Complex Dynamics: Hubbard Tree and Entropy

Presentation for Warwick SUMR

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University of Warwick, Aug 2024

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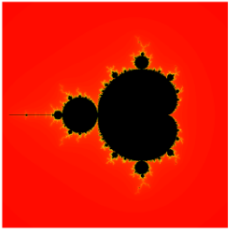
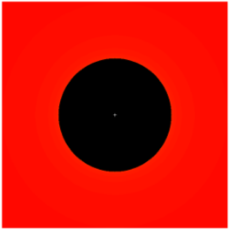
Introduction

Complex Dynamics Pictures: $f_c(z) = z^2 + c$



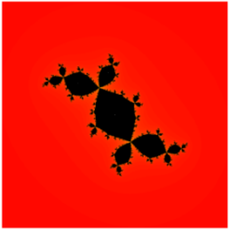
(A) Mandelbrot set



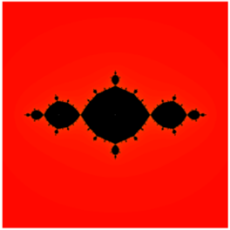
(B) $c = 0$

(C) $c = 0.295 + 0.06i$

(D) $c = -0.625 + 0.425i$



(E) $c \approx -0.122 + 0.745$



(F) $c = -1$

Fixed Points

We define a significant point: the fixed point and its basin of attraction:

Definition

Fixed point is a point that is mapped to itself by the function: $f(z) = z$

Definition

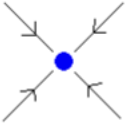
Basin of attraction of a fixed point z_0 is the set of all points z in the complex plane s.t: $lim_{n\to\infty} f^n(z) = z_0$, where f^n denotes the n-th iterate of f

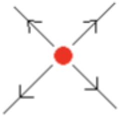
Definition

$\lambda = f'(z)$ is called the multiplier of f at fixed point z

We classify the fixed points according to λ as :

- repelling, if $|\lambda| > 1$ (red)
- neutral, if $|\lambda| = 1$
- attracting, if $|\lambda| < 1$ (blue)
- super-attracting, if $|\lambda| = 0$





Conjugation

Definition

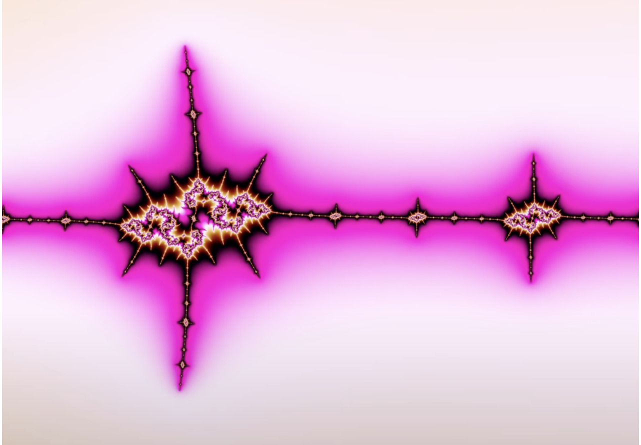
Two functions f and g are conjugate if there exists a homeomorphism (bijective, continuous, inverse continuous) ϕ such that $\phi(f(z)) = g(\phi(z))$

Boettcher's Theorem (super-attracting case)

Suppose f has a super-attracting fixed point at z_0:
$f(z) = z_0 + a_p(z - z_0)^p + ...,\ a_p \neq 0, p \geq 2.$
Then there is a conformal map $\gamma = \phi(z)$ of a neighborhood of z_0 onto a neighborhood of 0 which conjugates $f(z)$ to γ^p. The conjugating function is unique, up to multiplication by a $(p-1)$th root of unity.

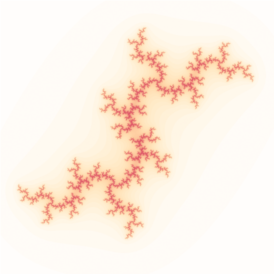
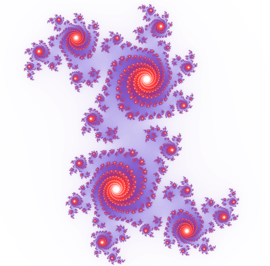
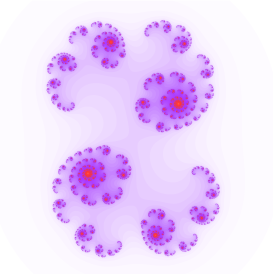
Julia set

What is a Julia set?



Julia set

Some more examples:

Julia set - First Method of Definition

Definition

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d > 1$. We call the filled-in Julia set of f the set K_f of points z such that $f^n(z) \not\to \infty$.

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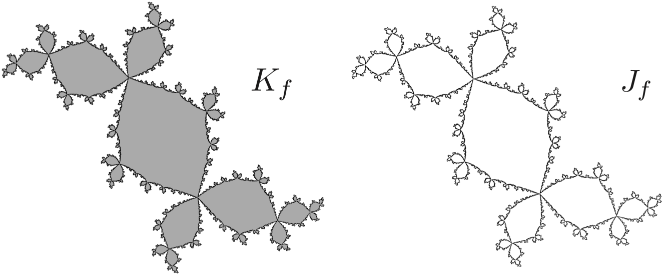

Figure: The filled-in Julia set and the Julia set of $f(z) = z^2 - 0.12256 + 0.74486i$

Julia set - Second Method of Definition

Definition

A family F is said to be normal on U if every sequence of functions f_n has a subsequence which **converges uniformly** on every compact subset of U.

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Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a polynomial, and let $f^n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be its n-fold iterate. Fixing some point $z_0 \in \widehat{\mathbb{C}}$, we have the following basic **dichotomy**:

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- If there exists some neighborhood U of z_0 so that the sequence of iterates $\{f^n\}$ restricted to U forms a normal family, then we say that z_0 belongs to the Fatou set F_f of f.

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- If there exists some neighborhood U of z_0 so that the sequence of iterates $\{f^n\}$ restricted to U forms a normal family, then we say that z_0 belongs to the Fatou set F_f of f.
- Otherwise, if no such neighborhood exists, we say that z_0 belongs to the Julia set J_f.
- Hence, $J_f = \widehat{\mathbb{C}} - F_f$.

Comparison of Fatou Set and Julia Set

Property	**Fatou Set F_f (or F)**	**Julia Set J_f (or J)**
Dynamics	Tame and predictable	Chaotic
Sensitivity	Not sensitive to small changes in initial conditions	Tiny perturbations lead to vastly different outcomes

Examples

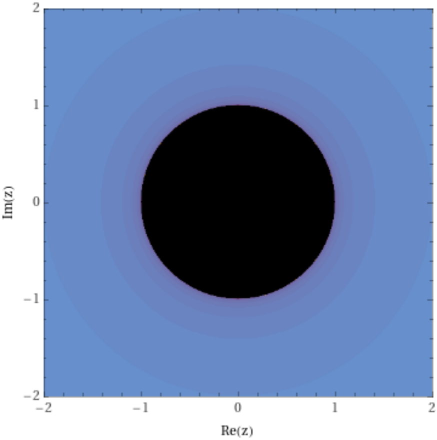


Figure: Julia set of $f(z) = z^2$

Examples

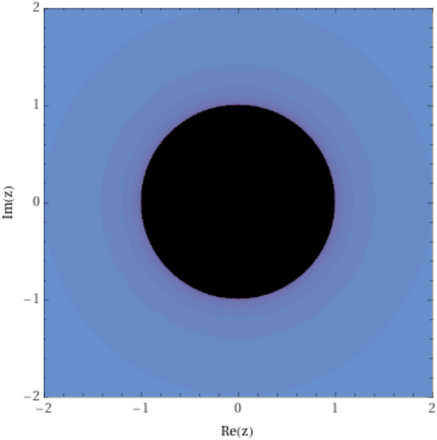
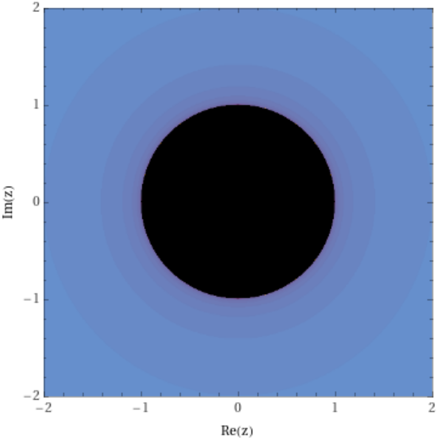


Figure: Julia set of $f(z) = z^2$

$f^n(z) = z^{2n}$
$|f^n(z)| \not\to \infty$ if and only if $|z| \leq 1$
$K_f = \overline{\mathbb{D}}$, $J_f = \mathbb{S}^1$

Examples



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$F = \{f^0, f^2, f^4, f^6, \ldots\}$
$F_f = \mathbb{D} \cup \{z : |z| > 1\}$
$J_f = \mathbb{S}^1$

Properties of Julia set

- Closed
- Bounded
- Compact
- Non-empty
- Completely invariant
  - $f(J) = J$
- Connectedness
  - J is **connected** if and only if the forward orbit of each finite critical point is bounded
  - If $f^n(q) \to \infty$ for each critical point q, then J is **totally disconnected**.

Hubbard tree: Motivation

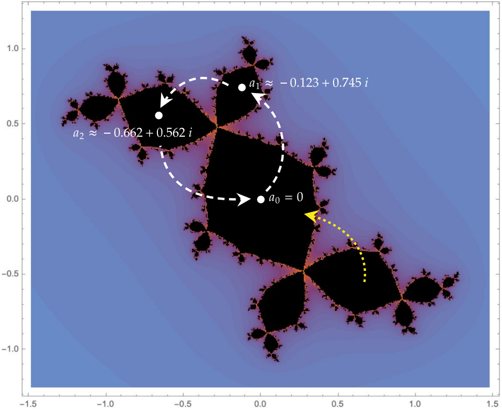
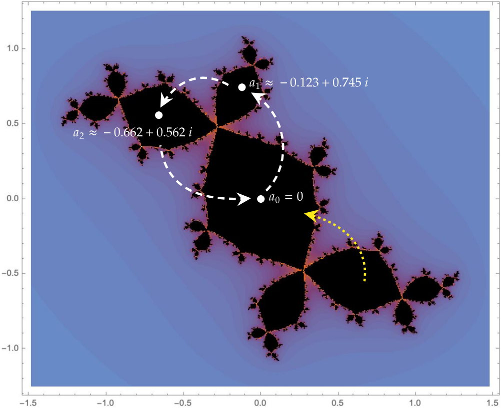


Figure: A filled Julia set and orbit of critical point

How f acts on J_f or K_f?

Hubbard tree: Motivation



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Too complicated...
We want a simpler, discrete invariant instead.

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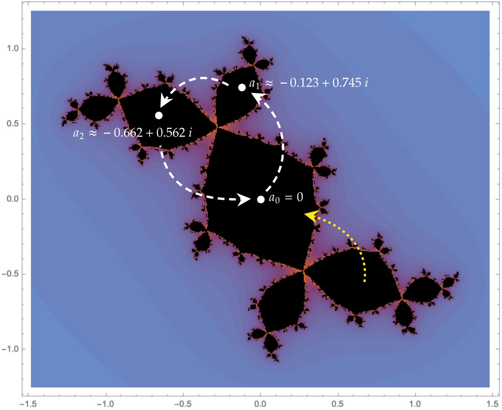


Figure: A filled Julia set and orbit of critical point

How f acts on J_f or K_f?

Too complicated...
We want a simpler, discrete invariant instead.

- the Hubbard tree!

PCF polynomials

Definition

A polynomial f is **postcritically finite (PCF)** if the postcritical orbit

$$\mathrm{PCO}_{\geq 0} = \bigcup_{z, f'(z)=0} \bigcup_{n \geq 0} \{f^n(z)\}$$

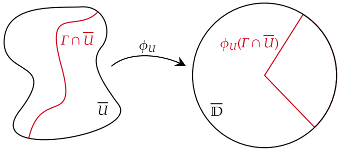
is a finite set.
Equivalently, each critical point of f is periodic or preperiodic.

Examples. For $f_c(z) = z^2 + c$, the only critical point is $z = 0$.

f_{-1}: $0 \mapsto -1 \mapsto 0 \mapsto \ldots$, periodic with period 2.
f_{-2}: $0 \mapsto -2 \mapsto 2 \mapsto 2 \mapsto \ldots$, preperiodic.
f_i: $0 \mapsto i \mapsto -1 + i \mapsto -i \mapsto -1 + i \mapsto \ldots$, preperiodic.

Allowable arcs

For each connected component U of $Int(K_f)$, we can find a biholomorphic $\phi_U : U \to \mathbb{D}$ by Riemann mapping theorem.
By Carathéodory's theorem, we can extend ϕ_U analytically to $\overline{U}$.
i.e. view $\overline{U}$ as the closed unit disk under a change of coordinates.



Definition

An arc $\Gamma \subset K_f$ is **allowable** if for any connected component U of $Int(K_f)$, $\phi_U(\Gamma \cap \overline{U})$ is in union of two rays in $\overline{\mathbb{D}}$.

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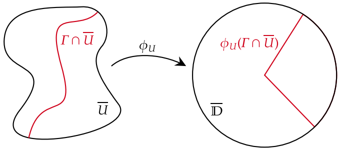

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An arc $\Gamma \subset K_f$ is **allowable** if for any connected component U of $Int(K_f)$, $\phi_U(\Gamma \cap \overline{U})$ is in union of two rays in $\overline{\mathbb{D}}$.

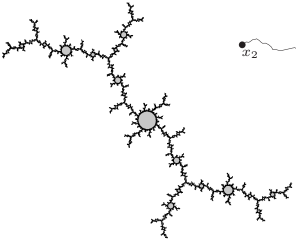
There is a **unique** allowable arc between any $x \neq y$ in K_f, denoted by $[x, y]_f$.

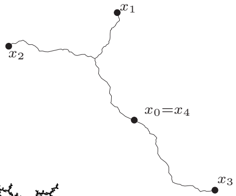
Hubbard trees

Definition

For a PCF f, the **Hubbard tree** H_f is the finite union of allowable arcs between distinct points in $\mathrm{PCO}_{\geq 0}$; i.e.

$$H_f = \bigcup_{x \neq y \in \mathrm{PCO}_{\geq 0}} [x, y]_f$$





Properties of Hubbard trees

- Indeed, one can show it is a **tree**.
- $f(H_f) \subset H_f$
- $f(V_f) \subset V_f$

Examples: the real case

Suppose PCF f is real and all critical points are real.
We can prove H_f consists of line segments and lies entirely in $\mathbb{R}$.

Examples: the real case

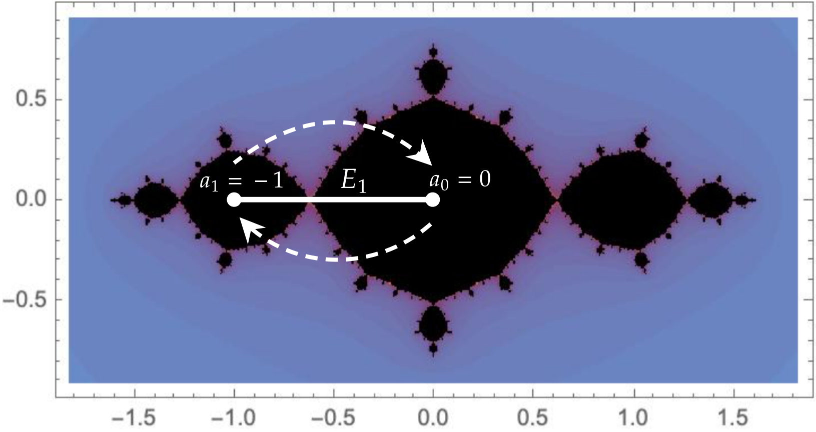


Figure: $f_{-1}(z) = z^2 - 1$

Examples: the real case

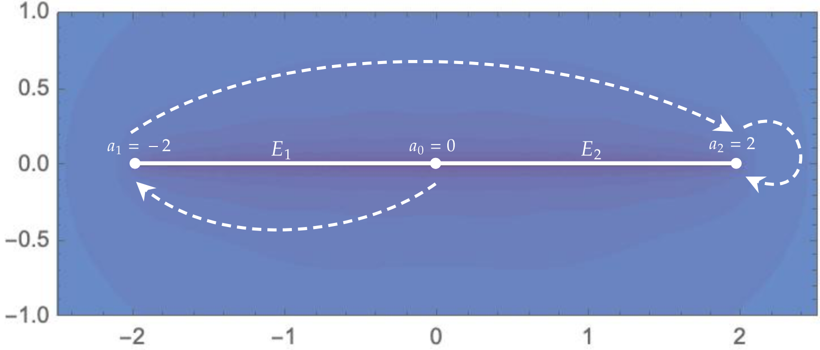


Figure: $f_{-2}(z) = z^2 - 2$

Two cases of PCF quadratics

For $f_c(z) = z^2 + c$, the only critical point is 0.

- 0 is periodic.
- 0 is preperiodic.

Example: the periodic case

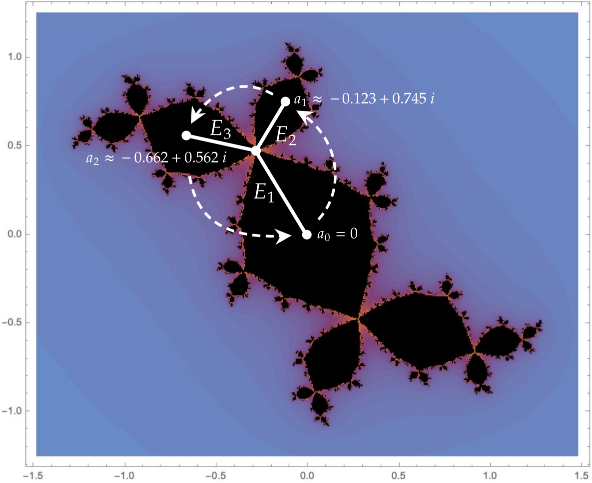


Figure: Julia set and Hubbard tree of f_c, $c \approx -0.123 + 0.745i$

0 has period 3.

- $Int(K_f) \neq \emptyset$.
- Superattracting cycle.

Example: the preperiodic case

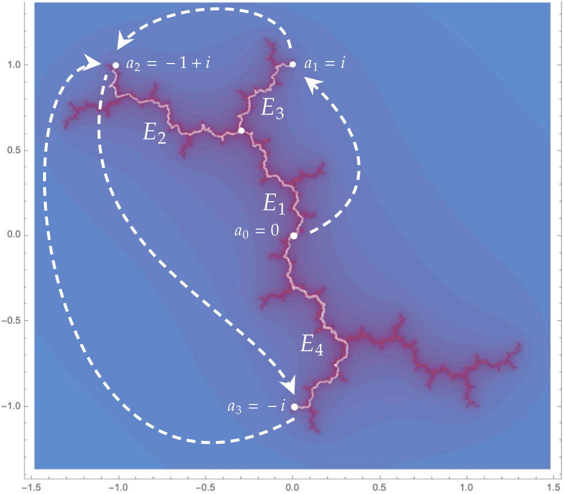


Figure: Julia set and Hubbard tree of $f_i(z) = z^2 + i$

- $Int(K_f) = \emptyset$.
- $H_f \subset J_f$.

Characterisation and realisation

[On Post Critically Finite Polynomials - Part Two: Hubbard Trees, Alfredo Poirier]

- Different conjugacy classes of PCF polynomials give rise to different Hubbard tree structures.
- Any expanding abstract tree can be realised as a Hubbard tree associated with a PCF polynomial (unique up to affine conjugation).

Entropy in Hubbard Trees

- **Definition of Entropy**
  - Measures the complexity of the dynamical system.
  - For Hubbard trees, it quantifies the exponential growth rate of distinct orbits.
- **Calculation of Entropy**
  - The forward orbit of the critical point, as well as the branched points of the tree, cut it into finitely many segments which form a **transition matrix** M.
  - Entropy $h = \log(\lambda_{\text{max}})$, where λ_{max} is the largest absolute value of an eigenvalue of M.

Example: Entropy of $f_i(z) = z^2 + i$

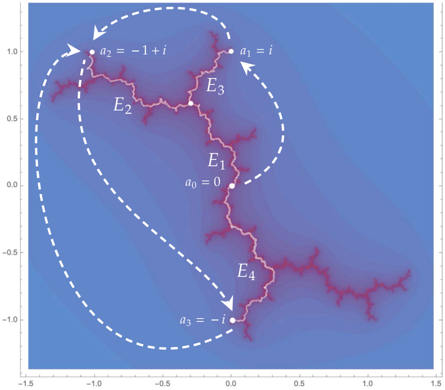


Figure: Hubbard Tree for $f_i(z) = z^2 + i$

Transition Relations:

- $E_1 \mapsto E_3$
- $E_2 \mapsto E_1 + E_4$
- $E_3 \mapsto E_2$
- $E_4 \mapsto E_2 + E_3$

Transition Matrix:

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Example: Entropy of $f_i(z) = z^2 + i$ (Continued)

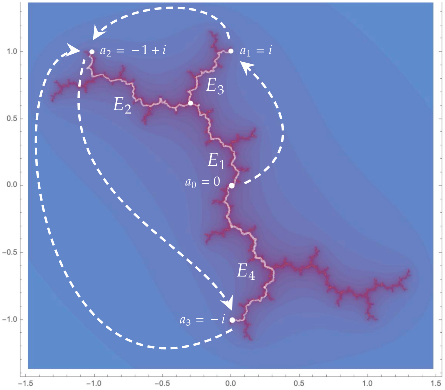


Figure: Hubbard Tree for $f_i(z) = z^2 + i$

Eigenvalues:

$\lambda_1 \approx 1.52138$

$\lambda_2 \approx -0.76069 + 0.857874i$

$\lambda_3 \approx -0.76069 - 0.857874i$

$\lambda_4 = 0$

Entropy Calculation:

$$h = \log(\lambda_1) \approx \log(1.52138) \approx 0.18223$$

Bounds on Entropy

We've proved these two bounds on entropy:

- $\lambda_{max} \geq 1$, and $h \geq 0$.
- For $J_f \not\subset H_f$, $h < \log d$, where $d = \deg f$.
- If $J_f = H_f$, $h = \log d$.

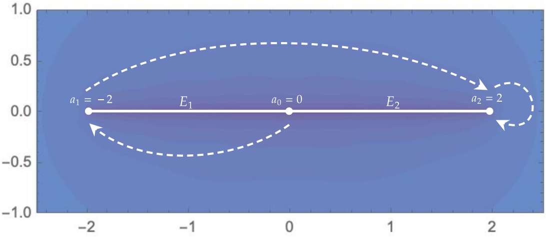


Figure: Hubbard Tree for $f_{-2}(z) = z^2 - 2$

Transition Relations:

- $E_1 \mapsto E_1 + E_2$
- $E_2 \mapsto E_1 + E_2$

Transition Matrix:

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Eigenvalues:

$$\lambda_1 = 2, \quad \lambda_2 = 0$$

Entropy Calculation:

$$h = \log(\lambda_1) = \log(2) = \log d$$