Perceptrons, Logical Functions, and the XOR problem

Deep Learning Pills #2

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Today we will explore what a Perceptron can do, what are its limitations, and we will prepare the ground to overreach these limits! Everything supported by graphs and code.

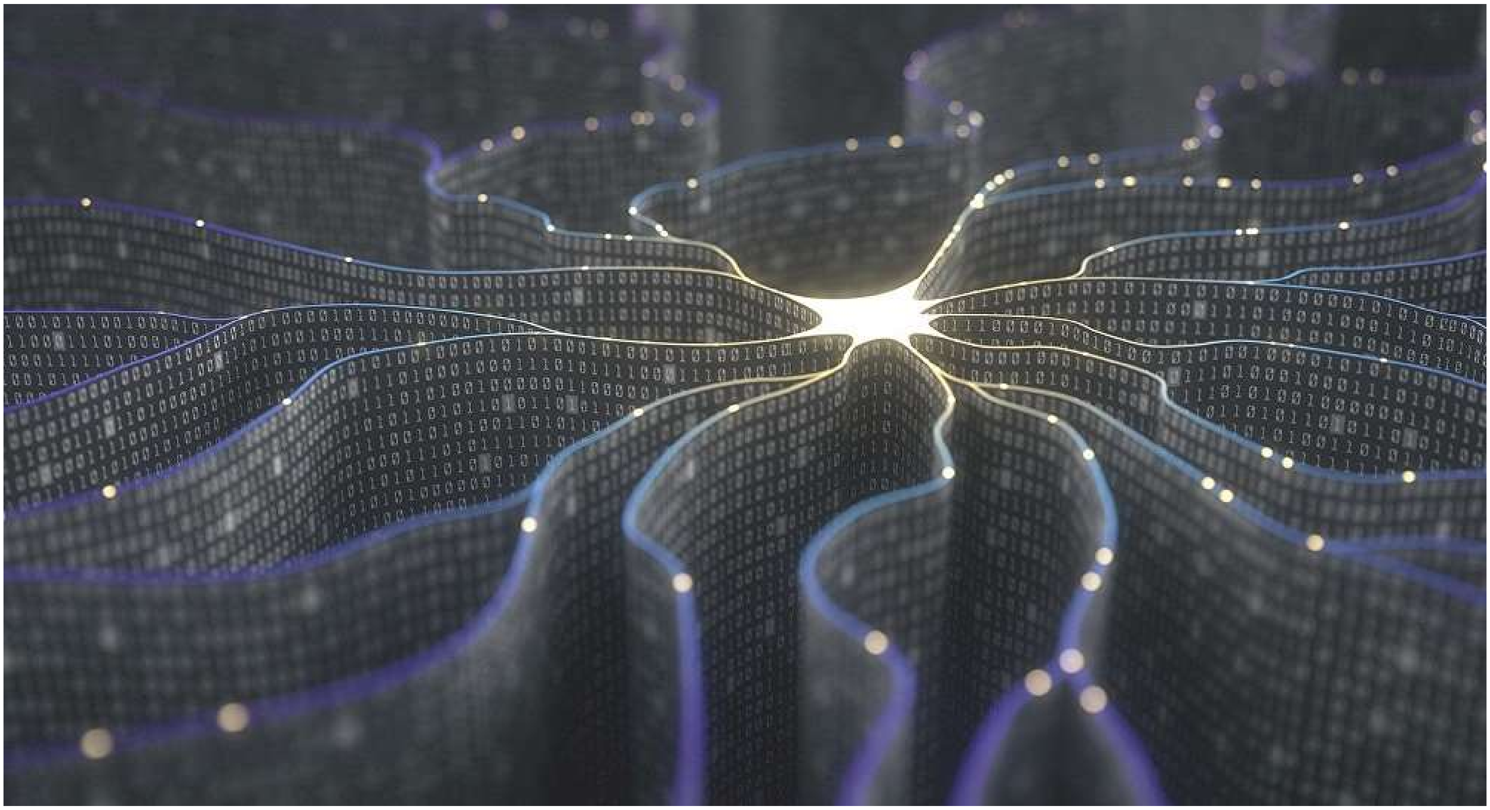
Elements from Deep Learning Pills #1

In Part 1 of this series, we introduced the Perceptron as a model that implements the following function:

$$\hat{y} = \Theta(w_1 x_1 + w_2 x_2 + \ldots + w_n x_n + b)$$
$$= \Theta(\mathbf{w} \cdot \mathbf{x} + b)$$

$$\text{where} \quad \Theta(v) = \begin{cases} 1 & \text{if } v \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

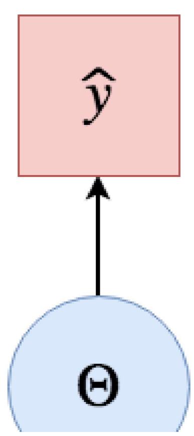
produced/predicted by the model. Soon, you will appreciate the ease of this notation.

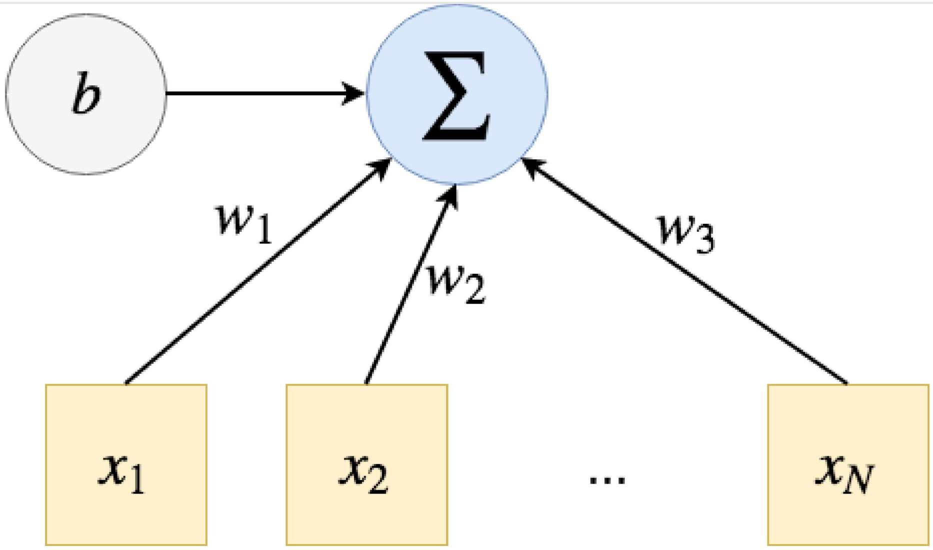


Computational Graph

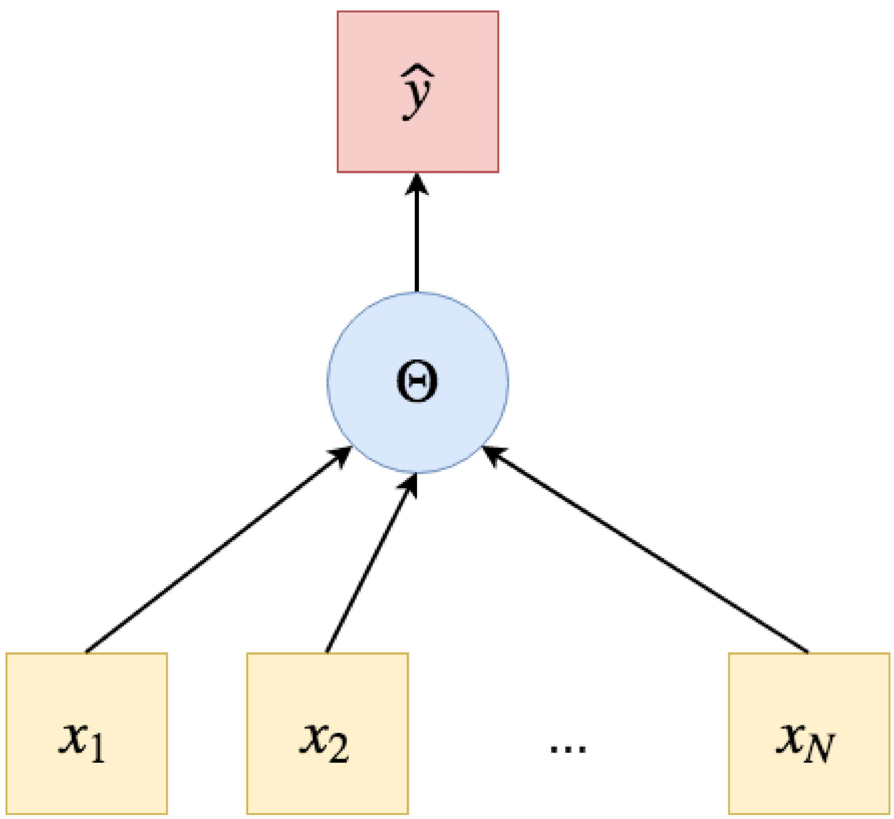
To visualize the **architecture** of a model, we use what is called **computational graph**: a directed graph which is used to represent a math function. Both variables and operations are nodes; variables are fed into operations and operations produce variables.

The computational graph of our perceptron is:





The Σ symbol represents the linear combination of the inputs ***x*** by means of the weights ***w*** and the bias *b*. Since this notation is quite heavy, from now on I will simplify the computational graph in the following way:



What can a perceptron do?

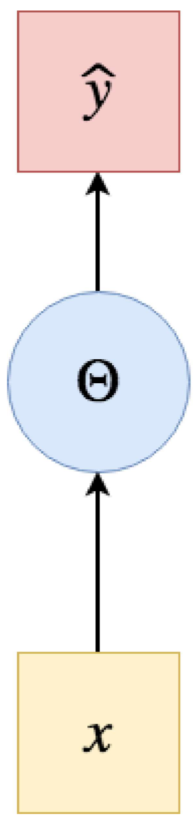
functions are a great starting point since they will bring us to a natural development of the theory behind the perceptron and, as a consequence, **neural networks**.

A	B	A AND B	A OR B	NOT A
False	False	False	False	True
False	True	False	True	True
True	False	False	True	False
True	True	True	True	False

NOT logical function

Let's start with a very simple problem:

> *Can a perceptron implement the NOT logical function?*



possible states: 0 and 1 (i.e., False and True): the Heaviside step function seems to fit our case since it produces a binary output.

With these considerations in mind, we can tell that, if there exists a perceptron which can implement the NOT(x) function, it would be like the one shown at left.
Given two parameters, *w* and *b*, it will perform the following computation:
$\hat{y} = \Theta(wx + b)$

The fundamental question is: do exist two values that, if picked as parameters, allow the perceptron to implement the NOT logical function? When I say that a *perceptron implements a function,* I mean that for each input in the function's domain the perceptron returns the same number (or vector) the function would return for the same input.
Back to our question: those values exist since we can easily find them: let's pick $w = -1$ and $b = 0.5$.

```
import numpy as np

def unit_step(v):
        """ Heavyside Step function. v must be a scalar """
        if v >= 0:
                return 1
        else:
                return 0

def perceptron(x, w, b):
    """ Function implemented by a perceptron with
                weight vector w and bias b """
        v = np.dot(w, x) + b
        y = unit_step(v)
        return y

def NOT_percep(x):
        return perceptron(x, w=-1, b=0.5)

print("NOT(0) = {}".format(NOT_percep(0)))
print("NOT(1) = {}".format(NOT_percep(1)))
```

not.py hosted with ❤ by GitHub view raw

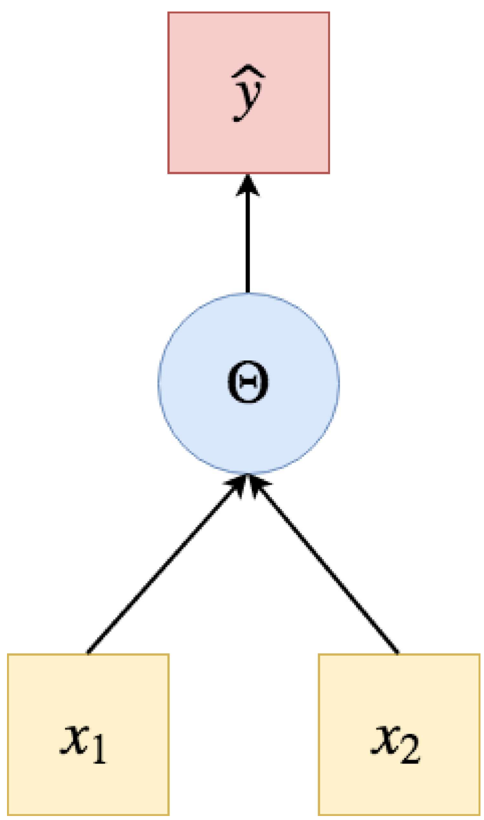
```
NOT(0) = 1
NOT(1) = 0
```

We conclude that the answer to the initial question is: yes, a perceptron can implement the NOT logical function; we just need to **properly set its parameters**. Notice that my solution isn't unique; in fact, solutions, intended as (w, b) points, are infinite for this particular problem! You can use your favorite one ;)

AND logical function

The next question is:

> *Can a perceptron implement the AND logical function?*



The AND logical function is a 2-variables function, *AND(x1, x2)*, with binary inputs and output.

This time, we have three parameters: *w1*, *w2,* and *b*.
Can you guess which are three values for these parameters which would allow the perceptron to solve the **AND problem**?

SOLUTION:
w1 = 1, w2 = 1, b = -1.5

```
def AND_percep(x):
    w = np.array([1, 1])
    b = -1.5
    return perceptron(x, w, b)

# Test
example1 = np.array([1, 1])
example2 = np.array([1, 0])
example3 = np.array([0, 1])
example4 = np.array([0, 0])

print("AND({}, {}) = {}".format(1, 1, AND_percep(example1)))
print("AND({}, {}) = {}".format(1, 0, AND_percep(example2)))
print("AND({}, {}) = {}".format(0, 1, AND_percep(example3)))
print("AND({}, {}) = {}".format(0, 0, AND_percep(example4)))
```

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And it prints:

```
AND(1, 1) = 1
AND(1, 0) = 0
AND(0, 1) = 0
AND(0, 0) = 0
```

OR logical function

OR(x1, x2) is a 2-variables function too, and its output is 1-dimensional (i.e., one number) and has two possible states (0 or 1). Therefore, we will use a perceptron with

SOLUTION:

w1 = 1, w2 = 1, b = -0.5

```python
def OR_percep(x):
    w = np.array([1, 1])
    b = -0.5
    return perceptron(x, w, b)

# Test
example1 = np.array([1, 1])
example2 = np.array([1, 0])
example3 = np.array([0, 1])
example4 = np.array([0, 0])

print("OR({}, {}) = {}".format(1, 1, OR_percep(example1)))
print("OR({}, {}) = {}".format(1, 0, OR_percep(example2)))
print("OR({}, {}) = {}".format(0, 1, OR_percep(example3)))
print("OR({}, {}) = {}".format(0, 0, OR_percep(example4)))
```

or.py hosted with ❤ by GitHub view raw

```
OR(1, 1) = 1
OR(1, 0) = 1
OR(0, 1) = 1
OR(0, 0) = 0
```

XOR — ALL (perceptrons) FOR ONE (logical function)

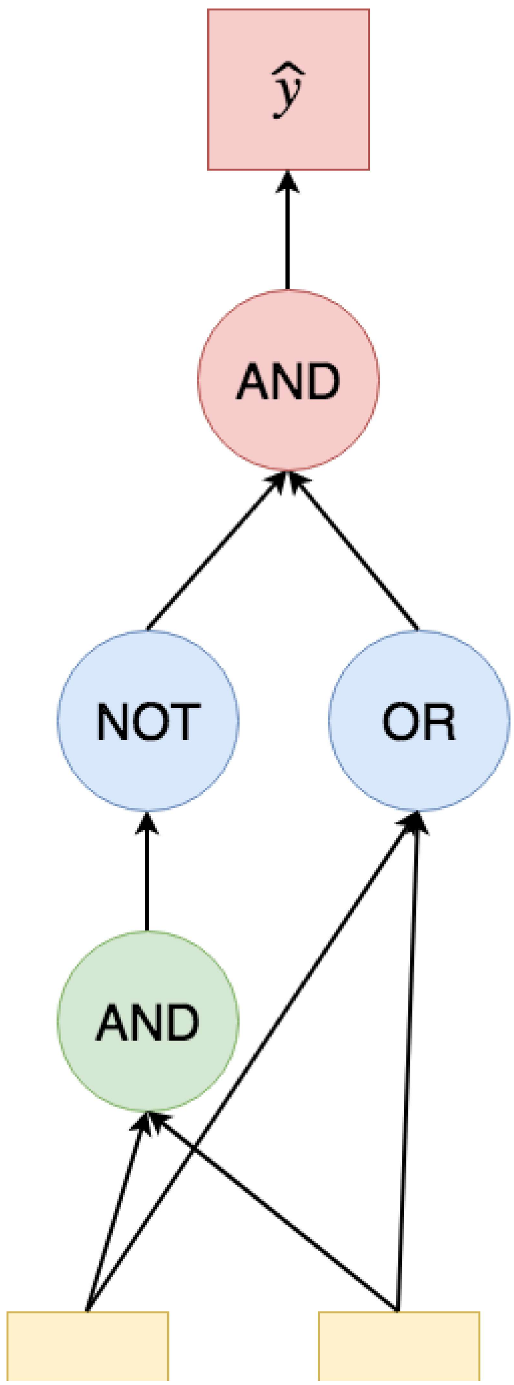
We conclude that a single perceptron with an Heaviside activation function can implement each one of the fundamental logical functions: NOT, AND and OR. They are called *fundamental* because any logical function, no matter how complex, can be obtained by a combination of those three. We can infer that, **if we appropriately connect the three perceptrons we just built, we can implement any logical function!** Let's see how:

> *How can we build a network of **fundamental logical perceptrons** so that it implements the XOR function?*

0	1	1
1	0	1
1	1	0

SOLUTION:

$$XOR(x1, x2) = AND(NOT(AND(x1, x2)), OR(x1, x2))$$



```python
def XOR_net(x):
    gate_1 = AND_percep(x)
    gate_2 = NOT_percep(gate_1)
    gate_3 = OR_percep(x)
    new_x = np.array([gate_2, gate_3])
    output = AND_percep(new_x)
    return output

print("XOR({}, {}) = {}".format(1, 1, XOR_net(example1)))
print("XOR({}, {}) = {}".format(1, 0, XOR_net(example2)))
print("XOR({}, {}) = {}".format(0, 1, XOR_net(example3)))
print("XOR({}, {}) = {}".format(0, 0, XOR_net(example4)))
```

xor_net.py hosted with ❤ by GitHub view raw

And the output is:

```
XOR(1, 1) = 0
XOR(1, 0) = 1
XOR(0, 1) = 1
XOR(0, 0) = 0
```

These are the predictions we were looking for! We just combined the three perceptrons above to get a more complex logical function.

Some of you may be wondering if, as we did for the previous functions, it is possible to find parameters' values for a single perceptron so that it solves the XOR problem all by itself.

I won't make you struggle too much looking for those three numbers, because it would be useless: the answer is that they do not exist. **Why?** The answer is that the XOR problem is not **linearly separable**, and we will discuss it in depth in the next chapter of this series!

perspective. In this way, every result we obtained today will get its natural and intuitive explanation.

If you liked this article, I hope you'll consider to give it some claps! Every clap is a great encouragement to me :) Also, feel free to get in touch with me on Linkedin!

See you very soon,
Frank

Machine Learning Artificial Intelligence Deep Learning Data Science