# NOTES ON GEOMETRY AND GROUPS

US

## 1. Introduction

Contents



## 2. Notation and things

## 3. Week 1

### 3.1. Group Actions.

**Definition 3.1** (Group Action)**.** Let $G$ be a group and $X$ be a set. We say that $G$ acts on $X$ or $G \circlearrowleft X$ if there is a map

$$G \times X \to X$$

$$g \mapsto g \cdot x$$

Such that

$$\forall x \in X, e \cdot x = x$$

where $e$ is the identity and

$$g \cdot (h \cdot x) = (gh) \cdot x$$

Furthermore if $G$ acts on $X$ there is a group homomorphism

$$\Theta : G \to Sym(X),$$

where $Sym(X)$ denotes the group of all bijections (symmetries) on the set $X$.

**Example 3.2.** We list some examples of group actions.

(1) Any group $G \circlearrowleft G$ by $g \cdot h = gh$, by definition any group acts on itself by left (right) multiplication;
(2) Similarly a group acts by itself bby conjugation $G \circlearrowleft G$ by $g \cdot h = ghg^{-1}$;
(3) $GL_n(\mathbb{R}) \circlearrowleft \mathbb{R}^n$ more generally one can get an action of $G$ on $G$ by any group automorphism.

*Date*: January 29, 2024.

(4) The Möbius group $SL_2(\mathbb{C}) \circlearrowleft \hat{\mathbb{C}}$, the Riemann sphere, by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z := \frac{az+b}{cz+d}$$

**Definition 3.3** (Faithful Group Action)**.** We say a group action $G \circlearrowleft X$ is **faithful** if $\forall g \in G/\{e\}\, \exists x \in X \; g \cdot x \neq x$. This definition is equivalen to saying that the homomorphism $\Theta : G \mapsto Sym(X)$ is injective.

**Example 3.4.** The Galois group $Gal(K/F)$ acts faithfully on the roots of the polynomial $f(x)$ for which $K$ is a splitting field

**Example 3.5.** The group of symmetries of an $n$-sided regular polygon $P$ (which is $D_n$) acts faithfully on $P$.

**Definition 3.6** (Orbits)**.** If $G \circlearrowleft X$ then the **orbit** of $x \in X$ is

$$Orb(x) = O(x) = G.x = \{y \in X \,|\, y = g.x,\, g \in G\}.$$

Orbits form equivalence relation on $X$ and we denote the quotient space $X/\sim$ by $X/G$. The equivalence relation is given as

$$x \sim y \iff \exists g \in G \,\text{such that}\, x = g.y,$$

or equivalently

$$x \in Orb(y) \iff \exists g \in G : x = g.y.$$

**Exercise 3.7.** Show the above is indeed an equivalence relation and that distinct orbits are disjoint.

**Example 3.8.** The group of $\frac{\pi}{4}$ rotations $G := \langle \begin{pmatrix} \cos(\frac{\pi}{4}) & \sin(\frac{\pi}{4}) \\ -\sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{pmatrix} \rangle$ acts on $\mathbb{R}^2$. For $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ the orbit $Orb(v)$ is the set of eight roots of unity identified with their elements of $\mathbb{R}^2$, see Figure 1.

**Definition 3.9** (Stabilisers)**.** For a group action $G \circlearrowleft X$ the **stabiliser** of $x \in X$ is

$$Stab(x) = \{g \in G \,|\, g \cdot x = x\} \subseteq G,$$

is the subset of $G$ which fixes the element $x$.

**Example 3.10.** The group $G = \langle \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \rangle \simeq \mathbb{Z}$ acts on $\mathbb{R}^2$

if $b \neq 0$ and $v = \begin{pmatrix} a \\ b \end{pmatrix}$ $v$ then $Stab(v) = \left\{ g^n \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \right\}$ which, as $2^n b$ must be equal to $b$, is just the identity element

$$g^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The orbit of $v$ is given as

$$Orb(v) = \left\{ \begin{pmatrix} a \\ 2^n b \end{pmatrix} | n \in \mathbb{Z} \right\}.$$

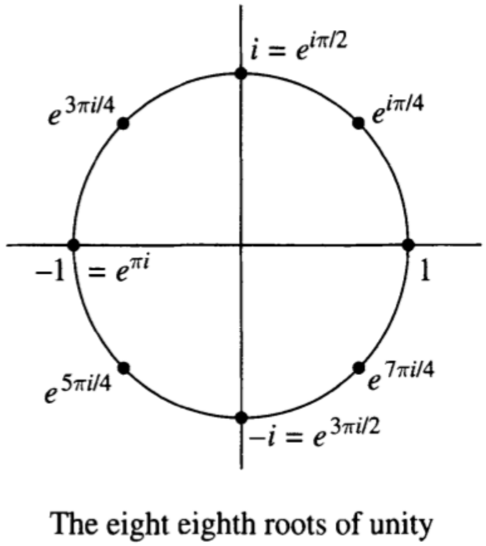


FIGURE 1. The orbit of $v = (1, 0)$ under the rotation by $\frac{\pi}{4}$.

**Theorem 3.11** (Orbit-Stabiliser)**.** $G \circlearrowleft X$ then

$$\alpha : G/Stab(x) \to Orb(x)$$

which is a bijection. Moreover, if G is a finite group $|G| < \infty$ then

$$|G| = |Stab(x)||Orb(x)|.$$

*Proof.* We want to show that the map: $gStab(x) \mapsto g.x$ is well defined and bijective. The map is well defined since if $a, b \in Stab(x)$ then:

$$ga.x = g.(a.x) = g.x = g.(b.x) = gb.x \ .$$

The map is surjective by definition and for the bijection part if $\alpha(gStab(x)) = \alpha(hStab(x))$ then we have:

$$\begin{aligned} &g.x = h.x \\ \iff &(h^{-1}g) \cdot x = x \\ \iff &h^{-1}g \in Stab(x) \ . \end{aligned}$$

Thus, $g = hs$ for $s \in Stab(x)$ and so $gStab(x) = hStab(x)$ ∎

**Lemma 3.12** (Burnside Lemma)**.** Let $G \circlearrowleft X$ be a group action and define $X^g = \{x \in X \mid g \cdot x = x\}$. Then there exists a bijection

$$G \times X/G \leftrightarrow \amalg_{g \in G} X^g$$

**Exercise 3.13.** Prove Burnside Lemma 3.12 for $|G| < \infty$ and find formula similar to the one in the orbit stabiliser.

**Definition 3.14.** Given a group action $G \circlearrowleft X$ a fundamental domain for $G \circlearrowleft X$ is a subset $F \subseteq X$ that contains exactly one point per orbit. Thus, we have

$$\forall x \in X \quad |\mathrm{Orb}(x) \cap F| = 1.$$

Equivalently, we can define it as the maximal set $F$ for which the projection $\pi : X \to X/G$ is injective.

We now consider certain examples of this.

**Example 3.15.** For our earlier considered group action generated by a $\frac{\pi}{4}$ rotation, we can consider $F$ as a wedge in the upper right quadrant (note this identification is not unique), that is

$$F = \{(x, y) \in \mathbb{R}^2 \ | \ 0 \leq x, 0 \leq y < x\}$$

**Remark 3.16.** An aside is that in most nice cases, there is a homeomorphism $\overline{F}/\!\sim \, \simeq X/G$.

We consider this identification in our next example

**Example 3.17.** Consider $\mathbb{Z} \circlearrowleft \mathbb{R}$ with the group action for any $n \in \mathbb{Z}$, $n \cdot x = x + n$. Here we have $\mathbb{Z} \cdot 0 \simeq \mathbb{Z}$ and we can choose $F = [0, 1)$, and with $\overline{F} = [0, 1]$ we can then say that

$$\mathbb{R} \,/\, \mathbb{Z} \simeq \frac{[0, 1]}{0 \sim 1}$$

a circle of length 1.

3.2. **Metric Spaces.**

**Definition 3.18.** A pair of a space $X$ and a map $d : X \times X \to \mathbb{R}_{\geq 0}$ together denoted $(X, d)$ is a metric space if

(1) $d(x, y) = d(y, x) \forall x, y \in X$
(2) $d(x, y) = 0 \iff x = y$
(3) $d(x, y) \leq d(x, z) + d(z, y) \forall x, y, z$, often called the triangle inequality.

**Example 3.19.** We list off some classic examples of metric spaces

- $(\mathbb{R}^n, || \cdot ||_2)$, where the (Euclidean) norm is $||x - y||_2 = \sqrt{\sum_i (x_i - y_i)^2}$
- $(\mathbb{R}^n, || \cdot ||_1)$, the taxicab metric is $||x - y||_1 = \sum_i |x_i - y_i|$
- $(\mathbb{S}^1, d_\theta)$ where $d(x, y) = \alpha$ the metric induced on the unit circle in $\mathbb{R}^2$

We also note certain other properties, if $(X, d)$ is a metric space, so too is $(X, \frac{d}{d+1}$ and this new metric is bounded.

**Exercise 3.20.** If $(X, d)$ is a metric space so is $(X, f(d))$ if $f$ is $C^0, f(0) = 0$, concave and strictly increasing, to start show it for differentiable $f$ with strictly positive derivative and $f'' < 0$.

**Definition 3.21.** A map $f : X \to X$ is an isometry if $d(f(x), f(y)) = d(x, y)$

**Example 3.22.** Any rotation, translation or reflection along an axis is an isometry of $(\mathbb{R}^2, \|\cdot\|_2)$ as is the composition of any of these.

**Exercise 3.23.** For $f$ an isometry, show this map is necessarily injective. Show that an isometry is not necessarily surjective. (Hint: needs too have infinite cardinality for this to be possible). Show that if $f$ is an isometry on a finite metric space then it is a bijection.

**Exercise 3.24.** For two isometries $f, g : \mathbb{R}^2 \to \mathbb{R}^2$, show that if $f$ and $g$ coincide on $p_1 \neq p_2 \neq p_3$ not all on a line, then the two maps are equal. Do the same for $\mathbb{R}^n$ and $n+1$ points.

**Example 3.25.** Here are some classical examples of isometries:
- $O_n(\mathbb{R})$ are isometries of $(\mathbb{R}^n, ||\cdot||_2)$.
- Translations are isometries of $(\mathbb{R}^n, ||\cdot||_2)$.
- Reflections on hyperplanes in $\mathbb{R}^n$.
- The group of symmetris of a regular $n$-polygon $D_n$ acts by isometries of $\mathbb{R}^2$.

3.3. **Geodesics.** In this short section we want to discuss what straight lines are in general metric spaces. In $\mathbb{R}^n$ we have the usual notion of straight lines.

**Example 3.26.** Given $x \neq y$ in $\mathbb{R}^2$, there exists line connecting them. We denote by$\gamma : [0,1] \to \mathbb{R}^2, \gamma(0) = x, \gamma(1) = y$ a smooth parametrisation of the line segment connecting them. Then, its length can be computed as

$$\ell_{||\cdot||_2}(\gamma) = \ell(\gamma) = \int_0^1 \left\|\dot{\gamma}\right\|_2 dt,$$

we can define the length as such since the metric is given by an inner products, this works in general for Riemmanian metric spaces. For example with the parametrisation

$$\gamma(t) = x + t(y-x) \Rightarrow l(\gamma) = d_{||\cdot||_2}(x,y).$$

**Definition 3.27.** A Geodesic $\gamma$ is a **locally** length minimising path $\gamma : [a,b] \to (X,d)$. That, is $\forall t \in (a,b) \exists \varepsilon > 0$:

$$\gamma : (t-\varepsilon, t+\varepsilon) \to (X,d)$$

is length minimising. This can be equivalently stated that given $\gamma$ there exists a constant $v > 0$ such that for the above quantifiers for all $s_1, s_2 \in (t-\varepsilon, t+\varepsilon)$ we have $d(\gamma(s_1), \gamma(s_2)) = v\,|s_2 - s_1|$.

When we do not have a Riemannian space the length of a path can be confuted as follows.

**Definition 3.28.** Given $\gamma : I \to (X,d)$, define $\ell(\gamma) = \sup_P \sum_{t_i \in P} d(\gamma(t_i), (t_{i+1}))$ for $P$ all finite partitions of $I$.

**Example 3.29.** Here are some examples given by various metrics
- For $\mathbb{S}^2 \subseteq (\mathbb{R}^3, ||\cdot||)$, a geodesic is just an arc of a great circle. That is, it can be thought as a path $\gamma : I \to \mathbb{R}^3, ||\gamma||_2 = 1$ that lies on a maximal circle. Thus, between any two (non-antipodal) points there are two geoedesics paths (the two possible arcs of a great circle) and between two antipodal points there are infinitely many.

- The Graph Metric, we have to decide edge lengths that preserve the triangle inequality. But we get $\gamma : I \to (X,d), t \mapsto x$ is a geodesic

Most of the examples we have seen are called length spaces, i.e. metric spaces $(X,d)$ such that between any two points they are connected by a geodesic realising the distance.

**Definition 3.30.** A metric space $(X,d)$ is a **geodesic length space** if $\forall x, y \in X$ there exists a geodesic $\gamma_{x,y}$ realising $d(x,y)$. If we have that $d(x,y) = \inf_{\gamma : x \to y} \ell(\gamma)$ then we say it is a length space.

**Example 3.31.** The following are all length spaces:

- Euclidean space $(\mathbb{R}^n, ||\cdot||_2)$, in which straight lines are the length minimising geodesics.
- Certain metric graphs, in particular if every vertex has only finitely many edges.
- Spherical space $(\mathbb{S}^n, d_\theta)$ with arcs of great circles realising the distance, however points can have infinitely many minimising geodeisics connecting them.
- Hyperbolic space $(\mathbb{H}, d_{\mathbb{H}})$.

**Remark 3.32.** Most metric spaces are not length spaces, in general the unique paths a space has are constant paths. For example take any collection of points in $\mathbb{R}^n$ with the induced metric.

**Exercise 3.33.** Let $X$ be the graph with two vertices $x, y$ connected by infinitely many edges $e_n$, $n \in \mathbb{N}$, such that, metrically, $e_n = [0, 1 + \frac{1}{n}]$. Show that the induced metric graph is a lenght space but not a geodesic one.

3.4. **Some Topology.**

**Definition 3.34.** (X,d) a set $O \subseteq X$ is open if $\forall x \in$ O, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq O$
$F \subseteq X$ is closed if $F^c$ is open.
Given $\{x_n\} \subseteq X, \bar{x}$ is an accumulation point if $\exists \{x_{n_k}\} \subseteq \{x_n\}$, $d(x_{n_k}, \bar{x}) \to 0$ as $k \to \infty$

**Example 3.35.** $(\mathbb{R}^2, ||\cdot||_2)$, $x_n = \begin{cases} (\frac{1}{n}, 0), \text{n even} \\ (n, 0), \text{n odd} \end{cases}$

**Definition 3.36.** Given $S \subseteq (X,d), y \in X$ is an accumulation point for S if $\exists \{S_n\} \subseteq S \backslash \{y\}$, s.t. $S_n \to y$.
The set of accumulation points for S is the frontier or boundary, denoted $\partial S$ or (Fr(S)). The closure of S is $S \cup \partial S$, this is equivalent to taking the smallest closed superset of S.

**Theorem 3.37** (On Unions and Intersections)**.** If $O_i$ is open $\forall\, i \in I$, then $O = \underset{i \in I}{\cup} O_i$ is open. If $F_i$ is closed $\forall i \in I$, then $F = \underset{i \in I}{F}_i$ is closed.

**Example 3.38.** Here are examples of open sets

- for $(\mathbb{R}^2, ||\cdot||_2)$ $B_\gamma(x)$ is open by definition,

- the set (x,y): $0 < x, \log x < y < \log x$ for any $(x, y)$ we can find an $\varepsilon$- ball around it between the two graphs;
- $\mathbb{R}^2 \backslash \mathbb{Z}^2$ is open for the same reasoning as the one above.

Here some closed sets:

- the closed ball of radius $r$, $B_r(x) \coloneqq \{y \in X : d(x, y) \leq r\}$;
- (x,y): $0 < x, -e^{-x} \leq y \leq e^{-x}$ for any $(x, y)$ we can find an $\varepsilon$- ball around it between the two graphs and the boundary is given by the graphs themselves;
- the set of points $\big\{(\frac{1}{n}, 0, 0), (0, 0)\big\}$ in $\mathbb{R}^2$, this is closed since it contains the origin which is the unique accumulation point;
- the Cantor ternary set, see https://en.wikipedia.org/wiki/Cantor_set. The reasoning here is that it is the countable intersection of closed sets $F_n$'s, which are the intervals one gets at level $n$ of the construction, see Figure 2.

FIGURE 2. The first 7-iteration of the middle third cantor set construction.

Some sets that are neither open nor closed:

- $[0, 1) \subseteq \mathbb{R}$;
- the set of rational numbers in $\mathbb{R}$, $\bar{\mathbb{Q}} = \mathbb{R}$;
- the square $[0, 1] \times [0, 1)$ in $\mathbb{R}^2$.

**Definition 3.39.** We say that a map $f : (X, d) \to (Y, d')$ is a **homeomorphism** if it is continuous and admits a continuous inverse.

3.5. **Isometries of $\mathbb{R}^n$.**

**Definition 3.40.** $\mathbb{R}^n$ shall refer to any metric space isometric to $(\mathbb{R}^n, || \cdot ||_2)$
$\mathrm{Isom}(\mathbb{R}^n) = \text{Affine group} \cong \underset{\in O(n)}{A_n} + \underset{\in \mathbb{R}^n}{b}$