DRAFT NOTES 0.01: SOLUTIONS TO PROBLEMS IN GROUPS AND GEOMETRY

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1. Prelimanaries: Motivation and References

2. Solutions to Exercises in Tom's Notes

Exercise 2.1. $x \sim x$ via identity, $x \sim y \Rightarrow y \sim x$ via inverse, $x \sim y, y \sim z \Rightarrow x \sim z$ as $x = g_1 g_2 z$, when $x = g_1 y, y = g_2 z$

Exercise 2.2. $|\amalg_{g \in G} X^g| = |\amalg_{x \in X} Stab(x)|$, by Orbit Stabiliser Theorem $|G/Stab(x)| = |Orb(x)|$, now by **Lagrange** (and not without it) we have $|Orb(x)| = |G|/|Stab(x)|$, so then $|\amalg_{x \in X} Stab(x)| = \amalg_{x \in X} |Stab(x)| = \sum_{x \in X} \frac{|G|}{|Orb(x)|} = |G| \sum_{O \in X/G} \sum_{x \in O} \frac{1}{|O|} = |G||x/G|$

Exercise 2.3. We can see that the problem unravels nicely if we try to use a minimum amount of assumptions at each point - this is a common strategy in analysis (see Tao's blog). $f(d(x,x)) = f(0) = 0$, $f(d(x,y)) > f(0) = 0, |f(d(x,y)) - f(d(y,x))| = |f(d(x,y)) - f(d(x,y))| = 0 \Rightarrow f(d(x,y)) = f(d(y,x)), f(d(x,y)) + f(d(y,z)) \geq f(d(x,y) + d(y,z)) \geq f(d(x,z))$

Exercise 2.4. Consider an arbitary projection from $\mathbb{R}^n \to \mathbb{R}^{n+1}$. The second part follows by pigeon-hole principle.

Exercise 2.5. It is clear that $n+1$ points uniquely define an isometry: providing full rigour evades me right now - though I have a general idea that one could prove the 2-d case using intersection of circles, and then use induction: this would be extremely tedious but irrefutable (one would have to prove for example the intersection of circles sufficiently close (circles not spheres! - the proof would fail if we considered the spherical version) is a lower-dimensional circle, this can probably be done by finding the hyperplane the two circles sit on and solving very annoying algebraic equations. This would allow one to then use induction.

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