

DRAFT NOTES 0.01: SOLUTIONS TO PROBLEMS IN GROUPS AND GEOMETRY

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1. PRELIMANARIES: MOTIVATION AND REFERENCES

2. SOLUTIONS TO EXERCISES IN TOM'S NOTES

Exercise 2.1. $x \sim x$ via identity, $x \sim y \Rightarrow y \sim x$ via inverse, $x \sim y, y \sim z \Rightarrow x \sim z$ as $x = g_1 g_2 z$, when $x = g_1 y, y = g_2 z$

Exercise 2.2. $|\Pi_{g \in G} X^g| = |\Pi_{x \in X} \text{Stab}(x)|$, by Orbit Stabiliser Theorem $|G/\text{Stab}(x)| = |\text{Orb}(x)|$, now by **Lagrange** (and not without it) we have $|\text{Orb}(x)| = |G|/|\text{Stab}(x)|$, so then $|\Pi_{x \in X} \text{Stab}(x)| = |\Pi_{x \in X} |\text{Stab}(x)|| = \sum_{x \in X} \frac{|G|}{|\text{Orb}(x)|} = |G| \sum_{O \in X/G} \sum_{x \in O} \frac{1}{|O|} = |G||X/G|$

Exercise 2.3. We can see that the problem unravels nicely if we try to use a minimum amount of assumptions at each point - this is a common strategy in analysis (see Tao's blog). $f(d(x, x)) = f(0) = 0$, $f(d(x, y)) \geq d(x, y) > 0$, $|f(d(x, y)) - f(d(y, x))| = |f(d(x, y)) - f(d(x, y))| = 0 \Rightarrow f(d(x, y)) = f(d(y, x))$, $f(d(x, y)) + f(d(y, z)) \geq f(d(x, y) + d(y, z)) \geq f(d(x, z))$

Exercise 2.4. Consider an arbitrary projection from $\mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$. The second part follows by pigeon-hole principle.

Exercise 2.5. It is clear that $n + 1$ points uniquely define an isometry: providing full rigour evades me right now - though I have a general idea that one could prove the 2-d case using intersection of circles, and then use induction: this would be extremely tedious but irrefutable (one would have to prove for example the intersection of circles sufficiently close (circles not spheres! - the proof would fail if we considered the spherical version) is a lower-dimensional circle, this can probably be done by finding the hyperplane the two circles sit on and solving very annoying algebraic equations. This would allow one to then use induction.