DRAFT NOTES 0.05: SOLUTIONS TO PROBLEMS IN COMMUTATIVE ALGEBRA

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1. Prelimanaries: Motivation and References

The central motivation behind our study of commutative algebra is to develop tools to handle algebraic geometry. For example, of central importance is Hilbert's Nullstellensatz,

Theorem 1.1. If a field $\mathbb{K}$ is algebraically closed, then for any ideal I in $K[x_1, ..., x_n]$, we have $I(Z(I)) = \sqrt{I}$

Here Z is a variety, $Z(I) = \{(a_1, ..., a_n) \in \mathbb{K}^n | f(a_1, ..., a_n) = 0 \, \forall f \in I\}$, and $\sqrt{I}$ refers to the radical of I, that is $\{a \in A | a^n \in I \, \exists n > 0\}$. You might notice that this appears to be a generalisation of the fundamental theorem of Algebra - in fact if we consider the field $\mathbb{K} = \mathbb{C}$ it is just the fundamental theorem of algebra for arbitrary polynomials. This is therefore a very useful tool, recall for example, the circle is the zero-set of $x^2 + y^2 - 1$, or the hyperboloid is the zero-set of $x^2 + y^2 - z^2 - 1$. In Algebraic geometry, we are particularly fascinated with the *loci* (that is the zero-sets of polynomials). There are many more tools that will be formed from our future study, but hopefully, this short example sets some inspiration.

More will be explained in my 'comments on Mozgovoy's Lecture Notes', but worth keeping in mind, for now, is that in commutative algebra, we often consider two types of rings: co-ordinate rings, and rings of algebraic integers in number fields. Throughout these notes, I will do my best to intervene with geometric examples, or better motivation for the given examples when possible, and hopefully, then the precise reasoning for each aspect of this course will reveal itself to you over time.

There are a total of five key texts that I have read (parts of, or all of) as part of writing these complementary notes, the first three should be read as Mozgovoy has recommended them, the fourth book provides rather good intuition as to the underlying framework of things, and the fifth one is of more interest if you want to do algebraic geometry next year:
Atiyah and Macdonald: Introduction to commutative algebra
Eisenbud: Commutative algebra
Matsumura: Commutative algebra
Cox, Little, O'Shea: Ideals, Varieties, and Algorithms An Introduction to Computational Algebraic Geometry and Commutative Algebra
Liu: Algebraic Geometry and Arithmetic Curves

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2. Solutions to Mozgovoy's Exercises

Exercise 1.5. Assume p = (x) is in some (y), then $x^n = y^m \, \exists m, n$...

Remark. I also have solved all the exercises in class, but I am positive at least some of these are going to be homework, and I'd like to avoid being disciplined over that! If you need advice just email me. I also have typed up notes on Mozgovoy's but these notes are personalised, and really only useful except for me explaining things, I might try to elaborate on them if I have time.

1. Solutions To Problems in Atiyah's Introduction To Commutative Algebra

1.1. Chapter 1.

Exercise 1.1. On first glance one should hopefully see 2 options: either we prove this directly or equivalently show that $(1 + x) = A = (1) \Leftrightarrow 1 + x \in A^x$. It turns out, both ways are nice, though the first is hardly anything instructive. Proving the first we use that $(1 + x)(1 + (-1)^1 x^1 + ... + (-1)^{n-1} x^{n-1}) = 1$, it is boring... Let us now use the second method: $x \in \mathcal{R} \Rightarrow x \in N(A) \subseteq J(A)$... Remark 1.22 of Sergey Mozgovoy's notes, and then: $\Rightarrow x \in J(A) \Rightarrow (1 + x) = A$... Lemma 1.24 of Sergey Mozgovoy's notes

Exercise 1.2. For the base case: r = 0 as $f \cdot f^{-1} = 1$, we have that $a_n b_m = 0$. Let us now consider arbitrary r ensures that $a_n^r b_{m-r+1} = 0$, it must then be that the coefficient of x^{m-n+r} in $f \cdot f^{-1}$ vanishes when $m - n + r \geq r$, that is to say: $a_n b_{m-r} + a_{n-1} b_{m-r+1} + ... = 0$, and multiplying with a_n^r, we have then that $a_n^{r+1} b_{m+r} + a_n^r a_{n-1} b_{m-r+1} + ... = 0$, by the inductive hypothesis all other elements are killed off, and so $a_n^{r+1} b_{m-r} = 0$, setting $r = m$, we have that a_n is nilpotent. The ($\Leftarrow$) side is immediate by exercise 1., we use induction on n for the ($\Rightarrow$) side, we know that a_n is nilpotent, and it must therefore be that $a_n x^n$ is nilpotent, and hence via exercise 1: $f - a_n x^n$ is invertible as well, then by the induction hypothesis (that it holds for those of a degree less), a_0 is invertible, and $a_1, ..., a_{n-1}$ are all nilpotent.
ii ($\Leftarrow$) If we have n elements, then f is nilpotent of degree $n(m - 1)$, where m is such that $a_i^m = 0 \, \forall i$. It is nilpotent because, to ensure we have every element has degree 1 at least, this requires f^1 clearly. To ensure degree 2 is essentially a combinatorial question, we just need to eliminate all possible degree 1 combinations (this will require putting it to the n-th power as the maximal combination is always when you just have all n elements multiplied together), so doing this just $m - 1$ times will be sufficient. ($\Rightarrow$) This follows by induction, when $n = 1$, it is quite immediate.$f - a_{N+1} x^{N+1}$ is nilpotent as well (as per ($\Leftarrow$) and the fact that the leading co-efficient must be nilpotent), and so invoking the induction hypothesis we have the $N + 1$ case.
iii We have that $fg = 0$ for some polynomial g, we pick g such that it's degree m is minimal. $a_n b_m = 0$ as we consider the leading co-efficients, so then $a_n g f = 0$, but $a_n g$ is of degree less than m - so no such minimal m can exist, so m must be equal to zero.
iv ($\Rightarrow$) $a_i b_j y$ can be generated by $a_i y$ or $b_j y$, we can then see that $(1) = (a_1 b_1, ... a_n b_m)$ which is a subset of $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$. ($\Leftarrow$) The contrapositive follows by

the same technique we used in the other direction. This is an extension of Gauss' Lemma (see Mozgovoy's FRM notes).

Exercise 1.4. This follows from $A[x]$ being a finitely generated algebra. (See theorem 5.26 in Mozgovoy's Notes). We provide an alternative proof using exercise 2: We have that $\mathcal{J} \supset \mathcal{R}$ (see Mozgovoy Remark 1.22), so we only need prove $\mathcal{R} \supset \mathcal{J}$. Assume $f(x) \in \mathcal{J}$, then $1 - f(x)g(x)$ is invertible, for all $g(x) \in A[x]$, let $g(x) = x$, i.e. $1 - f(x)x$ is a unit, and then by exercise 2.1 we can assume that $a_1, ...a_n$ are nilpotent. This means all the co-efficients of f are nilpotent, which means by exercise 2.2 we have that f itself is nilpotent.

Exercise 1.5. ($\Rightarrow$) $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i b_j x^{i+j}$ - this direction should be clear. ($\Leftarrow$) The inverse can be derived by long-division (divide 1 by our polynomial, the only inverse we require is an inverse of a_0) - there are (I would assume) other ways to find this fact, but this is the most elementary.
ii No, consider the counter-example: $f = \sum_i a_i x^i$ s.t. for each i we have $a_i^i = 0$ and $j < i \Rightarrow a_i^j \neq 0$.
iii This follows as per proposition 1.9 in Atiyah, combined with 5.i. The intuition behind this is that in the context of 5.i one may naturally consider that ideals consist of non-ideals. Proposition 1.9 is also a commonly used one, the identity $1 - r$ or $1 - xy$ is also commonly considered regarding rings.
v The mapping is a ring homomorphism, hence by lemma 1.14 (Mozgovoy) is prime

Exercise 1.6. I have still to solve this

Exercise 1.7. This is an exercise in Sergey's Notes, my solution is provided in the commentary

Exercise 1.8. This is just applying Zorn's lemma on decreasing chains rather than increasing chains (see Mozgovoy's proof of existence of maximal ideals)

Exercise 1.9. ($\Leftarrow$) Let us assume $x^m \in a, p_1, p_2, ... \Rightarrow x^m \in r(a)$. Let us assume $x^m \in r(a), p_1, p_2, ...$, so then $x^m x^n \in a$ by induction on n we have that $x^m \in r(a)$, so then $a = r(a)$. ($\Rightarrow$) Proposition 1.14 (Atiyah)

Exercise 1.10.

2. Further Resources For Learning Commutative Algebra