operators $\gamma^\dagger_k$ and $\eta^\dagger_k \equiv \gamma_{k-\pi}$ are still fermionic, $\varepsilon_k \equiv \sqrt{4h^2\cos^2(k)+V^2}$ and the unitary transformation matrix is

$$\begin{aligned} U_k &\equiv \frac{1}{\sqrt{2\varepsilon_k\left[\varepsilon_k + 2h\cos(k)\right]}} \begin{pmatrix} \varepsilon_k + 2h\cos(k) & -V \\ V & \varepsilon_k + 2h\cos(k) \end{pmatrix} \\ &\equiv \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix} = \cos\theta_k \mathbf{1} - i\sin\theta_k \boldsymbol{\sigma}_y = e^{-i\theta_k \boldsymbol{\sigma}_y}, \end{aligned} \tag{2.14}$$

with the property $\cos(2\theta_k) = \dfrac{2h\cos(k)}{\varepsilon_k}$, $\sin(2\theta_k) = \dfrac{V}{\varepsilon_k}$.

In summary, the above calculations have represented the hamiltonian in terms of b_n, a_n, f_k and $\{\gamma_k, \eta_k\}$ operators

$$\begin{aligned} H &= \sum_{i=1}^{L-1} h\left[a_i^\dagger a_{i+1} + h.c.\right] - he^{i\pi\mathcal{N}}\left[a_L^\dagger a_1 + h.c.\right] + \sum_{i=1}^{L} V(-1)^i a_i^\dagger a_i \\ &\to \sum_k^{\mathcal{K}^\pm} 2h\cos(k) f_k^\dagger f_k + V f_k^\dagger f_{k-\pi} = \sum_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \varepsilon_k \left[\gamma_k^\dagger\gamma_k - \gamma_{k-\pi}^\dagger\gamma_{k-\pi}\right] \\ &= \sum_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \varepsilon_k \left[\gamma_k^\dagger\gamma_k + \eta_k^\dagger\eta_k - 1\right]. \end{aligned} \tag{2.15}$$

The number operator is also represented as

$$\begin{aligned} \mathcal{N} &= \sum_{n=1}^{L} b_n^\dagger b_n = \sum_{n=1}^{L} a_n^\dagger a_n = \sum_k^{\mathcal{K}^\pm} f_k^\dagger f_k = \sum_k^{\mathcal{K}^\pm} \gamma_k^\dagger \gamma_k = \sum_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \gamma_k^\dagger\gamma_k + \gamma_{k-\pi}^\dagger\gamma_{k-\pi} \\ &= \sum_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \gamma_k^\dagger\gamma_k + \eta_k\eta_k^\dagger = \frac{L}{2} + \sum_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \gamma_k^\dagger\gamma_k - \eta_k^\dagger\eta_k, \end{aligned} \tag{2.16}$$

where the first few identities hold because the transformations $b_n \to a_n \to f_k \to \gamma_k$ are all unitary. The basic commutators among $\gamma_k^\dagger$ and $\eta_k^\dagger \equiv \gamma_{k-\pi}$ are

$$\begin{cases} \{\gamma_{k_1}, \gamma_{k_2}\} = \{\eta_{k_1}, \eta_{k_2}\} = 0. \\ \{\gamma_{k_1}, \gamma_{k_2}^\dagger\} = \{\eta_{k_1}, \eta_{k_2}^\dagger\} = \mathbf{1}\delta_{k_1,k_2} \\ \{\gamma_{k_1}, \eta_{k_2}\} = \{\gamma_{k_1}, \eta_{k_2}^\dagger\} = 0. \end{cases} \tag{2.17}$$

2.2.3 The half-filled ground state

The diagonalization above give us the 2-band hamiltonian and a known constant of motion $\mathcal{N}$

$$H^\pm = \sum_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \varepsilon_k\left[\gamma_k^\dagger\gamma_k + \eta_k^\dagger\eta_k - 1\right], \qquad \mathcal{N} = \frac{L}{2} + \sum_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \gamma_k^\dagger\gamma_k - \eta_k^\dagger\eta_k. \tag{2.18}$$

For the states $|\psi\rangle$ in the half-filled subspace, $\mathcal{N}|\psi\rangle = \dfrac{L}{2}|\psi\rangle$. These half-filled states can be in either the odd or even subspace of $\mathcal{N}$, depending on whether L divides by 4 or not. The entailed condition on these states is

$$\left[\sum_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \gamma_k^\dagger\gamma_k - \eta_k^\dagger\eta_k\right]|\psi\rangle = 0. \tag{2.19}$$

That is, the occupation numbers of the γ- and η-band are the same.

In this half-filled subspace, since $\varepsilon_k > 0\ \forall k$ (if $V \neq 0$), the ground state $|\Omega\rangle$ is the one with zero occupation in both bands.

$$\gamma_k^\dagger |\Omega\rangle = \eta_k^\dagger |\Omega\rangle = 0, \quad \forall k \in \mathcal{K}^\pm \cap \left[-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{2.20}$$

Since the minimum excitation energy is $\min(\varepsilon_k) = |V|$, $|\Omega\rangle$ can be called an *insulating state*. The ground state energy is therefore

$$E_0 = -\sum_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \varepsilon_k \quad \xrightarrow{L\to\infty} \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{L}{2\pi} dk \sqrt{4h^2\cos^2(k) + V^2}. \tag{2.21}$$

For the special case of $V = 0$, the excitation energy can be exactly zero if $\min(\varepsilon_k) = \varepsilon_{k^*} = 0$, leading to two degenerate ground states of $|\Omega\rangle$ and $\gamma_{k^*}^\dagger \eta_{k^*}^\dagger |\Omega\rangle$. In addition, the excitation energy ϵ_k is infinitesimally small in the thermodynamic limit $L \to \infty$, and $|\Omega\rangle$ is thus called a *superfluid state* (a system of bosons with gapless excitations).

2.3 The quenching process

To analyze the out-of-equilibrium properties of a system while maintaining its analytical tractability, a simple approach is to suddenly change the hamiltonian's parameters. This is call the quenching process. This section describes the quenching of the hamiltonian

$$H(t) = \sum_{n=1}^{L} h\left[b_n^\dagger b_{n+1} + e^{-i\phi} b_{n+1}^\dagger b_n\right] + V(t)(-1)^n b_n^\dagger b_n \tag{2.22}$$

with

$$V(t) = \begin{cases} V_i, & \text{if } t < 0 \\ V_f, & \text{if } t \geq 0. \end{cases} \tag{2.23}$$

For $t < t_0$ the system is prepared in the half-filled ground state $|\Omega_i\rangle$ and hence it will evolves as

$$|\Psi(t)\rangle = e^{-i\hat{H}_f t} |\Omega_i\rangle, \qquad \text{for } t \geq 0. \tag{2.24}$$

It should then be useful to express $|\Omega_i\rangle$ in terms of $|\Omega_f\rangle$, the half-filled ground state of H_f, to analyze the post-quench dynamics. Because of the condition (2.19) for the half-filled states, and the fact that changing $V(t)$ does not mix the Fourier modes at different k-points (see eq. 2.12), the general relation between $|\Omega_i\rangle$ and $|\Omega_f\rangle$ is

$$|\Omega_i\rangle = \mathfrak{N}\left[\prod_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} (1 + g_k \gamma_{k,f}^\dagger \eta_{k,f}^\dagger)\right] |\Omega_f\rangle, \tag{2.25}$$

where $\mathfrak{N} = \prod_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} 1/\sqrt{1+|g_k|^2}$ is the normalization constant and g_k are c-numbers to be determined. The relation among the pre-quench $\{\gamma_{k,i}\eta_{k,i}^\dagger\}$ and post-quench $\{\gamma_{k,f}\eta_{k,f}^\dagger\}$ are

$$\begin{pmatrix} \gamma_k \\ \eta_k^\dagger \end{pmatrix} = \begin{pmatrix} \gamma_k \\ \gamma_{k-\pi} \end{pmatrix} = U_k^\dagger \begin{pmatrix} f_k \\ f_{k-\pi} \end{pmatrix}$$

$$
\begin{aligned}
&\Rightarrow \begin{pmatrix} \gamma_{k,i} \\ \eta^\dagger_{k,i} \end{pmatrix} = U^\dagger_{k,i} \begin{pmatrix} f_k \\ f_{k-\pi} \end{pmatrix} = U^\dagger_{k,i} U_{k,f} \begin{pmatrix} \gamma_{k,f} \\ \eta^\dagger_{k,f} \end{pmatrix} = e^{i\theta_{k,i}\boldsymbol{\sigma}_y} e^{-i\theta_{k,f}\boldsymbol{\sigma}_y} \begin{pmatrix} \gamma_{k,f} \\ \eta^\dagger_{k,f} \end{pmatrix} \\
&\Rightarrow \begin{pmatrix} \gamma_{k,i} \\ \eta^\dagger_{k,i} \end{pmatrix} = e^{-i\Delta\theta_k \boldsymbol{\sigma}_y} \begin{pmatrix} \gamma_{k,f} \\ \eta^\dagger_{k,f} \end{pmatrix} = \begin{pmatrix} \cos\Delta\theta_k & -\sin\Delta\theta_k \\ \sin\Delta\theta_k & \cos\Delta\theta_k \end{pmatrix} \begin{pmatrix} \gamma_{k,f} \\ \eta^\dagger_{k,f} \end{pmatrix},
\end{aligned} \tag{2.26}
$$

where $\Delta\theta_k \equiv \theta_{k,f} - \theta_{k,i}$. Then the defining property gives

$$
\begin{aligned}
&\gamma_{k,i} |\Omega_i\rangle = \eta_{k,i} |\Omega_i\rangle = 0 \\
\Leftrightarrow &\begin{cases} \left[\cos(\Delta\theta_k)\gamma_{k,f} - \sin(\Delta\theta_k)\eta^\dagger_{k,f}\right] |\Omega_i\rangle = 0 \\ \left[\sin(\Delta\theta_k)\gamma^\dagger_{k,f} + \cos(\Delta\theta_k)\eta_{k,f}\right] |\Omega_i\rangle = 0. \end{cases}
\end{aligned} \tag{2.27}
$$

On the other hand, the general relation (2.25) gives

$$
\begin{cases}
\gamma_{k,f} |\Omega_i\rangle &= \gamma_{k,f} \left[\prod_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} (1 + g_k \gamma^\dagger_{k,f}\eta^\dagger_{k,f})\right] |\Omega_f\rangle = \left[\gamma_{k,f}, \prod_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} (1 + g_k \gamma^\dagger_{k,f}\eta^\dagger_{k,f})\right] |\Omega_f\rangle = g_k \eta^\dagger_{k,f} |\Omega_i\rangle \\
\eta_{k,f} |\Omega_i\rangle &= -g_k \gamma^\dagger_{k,f} |\Omega_i\rangle \qquad \text{(similarly)}
\end{cases}
$$

because i) $\gamma_{k,f} |\Omega_f\rangle = 0$, ii) the factors in the product are commuting, iii) $[\gamma_k, 1 + g_k\gamma^\dagger_k\eta^\dagger_k], = g_k\eta^\dagger_k$, and iv) $g_k\eta^\dagger_k |\Omega_f\rangle = g_k\eta^\dagger_k\, (1 + g_k\gamma^\dagger_{k,f}\eta^\dagger_{k,f}) |\Omega_f\rangle$. From these, the values of g_k are determined as

$$
[g_k \cos(\Delta\theta_k) - \sin(\Delta\theta_k)]\, \eta^\dagger_{k,f} |\Omega_i\rangle = 0 \quad \Leftrightarrow \quad g_k = \tan(\Delta\theta_k). \tag{2.28}
$$

Therefore, the post-quench state is

$$
|\Omega_i\rangle = \mathfrak{N} \prod_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \left[1 + \tan(\Delta\theta_k)\gamma^\dagger_{k,f}\eta^\dagger_{k,f}\right] |\Omega_f\rangle = \prod_{k\in[-\frac{\pi}{2},\frac{\pi}{2})}^{\mathcal{K}^\pm} \left[\cos(\Delta\theta_k) + \sin(\Delta\theta_k)\gamma^\dagger_{k,f}\eta^\dagger_{k,f}\right] |\Omega_f\rangle. \tag{2.29}
$$

Notation remarks

From now on, when dealing with the post-quench dynamics, only the operators $\gamma^\dagger_{k,f}, \eta^\dagger_{k,f}$ will be used. Therefore, the subscript f will be suppressed and implicitly understood.

The k-space region $\mathcal{K}^\pm \cap [-\frac{\pi}{2}, \frac{\pi}{2})$ will be written briefly as BZ. Furthermore, the figures will be plotted with only the even sectors of $\mathcal{N}$, i.e. with $\mathcal{K}^+$.

2.4 The generalized Gibbs ensemble (GGE)

The generalized Gibbs ensemble (GGE) is used to make the unbiased estimate of various observables based on the information of a certain number of constants of motion $\{\hat{\mathcal{I}}_n\}$. The GGE is constructed so as to maximize the Shannon entropy $S = -\operatorname{Tr}[\hat\rho \ln \hat\rho]$ with the constraint set by the expectation values $\{\langle\hat{\mathcal{I}}_n\rangle\}$ [11, 12]. The resulting density matrix is

$$
\hat\rho_{\text{GGE}} = \frac{1}{Z_{\text{GGE}}} e^{-\sum_n \lambda_n \hat{\mathcal{I}}_n}, \tag{2.30}
$$

$$
\text{with} \qquad \operatorname{Tr}\Big[\hat\rho_{\text{GGE}}\Big] = 1 \qquad \text{and} \qquad \operatorname{Tr}\Big[\hat\rho_{\text{GGE}}\hat{\mathcal{I}}_n\Big] = \mathcal{I}^0_n \qquad \forall n. \tag{2.31}
$$

The coefficients Z_{GGE} and λ_n are determined from the last two conditions.

For free field lattice models, the GGE are usually constructed with the conserved mode occupation numbers

[8, 13]. For the model in this project, they are $\{\hat{n}_{k,\gamma}, \hat{n}_{k,\eta}\} \equiv \{\gamma_k^\dagger\gamma_k, \eta_k^\dagger\eta_k\}$. The corresponding GGE is thus

$$\hat{\rho}_{\mathrm{GGE}} = \frac{1}{Z_{\mathrm{GGE}}}\exp\left(-\sum_k^{\mathrm{BZ}} \mu_{k,\gamma}\hat{n}_{k,\gamma} + \mu_{k,\eta}\hat{n}_{k,\eta}\right). \tag{2.32}$$

The constants μ_k are determined from the constraints as follows, with the calculation straight-forwardly done in the eigen-basis of the mode occupation operators.

$$Z_{\mathrm{GGE}} = \mathrm{Tr}\left[\exp\left(-\sum_k^{\mathrm{BZ}} \mu_{k,\gamma}\hat{n}_{k,\gamma} + \mu_{k,\eta}\hat{n}_{k,\eta}\right)\right] = \prod_k^{\mathrm{BZ}}(1+e^{-\mu_{k,\gamma}})(1+e^{-\mu_{k,\eta}}), \tag{2.33}$$

$$\mathrm{Tr}\Big[\hat{\rho}_{\mathrm{GGE}}\hat{n}_{k,\gamma}\Big] = \Big\langle \hat{n}_{k,\gamma}\Big\rangle \Leftrightarrow \frac{e^{-\mu_{k,\gamma}}}{1+e^{-\mu_{k,\gamma}}} = \langle\Omega_i|\,\gamma_k^\dagger\gamma_k\,|\Omega_i\rangle = \sin^2(\Delta\theta_k), \tag{2.34}$$

$$\mathrm{Tr}\Big[\hat{\rho}_{\mathrm{GGE}}\hat{n}_{k,\eta}\Big] = \Big\langle \hat{n}_{k,\eta}\Big\rangle \Leftrightarrow \frac{e^{-\mu_{k,\eta}}}{1+e^{-\mu_{k,\eta}}} = \langle\Omega_i|\,\eta_k^\dagger\eta_k\,|\Omega_i\rangle = \sin^2(\Delta\theta_k). \tag{2.35}$$

Therefore

$$\mu_{k,\gamma} = \mu_{k,\eta} = -\ln[\tan^2(\Delta\theta_k)]. \tag{2.36}$$

With the GGE constructed, it can be used to calculate the various stationary observables and compare with the result in pure states.

Chapter 3

The OTOC of the global staggered magnetization

This chapter studies the out of time order correlator (OTOC) of the total staggered magnetization $M(t)$, which is defined as

$$C(t) \equiv \left\langle [M(t), M(0)]^\dagger [M(t), M(0)] \right\rangle = -\left\langle [M(t), M(0)]^2 \right\rangle \tag{3.1}$$

where $\langle \cdots \rangle$ is the expectation value either in a specific state or in a statistical ensemble. In particular cases, in order to extract the interesting physics, $C(t)$ is substracted by the correlation's modulus squared [7, 15].

$$\mathcal{C}(t) = -\left[\left\langle [M(t), M(0)]^2 \right\rangle - \langle [M(t), M(0)] \rangle^2 \right] = C(t) + \chi^2(t) \tag{3.2}$$

where $\chi(t) \equiv \langle [M(t), M(0)] \rangle_i$ is the correlation function. After divided by the system size, $\mathcal{C}(t)$ becomes an intensive quantitive and is called the *OTOC density*, as in Ref. [7].

With the available expression for the post-quench magnetization from the previous chapter, the OTOC's stationary value in the post-quench state and also in the corresponding generalized Gibbs ensemble (GGE) are both calculated and compare with each other.

3.1 The observable: staggered magnetization

The total staggered magnetization is defined to be

$$\begin{aligned} M &:= -\sum_{n=1}^{L} (-1)^n \left(S_n^z + \frac{1}{2} \right) = \sum_{n=1}^{L} (-1)^n b_n^\dagger b_n = \sum_{n=1}^{L} (-1)^n a_n^\dagger a_n = \sum_{k}^{2\mathrm{BZ}} f_k^\dagger f_{k-\pi} \\ &= \sum_{k}^{\mathrm{BZ}} f_k^\dagger f_{k-\pi} + f_{k-\pi}^\dagger f_k = \sum_{k \in [-\frac{\pi}{2}, \frac{\pi}{2})}^{\mathcal{K}^\pm} \begin{pmatrix} f_k^\dagger \\ f_{k-\pi}^\dagger \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_k \\ f_{k-\pi} \end{pmatrix}. \end{aligned} \tag{3.3}$$

which in terms of the post-quench free fields becomes (using the transformation (2.14))

$$M = \sum_{k}^{\mathrm{BZ}} \begin{pmatrix} \gamma_k^\dagger \\ \eta_k \end{pmatrix}^T \begin{pmatrix} \sin 2\theta_k & \cos 2\theta_k \\ \cos 2\theta_k & -\sin 2\theta_k \end{pmatrix} \begin{pmatrix} \gamma_k \\ \eta_k^\dagger \end{pmatrix}. \tag{3.4}$$

The time evolution of $M(t)$for $t \geq 0$ is (with the f-subscript implied)

$$M(t) \equiv e^{i\hat{H}t} M e^{-i\hat{H}t} \tag{3.5}$$

is calculated below. Firstly, appplying the Baker–Campbell–Hausdorff formula gives

$$e^{i\hat{H}t}\mathcal{O}e^{-i\hat{H}t} = \sum_{n=0}^{\infty} \frac{\left[i\hat{H}t, \cdot\right]^n}{n!}\mathcal{O} \Rightarrow \begin{cases} e^{i\hat{H}t}\gamma_k^\dagger e^{-i\hat{H}t} & = \sum_{n=0}^{\infty} \frac{\left[i\hat{H}t, \cdot\right]^n}{n!}\gamma_k^\dagger = \sum_{n=0}^{\infty} \frac{(i\varepsilon_k t)^n}{n!}\gamma_k^\dagger = e^{i\varepsilon_k t}\gamma_k^\dagger \\ e^{i\hat{H}t}\eta_k^\dagger e^{-i\hat{H}t} & = \sum_{n=0}^{\infty} \frac{\left[i\hat{H}t, \cdot\right]^n}{n!}\gamma_k^\dagger = \sum_{n=0}^{\infty} \frac{(i\varepsilon_k t)^n}{n!}\eta_k^\dagger = e^{i\varepsilon_k t}\eta_k^\dagger \end{cases} \tag{3.6}$$

Therefore

$$M(t) = \sum_k^{\mathrm{BZ}} \begin{pmatrix} \gamma_k^\dagger \\ \eta_k \end{pmatrix}^T \begin{pmatrix} \sin 2\theta_k & e^{2i\varepsilon_k t}\cos 2\theta_k \\ e^{-2i\varepsilon_k t}\cos 2\theta_k & -\sin 2\theta_k \end{pmatrix} \begin{pmatrix} \gamma_k \\ \eta_k^\dagger \end{pmatrix} \tag{3.7}$$

$$= \sum_k^{\mathrm{BZ}} \sin(2\theta_k)\left[\gamma_k^\dagger\gamma_k - \eta_k\eta_k^\dagger\right] + \cos(2\theta_k)\left[e^{2i\varepsilon_k t}\gamma_k^\dagger\eta_k^\dagger + h.c.\right]. \tag{3.8}$$

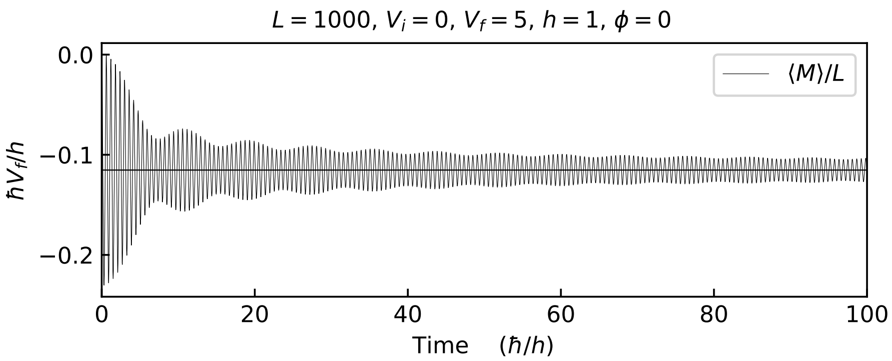
Secondly, with the post-quench expectation values calculated in appendix B,

$$\begin{cases} \langle\gamma_k^\dagger\gamma_k\rangle = \langle\eta_k^\dagger\eta_k\rangle = \sin^2(\Delta\theta_k) \\ \langle\gamma_k^\dagger\eta_k^\dagger\rangle = \sin(\Delta\theta_k)\cos(\Delta\theta_k), \end{cases}$$

the result for $M(t)$ is finally found

$$\langle M(t)\rangle = \sum_k^{\mathrm{BZ}} -\sin(2\theta_k)\cos(2\Delta\theta_k) + \cos(2\theta_k)\sin(2\Delta\theta_k)\cos(2\varepsilon_k t). \tag{3.9}$$

It is seen that the total magnetization reaches a stationary values after a long time, see the figure below.



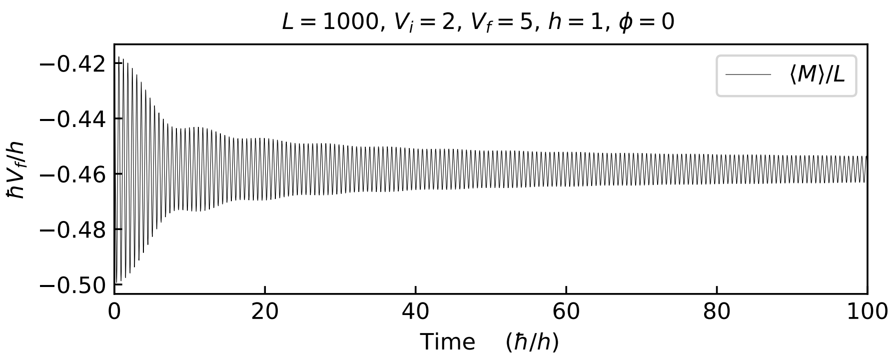


Figure 3.1: The expectation value of total staggered magnetization (density) after quenches, eq. (3.9).

3.2 The OTOC of $\hat{M}$ in the pure in the post-quench state

This section devotes to the calculation of the OTOC defined in eq. (3.2), which can be written in another way thanks to the hermiticity of $M(t)$

$$\mathcal{C}(t) = \left\langle [M(t), M(0)]^\dagger [M(t), M(0)] \right\rangle - \langle [M(t), M(0)] \rangle^* \langle [M(t), M(0)] \rangle . \tag{3.10}$$

This quantity can be seen as the centralized second moment of the commutator $i[M(t), M(0)]$.

Operator expression of $[M(t), M(0)]$

In order to make to calculation more readable, the calculated expression for $M(t)$ is schematically written in terms of the 2×2 matrix $\mathfrak{M}_k(t)$ whose elements are read from eq. (3.7).

$$M(t) = \sum_k^{\mathrm{BZ}} \begin{pmatrix} \gamma_k^\dagger \\ \eta_k \end{pmatrix}^T \begin{bmatrix} \mathfrak{M}_k(t) \end{bmatrix} \begin{pmatrix} \gamma_k \\ \eta_k^\dagger \end{pmatrix} \tag{3.11}$$

Although the commutator $[M(t), M(0)]$ is calculated terms by terms in a lengthy manner, its result appears in a compact form that is proved in appendix D. Using the formula (D.7), one gets

$$\begin{aligned}
[M(t), M(0)] &= \sum_k^{\mathrm{BZ}} \begin{pmatrix} \gamma_k^\dagger \\ \eta_k \end{pmatrix}^T \begin{bmatrix} \mathfrak{M}_k(t)\mathfrak{M}_k(0) - \mathfrak{M}_k(0)\mathfrak{M}_k(t) \end{bmatrix} \begin{pmatrix} \gamma_k \\ \eta_k^\dagger \end{pmatrix} \\
&= \sum_k^{\mathrm{BZ}} \begin{pmatrix} \gamma_k^\dagger \\ \eta_k \end{pmatrix}^T \begin{pmatrix} 2i\sin(2\varepsilon_k t)\cos^2(2\theta_k) & (1 - e^{2i\varepsilon_k t})\sin(4\theta_k) \\ -(1 - e^{-2i\varepsilon_k t})\sin(4\theta_k) & -2i\sin(2\varepsilon_k t)\cos^2(2\theta_k) \end{pmatrix} \begin{pmatrix} \gamma_k \\ \eta_k^\dagger \end{pmatrix} \\
&\equiv \sum_k^{\mathrm{BZ}} \begin{pmatrix} \gamma_k^\dagger \\ \eta_k \end{pmatrix}^T \begin{pmatrix} A_k(t) & B_k(t) \\ C_k(t) & D_k(t) \end{pmatrix} \begin{pmatrix} \gamma_k \\ \eta_k^\dagger \end{pmatrix},
\end{aligned} \tag{3.12}$$

where, for later brevity, the time-dependent coefficients $\{A_k, B_k, C_k, D_k\}$ are just in-place defined.

Operator expression of $[M(t), M(0)]^2$

In order to take the square of the commutator $[M(t), M(0)]$, the approach taken here is to write it as

$$[M(t), M(0)] = \sum_k^{\mathrm{BZ}} \begin{pmatrix} A_k \\ B_k \\ C_k \\ D_k \end{pmatrix}^T \begin{pmatrix} \gamma_k^\dagger \gamma_k \\ \gamma_k^\dagger \eta_k^\dagger \\ \eta_k \gamma_k \\ \eta_k \eta_k^\dagger \end{pmatrix} .$$

After that, the square can be taken in a nicely structured form, in which operators and c-numbers are kept separate.

$$\begin{aligned}
&[M(t), M(0)]^\dagger [M(t), M(0)] = -[M(t), M(0)]^2 \\
&= \sum_{k_1, k_2}^{\mathrm{BZ}} \begin{pmatrix} A_{k_1} \\ B_{k_1} \\ C_{k_1} \\ D_{k_1} \end{pmatrix}^\dagger \begin{pmatrix} \gamma_{k_1}^\dagger \gamma_{k_1} \\ \eta_{k_1} \gamma_{k_1} \\ \gamma_{k_1}^\dagger \eta_{k_1}^\dagger \\ \eta_{k_1} \eta_{k_1}^\dagger \end{pmatrix} \begin{pmatrix} \gamma_{k_2}^\dagger \gamma_{k_2} \\ \gamma_{k_2}^\dagger \eta_{k_2}^\dagger \\ \eta_{k_2} \gamma_{k_2} \\ \eta_{k_2} \eta_{k_2}^\dagger \end{pmatrix}^T \begin{pmatrix} A_{k_2} \\ B_{k_2} \\ C_{k_2} \\ D_{k_2} \end{pmatrix}
\end{aligned}$$

$$
= \sum_{k_1,k_2}^{\mathrm{BZ}} \begin{pmatrix} A_{k_1} \\ B_{k_1} \\ C_{k_1} \\ D_{k_1} \end{pmatrix}^{\dagger} \begin{pmatrix} \gamma^{\dagger}_{k_1}\gamma_{k_1}\gamma^{\dagger}_{k_2}\gamma_{k_2} & \gamma^{\dagger}_{k_1}\gamma_{k_1}\gamma^{\dagger}_{k_2}\eta^{\dagger}_{k_2} & \gamma^{\dagger}_{k_1}\gamma_{k_1}\eta_{k_2}\gamma_{k_2} & \gamma^{\dagger}_{k_1}\gamma_{k_1}\eta_{k_2}\eta^{\dagger}_{k_2} \\ \eta_{k_1}\gamma_{k_1}\gamma^{\dagger}_{k_2}\gamma_{k_2} & \eta_{k_1}\gamma_{k_1}\gamma^{\dagger}_{k_2}\eta^{\dagger}_{k_2} & \eta_{k_1}\gamma_{k_1}\eta_{k_2}\gamma_{k_2} & \eta_{k_1}\gamma_{k_1}\eta_{k_2}\eta^{\dagger}_{k_2} \\ \gamma^{\dagger}_{k_1}\eta^{\dagger}_{k_1}\gamma^{\dagger}_{k_2}\gamma_{k_2} & \gamma^{\dagger}_{k_1}\eta^{\dagger}_{k_1}\gamma^{\dagger}_{k_2}\eta^{\dagger}_{k_2} & \gamma^{\dagger}_{k_1}\eta^{\dagger}_{k_1}\eta_{k_2}\gamma_{k_2} & \gamma^{\dagger}_{k_1}\eta^{\dagger}_{k_1}\eta_{k_2}\eta^{\dagger}_{k_2} \\ \eta_{k_1}\eta^{\dagger}_{k_1}\gamma^{\dagger}_{k_2}\gamma_{k_2} & \eta_{k_1}\eta^{\dagger}_{k_1}\gamma^{\dagger}_{k_2}\eta^{\dagger}_{k_2} & \eta_{k_1}\eta^{\dagger}_{k_1}\eta_{k_2}\gamma_{k_2} & \eta_{k_1}\eta^{\dagger}_{k_1}\eta_{k_2}\eta^{\dagger}_{k_2} \end{pmatrix} \begin{pmatrix} A_{k_2} \\ B_{k_2} \\ C_{k_2} \\ D_{k_2} \end{pmatrix}. \tag{3.13}
$$

Expectation value of $[M(t), M(0)]$

With the basic expectation values calculated in appendix B, one gets

$$
\chi(t) = \langle [M(t), M(0)] \rangle = \sum_{k}^{\mathrm{BZ}} \begin{pmatrix} A_k \\ B_k \\ C_k \\ D_k \end{pmatrix}^{T} \begin{pmatrix} \langle \gamma^{\dagger}_k \gamma_k \rangle \\ \langle \gamma^{\dagger}_k \eta^{\dagger}_k \rangle \\ \langle \eta_k \gamma_k \rangle \\ \langle \eta_k \eta^{\dagger}_k \rangle \end{pmatrix} = \sum_{k}^{\mathrm{BZ}} \begin{pmatrix} A_k \\ B_k \\ C_k \\ D_k \end{pmatrix}^{T} \begin{pmatrix} \sin^2(\Delta\theta_k) \\ \frac{1}{2}\sin(2\Delta\theta_k) \\ \frac{1}{2}\sin(2\Delta\theta_k) \\ \cos^2(\Delta\theta_k) \end{pmatrix}. \tag{3.14}
$$

After substituting back the values of $\{A_k, B_k, C_k, D_k\}$ from eq. (3.12), it gives

$$
\chi(t) = \sum_{k}^{\mathrm{BZ}} -2i \sin(2\varepsilon_k t) \cos(2\theta_k) \cos(2\theta_k - \Delta\theta_k), \tag{3.15}
$$

which is correctly a purely imaginary number due to the commutator $[M(t), M(0)]$ being anti-hermitian.

Expectation value of $[M(t), M(0)]^2$ and the OTOC

Thanks to the deliberate structure in eq. (3.13) and taking the square of eq. (3.14), the OTOC comes out straight-forwardly as

$$
\mathcal{C}(t) = \langle [M(t), M(0)]^{\dagger} [M(t), M(0)] \rangle - \langle [M(t), M(0)] \rangle^{*} \langle [M(t), M(0)] \rangle
$$

$$
= \sum_{k_1,k_2}^{\mathrm{BZ}} \begin{pmatrix} A_{k_1} \\ B_{k_1} \\ C_{k_1} \\ D_{k_1} \end{pmatrix}^{\dagger} \begin{pmatrix} \langle \gamma^{\dagger}_{k_1}\gamma_{k_1}\gamma^{\dagger}_{k_2}\gamma_{k_2} \rangle^c & \langle \gamma^{\dagger}_{k_1}\gamma_{k_1}\gamma^{\dagger}_{k_2}\eta^{\dagger}_{k_2} \rangle^c & \langle \gamma^{\dagger}_{k_1}\gamma_{k_1}\eta_{k_2}\gamma_{k_2} \rangle^c & \langle \gamma^{\dagger}_{k_1}\gamma_{k_1}\eta_{k_2}\eta^{\dagger}_{k_2} \rangle^c \\ \langle \eta_{k_1}\gamma_{k_1}\gamma^{\dagger}_{k_2}\gamma_{k_2} \rangle^c & \langle \eta_{k_1}\gamma_{k_1}\gamma^{\dagger}_{k_2}\eta^{\dagger}_{k_2} \rangle^c & \langle \eta_{k_1}\gamma_{k_1}\eta_{k_2}\gamma_{k_2} \rangle^c & \langle \eta_{k_1}\gamma_{k_1}\eta_{k_2}\eta^{\dagger}_{k_2} \rangle^c \\ \langle \gamma^{\dagger}_{k_1}\eta^{\dagger}_{k_1}\gamma^{\dagger}_{k_2}\gamma_{k_2} \rangle^c & \langle \gamma^{\dagger}_{k_1}\eta^{\dagger}_{k_1}\gamma^{\dagger}_{k_2}\eta^{\dagger}_{k_2} \rangle^c & \langle \gamma^{\dagger}_{k_1}\eta^{\dagger}_{k_1}\eta_{k_2}\gamma_{k_2} \rangle^c & \langle \gamma^{\dagger}_{k_1}\eta^{\dagger}_{k_1}\eta_{k_2}\eta^{\dagger}_{k_2} \rangle^c \\ \langle \eta_{k_1}\eta^{\dagger}_{k_1}\gamma^{\dagger}_{k_2}\gamma_{k_2} \rangle^c & \langle \eta_{k_1}\eta^{\dagger}_{k_1}\gamma^{\dagger}_{k_2}\eta^{\dagger}_{k_2} \rangle^c & \langle \eta_{k_1}\eta^{\dagger}_{k_1}\eta_{k_2}\gamma_{k_2} \rangle^c & \langle \eta_{k_1}\eta^{\dagger}_{k_1}\eta_{k_2}\eta^{\dagger}_{k_2} \rangle^c \end{pmatrix} \begin{pmatrix} A_{k_2} \\ B_{k_2} \\ C_{k_2} \\ D_{k_2} \end{pmatrix}. \tag{3.16}
$$

where $\langle \gamma^{\dagger}_{k_1}\gamma_{k_1}\gamma^{\dagger}_{k_2}\gamma_{k_2} \rangle^c = \langle \gamma^{\dagger}_{k_1}\gamma_{k_1}\gamma^{\dagger}_{k_2}\gamma_{k_2} \rangle - \langle \gamma^{\dagger}_{k_1}\gamma_{k_1} \rangle \langle \gamma^{\dagger}_{k_2}\gamma_{k_2} \rangle$. The expectation values are taken in the post-quench state $|\Omega_i\rangle$ in eq. (2.29). For illustration, one of these matrix elements is calculated in appendix B. After obtaining all 16 terms, one gets

$$
\mathcal{C}(t) = \sum_{k}^{\mathrm{BZ}} \begin{pmatrix} A_k \\ B_k \\ C_k \\ D_k \end{pmatrix}^{\dagger} \begin{pmatrix} \alpha_k\beta_k & \beta_k\lambda_k & -\alpha_k\lambda_k & -\alpha_k\beta_k \\ \lambda_k\beta_k & -\lambda_k^2 + \beta_k & -\lambda_k\lambda_k & -\lambda_k\beta_k \\ -\lambda_k\alpha_k & -\lambda_k\lambda_k & -\lambda_k^2 + \alpha_k & \lambda_k\alpha_k \\ -\beta_k\alpha_k & -\beta_k\lambda_k & \alpha_k\lambda_k & \beta_k\alpha_k \end{pmatrix} \begin{pmatrix} A_k \\ B_k \\ C_k \\ D_k \end{pmatrix}. \tag{3.17}
$$

The parameters used above are $\alpha_k = \sin^2(\Delta\theta_k)$, $\beta_k = \cos^2(\Delta\theta_k)$, and $\lambda_k = \frac{1}{2}\sin(2\Delta\theta_k)$. After substituting back the values of $\{A_k, B_k, C_k, D_k\}$ from eq. (3.12), it gives

$$
\mathcal{C}(t) = \sum_{k}^{\mathrm{BZ}} 4\sin^2(\varepsilon_k t)\sin^2(4\theta_k) + 4\sin^2(2\varepsilon_k t)\cos^2(2\theta_k)\sin(2\Delta\theta_k)\sin(2\Delta\theta_k - 4\theta_k) \tag{3.18}
$$

$$
= \sum_{k}^{\mathrm{BZ}} 4\sin^4(\varepsilon_{k,f} t)\sin^2(4\theta_{k,f}) + 4\sin^2(2\varepsilon_{k,f} t)\cos^2(2\theta_{k,f})\sin^2(2\theta_{k_i}). \tag{3.19}
$$

3.3 Dynamical behaviors of the OTOC

This section analyzes the temporal behavior of the OTOC's

$$\mathcal{C}(t) = \sum_{k}^{\mathrm{BZ}} 4\sin^4(\varepsilon_{k,f}t)\sin^2(4\theta_{k,f}) + 4\sin^2(2\varepsilon_{k,f}t)\cos^2(2\theta_{k,f})\sin^2(2\theta_{k_i}),$$

which is a non-negative quantity as expected.

3.3.1 The envelop dynamics of the OTOC

The OTOC dynamics has the strong oscillations with frequencies in the range $[2\varepsilon_k, 4\varepsilon_k]$, with $k \in \mathrm{BZ}$. These fast oscillation interferes to create the slower variation of the envelop, which is can be studied numerically in this section. In particular, we analyze how the OTOC's envelop's lower bound increases at short time and reaches the stationary values at long time.

Firstly, the global OTOC saturates to a constant value at late time, similarly to what observed with the integrable transverse field Ising model in Ref. [7]. However, since in that reference, the OTOC of Majorana strings can still have linear growth, our model should also be further checked with similar observables.

Secondly, for the case of quenching from the superfluid to insulating phase, i.e. $V_i = 0, V_f \neq 0$, the OTOC increases as t^4. On the other hand, for the quenching between two insulating phases, i.e. $V_i \neq 0, V_f \neq 0$, the OTOC grows as t^2 only. In both cases, the OTOC approaches the stationary value with behavior of $t^{-1/2}$.

The OTOC dynamics are plotted in figure 3.2, with sample parameters of V_i and V_f.

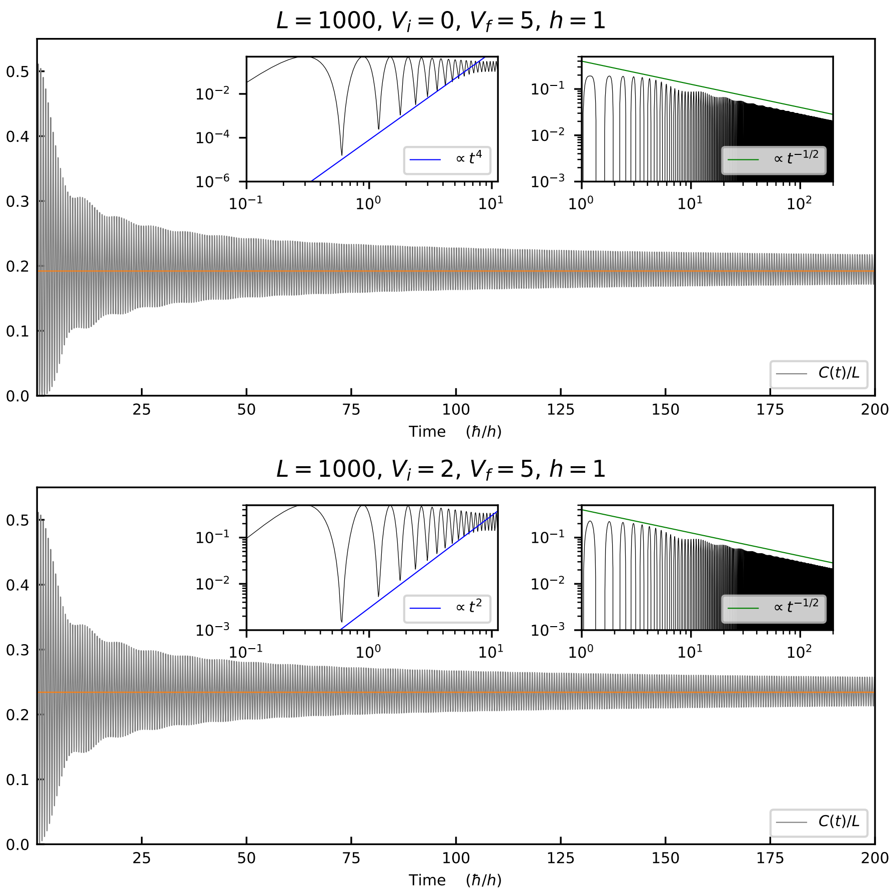


Figure 3.2: Upper: The evolution of the OTOC density $\mathcal{C}(t)/L$ for the superfluid-insulator transition, eq. (3.19). Lower: The evolution of the OTOC for the insulator-insulator transition. The orange line is the stationary value in eq. (3.20). The insets on the left are the same plots with log scale. The insets on the right are the plots of $\mathcal{C}(\infty) - \mathcal{C}(t)$.

3.3.2 The long-time average in thermodynamic limit

As calculated in the appendix C, the stationary behavior of the OTOC is

$$\begin{aligned}\overline{\mathcal{C}}\big|_{L\to\infty} &= \frac{L}{2\pi}\int_{-\pi/2}^{\pi/2} dk \frac{3}{2}\sin^2(4\theta_{k,f}) + 2\cos^2(2\theta_{k,f})\sin^2(2\theta_{k,i}) \\ &= \frac{3L}{2}\left|\frac{V_f}{2h}\right|\left[1+\left(\frac{V_f}{2h}\right)^2\right]^{-3/2} + L\frac{V_i^2}{V_f^2 - V_i^2}\left(\frac{|V_f|}{\sqrt{V_f^2+4h^2}} - \frac{|V_i|}{\sqrt{V_i^2+4h^2}}\right). \end{aligned} \tag{3.20}$$

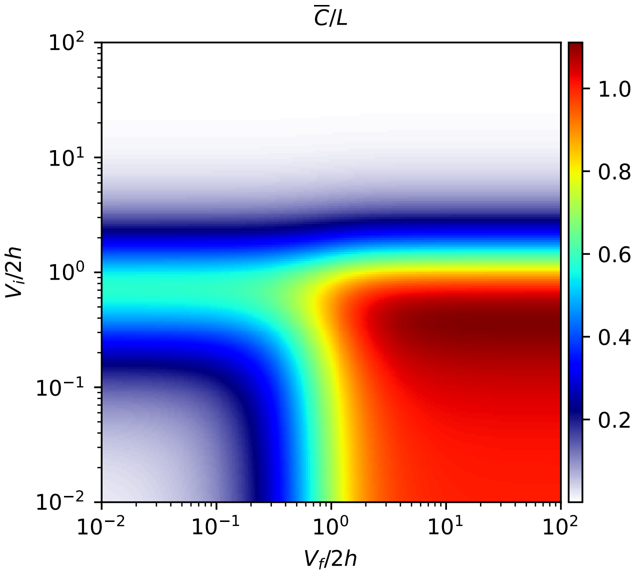


Figure 3.3: The stationary value of the OTOC for total staggered magnetization at different values of V_i and V_f, eq. (3.20).

It is seen that the OTOC density of the global operator M saturate to a constant value for a wide range of V_i and V_f. However, the stationary values are rather small for $\frac{V_i}{2h} > 10$. Besides, for the case of no quench, $V_i = V_f$, the OTOC still saturates to a non-zero value.

3.4 The estimated OTOC in the generalized Gibbs ensemble

With the generalized Gibbs ensemble (GGE)'s density matrix constructed in section 2.4, the OTOC's stationary value can be estimated by taking the GGE expectation values in eq. (3.16). Similarly to the pure state calculation, one need to obtain 16 matrix elements. The calculation for a typical term is done in appendix B. However in the GGE, most the terms are zeros. The main reason for that is the vanishing of $\langle \gamma_k^\dagger \eta_k^\dagger \rangle$ in the GGE, which is non-zero in the pure state. That is, the GGE constructed does not correctly describe the particle-hole pairs in our post-quench state.

The final result is

$$\mathcal{C}_{\mathrm{GGE}}(t) = \sum_k^{\mathrm{BZ}} \begin{pmatrix} A_k \\ B_k \\ C_k \\ D_k \end{pmatrix}^\dagger \begin{pmatrix} \alpha_k\beta_k & 0 & 0 & 0 \\ 0 & \beta_k^2 & 0 & 0 \\ 0 & 0 & \alpha_k^2 & 0 \\ 0 & 0 & 0 & \beta_k\alpha_k \end{pmatrix} \begin{pmatrix} A_k \\ B_k \\ C_k \\ D_k \end{pmatrix}, \tag{3.21}$$

which after substituting back the coefficients becomes

$$\mathcal{C}_{\text{GGE}}(t) = \sum_{k}^{\text{BZ}} 2\sin^2(\varepsilon_k t)\sin^2(4\theta_k)\left[1+\cos^2(2\Delta\theta_k)\right] + 2\sin^2(2\varepsilon_k t)\cos^4(2\theta_k)\sin^2(2\Delta\theta_{k_1}). \tag{3.22}$$

The stationary value of the OTOC in GGE

The long time average is taken by replacing $\sin^2(\omega t)$ with $\frac{1}{2}$

$$\begin{aligned}\overline{\mathcal{C}_{\text{GGE}}} &= \sum_{k}^{\text{BZ}} \sin^2(4\theta_k)\left[1+\cos^2(2\Delta\theta_k)\right] + \cos^4(2\theta_k)\sin^2(2\Delta\theta_{k_1}) \\ &= \sum_{k}^{\text{BZ}} 2\sin^2(4\theta_k) - 2\left[\sin^2(4\theta_k) - \cos^4(2\theta_k)\right]\sin^2(2\Delta\theta_k) \\ &= \sum_{k}^{\text{BZ}} 2\frac{4\alpha_f^2\cos^2 k}{(\cos^2 k+\alpha_f^2)^2} - 2\left[\frac{4\alpha_f^2\cos^2 k}{(\cos^2 k+\alpha_f^2)^2} - \frac{\cos^4 k}{(\cos^2 k+\alpha_f^2)^2}\right]\frac{(\alpha_f-\alpha_i)^2\cos^2 k}{(\cos^2 k+\alpha_f^2)(\cos^2 k+\alpha_i^2)},\end{aligned} \tag{3.23}$$

where $\alpha_{i,f} \equiv \dfrac{V_{i,f}}{2h}$. Then, in the thermodynamic limit $L \to \infty$, (see appendix C)

$$\begin{aligned}\frac{1}{L}\,\overline{\mathcal{C}_{\text{GGE}}}\Big|_{L\to\infty} =& \frac{2\alpha_f}{\left(\alpha_f^2+1\right)^{3/2}} + \frac{4\alpha_f^2\alpha_i^3+\alpha_i^5}{2\sqrt{\alpha_i^2+1}(\alpha_i-\alpha_f)(\alpha_f+\alpha_i)^3} \\ &+ \frac{16\alpha_f^7\left(2\alpha_i^2-1\right)+\alpha_f^5\left(8\alpha_i^4+76\alpha_i^2-1\right)+2\alpha_f^3\alpha_i^2\left(10\alpha_i^2+7\right)+27\alpha_f\alpha_i^4}{16\left(\alpha_f^2+1\right)^{5/2}(\alpha_f-\alpha_i)(\alpha_f+\alpha_i)^3}.\end{aligned} \tag{3.24}$$

The plot of the OTOC's stationary value in GGE is shown below. It is seen that the GGE only correctly predicts the OTOC's stationary value only for some special values of $\{V_i, V_f\}$. As seen in the calculation, one of the reason is that the GGE does not describe well the particle hole pair $\langle\gamma_k^\dagger\eta_k^\dagger\rangle$ in the half-filled post-quench state. For most of the cases, the relative error can be as large as 50%, and can be either negative or positive. The difference is particularly high for superfluid-insulator transition, i.e. for $V_i \to 0$.

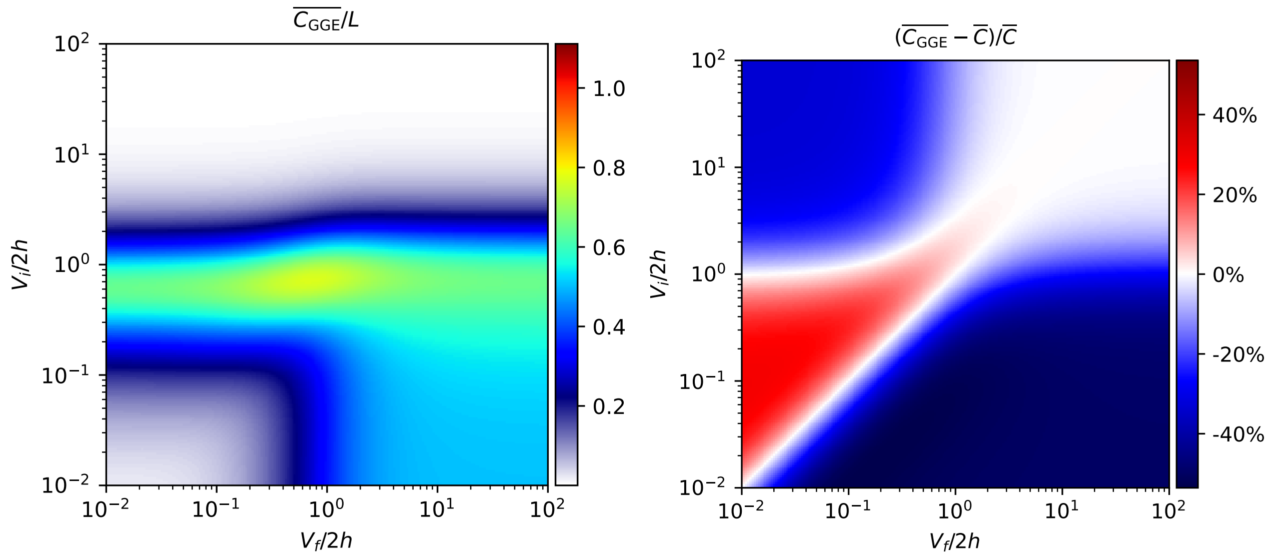


Figure 3.4: Left: The estimated stationary value of the total staggered magnetization's OTOC, calculated in the GGE, eq. (3.24). For visual comparison, the color-map is chosen to be the same as that of figure 3.3. Right: the relative errors of the estimated values in the left figure.

Chapter 4

The OTOC of the local staggered magnetization

This chapter studies the OTOC of the local staggered magnetization in the same direction with the external field, between the site n and site 0

$$\mathcal{C}(t) = -\left\langle [M_n(t), M_0(0)]^2 \right\rangle + \langle [M(t), M_0(0)] \rangle^2 .$$

A similar quantity for the transverse Ising model was studied in Ref. [14]. The calculation will show how the correlation propagates along the spin chain and how it decays over time.

4.1 The observable: local staggered magnetization

The investigated hamiltonian has a unit size containing two spins, therefore it is relevant to define the local staggered magnetization at site $n \in \{1, 2, ..., L/2\}$ as

$$\mathcal{M}_n \equiv S^z_{2n-1} - S^z_{2n} = a^\dagger_{2n} a_{2n} - a^\dagger_{2n-1} a_{2n-1}. \tag{4.1}$$

To represent $\mathcal{M}_n$ in terms of the free fields $\{\gamma^\dagger_k, \eta^\dagger_k\}$, one needs to apply the Fourier transform, divide the Brillouine zone, and perform the unitary rotation as used in the hamiltonian diagonalization, (see eq. (2.15)).

$$\begin{aligned}
\mathcal{M}_n &= \frac{1}{L} \sum_{k_1,k_2}^{2\text{BZ}} e^{-i(k_1-k_2)2n} f^\dagger_{k_1} f_{k_2} - e^{-i(k_1-k_2)(2n-1)} f^\dagger_{k_1} f_{k_2} \equiv \frac{1}{L} \sum_{k_1,k_2}^{2\text{BZ}} \mathcal{F}^n_{k_1,k_2} f^\dagger_{k_1} f_{k_2} \\
&= \frac{1}{L} \sum_{k_1,k_2}^{\text{BZ}} \mathcal{F}^n_{k_1,k_2} f^\dagger_{k_1} f_{k_2} + \mathcal{F}^n_{k_1,k_2-\pi} f^\dagger_{k_1} f_{k_2-\pi} + \mathcal{F}^n_{k_1-\pi,k_2} f^\dagger_{k_1-\pi} f_{k_2} + \mathcal{F}^n_{k_1-\pi,k_2-\pi} f^\dagger_{k_1-\pi} f_{k_2-\pi} \\
&\equiv \frac{1}{L} \sum_{k_1,k_2}^{\text{BZ}} \begin{pmatrix} f^\dagger_{k_1} \\ f^\dagger_{k_1-\pi} \end{pmatrix}^T \begin{pmatrix} \mathcal{F}^n_{k_1,k_2} & \mathcal{G}^n_{k_1,k_2} \\ \mathcal{G}^n_{k_1,k_2} & \mathcal{F}^n_{k_1,k_2} \end{pmatrix} \begin{pmatrix} f_{k_2} \\ f_{k_2-\pi} \end{pmatrix} \\
&= \frac{1}{L} \sum_{k_1,k_2}^{\text{BZ}} \begin{pmatrix} \gamma^\dagger_{k_1} \\ \eta_{k_1} \end{pmatrix}^T e^{i\theta_{k_1}\boldsymbol{\sigma}_y} \begin{pmatrix} \mathcal{F}^n_{k_1,k_2} & \mathcal{G}^n_{k_1,k_2} \\ \mathcal{G}^n_{k_1,k_2} & \mathcal{F}^n_{k_1,k_2} \end{pmatrix} e^{-i\theta_{k_2}\boldsymbol{\sigma}_y} \begin{pmatrix} \gamma_{k_2} \\ \eta^\dagger_{k_2} \end{pmatrix} \\
&= \quad \frac{1}{L} \sum_{k_1,k_2}^{\text{BZ}} \mathcal{F}^n_{k_1,k_2} \begin{pmatrix} \gamma^\dagger_{k_1} \\ \eta_{k_1} \end{pmatrix}^T \begin{pmatrix} \cos(\theta_{k_1} - \theta_{k_2}) & \sin(\theta_{k_1} - \theta_{k_2}) \\ -\sin(\theta_{k_1} - \theta_{k_2}) & \cos(\theta_{k_1} - \theta_{k_2}) \end{pmatrix} \begin{pmatrix} \gamma_{k_2} \\ \eta^\dagger_{k_2} \end{pmatrix}
\end{aligned}$$

$$+\frac{1}{L}\sum_{k_1,k_2}^{\mathrm{BZ}}\mathcal{G}^n_{k_1,k_2}\begin{pmatrix}\gamma^\dagger_{k_1}\\ \eta_{k_1}\end{pmatrix}^T\begin{pmatrix}\sin(\theta_{k_1}+\theta_{k_2}) & \cos(\theta_{k_1}+\theta_{k_2})\\ \cos(\theta_{k_1}+\theta_{k_2}) & -\sin(\theta_{k_1}+\theta_{k_2})\end{pmatrix}\begin{pmatrix}\gamma_{k_2}\\ \eta^\dagger_{k_2}\end{pmatrix}. \tag{4.2}$$

where

$$\begin{cases}\mathcal{F}^n_{k_1,k_2} & \equiv e^{-i(k_1-k_2)2n}-e^{-i(k_1-k_2)(2n-1)}=e^{-i(k_1-k_2)2n}[1-e^{i(k_1-k_2)}]\\ \mathcal{G}^n_{k_1,k_2} & \equiv \mathcal{F}^n_{k_1,k_2-\pi}=\mathcal{F}^n_{k_1-\pi,k_2}\\ & =e^{-i(k_1-k_2)2n}+e^{-i(k_1-k_2)(2n-1)}=e^{-i(k_1-k_2)2n}[1+e^{i(k_1-k_2)}]\end{cases} \tag{4.3}$$

satisfying $(\mathcal{F}^n_{k_1,k_2})^*=\mathcal{F}^n_{k_2,k_1}$ and $(\mathcal{G}^n_{k_1,k_2})^*=\mathcal{G}^n_{k_2,k_1}$. This operator's time dependence $\mathcal{M}_n(t)=e^{iHt}\mathcal{M}_n e^{-iHt}$ is then obtained by the mapping in eq. (3.6): $\{\gamma^\dagger_k,\eta^\dagger_k\}\mapsto\{e^{i\varepsilon_k t}\gamma^\dagger_k,e^{i\varepsilon_k t}\eta^\dagger_k\}$.

Expectation value

In order to calculate the expectation value of $\mathcal{M}_n(t)$ in the post-quench state $|\Omega_i\rangle$, one need to use the following identities

$$\begin{cases}\left\langle\gamma^\dagger_{k_1}\gamma_{k_2}\right\rangle_i=\delta_{k_1,k_2}\left\langle\gamma^\dagger_{k_1}\gamma_{k_1}\right\rangle_i=\delta_{k_1,k_2}\sin^2(\Delta\theta_{k_1})\\ \left\langle\eta_{k_1}\gamma_{k_2}\right\rangle_i=\left\langle\gamma^\dagger_{k_1}\eta^\dagger_{k_2}\right\rangle_i=\delta_{k_1,k_2}\left\langle\gamma^\dagger_{k_1}\eta^\dagger_{k_1}\right\rangle_i=\delta_{k_1,k_2}\frac{1}{2}\sin(2\Delta\theta_{k_1})\end{cases} \tag{4.4}$$

where the δ_{k_1,k_2} are due to the form of $|\Omega_i\rangle$ in which $\gamma^\dagger_k$ and $\eta^\dagger_k$ exist in pairs. It can be then seen that $\langle\mathcal{M}_n(t)\rangle_i$'s expectation value is consistent with the previous result for the total staggered magnetization, which is just the average of the latter, due to translational symmetry.

$$\langle\mathcal{M}_n(t)\rangle_i=\frac{1}{L}\sum_k^{\mathrm{BZ}}\mathcal{G}^n_{k,k}\left\langle\begin{pmatrix}\gamma^\dagger_k(t)\\ \eta_k(t)\end{pmatrix}^T\begin{pmatrix}\sin(2\theta_k) & \cos(2\theta_k)\\ \cos(2\theta_k) & -\sin(2\theta_k)\end{pmatrix}\begin{pmatrix}\gamma_k(t)\\ \eta^\dagger_k(t)\end{pmatrix}\right\rangle_i=\frac{2}{L}\langle M(t)\rangle_i \tag{4.5}$$

because $\mathcal{F}^n_{k,k}=0$, $\mathcal{G}^n_{k,k}=2$ independently of n and comparing with eq. (3.7).

4.2 The OTOC of $\hat{M}_n$ in the post-quench state

This section devotes to the calculation of the OTOC defined similarly to that in eq.(3.2)

$$\mathcal{C}(t)=-\left\langle[M_n(t),M_0(0)]^2\right\rangle+\langle[M(t),M_0(0)]\rangle^2. \tag{4.6}$$

This quantity can be seen as the centralized second moment of the commutator $i[M_n(t),M_0(0)]$.

Operator expression of $[M(t),M]$

We want to calculate the commutator in both space and time $[M_n(t),M_0(0)]$ [1]. Thanks to its generality, the calculation method for the global magnetization in section 3.2 can also be applied in this case. Firstly, $M_n(t)$ is written schematically as

$$M_n(t)=\frac{1}{L}\sum_{k_1,k_2}^{\mathrm{BZ}}\begin{pmatrix}\gamma^\dagger_{k_1}\\ \eta_{k_1}\end{pmatrix}^T\begin{pmatrix}A^n_{k_1,k_2}(t) & B^n_{k_1,k_2}(t)\\ C^n_{k_1,k_2}(t) & D^n_{k_1,k_2}(t)\end{pmatrix}\begin{pmatrix}\gamma_{k_2}\\ \eta^\dagger_{k_2}\end{pmatrix}\equiv\frac{1}{L}\sum_{k_1,k_2}^{\mathrm{BZ}}\begin{pmatrix}\gamma^\dagger_{k_1}\\ \eta_{k_1}\end{pmatrix}^T\left[\mathfrak{M}^n_{k_1,k_2}(t)\right]\begin{pmatrix}\gamma_{k_2}\\ \eta^\dagger_{k_2}\end{pmatrix}. \tag{4.7}$$

[1] With the current convention, $M_0\equiv M_{L/2}$.

where

$$\begin{cases} A^n_{k_1,k_2}(t) = e^{\;\;i(\varepsilon_{k_1}-\varepsilon_{k_2})t}\left[\;\;\mathcal{F}^n_{k_1,k_2}\cos(\theta_{k_1}-\theta_{k_2})+\mathcal{G}^n_{k_1,k_2}\sin(\theta_{k_1}+\theta_{k_2})\right] \\ B^n_{k_1,k_2}(t) = e^{\;\;i(\varepsilon_{k_1}+\varepsilon_{k_2})t}\left[\;\;\mathcal{F}^n_{k_1,k_2}\sin(\theta_{k_1}-\theta_{k_2})+\mathcal{G}^n_{k_1,k_2}\cos(\theta_{k_1}+\theta_{k_2})\right] \\ C^n_{k_1,k_2}(t) = e^{-i(\varepsilon_{k_1}+\varepsilon_{k_2})t}\left[-\mathcal{F}^n_{k_1,k_2}\sin(\theta_{k_1}-\theta_{k_2})+\mathcal{G}^n_{k_1,k_2}\cos(\theta_{k_1}+\theta_{k_2})\right] \\ D^n_{k_1,k_2}(t) = e^{-i(\varepsilon_{k_1}-\varepsilon_{k_2})t}\left[\;\;\mathcal{F}^n_{k_1,k_2}\cos(\theta_{k_1}-\theta_{k_2})-\mathcal{G}^n_{k_1,k_2}\sin(\theta_{k_1}+\theta_{k_2})\right]. \end{cases} \tag{4.8}$$

with the properties $A^n_{k_1,k_2}(t) = \left[A^n_{k_2,k_1}(t)\right]^*, B^n_{k_1,k_2}(t) = \left[C^n_{k_2,k_1}(t)\right]^*, D^n_{k_1,k_2}(t) = \left[D^n_{k_2,k_1}(t)\right]^*$. Then, using the formula (D.7) gives

$$\begin{aligned} &[M_n(t), M_0(0)] \\ &= \frac{1}{L^2}\sum_{k_1,k_2,k'}^{\mathrm{BZ}} \begin{pmatrix} \gamma^\dagger_{k_1} \\ \eta_{k_1} \end{pmatrix}^T \left[\mathfrak{M}^n_{k_1,k'}(t)\mathfrak{M}^0_{k',k_2}(0) - \mathfrak{M}^0_{k_1,k'}(0)\mathfrak{M}^n_{k',k_2}(t)\right] \begin{pmatrix} \gamma_{k_2} \\ \eta^\dagger_{k_2} \end{pmatrix} \\ &\equiv \frac{1}{L^2}\sum_{k_1,k_2}^{\mathrm{BZ}} \begin{pmatrix} \gamma^\dagger_{k_1} \\ \eta_{k_1} \end{pmatrix}^T \begin{pmatrix} E^n_{k_1,k_2}(t) & F^n_{k_1,k_2}(t) \\ G^n_{k_1,k_2}(t) & H^n_{k_1,k_2}(t) \end{pmatrix} \begin{pmatrix} \gamma_{k_2} \\ \eta^\dagger_{k_2} \end{pmatrix}. \end{aligned} \tag{4.9}$$

Where the $\{n,t\}$-dependent coefficients $\{E_{k_1,k_2}, F_{k_1,k_2}, G_{k_1,k_2}, H_{k_1,k_2}\}$ are simply obtained from 2×2 matrix multiplications, with the summation over k', and are combinations of $\{A_{k_1,k_2}, B_{k_1,k_2}, C_{k_1,k_2}, D_{k_1,k_2}\}$ in eq. (4.8).

$$\begin{aligned} \begin{pmatrix} E^n_{k_1,k_2}(t) & F^n_{k_1,k_2}(t) \\ G^n_{k_1,k_2}(t) & H^n_{k_1,k_2}(t) \end{pmatrix} = \sum_{k'}^{\mathrm{BZ}} &\begin{pmatrix} A^n_{k_1,k'}(t) & B^n_{k_1,k'}(t) \\ C^n_{k_1,k'}(t) & D^n_{k_1,k'}(t) \end{pmatrix} \begin{pmatrix} A^0_{k',k_2}(0) & B^0_{k',k_2}(0) \\ C^0_{k',k_2}(0) & D^0_{k',k_2}(0) \end{pmatrix} \\ - &\begin{pmatrix} A^0_{k_1,k'}(0) & B^0_{k_1,k'}(0) \\ C^0_{k_1,k'}(0) & D^0_{k_1,k'}(0) \end{pmatrix} \begin{pmatrix} A^n_{k',k_2}(t) & B^n_{k',k_2}(t) \\ C^n_{k',k_2}(t) & D^n_{k',k_2}(t) \end{pmatrix}. \end{aligned} \tag{4.10}$$

Operator expression of $[M_n(t), M_0(0)]^2$

Similarly for the case of global magnetization, in order to take the square of the commutator $[M_n(t), M_0(0)]$, the approach taken here is to write it as

$$[M_n(t), M_0(0)] = \frac{1}{L^2}\sum_{k_1,k_2}^{\mathrm{BZ}} \begin{pmatrix} E_{k_1,k_2} \\ F_{k_1,k_2} \\ G_{k_1,k_2} \\ H_{k_1,k_2} \end{pmatrix}^T \begin{pmatrix} \gamma^\dagger_{k_1}\gamma_{k_2} \\ \gamma^\dagger_{k_1}\eta^\dagger_{k_2} \\ \eta_{k_1}\gamma_{k_2} \\ \eta_{k_1}\eta^\dagger_{k_2} \end{pmatrix}.$$

After that, the square of the commutator easily comes out as

$$\begin{aligned} &[M_n(t), M_0(0)]^\dagger [M_n(t), M_0(0)]^2 = -[M_n(t), M_0(0)]^2 \\ &= \frac{1}{L^4}\sum_{k_1,k_2,k_3,k_4}^{\mathrm{BZ}} \begin{pmatrix} E_{k_2,k_1} \\ F_{k_2,k_1} \\ G_{k_2,k_1} \\ H_{k_2,k_1} \end{pmatrix}^\dagger \begin{pmatrix} \gamma^\dagger_{k_1}\gamma_{k_2} \\ \eta_{k_1}\gamma_{k_2} \\ \gamma^\dagger_{k_1}\eta^\dagger_{k_2} \\ \eta_{k_1}\eta^\dagger_{k_2} \end{pmatrix} \begin{pmatrix} \gamma^\dagger_{k_3}\gamma_{k_4} \\ \gamma^\dagger_{k_3}\eta^\dagger_{k_4} \\ \eta_{k_3}\gamma_{k_4} \\ \eta_{k_3}\eta^\dagger_{k_4} \end{pmatrix}^T \begin{pmatrix} E_{k_3,k_4} \\ F_{k_3,k_4} \\ G_{k_3,k_4} \\ H_{k_3,k_4} \end{pmatrix} \\ &= \frac{1}{L^4}\sum_{k_1,k_2,k_3,k_4}^{\mathrm{BZ}} \begin{pmatrix} E_{k_2,k_1} \\ F_{k_2,k_1} \\ G_{k_2,k_1} \\ H_{k_2,k_1} \end{pmatrix}^\dagger \begin{pmatrix} \gamma^\dagger_{k_1}\gamma_{k_2}\gamma^\dagger_{k_3}\gamma_{k_4} & \gamma^\dagger_{k_1}\gamma_{k_2}\gamma^\dagger_{k_3}\eta^\dagger_{k_4} & \gamma^\dagger_{k_1}\gamma_{k_2}\eta_{k_3}\gamma_{k_4} & \gamma^\dagger_{k_1}\gamma_{k_2}\eta_{k_3}\eta^\dagger_{k_4} \\ \eta_{k_1}\gamma_{k_2}\gamma^\dagger_{k_3}\gamma_{k_4} & \eta_{k_1}\gamma_{k_2}\gamma^\dagger_{k_3}\eta^\dagger_{k_4} & \eta_{k_1}\gamma_{k_2}\eta_{k_3}\gamma_{k_4} & \eta_{k_1}\gamma_{k_2}\eta_{k_3}\eta^\dagger_{k_4} \\ \gamma^\dagger_{k_1}\eta^\dagger_{k_2}\gamma^\dagger_{k_3}\gamma_{k_4} & \gamma^\dagger_{k_1}\eta^\dagger_{k_2}\gamma^\dagger_{k_3}\eta^\dagger_{k_4} & \gamma^\dagger_{k_1}\eta^\dagger_{k_2}\eta_{k_3}\gamma_{k_4} & \gamma^\dagger_{k_1}\eta^\dagger_{k_2}\eta_{k_3}\eta^\dagger_{k_4} \\ \eta_{k_1}\eta^\dagger_{k_2}\gamma^\dagger_{k_3}\gamma_{k_4} & \eta_{k_1}\eta^\dagger_{k_2}\gamma^\dagger_{k_3}\eta^\dagger_{k_4} & \eta_{k_1}\eta^\dagger_{k_2}\eta_{k_3}\gamma_{k_4} & \eta_{k_1}\eta^\dagger_{k_2}\eta_{k_3}\eta^\dagger_{k_4} \end{pmatrix} \begin{pmatrix} E_{k_3,k_4} \\ F_{k_3,k_4} \\ G_{k_3,k_4} \\ H_{k_3,k_4} \end{pmatrix}. \end{aligned} \tag{4.11}$$

Expectation value of $[M_n(t), M_0(0)]^2$ and the OTOC

Thanks to the well-structured formula above, the value of the OTOC comes out naturally from eq. (4.11), similarly to the case of the global OTOC.

$$
\begin{aligned}
\mathcal{C}_n(t) &= \langle [M_n(t), M_0(0)]^\dagger [M_n(t), 0(0)] \rangle - \langle [M_n(t), M_0(0)] \rangle^* \langle [M_n(t), M_0(0)] \rangle \\
&= \frac{1}{L^4} \sum_{\substack{k_1,k_2\\k_3,k_4}}^{\mathrm{BZ}} \begin{pmatrix} E_{k_2,k_1} \\ F_{k_2,k_1} \\ G_{k_2,k_1} \\ H_{k_2,k_1} \end{pmatrix}^\dagger \begin{pmatrix}
\langle \gamma^\dagger_{k_1}\gamma_{k_2}\gamma^\dagger_{k_3}\gamma_{k_4}\rangle^c & \langle \gamma^\dagger_{k_1}\gamma_{k_2}\gamma^\dagger_{k_3}\eta^\dagger_{k_4}\rangle^c & \langle \gamma^\dagger_{k_1}\gamma_{k_2}\eta_{k_3}\gamma_{k_4}\rangle^c & \langle \gamma^\dagger_{k_1}\gamma_{k_2}\eta_{k_3}\eta^\dagger_{k_4}\rangle^c \\
\langle \eta_{k_1}\gamma_{k_2}\gamma^\dagger_{k_3}\gamma_{k_4}\rangle^c & \langle \eta_{k_1}\gamma_{k_2}\gamma^\dagger_{k_3}\eta^\dagger_{k_4}\rangle^c & \langle \eta_{k_1}\gamma_{k_2}\eta_{k_3}\gamma_{k_4}\rangle^c & \langle \eta_{k_1}\gamma_{k_2}\eta_{k_3}\eta^\dagger_{k_4}\rangle^c \\
\langle \gamma^\dagger_{k_1}\eta^\dagger_{k_2}\gamma^\dagger_{k_3}\gamma_{k_4}\rangle^c & \langle \gamma^\dagger_{k_1}\eta^\dagger_{k_2}\gamma^\dagger_{k_3}\eta^\dagger_{k_4}\rangle^c & \langle \gamma^\dagger_{k_1}\eta^\dagger_{k_2}\eta_{k_3}\gamma_{k_4}\rangle^c & \langle \gamma^\dagger_{k_1}\eta^\dagger_{k_2}\eta_{k_3}\eta^\dagger_{k_4}\rangle^c \\
\langle \eta_{k_1}\eta^\dagger_{k_2}\gamma^\dagger_{k_3}\gamma_{k_4}\rangle^c & \langle \eta_{k_1}\eta^\dagger_{k_2}\gamma^\dagger_{k_3}\eta^\dagger_{k_4}\rangle^c & \langle \eta_{k_1}\eta^\dagger_{k_2}\eta_{k_3}\gamma_{k_4}\rangle^c & \langle \eta_{k_1}\eta^\dagger_{k_2}\eta_{k_3}\eta^\dagger_{k_4}\rangle^c
\end{pmatrix} \begin{pmatrix} E_{k_3,k_4} \\ F_{k_3,k_4} \\ G_{k_3,k_4} \\ H_{k_3,k_4} \end{pmatrix}.
\end{aligned} \tag{4.12}
$$

with the notation $\langle \gamma^\dagger_{k_1}\gamma_{k_2}\gamma^\dagger_{k_3}\gamma_{k_4}\rangle^c \equiv \langle \gamma^\dagger_{k_1}\gamma_{k_2}\gamma^\dagger_{k_3}\gamma_{k_4}\rangle - \langle \gamma^\dagger_{k_1}\gamma_{k_2}\rangle\langle \gamma^\dagger_{k_3}\gamma_{k_4}\rangle$. The expectation values are taken in the post-quench state $|\Omega_i\rangle$ in eq. (2.29). Luckily, after explicit calculation, all of these 16 matrix elements turns out to share the same simple form, for example

$$
\langle \gamma^\dagger_{k_1}\gamma_{k_2}\gamma^\dagger_{k_3}\gamma_{k_4}\rangle^c = \delta_{k_1,k_4}\delta_{k_2,k_3}\langle \gamma^\dagger_{k_1}\gamma_{k_4}\rangle\langle \gamma_{k_2}\gamma^\dagger_{k_3}\rangle = \delta_{k_1,k_4}\delta_{k_2,k_3}\alpha_{k_1}\beta_{k_2}. \tag{4.13}
$$

The explicit calculation for one of these terms are written down in appendix B. After these calculation, the result for the OTOC of the local magnetization is

$$
\mathcal{C}_n(t) = \frac{1}{L^4} \sum_{k_1,k_2}^{\mathrm{BZ}} \begin{pmatrix} E_{k_2,k_1} \\ F_{k_2,k_1} \\ G_{k_2,k_1} \\ H_{k_2,k_1} \end{pmatrix}^\dagger \begin{pmatrix}
\alpha_{k_1}\beta_{k_2} & \lambda_{k_1}\beta_{k_2} & -\alpha_{k_1}\lambda_{k_2} & -\lambda_{k_1}\lambda_{k_2} \\
\lambda_{k_1}\beta_{k_2} & \beta_{k_1}\beta_{k_2} & -\lambda_{k_1}\lambda_{k_2} & -\beta_{k_1}\lambda_{k_2} \\
-\alpha_{k_1}\lambda_{k_2} & -\lambda_{k_1}\lambda_{k_2} & \alpha_{k_1}\alpha_{k_2} & \lambda_{k_1}\alpha_{k_2} \\
-\lambda_{k_1}\lambda_{k_2} & -\beta_{k_1}\lambda_{k_2} & \lambda_{k_1}\alpha_{k_2} & \beta_{k_1}\alpha_{k_2}
\end{pmatrix} \begin{pmatrix} E_{k_2,k_1} \\ F_{k_2,k_1} \\ G_{k_2,k_1} \\ H_{k_2,k_1} \end{pmatrix}. \tag{4.14}
$$

The parameters used above are $\alpha_k = \sin^2(\Delta\theta_k)$, $\beta_k = \cos^2(\Delta\theta_k)$, and $\lambda_k = \frac{1}{2}\sin(2\Delta\theta_k)$. After substituting back the values of $\{E_k, F_k, G_k, H_k\}$ from eq. (4.10), the intensive quantity $\mathcal{C}_n(t)$ is plotted as a function of space and time in the next section.

4.3 Dynamical behavior of the local OTOC

In contrast to the global OTOC, which reaches a plateau, the local OTOC $\mathcal{C}_n(t)$ decays to zero, showing that there is no information scrambling for this observable. It is seen that this decay goes with the power-law behavior of t^{-1} (see figure 4.1), for all values of n. This result is similar to what observed in Ref. [14] (figure 1), which studied the transverse-field Ising model without quench. However, in that same reference, the local OTOC of operators that change Jordan-Wigner fermions' parity can reach a high plateau. It is worth exploring this direction when expanding the thesis.

Secondly, it is seen in figure 4.2 that the OTOC grows along the chain with a constant velocity equals the maximum group velocity of the free $\{\gamma^\dagger_k, \eta^\dagger_k\}$ quasi-particles, $v = \max(\frac{\partial\varepsilon_k}{\partial 2k})$. This light-cone effect is also universally observed in Ref. [14].

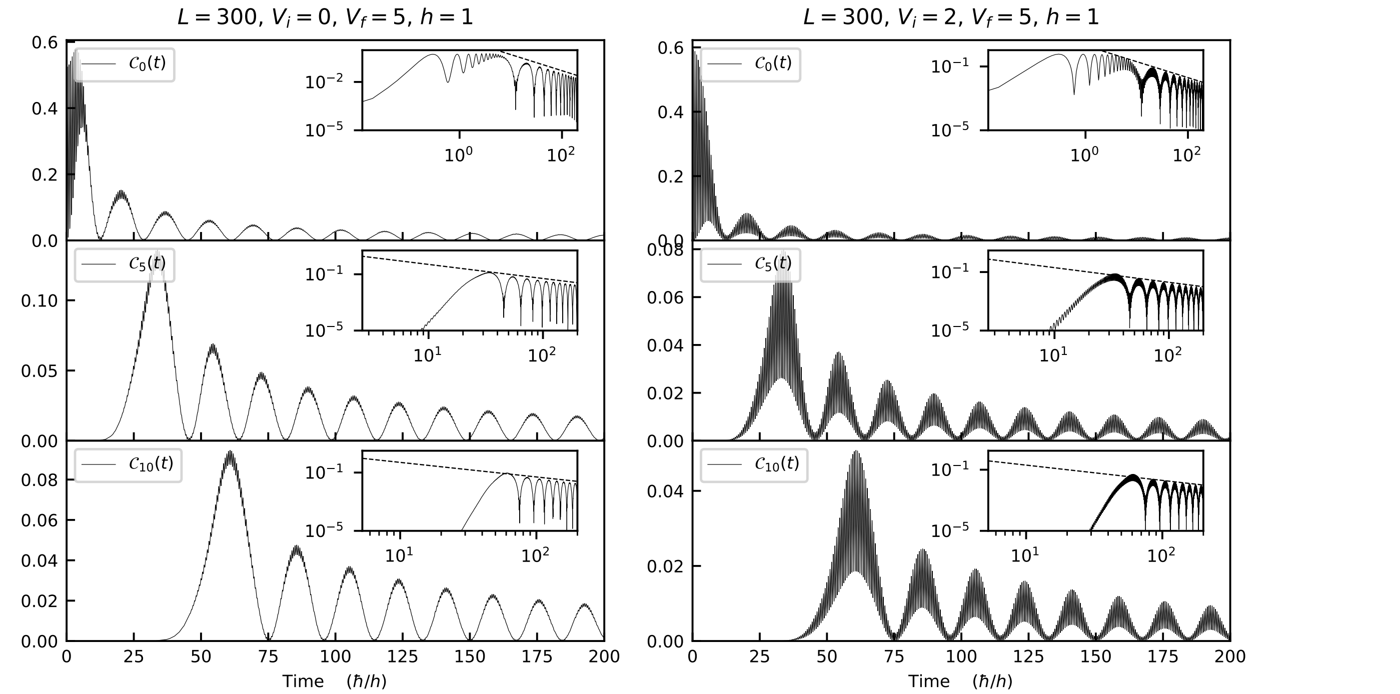


Figure 4.1: Plots of $\mathcal{C}_n(t)$, eq. (4.14), for several values of n. The system size $L = 300$ and $V_f = 5$. Left: Insulator-superfluid quench $V_i = 0$. Right: Insulator-insulator quench $V_i = 2$. The insets are the log-scale of the same plot, showing the power-law behavior, with the dashed lines guiding the eyes are of power t^{-1}.

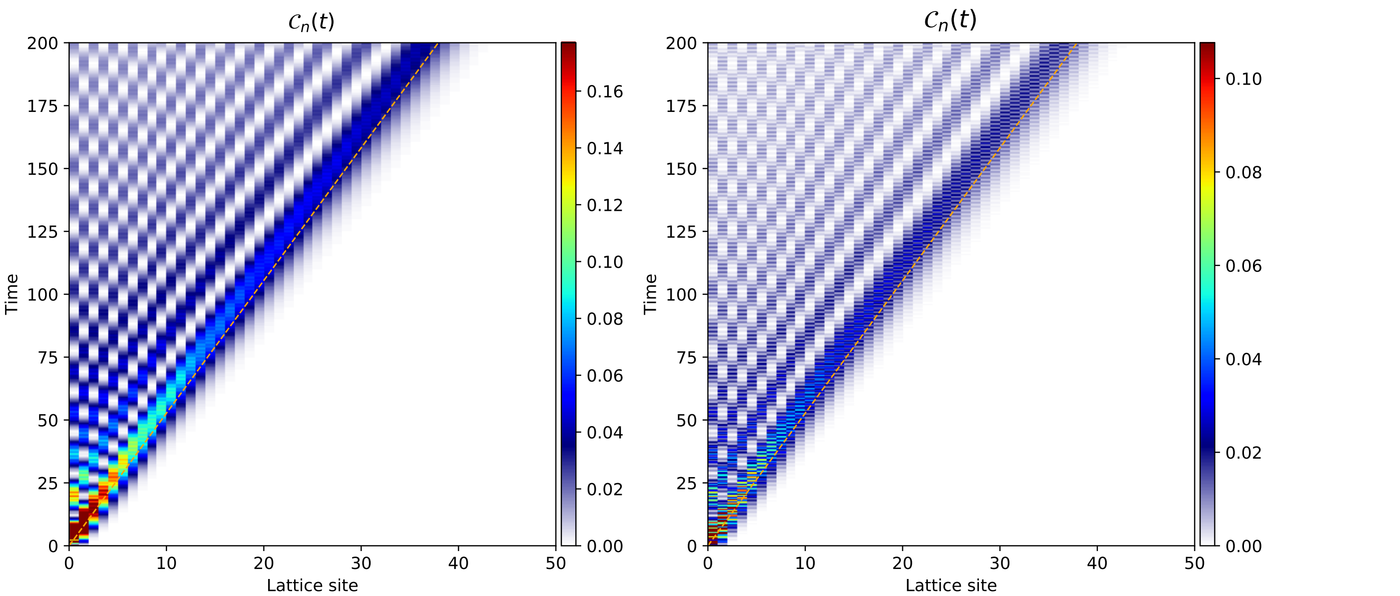


Figure 4.2: Plots of $\mathcal{C}_n(t)$, eq. (4.14), for system size $L = 300$ and $V_f = 5$. Left: Insulator-superfluid quench $V_i = 0$. Right: Insulator-insulator quench $V_i = 2$. The orange dashed line is the light-cone with maximal group velocity $v = \max(\dfrac{\partial \varepsilon_k}{\partial 2k})$.

Chapter 5

Summary

In summary, this thesis has analyzed the OTOC (density) of one-body observables in the XX chain with a staggered field (a two-band model), in the half-filled ground state undergoing a quench of the external field. With the total staggered magnetization, the OTOC density $\mathcal{C}(t)$'s envelop is observed to grow with power law of t^2 in an insulator-insulator and t^4 in an insulator-superfluid quench. It then reaches a constant values at late time in the thermodynamic limit, with a power of $t^{-1/2}$ in both kinds of quench. To develop the thesis further, is worth checking whether a composite operators which involve a string of fermions can have the OTOC grows linearly in our model, as reported in Ref. [7], figure 1 (b).

Besides, we have compared the stationary values of $\mathcal{C}(t)$ calculated in the pure state and in the generalized Gibbs ensemble. The generalized Gibbs ensemble (GGE), while works well with the normal correlation [8], does not provide a good prediction for the stationary value of $\mathcal{C}(t)$, which is a four-point correlation. One of the reason is that the GGE does not describe well the particle hole pair in the half-filled post-quench state.

With the local staggered magnetization, our result shows that the OTOC $\mathcal{C}_n(t)$ decays universally with power law of t^{-1} with all n, showing no signature of information scrambling. The correlation is seen to propagate along the chain with a constant velocity equal to the maximum group velocity of free quasi-particles. To expand the thesis, it is worth checking whether this is still the case for operators that change Jordan-Wigner fermions' parity, since it was reported not to in the transverse-field Ising model in Ref. [14].

Appendix A

Jordan-Wigner transformation

The Jordan-Wigner transformation is defined as

$$a_i^\dagger \coloneqq \exp\left\{ i\pi \sum_{j=1}^{i-1} b_j^\dagger b_j \right\} b_i^\dagger = \prod_{j=1}^{i-1} e^{i\pi b_j^\dagger b_j} b_i^\dagger, \quad i \in \{1, 2, ..., L\}. \tag{A.1}$$

The main properties of these operators are derived in below.

$$n_i \coloneqq b_i^\dagger b_i \tag{A.2}$$

$$\Rightarrow \begin{cases} n_i^\dagger = n_i, \quad n_i^2 = n_i, \quad n_i b_i = b_i^\dagger n_i = 0 \\ [n_i, n_j] = 0, \quad [n_i, b_j] = -\delta_{ij} b_i, \quad [n_i, b_j^\dagger] = \delta_{ij} b_i^\dagger. \end{cases} \tag{A.3}$$

$$\Rightarrow \begin{cases} e^{i\pi n_i} = e^{-i\pi n_i} = 1 + \sum_{k=1}^{\infty} \frac{(\pm i\pi n_i)^k}{k!} = 1 + n_i \sum_{k=1}^{\infty} \frac{(\pm i\pi)^k}{k!} = 1 + n_i(e^{\pm i\pi} - 1) = 1 - 2n_i \\ e^{i2\pi n_i} = e^{i\pi n_i} e^{i\pi n_i} = e^{i\pi n_i} e^{-i\pi n_i} = 1 \\ b_i e^{i\pi n_i} = b_i(1 - 2n_i) = b_i(1 - 2b_i^\dagger b_i) = b_i(1 - 2 + 2b_i b_i^\dagger) = -b_i = -e^{i\pi n_i} b_i \\ e^{i\pi n_i} b_i^\dagger = -b_i^\dagger = -b_i^\dagger e^{i\pi n_i} \\ \{b_i, e^{i\pi n_i}\} = \{b_i^\dagger, e^{i\pi n_i}\} = 0. \end{cases} \tag{A.4}$$

$$\Rightarrow \begin{cases} a_i^\dagger \coloneqq \exp\left\{ i\pi \sum_{j=1}^{i-1} n_j \right\} b_i^\dagger, \qquad a_i = \exp\left\{ i\pi \sum_{j=1}^{i-1} n_j \right\} b_i \\ a_i^\dagger a_i = b_i^\dagger b_i = n_i \\ b_i^\dagger = \exp\left\{ i\pi \sum_{j=1}^{i-1} n_j \right\} a_i^\dagger, \qquad b_i = \exp\left\{ i\pi \sum_{j=1}^{i-1} n_j \right\} a_i. \end{cases} \tag{A.5}$$

The $a_i^\dagger$ operators are also proven to be fermionic.

$$\begin{aligned} \{a_i, a_j^\dagger\} &= \{e^{i\pi S_{i-1}} b_i, e^{i\pi S_{j-1}} b_j^\dagger\} \qquad \text{with } S_i \coloneqq \sum_{k=1}^{i} n_k \\ &= e^{i\pi S_{i-1}} b_i e^{i\pi S_{j-1}} b_j^\dagger + e^{i\pi S_{j-1}} b_j^\dagger e^{i\pi S_{i-1}} b_i \\ &= \begin{cases} \overset{(i<j)}{=} \; e^{i\pi S_{i-1}} e^{i\pi S_{j-1}} (-b_i b_j^\dagger + b_j^\dagger b_i) \\ \overset{(i=j)}{=} \; e^{i\pi S_{i-1}} e^{i\pi S_{i-1}} (b_i b_i^\dagger + b_i^\dagger b_i) \end{cases} = \begin{cases} \overset{(i<j)}{=} \; e^{i\pi S_{i-1}} e^{i\pi S_{j-1}} 0 = 0 \\ \overset{(i=j)}{=} \; e^{i\pi S_{i-1}} e^{i\pi S_{i-1}} \mathbb{1} = \mathbb{1}. \end{cases} = \delta_{ij} \mathbb{1}. \end{aligned} \tag{A.6}$$

$$\begin{aligned} \{a_i, a_j\} &= \{e^{i\pi S_{i-1}} b_i, e^{i\pi S_{j-1}} b_j\} \\ &= e^{i\pi S_{i-1}} b_i e^{i\pi S_{j-1}} b_j + e^{i\pi S_{j-1}} b_j e^{i\pi S_{i-1}} b_i \end{aligned}$$

$$
= \begin{cases} \overset{(i<j)}{=} e^{i\pi S_{i-1}} e^{i\pi S_{j-1}} (-b_i b_j + b_j b_i) \\ \overset{(i=j)}{=} e^{i\pi S_{i-1}} e^{i\pi S_{i-1}} (b_i b_i + b_i b_i) \end{cases} = \begin{cases} \overset{(i<j)}{=} e^{i\pi S_{i-1}} e^{i\pi S_{j-1}} 0 = 0 \\ \overset{(i=j)}{=} e^{i\pi S_{i-1}} e^{i\pi S_{i-1}} 0 = 0. \end{cases} = 0. \tag{A.7}
$$

Appendix B

Some basic operator algebra

B.1 The normalization factor $\mathfrak{N}$

$$|\Omega_i\rangle \equiv \mathfrak{N}\left[\prod_k^{\text{BZ}}(1+g_k\gamma_{k,f}^\dagger\eta_{k,f}^\dagger)\right]|\Omega_f\rangle\,. \tag{B.1}$$

Since the factors in the product are commuting

$$\langle\Omega_i|\Omega_i\rangle = \mathfrak{N}^2\,\langle\Omega_f|\prod_k^{\text{BZ}}(1+g_k^*\eta_{k,f}\gamma_{k,f})(1+g_k\gamma_{k,f}^\dagger\eta_{k,f}^\dagger)\,|\Omega_f\rangle = 1. \tag{B.2}$$

Each factor gives (where $\square$ stands for commuting operators at different k-points)

$$\begin{aligned}
&\langle\Omega_f|\,\square(1+g_k^*\eta_{k,f}\gamma_{k,f})(1+g_k\gamma_{k,f}^\dagger\eta_{k,f}^\dagger)\square\,|\Omega_f\rangle\\
=&\langle\Omega_f|\,\square\left(1+g_k^*\eta_{k,f}\gamma_{k,f}+g_k\gamma_{k,f}^\dagger\eta_{k,f}^\dagger+|g_k|^2\eta_{k,f}\gamma_{k,f}\gamma_{k,f}^\dagger\eta_{k,f}^\dagger\right)\square^\dagger\,|\Omega_f\rangle\\
=&\langle\Omega_f|\,\square\left(1+|g_k|^2\eta_{k,f}\gamma_{k,f}\gamma_{k,f}^\dagger\eta_{k,f}^\dagger\right)\square^\dagger\,|\Omega_f\rangle\\
=&\langle\Omega_f|\,\square\left[1+|g_k|^2(1-\gamma_{k,f}^\dagger\gamma_{k,f})(1-\eta_{k,f}^\dagger\eta_{k,f})\right]\square^\dagger\,|\Omega_f\rangle\\
=&\langle\Omega_f|\,\square\left(1+|g_k|^2\right)\square^\dagger\,|\Omega_f\rangle\\
=&\left(1+|g_k|^2\right)\langle\Omega_f|\,\square\square^\dagger\,|\Omega_f\rangle\,.
\end{aligned} \tag{B.3}$$

Inductively, this leads to

$$\mathfrak{N}^2\prod_k^{\text{BZ}}\left(1+|g_k|^2\right) = 1 \Rightarrow \mathfrak{N} = \prod_k^{\text{BZ}} 1/\sqrt{1+|g_k|^2}. \tag{B.4}$$

B.2 Basic commutators

$$\begin{cases} [A,BC] & = [A,B]_{\mp}C \pm B[A,C]_{\mp} \\ & = [A,B]C + B[A,C] = \{A,B\}C - B\{A,C\} \\ [BC,A] & = B[C,A]_{\mp} \pm [B,A]_{\mp}C \\ & = B[C,A] + [B,A]C = B\{C,A\} - \{B,A\}C \\ \{A,BC\} & = [A,B]C + B\{A,C\} = \{A,B\}C - B[A,C] \\ \{BC,A\} & = B[C,A] + \{B,A\}C = B\{C,A\} - [B,A]C \\ [A,B]^\dagger & = -[A^\dagger, B^\dagger] \\ [AB,CD] & = A[B,CD] + [A,CD]B \\ & = A[B,C]D + AC[B,D] + [A,C]DB + C[A,D]]B \\ & = A\{B,C\}D - AC\{B,D\} + \{A,C\}DB - C\{A,D\}B \\ \{AB,CD\} & = 2ABCD - [AB,CD] = 2CDAB + [AB,CD] \end{cases} \tag{B.5}$$

$$\begin{cases} [\gamma_{k_1}^\dagger \gamma_{k_1}, \gamma_{k_2}^\dagger \eta_{k_2}^\dagger] = \gamma_{k_2}^\dagger [\gamma_{k_1}^\dagger \gamma_{k_1}, \eta_{k_2}^\dagger] + [\gamma_{k_1}^\dagger \gamma_{k_1}, \gamma_{k_2}^\dagger]\eta_{k_2}^\dagger = \delta_{k_1,k_2}\gamma_{k_2}^\dagger \eta_{k_2}^\dagger \\ [\gamma_{k_1}^\dagger \gamma_{k_1}, \eta_{k_2}\gamma_{k_2}] = -[\gamma_{k_1}^\dagger \gamma_{k_1}, \gamma_{k_2}^\dagger \eta_{k_2}^\dagger]^\dagger = -\delta_{k_1,k_2}\eta_{k_2}\gamma_{k_2} \end{cases} \tag{B.6}$$

$$\begin{cases} \{\gamma_{k_1}^\dagger \gamma_{k_1}, \gamma_{k_2}^\dagger \eta_{k_2}^\dagger\} = 2\gamma_{k_1}^\dagger \gamma_{k_1}\gamma_{k_2}^\dagger \eta_{k_2}^\dagger - [\gamma_{k_1}^\dagger \gamma_{k_1}, \gamma_{k_2}^\dagger \eta_{k_2}^\dagger] = 2\gamma_{k_1}^\dagger \gamma_{k_1}\gamma_{k_2}^\dagger \eta_{k_2}^\dagger - \delta_{k_1,k_2}\gamma_{k_2}^\dagger \eta_{k_2}^\dagger \\ \{\gamma_{k_1}^\dagger \gamma_{k_1}, \eta_{k_2}\gamma_{k_2}\} = 2\gamma_{k_1}^\dagger \gamma_{k_1}\eta_{k_2}\gamma_{k_2} - [\gamma_{k_1}^\dagger \gamma_{k_1}, \eta_{k_2}\gamma_{k_2}] = 2\gamma_{k_1}^\dagger \gamma_{k_1}\eta_{k_2}\gamma_{k_2} + \delta_{k_1,k_2}\eta_{k_2}\gamma_{k_2} \end{cases} \tag{B.7}$$

$$\begin{cases} [\gamma_{k_1}^\dagger \eta_{k_1}^\dagger, \gamma_{k_2}^\dagger \eta_{k_2}^\dagger] = \gamma_{k_1}^\dagger [\eta_{k_1}^\dagger, \gamma_{k_2}^\dagger \eta_{k_2}^\dagger] + [\gamma_{k_1}^\dagger, \gamma_{k_2}^\dagger \eta_{k_2}^\dagger]\eta_{k_1}^\dagger \\ = \gamma_{k_1}^\dagger \left(\{\eta_{k_1}^\dagger, \gamma_{k_2}^\dagger\}\eta_{k_2}^\dagger - \gamma_{k_2}^\dagger\{\eta_{k_1}^\dagger, \eta_{k_2}^\dagger\}\right) + \left(\{\gamma_{k_1}^\dagger, \gamma_{k_2}^\dagger\}\eta_{k_2}^\dagger - \gamma_{k_2}^\dagger\{\gamma_{k_1}^\dagger, \eta_{k_2}^\dagger\}\right)\eta_{k_1}^\dagger \\ = 0. \\ [\gamma_{k_1}^\dagger \eta_{k_1}^\dagger, \eta_{k_2}\gamma_{k_2}] = \gamma_{k_1}^\dagger [\eta_{k_1}^\dagger, \eta_{k_2}\gamma_{k_2}] + [\gamma_{k_1}^\dagger, \eta_{k_2}\gamma_{k_2}]\eta_{k_1}^\dagger \\ = \gamma_{k_1}^\dagger \left(\{\eta_{k_1}^\dagger, \eta_{k_2}\}\gamma_{k_2} - \eta_{k_2}\{\eta_{k_1}^\dagger, \gamma_{k_2}\}\right) + \left(\{\gamma_{k_1}^\dagger, \eta_{k_2}\}\gamma_{k_2} - \eta_{k_2}\{\gamma_{k_1}^\dagger, \gamma_{k_2}\}\right)\eta_{k_1}^\dagger \\ = \delta_{k_1,k_2}\left(\gamma_{k_1}^\dagger \gamma_{k_1} - \eta_{k_1}\eta_{k_1}^\dagger\right). \end{cases} \tag{B.8}$$

B.3 Some expectation values appearing the in the global OTOC

- $|\Omega\rangle = \left[\prod_k^{\mathrm{BZ}} (\cos(\Delta\theta_k) + \sin(\Delta\theta_k)\gamma_k^\dagger \eta_k^\dagger)\right]|0\rangle, \qquad \langle\hat{O}\rangle \equiv \langle\Omega|\,\hat{O}\,|\Omega\rangle$
- $\langle\gamma_k^\dagger \gamma_k\rangle = \sin^2(\Delta\theta_k) = \alpha_k$
- $\langle\eta_k \eta_k^\dagger\rangle = \cos^2(\Delta\theta_k) = \beta_k$
- $\langle\gamma_k^\dagger \eta_k^\dagger\rangle = \langle\eta_k\gamma_k\rangle = \sin(\Delta\theta_k)\cos(\Delta\theta_k) = \frac{1}{2}\sin(2\Delta\theta_k) = \lambda_k$
- $$\begin{aligned} \langle\gamma_{k_1}^\dagger \gamma_{k_1}\gamma_{k_2}^\dagger \eta_{k_2}^\dagger\rangle &= (1-\delta_{k_1,k_2})\langle\gamma_{k_1}^\dagger \gamma_{k_1}\gamma_{k_2}^\dagger \eta_{k_2}^\dagger\rangle_{k_1\neq k_2} + \delta_{k_1,k_2}\langle\gamma_{k_1}^\dagger \gamma_{k_1}\gamma_{k_1}^\dagger \eta_{k_1}^\dagger\rangle \\ &= (1-\delta_{k_1,k_2})\langle\gamma_{k_1}^\dagger \gamma_{k_1}\rangle\langle\gamma_{k_2}^\dagger \eta_{k_2}^\dagger\rangle + \delta_{k_1,k_2}\langle\gamma_{k_1}^\dagger \eta_{k_1}^\dagger\rangle \\ &= \langle\gamma_{k_1}^\dagger \gamma_{k_1}\rangle\langle\gamma_{k_2}^\dagger \eta_{k_2}^\dagger\rangle - \delta_{k_1,k_2}\left[\langle\gamma_{k_1}^\dagger \gamma_{k_1}\rangle\langle\gamma_{k_2}^\dagger \eta_{k_2}^\dagger\rangle - \langle\gamma_{k_1}^\dagger \eta_{k_1}^\dagger\rangle\right] \\ &= \langle\gamma_{k_1}^\dagger \gamma_{k_1}\rangle\langle\gamma_{k_2}^\dagger \eta_{k_2}^\dagger\rangle - \delta_{k_1,k_2}\left[\alpha_{k_1}\lambda_{k_1} - \lambda_{k_1}\right] \\ &= \langle\gamma_{k_1}^\dagger \gamma_{k_1}\rangle\langle\gamma_{k_2}^\dagger \eta_{k_2}^\dagger\rangle + \delta_{k_1,k_2}\beta_{k_1}\lambda_{k_1} \end{aligned} \tag{B.9}$$

Where

$$\begin{aligned}
&\langle \gamma^\dagger_{k_1}\gamma_{k_1}\gamma^\dagger_{k_2}\eta^\dagger_{k_2}\rangle_{k_1\neq k_2} \\
&= \langle 0|\,\Box^\dagger\Box[\cos\Delta\theta_{k_1}+\sin\Delta\theta_{k_1}\eta_{k_1}\gamma_{k_1}]\gamma^\dagger_{k_1}\gamma_{k_1}[\cos\Delta\theta_{k_1}+\sin\Delta\theta_{k_1}\gamma^\dagger_{k_1}\eta^\dagger_{k_1}]\times \\
&\qquad\qquad [\cos\Delta\theta_{k_2}+\sin\Delta\theta_{k_2}\eta_{k_2}\gamma_{k_2}]\gamma^\dagger_{k_2}\eta^\dagger_{k_2}[\cos\Delta\theta_{k_2}+\sin\Delta\theta_{k_2}\gamma^\dagger_{k_2}\eta^\dagger_{k_2}]\,|0\rangle \\
&= \langle 0|\,\Box^\dagger\Box[\sin^2\Delta\theta_{k_1}\ \eta_{k_1}\gamma_{k_1}\gamma^\dagger_{k_1}\gamma_{k_1}\gamma^\dagger_{k_1}\eta^\dagger_{k_1}][\sin\Delta\theta_{k_2}\cos\Delta\theta_{k_2}\eta_{k_2}\gamma_{k_2}\gamma^\dagger_{k_2}\eta^\dagger_{k_2}]\,|0\rangle \\
&= \langle 0|\,\Box^\dagger\Box[\sin^2\Delta\theta_{k_1}\ \eta_{k_1}\eta^\dagger_{k_1}\gamma_{k_1}\gamma^\dagger_{k_1}][\sin\Delta\theta_{k_2}\cos\Delta\theta_{k_2}\eta_{k_2}\eta^\dagger_{k_2}\gamma_{k_2}\gamma^\dagger_{k_2}]\,|0\rangle \\
&= \sin^2(\Delta\theta_{k_1})\frac{1}{2}\sin(2\Delta\theta_{k_2})\,\langle 0|\,\Box^\dagger\Box\,|0\rangle \\
&= \sin^2(\Delta\theta_{k_1})\frac{1}{2}\sin(2\Delta\theta_{k_2}) = \langle\gamma^\dagger_{k_1}\gamma_{k_1}\rangle\langle\gamma^\dagger_{k_2}\eta^\dagger_{k_2}\rangle.
\end{aligned}\tag{B.10}$$

with $\Box=\prod_{k\notin\{k_1,k_2\}}^{\mathrm{BZ}}[\cos(\Delta\theta_k)+\sin(\Delta\theta_k)\gamma^\dagger_k\eta^\dagger_k]$ commuting with the chain of operators at k_1 and k_2.

B.4 Some expectation values in the GGE

The expectation values of relevant operators in the GGE are

$$\langle\gamma^\dagger_k\gamma_k\rangle_{\mathrm{GGE}}=\langle\eta^\dagger_k\eta_k\rangle_{\mathrm{GGE}}=\sin^2(\Delta\theta_k)\qquad\text{(by construction)},\tag{B.11}$$
$$\langle\gamma^\dagger_k\eta^\dagger_k\rangle_{\mathrm{GGE}}=0\qquad\text{(unpaired creation and annihilation operators)},\tag{B.12}$$
$$\langle\gamma^\dagger_{k_1}\eta^\dagger_{k_1}\gamma^\dagger_{k_2}\eta^\dagger_{k_2}\rangle_{\mathrm{GGE}}=\langle\gamma^\dagger_{k_1}\eta^\dagger_{k_1}\gamma^\dagger_{k_2}\gamma_{k_2}\rangle_{\mathrm{GGE}}=\langle\eta_{k_2}\eta^\dagger_{k_2}\gamma^\dagger_{k_1}\eta^\dagger_{k_1}\rangle_{\mathrm{GGE}}=0\qquad\text{(similarly)},\tag{B.13}$$
$$\langle\gamma^\dagger_{k_1}\eta^\dagger_{k_1}\eta_{k_2}\gamma_{k_2}\rangle_{\mathrm{GGE}}=\delta_{k_1,k_2}\sin^4(\Delta\theta_{k_1}),\tag{B.14}$$
$$\langle\gamma^\dagger_{k_1}\gamma_{k_1}\eta^\dagger_{k_2}\eta_{k_2}\rangle_{\mathrm{GGE}}=\sin^2(\Delta\theta_{k_1})\sin^2(\Delta\theta_{k_2}),\tag{B.15}$$
$$\langle\gamma^\dagger_{k_1}\gamma_{k_1}\gamma^\dagger_{k_2}\gamma_{k_2}\rangle_{\mathrm{GGE}}=(1-\delta_{k_1,k_2})\sin^2(\Delta\theta_{k_1})\sin^2(\Delta\theta_{k_2})+\delta_{k_1,k_2}\sin^2(\Delta\theta_{k_1})=\tag{B.16}$$
$$\langle\eta^\dagger_{k_1}\eta_{k_1}\eta^\dagger_{k_2}\eta_{k_2}\rangle_{\mathrm{GGE}}=\sin^2(\Delta\theta_{k_1})\sin^2(\Delta\theta_{k_2})+\delta_{k_1,k_2}\sin^2(\Delta\theta_{k_1})\cos^2(\Delta\theta_{k_1}).\tag{B.17}$$

B.5 Some expectation values in calculating the local OTOC

$$\begin{aligned}
\langle\gamma^\dagger_{k_1}\gamma_{k_2}\gamma^\dagger_{k_3}\eta^\dagger_{k_4}\rangle = &\ \ \delta_{k_1,k_2}\delta_{k_3,k_4}(1-\delta_{k_1,k_3})\langle\gamma^\dagger_{k_1}\gamma_{k_1}\gamma^\dagger_{k_3}\eta^\dagger_{k_3}\rangle_{k_1\neq k_3} \\
&+\delta_{k_1,k_3}\delta_{k_2,k_4}(1-\delta_{k_1,k_2})\langle\gamma^\dagger_{k_1}\gamma_{k_2}\gamma^\dagger_{k_1}\eta^\dagger_{k_2}\rangle_{k_1\neq k_2} \\
&+\delta_{k_1,k_4}\delta_{k_2,k_3}(1-\delta_{k_1,k_2})\langle\gamma^\dagger_{k_1}\gamma_{k_2}\gamma^\dagger_{k_2}\eta^\dagger_{k_1}\rangle_{k_1\neq k_2} \\
&+\delta_{k_1,k_2}\delta_{k_3,k_4}\delta_{k_1,k_3}\langle\gamma^\dagger_{k_1}\gamma_{k_1}\gamma^\dagger_{k_1}\eta^\dagger_{k_1}\rangle \\
= &\ \ \delta_{k_1,k_2}\delta_{k_3,k_4}(1-\delta_{k_1,k_3})\langle\gamma^\dagger_{k_1}\gamma_{k_1}\rangle\langle\gamma^\dagger_{k_3}\eta^\dagger_{k_3}\rangle \\
&-\delta_{k_1,k_3}\delta_{k_2,k_4}(1-\delta_{k_1,k_2})\langle\gamma^\dagger_{k_1}\gamma^\dagger_{k_1}\rangle\langle\gamma_{k_2}\eta^\dagger_{k_2}\rangle \\
&+\delta_{k_1,k_4}\delta_{k_2,k_3}(1-\delta_{k_1,k_2})\langle\gamma^\dagger_{k_1}\eta^\dagger_{k_1}\rangle\langle\gamma_{k_2}\gamma^\dagger_{k_2}\rangle \\
&+\delta_{k_1,k_2}\delta_{k_3,k_4}\delta_{k_1,k_3}\langle\gamma^\dagger_{k_1}\gamma_{k_1}\gamma^\dagger_{k_1}\eta^\dagger_{k_1}\rangle \\
= &\ \ \delta_{k_1,k_2}\delta_{k_3,k_4}\langle\gamma^\dagger_{k_1}\gamma_{k_1}\rangle\langle\gamma^\dagger_{k_3}\eta^\dagger_{k_3}\rangle \\
&-\delta_{k_1,k_3}\delta_{k_2,k_4}\langle\gamma^\dagger_{k_1}\gamma^\dagger_{k_1}\rangle\langle\gamma_{k_2}\eta^\dagger_{k_2}\rangle \\
&+\delta_{k_1,k_4}\delta_{k_2,k_3}\langle\gamma^\dagger_{k_1}\eta^\dagger_{k_1}\rangle\langle\gamma_{k_2}\gamma^\dagger_{k_2}\rangle \\
&+\delta_{k_1,k_2}\delta_{k_3,k_4}\delta_{k_1,k_3}\left[\langle\gamma^\dagger_{k_1}\eta^\dagger_{k_1}\rangle-\langle\gamma^\dagger_{k_1}\gamma_{k_1}\rangle\langle\gamma^\dagger_{k_1}\eta^\dagger_{k_1}\rangle+\langle\gamma^\dagger_{k_1}\gamma^\dagger_{k_1}\rangle\langle\gamma_{k_1}\eta^\dagger_{k_1}\rangle-\langle\gamma^\dagger_{k_1}\eta^\dagger_{k_1}\rangle\langle\gamma_{k_1}\gamma^\dagger_{k_1}\rangle\right]
\end{aligned}\tag{B.18}$$

$$\langle\gamma^\dagger_{k_1}\gamma_{k_2}\rangle\langle\gamma^\dagger_{k_3}\eta^\dagger_{k_4}\rangle=\delta_{k_1,k_2}\delta_{k_3,k_4}\langle\gamma^\dagger_{k_1}\gamma_{k_1}\rangle\langle\gamma^\dagger_{k_3}\eta^\dagger_{k_3}\rangle\tag{B.19}$$

Therefore

$$\begin{aligned}
\langle\gamma_{k_1}^\dagger\gamma_{k_2}\gamma_{k_3}^\dagger\eta_{k_4}^\dagger\rangle^c =&\delta_{k_1,k_4}\delta_{k_2,k_3}\lambda_{k_1}\beta_{k_2} + \delta_{k_1,k_2}\delta_{k_3,k_4}\delta_{k_1,k_3}\left(\lambda_{k_1} - \alpha_{k_1}\lambda_{k_1} - \lambda_{k_1}\beta_{k_1}\right)\\
=&\delta_{k_1,k_4}\delta_{k_2,k_3}\lambda_{k_1}\beta_{k_2}\\
=&\delta_{k_1,k_4}\delta_{k_2,k_3}\langle\gamma_{k_1}^\dagger\eta_{k_4}^\dagger\rangle\langle\gamma_{k_2}\gamma_{k_3}^\dagger\rangle.
\end{aligned} \tag{B.20}$$

Appendix C

Some integrals

C.1 To calculate the OTOC in pure state

$$\overline{\mathcal{C}(t)}\Big|_{L\to\infty} = \frac{L}{2\pi}\int_{-\pi/2}^{\pi/2} dk \frac{3}{2}\sin^2(4\theta_{k,f}) + 2\cos^2(2\theta_{k,f})\sin^2(2\theta_{k,i}). \tag{C.1}$$

Steps by steps, the first term is

$$\bullet \frac{3}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dk}{2\pi}\sin^2(4\theta_k) = \frac{3\alpha^2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dk \frac{\cos^2(k+\phi)}{[\cos^2(k+\phi)+\alpha^2]^2} \quad (\text{with } \alpha \equiv \frac{V_f}{2h})$$

$$=\frac{6\alpha^2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dk \frac{1+\cos 2(k+\phi)}{[\cos 2(k+\phi)+\beta]^2} \qquad (\text{with } \beta \equiv 1+2\alpha^2)$$

$$=\frac{3\alpha^2}{\pi}\int_{-\pi}^{\pi} dx \frac{1+\cos x}{[\cos x+\beta]^2} = \frac{3\alpha^2}{\pi}\oint_{|z|=1} \frac{dz}{iz}\frac{1+\frac{z+z^{-1}}{2}}{\left[\frac{z+z^{-1}}{2}+\beta\right]^2}$$

$$=\frac{6\alpha^2}{\pi}\oint_{|z|=1} \frac{dz}{i}\frac{(z+1)^2}{(z^2+2\beta z+1)^2} = \frac{6\alpha^2}{\pi}\frac{2\pi i}{i}\frac{1}{2(\beta+1)\sqrt{\beta^2-1}}$$

$$=6\alpha^2\frac{1}{(2+2\alpha^2)\sqrt{4\alpha^4+4\alpha^2}} = \frac{3|\alpha|}{2}\frac{1}{(1+\alpha^2)^{3/2}} = \frac{3}{2}\left|\frac{V_f}{2h}\right|\left[1+\left(\frac{V_f}{2h}\right)^2\right]^{-3/2}. \tag{C.2}$$

The second term is

$$\bullet \int_{-\pi/2}^{\pi/2} dk\cos^2(2\theta_{k,f})\sin^2(2\theta_{k,i}) = \int_{-\pi/2}^{\pi/2} dk\frac{\cos^2(k+\phi)}{\cos^2(k+\phi)+\alpha_f^2}\frac{\alpha_i^2}{\cos^2(k+\phi)+\alpha_i^2}$$

$$= \frac{\alpha_i^2}{\alpha_f^2-\alpha_i^2}\int_{-\pi/2}^{\pi/2} dk\cos^2 k\left[\frac{1}{\cos^2 k+\alpha_i^2} - \frac{1}{\cos^2 k+\alpha_f^2}\right]. \qquad (\text{with } \alpha_{i,f} \equiv \frac{V_{i,f}}{2h})$$

$$\bullet \int_{-\pi/2}^{\pi/2} dk\frac{\cos^2 k}{\cos^2 k+\alpha^2} = \int_{-\pi/2}^{\pi/2} dk\frac{1+\cos 2k}{\cos 2k+\beta} \qquad (\text{with } \beta_{i,f} \equiv 1+2\alpha_{i,f}^2)$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} dx\frac{1+\cos x}{\cos x+\beta} = \frac{1}{2}\oint_{|z|=1}\frac{dz}{iz}\frac{(z+1)^2}{z^2+2\beta z+1} = \pi\left[1+\frac{\left(\sqrt{\beta^2-1}-\beta+1\right)^2}{2\sqrt{\beta^2-1}\left(\sqrt{\beta^2-1}-\beta\right)}\right]$$

$$= \pi\left(1-\frac{|\alpha|}{\sqrt{\alpha^2+1}}\right).$$

$$\bullet \Rightarrow \int_{-\pi/2}^{\pi/2} dk\cos^2(2\theta_{k,f})\sin^2(2\theta_{k,i}) = \frac{\pi\alpha_i^2}{\alpha_f^2-\alpha_i^2}\left(\frac{|\alpha_f|}{\sqrt{\alpha_f^2+1}} - \frac{|\alpha_i|}{\sqrt{\alpha_i^2+1}}\right)$$

$$= \frac{\pi V_i^2}{V_f^2 - V_i^2}\left(\frac{|V_f|}{\sqrt{V_f^2+4h^2}} - \frac{|V_i|}{\sqrt{V_i^2+4h^2}}\right). \tag{C.3}$$

Therefore

$$\overline{\mathcal{C}(t)}\Big|_{L\to\infty} = \frac{3L}{2}\left|\frac{V_f}{2h}\right|\left[1+\left(\frac{V_f}{2h}\right)^2\right]^{-3/2} + L\frac{V_i^2}{V_f^2-V_i^2}\left(\frac{|V_f|}{\sqrt{V_f^2+4h^2}} - \frac{|V_i|}{\sqrt{V_i^2+4h^2}}\right). \tag{C.4}$$

C.2 To calculate the OTOC in GGE

In the thermodynamic limit $L \to \infty$,

$$\begin{aligned}
\overline{\mathcal{C}_{\mathrm{GGE}}}\Big|_{L\to\infty} =& \frac{L}{2\pi}\int_{-\pi/2}^{\pi/2} dk \quad 8\alpha_f^2 \frac{\cos^2 k}{(\cos^2 k + \alpha_f^2)^2} \\
&- \frac{L}{2\pi}\int_{-\pi/2}^{\pi/2} dk \quad 4\alpha_f^2(\alpha_f-\alpha_i)^2 \frac{\cos^4 k}{(\cos^2 k+\alpha_f^2)^3(\cos^2 k+\alpha_i^2)} \\
&+ \frac{L}{2\pi}\int_{-\pi/2}^{\pi/2} dk \quad (\alpha_f-\alpha_i)^2 \frac{\cos^6 k}{(\cos^2 k+\alpha_f^2)^3(\cos^2 k+\alpha_i^2)} \\
=& \frac{L}{2\pi}\int_{-\pi/2}^{\pi/2} dk \quad 16\alpha_f^2 \frac{1+\cos 2k}{(\cos 2k+\beta_f)^2} \\
&- \frac{L}{2\pi}\int_{-\pi/2}^{\pi/2} dk \quad 16\alpha_f^2(\alpha_f-\alpha_i)^2 \frac{(1+\cos 2k)^2}{(\cos 2k+\beta_f)^3(\cos 2k+\beta_i)} \\
&+ \frac{L}{2\pi}\int_{-\pi/2}^{\pi/2} dk \quad 2(\alpha_f-\alpha_i)^2 \frac{(1+\cos 2k)^3}{(\cos 2k+\beta_f)^3(\cos 2k+\beta_i)} \\
=& \frac{L}{2\pi}\int_{-\pi}^{\pi} dx \quad 8\alpha_f^2 \frac{1+\cos x}{(\cos x+\beta_f)^2} \\
&- \frac{L}{2\pi}\int_{-\pi}^{\pi} dx \quad 8\alpha_f^2(\alpha_f-\alpha_i)^2 \frac{(1+\cos x)^2}{(\cos x+\beta_f)^3(\cos x+\beta_i)} \\
&+ \frac{L}{2\pi}\int_{-\pi}^{\pi} dx \quad (\alpha_f-\alpha_i)^2 \frac{(1+\cos x)^3}{(\cos x+\beta_f)^3(\cos x+\beta_i)} \\
=& \frac{L}{2\pi}\oint_{|z|=1} \frac{dz}{iz} \quad 8\alpha_f^2 \frac{2z(z+1)^2}{(z^2+2\beta_f z+1)^2} \\
&- \frac{L}{2\pi}\oint_{|z|=1} \frac{dz}{iz} \quad 8\alpha_f^2(\alpha_f-\alpha_i)^2 \frac{(2z)^2(z+1)^4}{(z^2+2\beta_f z+1)^3(z^2+2\beta_i z+1)} \\
&+ \frac{L}{2\pi}\oint_{|z|=1} \frac{dz}{iz} \quad (\alpha_f-\alpha_i)^2 \frac{(2z)(z+1)^6}{(z^2+2\beta_f z+1)^3(z^2+2\beta_i z+1)}
\end{aligned} \tag{C.5}$$

- $\frac{1}{2\pi}\oint_{|z|=1} \frac{dz}{iz} \quad \frac{2z(z+1)^2}{(z^2+2\beta_f z+1)^2} = \frac{1}{4\alpha_f\left(\alpha_f^2+1\right)^{3/2}}$

- $\frac{1}{2\pi}\oint_{|z|=1} \frac{dz}{iz} \quad \frac{(2z)^2(z+1)^4}{(z^2+2\beta_f z+1)^3(z^2+2\beta_i z+1)}$

$$= -\frac{\alpha_i^3}{4\sqrt{\alpha_i^2+1}\left(\alpha_i^2-\alpha_f^2\right)^3}$$

$$
- \frac{\left(\alpha_f - \sqrt{\alpha_f^2+1}\right)^2 \left(-8\alpha_f^4 - 8\alpha_f^2 + 4\sqrt{\alpha_f^2+1}\alpha_f + 8\sqrt{\alpha_f^2+1}\alpha_f^3 - 1\right)\left(\alpha_f^6\left(8\alpha_i^2-4\right) + \alpha_f^4\left(20\alpha_i^2-1\right) + 6\alpha_f^2\alpha_i^2 + 3\alpha_i^4\right)}{32\alpha_f\left(\alpha_f^2+1\right)^{5/2}\left(-2\alpha_f^2 + 2\sqrt{\alpha_f^2+1}\alpha_f - 1\right)^3\left(\alpha_f^2-\alpha_i^2\right)^3}
$$

$$
\bullet \frac{1}{2\pi}\oint\limits_{|z|=1}\frac{dz}{iz} \quad \frac{(2z)(z+1)^6}{(z^2+2\beta_f z+1)^3(z^2+2\beta_i z+1)}
$$

$$
\begin{aligned}
=&\frac{\alpha_i^5}{2\sqrt{\alpha_i^2+1}\left(\alpha_i^2-\alpha_f^2\right)^3} \\
&+ \frac{\alpha_f\left(8\alpha_f^4 + 8\alpha_f^2 - 4\sqrt{\alpha_f^2+1}\alpha_f - 8\sqrt{\alpha_f^2+1}\alpha_f^3 + 1\right)^2\left(\alpha_f^4\left(8\alpha_i^4 - 4\alpha_i^2 + 3\right) + 10\alpha_f^2\alpha_i^2\left(2\alpha_i^2-1\right) + 15\alpha_i^4\right)}{16\left(\alpha_f^2+1\right)^{5/2}\left(2\alpha_f^2 - 2\sqrt{\alpha_f^2+1}\alpha_f + 1\right)^4\left(\alpha_f^2-\alpha_i^2\right)^3}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{1}{L}\,\overline{\mathcal{C}_{\mathrm{GGE}}}\Big|_{L\to\infty} =&\frac{2\alpha_f}{\left(\alpha_f^2+1\right)^{3/2}} + \frac{4\alpha_f^2\alpha_i^3 + \alpha_i^5}{2\sqrt{\alpha_i^2+1}(\alpha_i-\alpha_f)(\alpha_f+\alpha_i)^3} \\
&+ \frac{16\alpha_f^7\left(2\alpha_i^2-1\right) + \alpha_f^5\left(8\alpha_i^4 + 76\alpha_i^2 - 1\right) + 2\alpha_f^3\alpha_i^2\left(10\alpha_i^2+7\right) + 27\alpha_f\alpha_i^4}{16\left(\alpha_f^2+1\right)^{5/2}(\alpha_f-\alpha_i)(\alpha_f+\alpha_i)^3}
\end{aligned} \tag{C.6}
$$

Appendix D

A formula for the correlator

For an operator $M(t)$ that has the following structure, its correlator is easily calculated by a formula proved in this appendix.

$$M_n(t) = \frac{1}{L}\sum_{k_1,k_2}^{\mathrm{BZ}} \begin{pmatrix} \gamma^\dagger_{k_1} \\ \eta_{k_1} \end{pmatrix}^T \begin{pmatrix} A^n_{k_1,k_2}(t) & B^n_{k_1,k_2}(t) \\ C^n_{k_1,k_2}(t) & D^n_{k_1,k_2}(t) \end{pmatrix} \begin{pmatrix} \gamma_{k_2} \\ \eta^\dagger_{k_2} \end{pmatrix} \tag{D.1}$$

$$= \frac{1}{L}\sum_{k_1,k_2}^{\mathrm{BZ}} A^n_{k_1,k_2}(t)\gamma^\dagger_{k_1}\gamma_{k_2} + B^n_{k_1,k_2}(t)\gamma^\dagger_{k_1}\eta^\dagger_{k_2} + C^n_{k_1,k_2}(t)\eta_{k_1}\gamma_{k_2} + D^n_{k_1,k_2}(t)\eta_{k_1}\eta^\dagger_{k_2} \tag{D.2}$$

Steps by steps

$$\left[\gamma^\dagger_{k_1}\gamma_{k_2}, M_0(0)\right] = \frac{1}{L}\sum_{k'}\left(A^0_{k_2,k'}\gamma^\dagger_{k_1}\gamma_{k'} + B^0_{k_2,k'}\gamma^\dagger_{k_1}\eta^\dagger_{k'}\right) - \left(A^0_{k',k_1}\gamma^\dagger_{k'}\gamma_{k_2} + C^0_{k',k_1}\eta_{k'}\gamma_{k_2}\right) \tag{D.3}$$

$$\left[\gamma^\dagger_{k_1}\eta^\dagger_{k_2}, M_0(0)\right] = \frac{1}{L}\sum_{k'}\left(C^0_{k_2,k'}\gamma^\dagger_{k_1}\gamma_{k'} + D^0_{k_2,k'}\gamma^\dagger_{k_1}\eta^\dagger_{k'}\right) - \left(C^0_{k',k_1}\eta_{k'}\eta^\dagger_{k_2} + A^0_{k',k_1}\gamma^\dagger_{k'}\eta^\dagger_{k_2}\right) \tag{D.4}$$

$$\left[\eta_{k_1}\gamma_{k_2}, M_0(0)\right] = \frac{1}{L}\sum_{k'}\left(B^0_{k_2,k'}\eta_{k_1}\eta^\dagger_{k'} + A^0_{k_2,k'}\eta_{k_1}\gamma_{k'}\right) - \left(B^0_{k',k_1}\gamma^\dagger_{k'}\gamma_{k_2} + D^0_{k',k_1}\eta_{k'}\gamma_{k_2}\right) \tag{D.5}$$

$$\left[\eta_{k_1}\eta^\dagger_{k_2}, M_0(0)\right] = \frac{1}{L}\sum_{k'}\left(D^0_{k_2,k'}\eta_{k_1}\eta^\dagger_{k'} + C^0_{k_2,k'}\eta_{k_1}\gamma_{k'}\right) - \left(D^0_{k',k_1}\eta_{k'}\eta^\dagger_{k_2} + B^0_{k',k_1}\gamma^\dagger_{k'}\eta^\dagger_{k_2}\right) \tag{D.6}$$

Therefore

$$\begin{aligned}
&[M_n(t), M_0(0)] \\
= &\frac{1}{L^2}\sum_{k_1,k_2,k'}^{\mathrm{BZ}} \quad \gamma^\dagger_{k_1}\gamma_{k'}\left[A^n_{k_1,k_2}(t)A^0_{k_2,k'} + B^n_{k_1,k_2}(t)C^0_{k_2,k'}\right] + \gamma^\dagger_{k_1}\eta^\dagger_{k'}\left[A^n_{k_1,k_2}(t)B^0_{k_2,k'} + B^n_{k_1,k_2}(t)D^0_{k_2,k'}\right] \\
&\qquad + \eta_{k_1}\gamma_{k'}\left[C^n_{k_1,k_2}(t)A^0_{k_2,k'} + D^n_{k_1,k_2}(t)C^0_{k_2,k'}\right] + \eta_{k_1}\eta^\dagger_{k'}\left[C^n_{k_1,k_2}(t)B^0_{k_2,k'} + D^n_{k_1,k_2}(t)D^0_{k_2,k'}\right] \\
&- \frac{1}{L^2}\sum_{k_1,k_2,k'}^{\mathrm{BZ}} \quad \gamma^\dagger_{k'}\gamma_{k_2}\left[A^n_{k_1,k_2}(t)A^0_{k',k_1} + C^n_{k_1,k_2}(t)B^0_{k',k_1}\right] + \gamma^\dagger_{k'}\eta^\dagger_{k_2}\left[B^n_{k_1,k_2}(t)A^0_{k',k_1} + D^n_{k_1,k_2}(t)B^0_{k',k_1}\right] \\
&\qquad + \eta_{k'}\gamma_{k_2}\left[A^n_{k_1,k_2}(t)C^0_{k',k_1} + C^n_{k_1,k_2}(t)D^0_{k',k_1}\right] + \eta_{k'}\eta^\dagger_{k_2}\left[B^n_{k_1,k_2}(t)C^0_{k',k_1} + D^n_{k_1,k_2}(t)D^0_{k',k_1}\right] \\
= &\frac{1}{L^2}\sum_{k_1,k_2,k'}^{\mathrm{BZ}} \quad \gamma^\dagger_{k_1}\gamma_{k_2}\left[A^n_{k_1,k'}(t)A^0_{k',k_2} + B^n_{k_1,k'}(t)C^0_{k',k_2} - A^n_{k',k_2}(t)A^0_{k_1,k'} - C^n_{k',k_2}(t)B^0_{k_1,k'}\right] \\
&\qquad + \gamma^\dagger_{k_1}\eta^\dagger_{k_2}\left[A^n_{k_1,k'}(t)B^0_{k',k_2} + B^n_{k_1,k'}(t)D^0_{k',k_2} - B^n_{k',k_2}(t)A^0_{k_1,k'} - D^n_{k',k_2}(t)B^0_{k_1,k'}\right]
\end{aligned}$$

$$+ \eta_{k_1}\gamma_{k_2}\left[C^n_{k_1,k'}(t)A^0_{k',k_2} + D^n_{k_1,k'}(t)C^0_{k',k_2} - A^n_{k',k_2}(t)C^0_{k_1,k'} - C^n_{k',k_2}(t)D^0_{k_1,k'}\right]$$

$$+ \eta_{k_1}\eta^\dagger_{k_2}\left[C^n_{k_1,k'}(t)B^0_{k',k_2} + D^n_{k_1,k'}(t)D^0_{k',k_2} - B^n_{k',k_2}(t)C^0_{k_1,k'} - D^n_{k',k_2}(t)D^0_{k_1,k'}\right]$$

Therefore

$$\begin{aligned}
[M_n(t), M_0(0)] = \frac{1}{L^2}\sum_{k_1,k_2,k'}^{\mathrm{BZ}} &\begin{pmatrix}\gamma^\dagger_{k_1}\\ \eta_{k_1}\end{pmatrix}^T \begin{pmatrix}A^n_{k_1,k'}(t) & B^n_{k_1,k'}(t)\\ C^n_{k_1,k'}(t) & D^n_{k_1,k'}(t)\end{pmatrix}\begin{pmatrix}A^0_{k',k_2} & B^0_{k',k_2}\\ C^0_{k',k_2} & D^0_{k',k_2}\end{pmatrix}\begin{pmatrix}\gamma_{k_2}\\ \eta^\dagger_{k_2}\end{pmatrix}\\
&- \begin{pmatrix}\gamma^\dagger_{k_1}\\ \eta_{k_1}\end{pmatrix}^T \begin{pmatrix}A^0_{k_1,k'} & B^0_{k_1,k'}\\ C^0_{k_1,k'} & D^0_{k_1,k'}\end{pmatrix}\begin{pmatrix}A^n_{k',k_2}(t) & B^n_{k',k_2}(t)\\ C^n_{k',k_2}(t) & D^n_{k',k_2}(t)\end{pmatrix}\begin{pmatrix}\gamma_{k_2}\\ \eta^\dagger_{k_2}\end{pmatrix}.
\end{aligned} \tag{D.7}$$

References

[1] B. Swingle. “Unscrambling the physics of out-of-time-order correlators”. In: Nature Physics **14**.10 (2018), 988–990.

[2] M. Gärttner et al. “Measuring out-of-time-order correlations and multiple quantum spectra in a trapped-ion quantum magnet”. In: Nature Physics **13**.8 (2017), 781–786.

[3] J. Li et al. “Measuring Out-of-Time-Order Correlators on a Nuclear Magnetic Resonance Quantum Simulator”. In: Physical Review X **7**.3 (2017).

[4] D. A. Roberts and D. Stanford. “Diagnosing Chaos Using Four-Point Functions in Two-Dimensional Conformal Field Theory”. In: Physical Review Letters **115**.13 (2015).

[5] D. A. Roberts and B. Swingle. “Lieb-Robinson Bound and the Butterfly Effect in Quantum Field Theories”. In: Physical Review Letters **117**.9 (2016).

[6] R.-Q. He and Z.-Y. Lu. “Characterizing many-body localization by out-of-time-ordered correlation”. In: Phys. Rev. B **95** (5 2017), 054201.

[7] I. Kukuljan, S. Grozdanov, and T. Prosen. “Weak quantum chaos”. In: Physical Review B **96**.6 (2017).

[8] L. Vidmar and M. Rigol. “Generalized Gibbs ensemble in integrable lattice models”. In: Journal of Statistical Mechanics: Theory and Experiment **2016**.6 (2016), 064007.

[9] M. Girardeau. “Relationship between Systems of Impenetrable Bosons and Fermions in One Dimension”. In: Journal of Mathematical Physics **1**.6 (1960), 516–523.

[10] B. Paredes et al. “Tonks–Girardeau gas of ultracold atoms in an optical lattice”. In: Nature **429**.6989 (2004), 277–281.

[11] M. Rigol et al. “Relaxation in a Completely Integrable Many-Body Quantum System: An Ab Initio Study of the Dynamics of the Highly Excited States of 1D Lattice Hard-Core Bosons”. In: Phys. Rev. Lett. **98** (5 2007), 050405.

[12] E. T. Jaynes. “Information Theory and Statistical Mechanics”. In: Physical Review **106**.4 (1957), 620–630.

[13] F. H. L. Essler and M. Fagotti. “Quench dynamics and relaxation in isolated integrable quantum spin chains”. In: Journal of Statistical Mechanics: Theory and Experiment **2016**.6 (2016), 064002.

[14] C.-J. Lin and O. I. Motrunich. “Out-of-time-ordered correlators in a quantum Ising chain”. In: Phys. Rev. B **97** (14 2018), 144304.

[15] M. Brenes et al. “Out-of-time-order correlations and the fine structure of eigenstate thermalization”. In: Phys. Rev. E **104** (3 2021), 034120.